Optimal Auction Design under Non-Commitment

Vasiliki Skreta*

University of Minnesota &

Federal Reserve Bank of Minneapolis

skreta@econ.umn.edu

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Abstract

This paper characterizes the optimal auction in a two-period model under non-commitment. In the first period, a risk-neutral seller designs a mechanism to sell an indivisible object. If no trade takes place, the seller cannot commit not to try to sell the object in the second period. Assuming independent private values and risk neutral buyers we show that the seller can implement a revenue maximizing allocation rule by running a 'Myerson' auction with buyer-specific cutoffs in each period. A buyer can either claim a type above his/her cut-off or claim the lowest possible type. If no buyer claims a value above his/her cutoff, no trade takes place in the first period, and the seller runs a 'Myerson' auction in the second period with lower cutoffs. If the buyers are ex-ante symmetric, this rule can be implemented by a sequence of second or first price auctions with a reservation price in each period. The reservation price decreases overtime. The paper also develops a general procedure to characterize the optimal dynamic incentive schemes under non-commitment in asymmetric information environments with multiple agents, when types are drawn from a continuum. *Keywords: mechanism design, optimal auctions, sequential rationality. JEL Classification Codes: C72, D44, D82.*

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1 Introduction

The classical work on optimal auctions (see Myerson (1981) and Riley and Samuelson (1981)) characterizes the revenue-maximizing allocation mechanism for a risk-neutral seller who owns one object and faces a fixed number of buyers whose valuation is private information. An important assumption in these papers is that the seller can commit to withdraw the item from the market in the event that it is not sold.[1] In other words, the seller is free to employ any mechanism to sell the object but once it is determined the seller should respect it forever. The assumption that the seller can commit to the outcome of the mechanism is far-fetched and often not met in reality. Auction houses very seldom remove from the market items that remain unsold. For instance, Christies in Chicago auctions the same bottles of wine that failed to sell in earlier auctions. The US government re-auctions properties that fail to sell: lumber tracts, oil tracts and real estate are put up for a new auction if no bidder bids above the reserve price.[2] As Porter (1995) reports, 46.8 percent of the oil and gas tracts with rejected high bids were put up for a new auction. The mean time elapsed between the first and the second auction is 2.7 years.

The inability of a seller to commit to a given institution in the event that it fails to realize all gains of trade, has been studied extensively in the durable good monopolist literature, (Bulow (1982), Gul-Sonnenschein and Wilson (1986), Stokey (1981)), and by a more recent paper in an auction set-up by McAfee and Vincent (1997). A crucial assumption in these papers is that the seller's action in each stage is restricted to be out of a specific class. The seller chooses prices in the durable goods case, and reservation prices in the paper by McAfee and Vincent. It is interesting and relevant to investigate what is the optimal procedure among all possible ones. This paper obtains a characterization in the following environment.

There is a risk neutral seller who owns a single object and faces I risk neutral buyers, whose valuation is private information and is drawn from a continuum. Moreover valuations are private and independently distributed across buyers. There are 2 periods and the buyers and the seller discount the future with the same discount factor. At the beginning of each period the seller proposes a mechanism to sell the object. If the object is sold at the end of the first period, the game ends otherwise the seller returns the next period and offers a new mechanism. The game ends after 2 periods even if the object remains unsold. This is the simplest possible environment that allows us to examine the effect of having limited commitment on the optimal mechanism. We show that the seller will maximize expected discounted revenue by running at t=1 a 'Myerson' auction with buyer-specific cutoffs. A 'Myerson' auction assigns the object to the buyer with the highest virtual valuation if it is above a cut-off. A buyer can either claim a type above his/her

[1] No sale is not the only form of inefficiency of the classical optimal auction. Sometimes is allocates the object inefficiently, thus leaving open resale opportunities for the new owner. A recent paper by Zheng (2002) studies optimal auctions optimum given resale.

[2] These examples are also mentioned in McAfee and Vincent (1997).

cut-off or claim the lowest possible type. If no bidder claims a value above his/her cut-off, no trade takes place in the first period and the seller runs a 'Myerson' auction in the second period.

This is the first work that studies a mechanism design problem under non-commitment in a multi-agent asymmetric information environment. Mechanism design under non-commitment is notoriously difficult even in single agent environments. In our multi-agent environment the main difficulties are two.

The first difficulty is due to the fact that one cannot use the revelation principle. Our first insight is a general method to characterize the optimal mechanism in environments where the revelation principle is of no help. How can one characterize the optimal mechanisms, when one potentially needs to consider mechanisms with arbitrary messages? The idea is to characterize equilibrium outcomes. Outcomes is all that matters for payoffs and in mechanism design we care primarily for payoffs. This method of solving dynamic mechanism design problems under non-commitment was developed in Skreta (2004). As the current paper illustrates, it can be used in multi-agent environments and potentially it can be used to characterize equilibrium outcomes of other solution concepts.

The second difficulty is specific to multi-agent environments. In those environments one has to address the possibility that the mechanism designer knows more about the agents, then the agents do about their opponents at the end of the first period. For instance, if the seller used a sealed-bid mechanism at $t = 1$ she has observed the bids of all the buyers, but buyer i has not observed the bids of his opponents. In other words, the seller at $t = 2$ becomes an *informed principal*, since she possesses information that is not available to the agents.[3] The seller at $t = 1$ controls to a very large extend the amount of information that is released to the buyers, since she can determine by the mechanism she employs how much more she will know relative to the buyers at $t = 2$. In other words, in our setup one of the seller's choices is to determine the optimal amount of information that should be revealed to the buyers at the beginning of the second period.[4] Moreover it is possible that the seller releases different information to different buyer. Then it is possible that at $t = 2$, the beliefs of buyer A about buyer B's valuation may be different from the ones of buyer C. That is, at $t = 2$ beliefs maybe part of the players private information. Also, the seller by revealing certain information may be able to introduce correlation in the buyers' beliefs about each other's valuation at the beginning of $t = 2$. In general institutions are not only characterized by the allocations

[3] There is some work on mechanism design by an informed principal by Maskin and Tirole (1990) and (1992). Those papers consider the *single agent* case in two scenarios, the one where the principal's private information does not affect the payoff of the agent and the case where is does. In our problem, the seller becomes informed endogenously, since she obtains her private information by interacting with the buyers in period $t = 1$. This information affects $i's$ expected payoffs, hence we are in a common value setting according to the terminology of Maskin and Tirole (1992). The question of mechanism design by an informed principal that faces multiple agents, is an important yet open and challenging question.

[4] This does not arise in a single agent framework under non-commitment since there the seller does never have more information than the buyer.

they lead to, but also by the amount of information they release to their participants. In order to capture these issues a mechanism in this environment consists of three objects: a game form, a communication device (mediator), whose purpose is to coordinate play, and an information disclosure policy whose purpose is to capture the degree of transparency of an institution. A buyer can always choose not participate in a mechanism if he wishes to do so.

The solution is characterized under the following assumptions (i) the seller at each stage observes the actions chosen by the buyers and whether trade took place or not, (ii) the distribution of valuations has a strictly positive and continuous density and satisfies the monotone hazard rate property, (iii) (Assumption O) the history where all buyers choose not to participate in the first-period mechanism becomes common knowledge (vi) a buyer chooses not to participate if indifferent, (v) buyers employ pure strategies.

It is very natural to assume that if all buyers walk away without participating in the mechanism that the seller proposed at $t = 1$, this is observed by everyone. This is the only restriction imposed on the information disclosure policy employed by the seller. In other words, we assume that the seller employs an information disclosure policy that makes public information the event that *all* buyers have rejected the mechanism that she proposed at $t = 1$, (Assumption O). Given this relatively minor restriction we characterized the revenue maximizing PBE. No other restrictions were imposed on the information disclosure policy. The seller can send whatever messages she wishes in all other circumstances, which can lead to a second period problem with correlation and with beliefs being private information. Moreover, no assumptions are made about whether the seller commits to tell the truth, or not, or reveal part of the truth via her information disclosure policy. We characterized the optimal mechanism without having to deal directly with all these complications by doing the following. We look at outcomes that arise from assessments where the players' strategy profile is restricted to be sequentially rational only after all buyers rejected the mechanism at $t = 1$. Clearly this set of outcomes contains all the ones that arise at a PBE, since at a PBE players behave sequentially rationally at each node, and hence after the history where all buyers rejected the first-period mechanism, M^1. We call this the set of conditionally sequentially rational outcomes, CSR(all reject). It turns out, that at the revenue maximizing outcome out of this larger set, trade will take place with probability 1 at $t = 1$ unless everybody rejects M^1. Put differently, the optimal outcome for the seller out of this larger set is a PBE outcome. In other words, we solved a program ignoring most of the sequential rationality constraints - we just required sequential rationality after one history - the one where everybody rejects M^1. It turned out that the optimum satisfies all constraints - since the only case that we move on to $t = 2$ is whenever everybody rejects M^1.

Apart from the optimal auction literature this work is related to the literature of dynamic mechanism design under non-commitment. The early papers on dynamic mechanism design, (Freixas, Guesnerie and Tirole (1985), FGT, Hart and Tirole (1988), Laffont and Tirole (1988), LT), establish that under non-commitment the principal cannot appeal to the standard revelation principle in order to characterize the optimal mechanism. This makes the characterization of the optimal contract very difficult,[5] and is the main reason why the research on mechanism design in dynamic settings under non-commitment has not progressed much. For this reason, FGT (1985) characterize the optimal incentive schemes among the class of *linear* incentive schemes. LT (1988) consider arbitrary schemes but examine only special classes of equilibria, namely pooling and partition equilibria. A remarkable result is derived in a recent paper by Bester and Strausz (2001), BS. They show that when the principal faces *one* agent whose type space is *finite,* she can, without loss of generality, restrict attention to mechanisms where the message space has the same cardinality as the type space. As BS illustrate, in order to find the optimal mechanism one has to check which incentive compatibility constraints are binding. In an environment with limited commitment, constraints may be binding 'upwards' and 'downwards'. Even if one could obtain an analog of the BS result for the continuum type case, it does not seem straightforward to generalize the procedure of checking which incentive compatibility constraints are binding. Moreover, as Bester and Strausz (2000) report, their version of the revelation principle for environments of limited commitment does *not* extend to the case that the principal faces *many* agents. This paper provides a method that does.

The paper is structured as follows. The environment under consideration is described in Section 2. Section 3 outlines our method for characterizing the optimal mechanism under non-commitment. The main analysis and results of this work can be found in Section 4, which is the core of the paper. Section 5 illustrates how to calculate the optimal auction. Concluding remarks are in Section 6. All proofs that are not in the main text can be found in the Appendix.

2 The Environment

A risk neutral seller owns a unit of an indivisible object, and faces I risk neutral buyers. The seller's valuation for the object is zero, whereas that of buyer i is drawn from and interval $[a_i, b_i]$ according to a strictly positive and continuous density f_i. A buyer's valuation v_i is private and independently distributed across buyers. Time is discrete and the game lasts two periods, $t = 1, 2$. The buyers and the seller discount the future with the same discount factor δ. All elements of the game except the realization of the buyers' valuations are common knowledge. The seller's goal is to maximize expected discounted revenue. The buyers aim to maximize expected surplus.

[5] See the discussion in Laffont and Tirole (1993), Ch. 9, and Salanie (1998), Ch.6.

Notation

$$\begin{aligned}
V_i &= [a_i, b_i],\ i \in I,\ \text{denotes the set of buyer } i's \text{ all possible valuations} \\
V &= V_1 \times V_2 \times ... \times V_I,\ \ \text{denotes the set} \\
&\quad \text{of all possible vectors of valuations of all the buyers} \\
V_{-i} &= V_1 \times .. \times V_{i-1} \times V_{i+1} ... \times V_I,\ \text{stands for the set of} \\
&\quad \text{all possible vectors of valuations of } I \backslash \{i\}. \\
v &= (v_1, v_2, ..., v_I),\ \text{denotes a vector of valuations of all the buyers} \\
v_{-i} &= (v_1, v_2, ..., v_{i-1}, v_{i+1}, .., v_I),\ \text{denotes a vector of valuations of } I \backslash \{i\}. \\
f &= f_1 \times f_2 \times ... \times f_I,\ \text{denotes the joint pdf of } v \text{ on } V. \\
f_{-i} &= f_1 \times .. \times f_{i-1} \times f_{i+1} ... \times f_I,\ \text{denotes the joint pdf of } v_{-i} \text{ on } V_{-i}.
\end{aligned}$$

Timing

$t = 1$

seller proposes a "mechanism" - buyers choose their actions- TRADE - Game ends - NO TRADE go to $t = 2$

$t = 2$

seller proposes a "mechanism"$-$ buyers choose their actions - Game ends

A mechanism, M^t, consists of a *game form*, a *communication system* (mediator) and an *information disclosure policy.*

Definition 1 *(Game Form) A* game form $G^t = (S^t, \gamma^t)$ *consists of a set of actions* $S^t = S_1^t \times S_2^t \times ... \times S_I^t$ *available to the buyers and an outcome function* $\gamma^t : S^t \to [0,1]^I \times \mathbb{R}^I$.

The outcome specified via $\gamma^t(s^t)$, where $s^t = (s_1^t, s_2^t, ..., s_I^t)$, is a probability that each buyer obtains the good, $r^t(s^t) = [r_1(s^t), r_2(s^t), ..., r_I(s^t)]$, such that $\sum_{i=1}^I r_i^t(s^t) \leq 1$ and $r_i^t(s^t) \geq 0$ and an expected payment for each buyer $z^t(s^t) = [z_1^t(s^t), z_2^t(s^t), ..., z_I(s^t)]$.

Definition 2 *(Communication System) Let* B_i^t *denote the set of reports buyer* i *can send into the communication system and* N_i^t *set of messages buyer* i *can receive from the communication system. A communication system is a maps a vector of reports of the buyers to a vector of messages.*

The purpose of a communication system is to coordinate play

Example: Buyer 1: β_1 =my valuation is low; Buyer 2: β_2 =my valuation is 5, $n_1(\beta_1, \beta_2)$ =bid 1, $n_2(\beta_1, \beta_2)$ =bid 4.

Definition 3 *(Information Disclosure Policy) A information disclosure policy is a mapping from the vector of actions chosen by the buyers to a vector of messages, one for each buyer, or* $D^t : S^t \to \Delta(\Lambda^t)$ *where* $\Lambda^t := \times_{i \in K} \Lambda_i^t$ *and* Λ_i^t *is the set of messages that the seller can send to buyer* i*.*

Purpose: Capture that two different mechanisms differ in amount of information they release to participants. In a multi-agent dynamic problem under non-commitment, an important feature of an institution is the amount of information agents acquire from their interaction at $t = 1$, since information will affect their beliefs about their opponents, which will in turn affect their future interaction $t = 2$. And two different mechanisms that implement the same allocation may release different amounts of information to their participants. For instance, in the case of symmetric buyers, the symmetric equilibrium of a second price sealed bid auction and the symmetric equilibrium of an English auction will allocate the good in the same way, but in the SPA buyers observe only who won the object, whereas in an English Auction participants observe the drop-out prices of everyone. Information revelation at $t = 1$ may be very important for the interaction of the buyers at $t = 2$ and consequently for the revenue that the seller can expect to extract.

A buyer can always choose not to participate in a mechanism. We model this by assuming that every game form that the seller can propose, contains an action $s_i = 0_i$ for all i such that

$$r_i(0_i, s_{-i}) = 0 \text{ and } z_i(0_i, s_{-i}) = 0 \text{ for all } s_{-i}.$$

If a buyer chooses 0_i he does not get the object and he does not pay anything no matter what the other buyers do. For instance if the mechanism is a FPA with a reserve price, then submitting a bid below the reserve price implies that no matter what your opponents do, you will not get the object and will not pay anything.

Definition 4 *We say that buyer* i rejects M^1 *if he chooses* 0_i *at* $t = 1$*.*

Let $\mathcal{M}$ denote the set of all possible mechanisms. The seller's information set in period t is identified with an element of H_S^t, where H_S^t is the set of all feasible histories at date t. Similarly, buyer i's information set is an element of $H_{B(i)}^t$. An element of H_S^t, $(H_{B(i)}^t)$, is denoted by h_S^t, (h_i^t). A strategy for the seller, σ_S, consists of a sequence of maps from H_S^t to $\mathcal{M}$. A *pure communication strategy* of buyer i, $\sigma_{B(i)}$, consists of sequence of 2 mappings: a mapping from $V^i \times H_{B(i)}^t$ to a report, B_i^t, and a mapping from $V^i \times H_{B(i)}^t \times N_i^t$ to an action. The set of feasible actions for a buyer at $t = 1, 2$ is determined by the mechanism that the seller offers. We use σ_B to denote the strategies used by all buyers that is, $\sigma_B = (\sigma_B^1, \sigma_B^2, ..., \sigma_B^I)$. A strategy profile $\sigma = (\sigma_j)_{j=S,B}$, specifies a strategy for each player. A belief system, μ, maps H_S^t to the set of probability distributions over V.

We make the following assumptions:

(i) Buyers employ pure Strategies

(ii) The seller observes the *actions,* that is the vector s, chosen by the buyers and whether trade took place or not.

(iii) The history when all choose not to participate at $t = 1$ becomes common knowledge, (Assumption O).

(iv) A buyer does not participate if indifferent

(v) Assumption MHR. Each buyer's virtual valuation

$$v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}$$

is strictly increasing in v_i

Our aim is to characterize the maximum expected revenue that the seller can guarantee at a PBE. As usual we require that strategies yield a BNE, not only for the whole game, but also for the continuation game starting at each t after every history.

3 The Methodology

3.1 The revelation principle is of no help

As it is well known, the question in a mechanism design problem is to find the optimal, given a criterion, institution among all possible ones. This is quite a task because the analyst typically cannot even describe the set of all possible institutions. The revelation principle provides a parsimonious way to characterize the set of all social choice functions that can be implemented by a Bayes-Nash equilibrium of a game where the principal's strategy space is the set of all possible mechanisms. It points out that this set is equal to the one that can be implemented at a Bayes-Nash equilibrium of the simpler version of the original game where the principal's strategy space is the set of direct mechanisms The characterization provided is complete, in the sense that necessary and sufficient conditions for feasibility are provided. In a dynamic setup under non-commitment one is interested in what can be implemented in a PBE. For this solution concept there is no result analogous to the revelation principle that provides necessary and sufficient conditions for implementability. And, as it was realized in the earlier literature on mechanism design under non-commitment, (LT 1988), one cannot use the standard revelation principle in each period.

To see why, suppose that at period one the seller employs a direct revelation mechanism, buyers have claimed their true valuations, and according to this mechanism no trade takes place. If the seller behaves sequentially rationally, she will try to sell the object at $t = 2$ using a different mechanism. And in the case that the buyers have revealed their true valuations at $t = 1$, the seller has complete information at $t = 2$. She can therefore use this information to extract all the surplus from the highest valuation buyer. In this situation buyers will have an incentive to manipulate the seller's beliefs. One would expect that they will not always reveal their valuations truthfully at the beginning of the relationship. The seller, since she does not have commitment power, cannot play the role of the "machine" that exogenously specifies the direct revelation game that implements an equilibrium of some general game.

What we are interested in here is the set of social choice functions that can be implemented at a PBE of the game. (Clearly, this is a subset of the $BNE-$implementable social choice functions.) In particular, we are interested in the one that generates maximal revenue for the seller. One could obtain that by brute force: the mechanisms employed at $t = 2$ depend on the seller's posterior and the information that buyers have at that stage. Along the equilibrium path, posteriors are determined by Bayes rule from the buyers' strategies, the mechanism proposed by the seller, the actions chosen by the buyers, and the seller's information disclosure policy. There can be infinitely many vectors of choices at $t = 1$ that end up in no trade since lotteries are allowed. Each of these choices leads to a different posterior which determines the optimal period-2 mechanism. And at the optimum the mechanism at $t = 1$ must be optimally chosen taking into account not only revenue at $t = 1$ but also what beliefs the seller and the buyers will have after each history where there is no trade at $t = 1$, which in turn will determine the optimal mechanism for $t = 2$. This is very complicated. We choose to proceed indirectly by looking at outcomes of the whole game.

3.2 Allocation Rules and Payment Rules

Given a strategy profile, $(\sigma_S, \sigma_B^1, \sigma_B^2, ..., \sigma_B^I)$ and a belief system, μ, we can calculate for each realization of $v = (v_1, v_2, ..., v_I)$ the ex-ante probability that buyer i obtains the object, $p_i(\sigma, \mu)(v_i, v_{-i})$ and the ex-ante expected discounted payment is denoted by $x_i(\sigma, \mu)(v_i, v_{-i})$. This is the set of expected discounted outcomes of the game given (σ, μ).[6] The rule $p_i(\sigma, \mu)(v)$, sometimes abbreviated as p_i, maps V to probabilities, and denotes the expected, discounted probability that buyer i will obtain the object given (σ, μ) when the

[6] We need to include the belief system in the arguments of p and x because it is part of the equilibrium concepts we will examine.

realization of the buyers' valuations is v. We will call it *allocation rule.* It is formally defined as

$$p_i(\sigma,\mu)(v) = \sum_{t=1}^{2} \delta^{t-1} \left[1_{\{\text{trade with } i \text{ at } t\}} | (\sigma,\mu), v \right].$$

Allocation rules will play a central role in our analysis. It is possible that different strategy profiles lead to the same allocation rule. The rule $x_i(\sigma,\mu)(v)$, sometimes abbreviated as $x_i(v)$, maps V into $\mathbb{R}$ and we will call it *payment rule.* It is formally defined as

$$x_i(\sigma,\mu)(v) = \sum_{t=1}^{2} \delta^{t-1} \left[1_{\{\text{trade with } i \text{ at } t\}} \cdot \{i\text{'s expected payment at } t\} | (\sigma,\mu), v \right].$$

In the environment under consideration, a buyer is uncertain about the outcome of an action he chooses because it depends on the actions chosen by the other buyers. Given the actions of the other buyers, $i's$ action leads to an outcome which is a lottery; the outcome of a vector of actions - one action for each buyer - is a probability that each buyer obtains the object and an expected payment. When buyer i takes an action he does not know the actions of the other buyers, thus he can only determine a probability distribution over lotteries associated with each of his potential actions. These probability distributions depend on the other buyers' strategies and on $i's$ beliefs about their valuations. In order to obtain i's ex-ante payoff from his point of view we have to calculate the expectations of p_i and x_i over v_{-i} which are given by,

$$\begin{aligned} \bar{p}_i(\sigma,\mu)(v_i) &= \int_{V_{-i}} p_i(\sigma,\mu)(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} \\ \text{and } \bar{x}_i(\sigma,\mu)(v_i) &= \int_{V_{-i}} x_i(\sigma,\mu)(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i}. \end{aligned} \tag{1}$$

Then given σ_S, σ_B^{-i} buyer $i's$ expected discounted payoff from $\sigma_B^i(v_i)$ is given by

$$U^i_{\sigma,\mu}(v_i, \sigma_B^i(v_i)) = \bar{p}_i(\sigma,\mu)(v_i) v_i - \bar{x}_i(\sigma,\mu)(v_i).$$

Often we will omit (σ,μ) from the arguments of p_i, x_i, $\bar{p}_i$, $\bar{x}_i$, we just write for instance $p_i(v)$.

3.3 The Procedure

We start by examining the outcomes that arise from arbitrary strategy profiles. Then we examine how the structure of the outcomes of the game will be affected by restrictions on (σ,μ) dictated by a solution concept, in our case PBE. In other words, a solution concept imposes restrictions on (σ,μ), which in turn translate to restrictions on p and x. Our objective is to characterize outcomes arising from assessments, (σ,μ), that consist Perfect Bayesian Equilibria of this game and choose the one that maximizes revenue for the seller. We first look at the restrictions imposed on the allocation rule p by requiring (σ, μ) to be a Bayes-Nash Equilibrium, BNE, of the game. We show that if σ_B^i is a best response to σ_S and to σ_B^{-i}, then the expectation of p over v_{-i} , that is $\bar{p}_i$, is increasing in v_i, and we can write the seller's expected revenue as a linear function of p. Our objective then reduces to identifying a PBE, that implements an allocation rule p^* which maximizes expected discounted revenue among all allocation rules implemented by a PBE of the game. But the set of PBE implementable allocations is very difficult to characterize because beliefs may be private information at $t = 2$. What we do is to focus on a superset of these allocations. We look at allocations implemented by strategy profiles where the seller behaves sequentially rationally at $t = 2$ only after the history where all buyers rejected the first period mechanism. This is the set of conditionally sequentially rational allocations at the history all reject, CSR(all reject).[7] It is relatively straightforward to characterize this set and it turns out that the optimal allocation among CSR(all reject) is implemented by an assessment that is a PBE.

4 The Optimal Mechanism Under Non-Commitment

4.1 Necessary Conditions at a BNE

Our goal is to investigate the properties of allocation rules that are implemented by a Perfect Bayesian Equilibrium, PBE. In this subsection we look at the restrictions on p imposed by requiring (σ, μ) to be a Bayes-Nash Equilibrium, BNE, of the game. In the following subsection we will study implications of the requirement that (σ, μ) yields a BNE for the continuation game starting at each t after any history, (sequential rationality).

A PBE is a BNE so first we look at Necessary Conditions at a BNE.

Lemma 1 ***(Myerson (1981)).*** *If p, x are implemented by a BNE the following conditions must hold:*

(i) $\bar{p}_i(v_i)$ is increasing in v_i
(ii) $U^i_{\sigma,\mu}(v_i, \sigma_{B(i)}(v_i)) = \int_{a_i}^{v_i} \bar{p}_i(s_i)ds_i + U^i_{\sigma,\mu}(a_i, \sigma_{B(i)}(a_i))$
(iii) $U^i_{\sigma,\mu}(a_i, \sigma_{B(i)}(a_i)) \geq 0$ and (iv) $\Sigma_{i\in I} p_i(v) \leq 1$, $p_i(v) \geq 0$ for all i.

After a few familiar steps expected revenue for the seller at an assessment that implements p, can be written as

$$R(p) = \int_V \sum_{i\in K} p_i(v) \left[v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)} \right] f(v)dv - \sum_{i\in K} U^i_{\sigma,\mu}(\sigma^i_B(a_i), a_i). \tag{2}$$

[7] This is approximately accurate, in the sense that we obtain a consequence of sequential rationality after other histories at $t = 2$, without though characterizing the optimal $t = 2$ mechanism at those histories.

Summarizing, from the above analysis it follows that if for all $i \in K$, $i's$ strategy is a best response to σ_S and σ_B^{-i}, $\bar{p}_i$ is an increasing function of v_i and expected revenue for the seller from employing σ_S is determined solely by $p_i's$ and by $\sum_{i\in K} U^i_{\sigma,\mu}(\sigma^i_B(a_i), a_i)$.

A Benchmark: The Solution under Commitment (Myerson (1981))

Let Φ_i denote buyer $i's$ virtual valuation, which is given by

$$\Phi_i(v_i) = v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)}. \tag{3}$$

We will assume that $\Phi_i's$ are strictly increasing in v_i. A sufficient condition for this is that f_i satisfies the monotone hazard rate assumption.

Assumption MHR is equivalent to the following condition

$$f_i'(v_i)[1 - F_i(v_i)] \geq -f_i^2(v_i). \tag{4}$$

The monotone hazard rate assumption is standard in the literature of mechanism design.

Maximizing (2) subject to the constraint that $\bar{p}_i$ is increasing in v_i gives the revenue maximizing allocation rule among all BNE implementable ones. We maximize (2) pointwise ignoring for the moment the constraint that $\bar{p}_i$ is increasing in v_i. Let ξ_i^1 denote the solution of $\Phi_i(v_i) = 0$, which given MHR, if it exists, it will be unique; if $\Phi_i(v_i) > 0$ for all v_i than set $\xi_i = a_i$ and define

$$\Xi^1 = \times_{i\in I}[a_i, \xi_i^1]. \tag{5}$$

This is the region V where the virtual valuations of all buyers are negative. The optimal BNE implementable allocation rule assigns the object to the buyer with the highest virtual valuation, if his virtual valuation is non-negative, that is

$$\begin{aligned} \text{for } v &\in \Xi^1 \; p_i^C(v) = 0 \text{ for all } i \\ \text{for } v &\in V\backslash\Xi^1 \; p_i^C(v) = 1 \text{ for } i = I^C(v,f) \\ \text{and } p_j^C(v) &= 0 \text{ and } j \neq I^C(v,f), \end{aligned} \tag{6}$$

where $I^C(v,f)$ stands for the buyer that has the highest virtual valuation at vector v, that is $I^C(v,f)_{i\in I} \in \arg\max \Phi_i(v_i)$.[8] Given v_{-i} if buyer i obtains the object when his valuation is v_i, he also obtains the object with valuation $v_i' > v_i$. Therefore p^C is such that $\bar{p}_i$ is increasing in v_i. This allocation rule maximizes the first term of (2) and by setting the payment rule by

$$x_i(v) = p_i(v)v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i,$$

[8] Given MHR ties occur with probability zero, and can be broken arbitrarily.

makes the second term equal to zero. As is it well known from the analysis in Myerson (1981), when the buyers are ex-ante symmetric, in the regular case, this allocation and payment rule can be implemented at the symmetric equilibrium of a first price auction or a second price auction with a reservation price ξ^1.

4.2 The Seller's Second-Period Problem

In order to identify the restrictions that sequential rationality imposes on p, we need to study the seller's behavior at the final period of the game. At a PBE (σ, μ), at a continuation game that starts at $t = 2$ after a history where trade did not occur at $t = 1$ the seller's strategy specifies a mechanism that implements an allocation that maximizes her expected revenue among all allocations that can be implemented at a BNE of this continuation game.

In the case that the buyers' valuations are fully revealed after some history h_S^2, the seller's problem at $t = 2$ is trivial. She names a price equal to the highest valuation and extracts all the surplus from the bidder whose valuation is the highest. In what follows we analyze the case where the seller is uncertain about the buyers' valuations at the beginning of period $t = 2$. We will use r^2 and z^2 to denote respectively the allocation rule and payment rule implemented by the seller's and the buyers' actions at a continuation game at $t = 2$. The seller's continuation strategy at $t = 2$ is the mechanism that she proposes. A continuation strategy of a buyer is a mapping from his type, which consists of everything that is not common knowledge at that information set, to actions. In case that the seller employs a public information disclosure policy at $t = 1$, $i's$ type will just consist of his valuation v_i; if the seller employs a private information disclosure policy at $t = 1$, then $i's$ type will consist of v_i, and the message he received from the seller at $t = 1$, λ_i. Hence given a strategy profile of the continuation game that starts at $t = 2$ the allocation rule and the payment rule, r^2 and z^2 are in general going to be functions of valuations and messages:

$$r_i^2(v_i, v_{-i}, \lambda_i, \lambda_{-i}) \text{ and } z_i^2(v_i, v_{-i}, \lambda_i, \lambda_{-i}),$$

where $v_i \in V_i$, $v_{-i} \in V_{-i}$; $\lambda = (\lambda_i, \lambda_{-i}) \in \Lambda$. Let

$$\begin{aligned} \bar{r}_i^2(v_i, \lambda_i) &= E_{v_{-i}} E_{\lambda_{-i}} \left[r_i^2(v_i, v_{-i}, \lambda_i, \lambda_{-i}) \,|v_i, \lambda_i \right] \text{ and} \\ \bar{z}_i^2(v_i, \lambda_i) &= E_{v_{-i}} E_{\lambda_{-i}} \left[z_i^2(v_i, v_{-i}, \lambda_i, \lambda_{-i}) \,|v_i, \lambda_i \right] \end{aligned} \tag{7}$$

Buyer $i's$ expected payoff at the beginning of $t = 2$ given actions that implement (r^2, z^2), is given by

$$u_i^2(r^2, z^2, v_i) \equiv \bar{r}_i^2(v_i, \lambda_i)v_i - \bar{z}_i^2(v_i, \lambda_i).$$

For notational simplicity we ignore the superscript 2 from now on. Since we move on to $t = 2$ only if there is no trade at $t = 1$, all that follows is conditional on no trade at $t = 1$.

Given an assessment (σ, μ), a mechanism M^1 determines a joint distribution over messages that each buyer may receive. This is common knowledge, since the mechanism is observed by all buyers. To see this consider a typical history for buyer i at $t = 2$. It consists of the mechanism that the seller proposed at $t = 1$, M^1, the action that he chose at $t = 1$, s_i^1, and a message that he received from the seller, $\hat{\lambda}_i$. A message $\hat{\lambda}_i$ can be mapped via the information disclosure policy to a set of possible vectors of messages for the other agents, call it $\Lambda_{-i}(\hat{\lambda}_i) = \left\{ \lambda_{-i} \in \Lambda_{-i} : \lambda = (\hat{\lambda}_i, \lambda_{-i}) \,\middle|\, M^1, C, \sigma \right\}$. An element of $\Lambda_{-i}(\lambda_i)$, call it $\hat{\lambda}_{-i}$ can be again, using the information disclosure policy, mapped back to a set of possible vectors of actions $S_{-i}(\hat{\lambda}_{-i}) = \left\{ s_{-i} \in S_{-i} : \ \lambda_{-i}(s_{-i}) = \hat{\lambda}_{-i} \,\middle|\, M^1, C, \sigma \right\}$. Each of these possible vectors of actions can be mapped back to a set of possible valuations using the strategy profile: $V(\hat{s}_{-i}) = \{v_{-i} \in V_{-i} : \ s_{-i}(v_{-i}) = \hat{s}_{-i}\}$. Given $\hat{\lambda}_i$ the set of possible vectors of v_{-i} is given by $\mathcal{V}_{-i}(\hat{\lambda}_i) = \cup_{\lambda_{-i} \in \Lambda_{-i}(\hat{\lambda}_i)} \left[\cup_{s_{-i} \in S_{-i}(\lambda_{-i})} V(s_{-i}) \right]$. Hence $i's$ posterior distribution over the valuations of $-i$ depends on M^1, s_1 on λ_i and on the strategy profile- which of course describes the seller's information disclosure policy and the actions of the other buyers' at $t = 1$. The same holds for all agents, that is, at a given history and in particular, given a message they have received from the seller, we can calculate their beliefs about their opponents' valuations. Hence if i wants to determine $j's$ beliefs over beliefs of his opponents -which as just said, are determined among others, by the message j receives via the seller's information disclosure policy - buyer i has to assess given a particular message, say λ_i, what message j has received and with what probability. Let G_i^S denote the seller's posterior about $i's$ valuation at the beginning of $t = 2$ after a history h_S^2 where trade has not taken place at $t = 1$; G^S the seller's joint posterior of v and G_{-i}^S denote the seller's beliefs over V_{-i}. Also, let G^i denote $i's$ joint posterior about v; G_{-i}^i denote $i's$ posterior about v_{-i} , given a history h_i^2 and so forth.

When type v_i of buyer i chooses at $t = 2$ the action specified by (σ, μ) for type v_i', then his expected payoff at $t = 2$ is given by

$$u_i^2(r, z, v_i) \equiv \bar{r}_i(v_i', \lambda_i) v_i - \bar{z}_i(v_i', \lambda_i).$$

When we keep λ_i fixed, we omit it and write

$$u_i^2(r, z, v_i) \equiv \bar{r}_i(v_i') v_i - \bar{z}_i(v_i').$$

Lemma 2 *If at an information set the continuation strategy of i is a best response to the continuation strategies of $-i$ and to the seller's continuation strategy, then $\bar{r}_i$ is increasing in v_i.*

The proof of this Lemma is identical to the step of establishing the monotonicity of $\bar{p}_i$ in the proof of Lemma 1.

Lemma 3 *Let Y_i denote the support of G_i^S. Suppose that there exist v_i^L, v_i^H on the boundary of Y_i, such that $(v_i^L, v_i^H) \cap Y = \emptyset$. Then if M^2 is optimally chosen it must hold that $\bar{r}_i(v_i^H) v_i^H - \bar{z}_i(v_i^H) = \bar{r}_i(v_i^L) v_i^H - \bar{z}_i(v_i^L)$.*

In the remaining of this section we look at the continuation game after the history where all buyers reject M^1 at $t = 1$.

The solution at $t = 2$ after the history where all buyers reject M^1.

Given Assumption O all buyers have observed this event and this is common knowledge. This implies that the beliefs of i and the seller about $-i$ are common and are common knowledge and this is true for all i and all $-i$. Consider a PBE (σ, μ) and a history along the equilibrium path h_S^2, where all buyers rejected M^1. From now on we use Y_i to denote the set of types of i that reject M^1 given (σ, μ). We assume that Y_i is measurable and it has strictly positive measure.

Given that along the equilibrium path the seller and the buyers use Bayes' rule and the strategy profile to update beliefs, the closure of Y_i is the support of the seller's and $-i$'s beliefs at $t = 2$ about v_i. Then, the probability density function of v_i given h_S^2 is given by

$$g_i(v_i) = \begin{cases} \frac{f_i(v_i)}{\int_{Y_i} f_i(s_i)ds_i} \text{ if } v_i \in Y_i \\ 0 \text{ otherwise} \end{cases} . \tag{8}$$

Since we are looking at the case where posterior beliefs are common this is the seller's posterior as well as $j's$, $j \in K$ and $j \neq i$. The closure of $Y = Y_1 \times ... \times Y_I$ is the support of the seller's joint posterior at $t = 2$ when all buyers reject.[9] Given Assumption O, posterior beliefs after all buyers reject are independent, and the density is given by $g(v) = g_1(v_1) \times g_2(v_2) \times ... \times g_I(v_I)$. The game ends at period $t = 2$, hence the seller's problem at $t = 2$ is isomorphic to a static problem under commitment. Hence, the seller can, without loss of generality, choose M^2 among the class of direct revelation mechanisms, (DRM), that are incentive compatible, (IC), and individually rational, (IR). The set of types at $t = 2$ is the support of the posterior. A DRM consists of two mappings $r^2 : Y \to [0,1]^I$ and $z^2 : Y \to \mathbb{R}_+^I$; $r^2(v) = (r_1^2(v), r_2^2(v), ..., r_I^2(v))$ specifies the probability of obtaining the object for each buyer, if the buyers have claimed $v = (v_1, v_2, ..., v_I)$, and $z^2(v) = (z_1^2(v), z_2^2(v), ..., z_I^2(v))$ specifies the corresponding expected payment. From now on we will omit the superscript 2.

Consider the seller's problem after the history where all buyers have rejected M^1. At $t = 2$ the mechanism that the seller will employ according to her equilibrium strategy, denoted by $M^2(h_S^2)$, must solve **Program 1**:

$$\max_{r,z \in DRM} \int_Y \sum_{i \in I} z_i(v) g(v) dv, \tag{R$_1$}$$

subject to incentive compatibility and individual rationality constraints

[9] From now on when we write Y_i and Y we will actually mean their corresponding closures.

$$\begin{aligned} \bar{r}_i(v_i)v_i - \bar{z}_i(v_i) &\geq \bar{r}_i(\tilde{v}_i)v_i - \bar{z}_i(\tilde{v}_i), \quad \text{(IC)} \\ \text{for all } v_i, \tilde{v}_i &\in Y_i, \text{ and } i \in I \\ \bar{r}_i(v_i)v_i - \bar{z}_i(v_i) &\geq 0 \text{ for all } v_i \in Y_i \text{ and } i \in I, \quad \text{(IR)} \end{aligned}$$

where

$$\bar{r}_i(v_i) = \int_{Y_{-i}} r_i(v_i, v_{-i}) g_{-i}(v_{-i}) dv_{-i} \tag{9}$$

and $\bar{z}_i(v_i) = \int_{Y_{-i}} z_i(v_i, v_{-i}) g_{-i}(v_{-i}) dv_{-i}$.

Buyer $i's$ expected payoff at the continuation game that starts at $t = 2$ when the seller employs the DRM (r, z) and his valuation is v_i is given by $u_i^2(r, z, v_i) = \bar{r}_i(v_i)v_i - \bar{z}_i(v_i)$.

Since the seller may use any mechanism at $t = 1$ Y_i, may not be a convex subset of the real line as it is usually assumed in the mechanism design literature under commitment. Program 1 differs from a standard static problem in that the type space is not necessarily an interval. We consider a version of Program 1, where the mechanism that the seller employs must satisfy IC and IR on the convex hull of Y. Let $\bar{Y}_i$ denote the convex hull of Y_i and $\bar{Y}$ the convex hull of Y.

Program 2:

$$\max_{(r,z) \in DRM} \int_{\bar{Y}} \sum_{i \in I} z_i(v) g(v) dv, \quad (\mathrm{R}_1^E)$$

subject to incentive compatibility and individual rationality constraints

$$\begin{aligned} \bar{r}_i(v_i)v_i - \bar{z}_i(v_i) &\geq \bar{r}_i(\tilde{v}_i)v_i - \bar{z}_i(\tilde{v}_i), \quad (\mathrm{IC}^E) \\ \text{for all } v_i, \tilde{v}_i &\in \bar{Y}_i, \text{ and } i \in I \\ \bar{r}_i(v_i)v_i - \bar{z}_i(v_i) &\geq 0 \text{ for all } v_i \in \bar{Y}_i \text{ and } i \in I. \quad (\mathrm{IR}^E) \end{aligned}$$

Let R_2 denote the seller's expected revenue at the beginning of $t = 2$.

Proposition 1 *Suppose that posterior beliefs are common and buyers valuations are independent. Let M^2 denote the solution of Program 1 and $\hat{M}^2$ denote the solution of Program 2. Then*

$$R_2(M^2) = R_2(\hat{M}^2).$$

With the help of Lemma 3, Proposition 1 follows from Proposition 1 in Skreta (2003).

From Proposition 1 it follows that it is without any loss to require the mechanism to satisfy IC and IR on the convex hull of Y_i. The solution of this problem follows Myerson closely - with some small modifications in order to take care of the complications that arise from having $g_i(v_i) = 0$ for some $v_i's$.[10]

Proposition 2 *Suppose that Assumption O and 3 hold, and suppose that r solves*

$$\max \int_Y \sum_{i \in I} r_i(v) \left[v_i g_i(v_i) - (1 - G_i(v_i)) \right] g_{-i}(v_{-i}) dv, \tag{10}$$

subject to $\bar{r}_i(v_i)$ is increasing in v_i for all i,

$$\Sigma_{i \in K} r_i(v) \leq 1; \textit{ and } r_i(v) \geq 0 \textit{ for all } i \in I$$

and suppose that

$$z_i(v) = r_i(v) v_i - \int_{a_i}^{v_i} r_i(v_{-i}, s_i) ds_i, \textit{ for all } i \in I. \tag{11}$$

Then r, z is the optimal auction mechanism at $t = 2$.

Proof. Let R_2 denote the seller's revenue at the continuation game that starts after the history that all the buyers rejected M^1. It is given by

$$\begin{aligned} R_2(r, z) &= \Sigma_{i \in I} \int_Y z_i(v) g(v) dv \\ &= \Sigma_{i \in I} \int_Y r_i(v) v_i g(v) dv + \Sigma_{i \in I} \int_Y \left(z_i(v) - r_i(v) v_i \right) g(v) dv. \end{aligned}$$

Using Lemma 2 in Myerson (1981) it can be rewritten as

$$\int_Y \Sigma_{i \in I} r_i(v) \left[v_i g_i(v_i) - (1 - G_i(v_i)) \right] g_{-i}(v_{-i}) dv - \Sigma_{i \in I} u_i^2(r, z, a_i). \tag{12}$$

Define

$$\phi_i(v_i, v_{-i}) = \left[v_i g_i(v_i) - (1 - G_i(v_i)) \right] g_{-i}(v_{-i}), \tag{13}$$

and

$$J_i(v_i) = \left[v_i g_i(v_i) - (1 - G_i(v_i)) \right] \tag{14}$$

Using (12), (13) can be rewritten as

$$R_2(r, z) = \int_Y \Sigma_{i \in I} r_i(v) \phi_i(v_i, v_{-i}) dv - \Sigma_{i \in K} u_i^2(r, z, a_i). \tag{15}$$

[10] In particular, because the expression $v_i - \frac{1 - Gi(v_i)}{g_i(v_i)}$ is not always well defined we use $J_i(v_i) = v_i g_i(v_i) - [1 - G_i(v_i)]$ instead. Moreover, in the case that J_i fails to be strictly increasing, (this is the "general case" in Myerson), we have to take into account the possibility that $g_i(v_i) = 0$ when we obtain the "ironed" J.

The seller's problem reduces to maximizing (15) subject to $\bar{r}_i(v_i)$ is increasing in v_i for all i and $\Sigma_{i\in I} r_i(v) \leq 1$. Also when z_i is given by (11) then $u_i^2(r, z, a_i) = 0$ for all i and (15) reduces to

$$R_2(r, z) = \int_Y \Sigma_{i\in I} r_i(v)\phi_i(v_i, v_{-i})dv.$$

■

Regular Case. Suppose that J_i is strictly increasing in v_i. Let $\xi_i^2(Y_i)$ denote the solution of $J_i(v_i) = 0$, $i \in I$; if $J_i(v_i) > 0$ for all $v_i \in Y_i$ then let $\xi_i^2(Y_i)$ denote the smallest element of Y_i, then let $\Xi_i(Y_i) = \left\{v_i \in Y_i : v_i \leq \xi_i^2(Y_i)\right\}$. This set contains all elements of Y_i such that $i's$ posterior virtual valuation is negative and the set $\Xi(Y)$ defined as

$$\Xi(Y) = \times_{i\in I}\Xi_i(Y_i) \tag{16}$$

and contains all elements of Y such that the posterior virtual valuation of all buyers are negative. The optimal allocation rule assigns the object to the buyer that has the highest posterior virtual valuation if it is non-negative, that is

$$\begin{aligned}
\text{for } v &\in \Xi(Y) \text{ set } r_i(v) = 0 \text{ for all } i \in I; \\
\text{for } v &\in Y\backslash\Xi(Y) \text{ set } r_i(v) = 1 \text{ if } i = I(v, g) \\
\text{and } r_j(v) &= 0 \text{ if } j \neq I(v, g); \\
&\quad \text{the payment given by (11),}
\end{aligned}$$

where $I(v, g)$ denotes the buyer with the highest posterior virtual valuation, that is

$$I(v, g) \in \arg\max_{i\in I} \phi_i(v_i, v_{-i}). \tag{17}$$

If J_i is strictly increasing in i, and $v_i > \tilde{v}_i$, buyer i will win with v_i whenever he wins with $\tilde{v}_i$, hence $\bar{r}_i(v_i) \geq \bar{r}_i(\tilde{v}_i)$. The optimal auction in the regular case assigns the object to the buyer with the highest ϕ_i, which we denote by $I(v, g)$, if this is non-negative. Note that $\phi_i's$ in our framework play a role similar to the virtual valuations in the standard framework where $g_i > 0$.

General Case. In the general case the optimal auction assigns the object to one of the buyers with highest 'ironed' $\hat{\phi}_i$ if it is non-negative. The proof of this and the derivation of the "ironed" virtual valuation for the case that g_i is not necessarily positive, (as is the case in the analysis of Myerson (1981)), can be found in the Appendix.

Given the results obtained in this section we proceed to derive necessary conditions that an allocation rule satisfies if it is implemented by a PBE.

4.3 Necessary Conditions at a PBE

Fix a PBE. At a PBE σ_B^i is a best response to σ_B^{-i} and to σ_S. From Lemma 1 we have the restriction that p must be such that $\bar{p}_i$, (defined in (1)) be increasing in v_i. The allocation rule must also satisfy feasibility constraints $p_i(v) \geq 0$ and $\Sigma_{i \in I} p_i(v) \leq 1$. Moreover, at a PBE the seller's strategy has to be optimal given her beliefs at the beginning of $t = 2$ after every history where trade has not taken place at $t = 1$, and these beliefs must be derived from the buyers' strategies using Bayes' rule. We have shown that given Assumption O, when all the buyers reject M^1, the seller will employ a mechanism at $t = 2$ that assigns the object to the buyer with the highest ('ironed') virtual valuation if it is non-negative. From the above observations it follows that if an allocation rule is implemented by a PBE it belongs in $\mathcal{P}$:

Proposition 3 *Suppose that Assumptions O and 3 hold. If an allocation rule p is implemented at a* PBE, *then it belongs in* $\mathcal{P}$, *where*
$$\mathcal{P} = \left\{ \begin{array}{c} p_i : V \to [0,1],\ i \in K \ \textit{such that} \\ \bar{p}_i \ \textit{is increasing in } v_i \ \textit{and} \\ p_i(v) = 0,\ \textit{for all } i \in K \ \textit{and } v \in \Xi(Y), \\ p_i(v) = \delta \ ,\ \textit{if } i = I(v,g) \ \textit{and } p_j(v) = 0 \ \textit{for } j \neq I(v,g),\ \textit{and } v \in Y \backslash \Xi(Y) \\ 0 \leq \sum_{i \in I} p_i(v) \leq 1,\ \textit{and } v \in V \backslash Y \\ \textit{for some } Y \in V;\ \Xi(Y) \ \textit{given by (16)}, \\ I(v,g) \ \textit{defined in } (17) \ \textit{and } g(v) = \left\{ \begin{array}{c} \frac{f(v)}{\int_Y f(s)ds} \ \textit{if } v \in Y \\ 0 \ \textit{otherwise} \end{array} \right. . \end{array} \right\}.$$

Note that $\mathcal{P}$ is actually a superset of allocation rules that can be implemented by assessments that are $PBE's$.

4.4 The Revenue Maximizing PBE

In this section we solve

$$\max_{p \in \mathcal{P}} R(p), \tag{18}$$

where $R(p) = \int_V \sum_{i \in I} p_i(v) \left[v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)} \right] f(v) dv$. As already noted $\mathcal{P}$ is a superset of the set of $PBE-$ implementable allocation rules, but it will turn out that a solution to (18) is PBE implementable.

Note first that p^C, the allocation rule that maximizes expected revenue under commitment, is not feasible under non-commitment, since $p^C \notin \mathcal{P}$. We search among functions p that are elements of $\mathcal{P}$. For our purposes all equilibria that lead to the same p will be considered as equivalent since they guarantee the same expected revenue for the seller.

First we establish that the seller can restrict attention to a subset of $\mathcal{P}$ which we call $\mathcal{P}^*$. $\mathcal{P}^*$ is a subset of $\mathcal{P}$ with the following characteristics: the set of types that reject is convex, that is, and of the form $Y = [a_1, \bar{v}_1] \times ... \times [a_I, \bar{v}_I]$ for some $(\bar{v}_1, ..., \bar{v}_I)$ where $\bar{v}_i \geq \xi_i$, and for every $v \in V \backslash Y$ the object is assigned with probability one to the buyer with highest virtual valuation. As we will later demonstrate the symmetric equilibrium of a sequence of second or first price auctions implement allocation rules in $\mathcal{P}^*$.

Definition 5 $\mathcal{P}^*= \left\{ \begin{array}{c} p_i : V \rightarrow [0,1],\ i \in I \ \ \textit{such that} \\ p_i(v) = 0,\ \textit{for all } i \in K \textit{ and } v \in \Xi(Y),\ \textit{where } \Xi(Y) \textit{ is given by (??)} \\ p_i(v) = \delta\ ,\ \textit{if } i = I(v,g) \textit{ and } p_j(v) = 0 \\ \textit{for } j \neq I(v,g) \textit{ and } v \in Y \backslash \Xi(Y) \\ p_i(v) = 1,\ \textit{if } i = I(v,f) \textit{ and } p_j(v) = 0 \\ \textit{for } j \neq I(v,f) \textit{ and } v \in V \backslash Y \\ \textit{where } I(v,f) \in \arg\max_{\{i \in I : v_i \in V_i \backslash Y_i\}} \Phi_i(v_i),\ \textit{and } I(v,g) \textit{ defined in (17);} \\ Y = [a_1, \bar{v}_1] \times ... \times [a_I, \bar{v}_I] \textit{ for some } (\bar{v}_1, ..., \bar{v}_I) \textit{ and } \Xi^1 \subset Y \textit{ and} \\ g(v) = \left\{ \begin{array}{c} \frac{f(v)}{\int_Y f(s)ds}\ \textit{if } v \in Y \\ 0\ \textit{otherwise} \end{array} \right. . \end{array} \right\}$

Lemma 4 *$\mathcal{P}^*$ is a subset of $\mathcal{P}$*

Proof. Take a $p \in \mathcal{P}^*$ we would like to verify that $p \in \mathcal{P}$. A moment's look at the definitions of $\mathcal{P}$ and $\mathcal{P}^*$ will convince the reader that the only thing we need to verify is that $\bar{p}_i$ is increasing in v_i.

First observe that for $v \in V \backslash Y$ there exist $j \in I$ such that $p_j(v) = 1$ which implies that for every such v trade take place with probability 1 at $t = 1$. The only possibility that we move on at $t = 2$ is when every buyer rejects, that is when $v \in Y$. This implies that for $v_i \in [a_i, \bar{v}_i]$ $\bar{p}_i(v_i) = \delta \bar{r}_i(v_i)$, where $\bar{r}_i$ is given by (9), and as we mentioned when we derived the solution of the seller's problem at the beginning of the final period of the game after all buyers reject, $\bar{r}_i(v_i)$ is increasing in v_i. So for $v_i \in [a_i, \bar{v}_i]$ $\bar{p}_i$ is increasing in v_i. Now for $v_i \in (\bar{v}_i, b_i]$ i participates in M^1. The monotonicity of $\bar{p}_i$ for $v_i \in (\bar{v}_i, b_i]$ follows from the fact that given MHR for all vectors v such that v_i has the highest virtual valuation- that is $i \in I(v,f)-$, type $v_i' \geq v_i$ also does, that is $p_i(v_i, v_{-i}) = p_i(v_i', v_{-i}) = 1$ for all v_{-i}. That is, p_i is increasing in v_i for $v_i \in (\bar{v}_i, b_i]$, which clearly ensures that $\bar{p}_i$ is increasing in $v_i \in (\bar{v}_i, b_i]$. So far we have established that $\bar{p}_i$ is increasing for $v_i \in [a_i, \bar{v}_i]$ and for $v_i \in (\bar{v}_i, b_i]$. Lastly we should verify that $\bar{p}_i$ does not drop at $\bar{v}_i$. We take the convention that $\bar{v}_i$ rejects M^1. This is without loss since $\bar{v}_i$ is indifferent between accepting and rejecting and moreover it is of measure zero. Since $\bar{v}_i \in Y_i$ we have that $\bar{p}_i(\bar{v}_i) \leq \delta F_{-i}(\bar{v}_{-i})$. The reason for this is that $\bar{v}_i$ can only obtain the object only if everybody rejects which happens with probability $F_{-i}(\bar{v}_{-i})$. So even if he gets the object with probability 1 at $t = 2$, once everybody has rejected, from the ex-ante point of view it must hold $\bar{p}_i(\bar{v}_i) \leq \delta F_{-i}(\bar{v}_{-i})$. In other words, the upper bound for $\bar{p}_i(\bar{v}_i)$ is $\delta F_{-i}(\bar{v}_{-i})$. Now type $\bar{v}_i + \varepsilon$ where $\varepsilon > 0$ and arbitrarily small, accepts M^1 so he gets the object with probability one, at least when everybody else has rejected. This occurs with probability $F_{-i}(\bar{v}_{-i})$. Hence it holds that $\bar{p}_i(\bar{v}_i + \varepsilon) \geq F_{-i}(\bar{v}_{-i})$. Therefore $\bar{p}_i$ is increasing in v_i. This implies that $\mathcal{P}^* \subset \mathcal{P}$. ■

Assumption TBR: Tie-Breaking Rule (TBR): Suppose that for all $i \in I$, i rejects M^1 whenever he is indifferent between participating and rejecting.

The Proposition that follows establishes that maximizing R over $\mathcal{P}$ is equivalent to maximizing R over $\mathcal{P}^*$. This is done by showing that for each $p \in \mathcal{P}$ there exists $\hat{p} \in \mathcal{P}^*$ that generates higher expected revenue.

The main steps in establishing Proposition 4 are as follows.

Step 1: Assumung that i rejects M^1 if indifferent then types where i rejects are $[a_i, \bar{v}_i]$ for some $\bar{v}_i \in [a_i, b_i]$ (Lemma 6 which employs Lemma 5)

Step 2. At a revenue maximizing allocation rule out of $\mathcal{P}$, the cutoff $\bar{v}_i$ is greater or equal to the point where a buyer's virtual valuation is zero.

Intuition: The seller does not want to trade with strictly positive probability with types of buyer i where his virtual valuation is strictly negative. (Lemma

Step 3:Show that for every $p \in \mathcal{P}^{CSR}$ there exists $\hat{p} \in \mathcal{P}^*$ such that $R(\hat{p}) \geq R(p)$, so

$$\max_{p \in \mathcal{P}^{CSR}} R(p) = \max_{p \in \mathcal{P}^*} R(p)$$

Proposition 4 *Suppose that Assumptions O, 3, TBR and MHR hold. Then for each $p \in \mathcal{P}$, there exists $\hat{p} \in \mathcal{P}^*$ that raises higher expected revenue for the seller, that is $R(\hat{p}) \geq R(p)$, so*

$$\max_{p \in \mathcal{P}^*} R(p) = \max_{p \in \mathcal{P}} R(p).$$

and is established with the help of the following two Lemmas:

Lemma 5 *Consider a PBE (σ, μ) and let Y_i denote the set of types of buyer i that reject M^1. Let $\bar{Y}_i$ denote its convex hull. Then for $v_i \in \bar{Y}_i \backslash Y_i$ it holds that $\bar{p}_i(v_i) = \bar{p}_i(\hat{v}_i)$ where $\hat{v}_i = \sup\{\tilde{v}_i \in Y_i \text{ s.t. } \tilde{v}_i \leq v_i\}$.*

Lemma 6 *Given TBR then for all i, that is the set of types that reject M^1, is convex.*

Proof. Consider a strategy profile where the set of types that reject M^1, that is Y_i, is not convex. From the previous Lemma it follows for all i when $v_i \in \bar{Y}_i \backslash Y_i$ buyer i is indifferent between accepting and rejecting M^1. Given the tie-breaking rule, then for $v_i \in \bar{Y}_i \backslash Y_i$, buyer i will reject. All types in $\bar{Y}_i$ reject, which is by definition convex. ■

This Lemma relies on the tie-breaking rule. Ideally one should show that the seller prefers to have ties broken the way suggested above. This result can be established when the seller faces just one buyer, but we have so far been unable to establish or provide a counter-example to it in the multi-buyer case.

Lemma 7 *At a revenue maximizing allocation rule out of $\mathcal{P}$, the cutoff, $\bar{v}_i$, is greater or equal to the point where a buyer's virtual valuation is zero, that is*

$$\bar{v}_i \geq \xi_i.$$

From Lemma 7 it follows that at an optimal element of $\mathcal{P}$ we have that $\Xi \subset Y$. It is then easy to show that an allocation rule that for each $v \in V \backslash Y$ assigns the object to the buyer with the highest virtual valuation is the best we can hope for.

Theorem 1 *Suppose that Assumptions O, 3, TBR and MHR hold. Then, at the revenue maximizing PBE, the seller at $t = 1$ employs a mechanism where each buyer can either claim a value above a buyer specific cut-off, or claim the lowest possible value, (reject). The object is awarded with probability one to the buyer to with the highest virtual valuation among all buyers who claimed a value above the cut-off. If no buyer claims a value above the cut-off no trade take place at $t = 1$ and we move on to $t = 2$ where the seller employs a direct revelation mechanism that assigns the object to the buyer with the highest posterior virtual valuation if it is non-negative.*

Proof. From Proposition 4 we know that given Assumption O, the seller can look for the PBE that generates the highest expected revenue, among ones that implement allocation rules in $\mathcal{P}^*$. The proof of the theorem consists of the following three steps: First we describe for any element of $p \in \mathcal{P}^*$ a strategy profile that implements p. Second we choose the element of $\mathcal{P}^*$ that is optimal for the seller, call it p^* and in step three we verify that an assessment that implements p^* is a PBE of the game under consideration.

Step 1: **Implementation of elements of $\mathcal{P}^*$:**

A strategy profile that implements an element of $\mathcal{P}^*$ is as follows: The seller proposes $M^1 = (S^1, \gamma^1)$, where $S_i^1 = \{a_i\} \cup [\bar{v}_i, b_i]$ and γ_i^1 is as follows

$$\begin{aligned} r_i^1(a_i, v_{-i}) &= 0 \\ r_i^1(v_i, v_{-i}) &= 1 \text{ if } i = I(v, f),\ v_i \in [\bar{v}_i, b_i] \\ &= 0 \text{ otherwise, that is for } i \neq I(v, f), \end{aligned}$$

where

$$I(v, f) \in \arg \max_{\{i \in I : v_i \in V_i \backslash Y_i\}} \Phi_i(v_i). \tag{19}$$

Payments:

$$\begin{aligned}
z_i^1(a_i, v_{-i}) &= 0 \\
z_i^1(v_i, v_{-i}) &= c_i^1(v_{-i}),\ v_{-i} \neq a_{-i};\ \text{if } i = I(v, f) \\
&= 0 \text{ otherwise, that is for } i \neq I(v, f) \text{ and} \\
z_i^1(v_i, a_{-i}) &= Z_i^1 \text{ where} \\
Z_i^1 &= \frac{1}{F_{-i}(\bar{v}_{-i})}\left[\int_{A_i(\bar{v}_i)} \bar{v}_i f_{-i}(v_{-i})dv_{-i} + \int_{Y_{-i}\backslash A_i(\bar{v}_i)} \left[(1-\delta)\bar{v}_i + \delta c_i^2(v_{-i})\right] f_{-i}(v_{-i})dv_{-i}\right]
\end{aligned}$$

and

$$c_i^1(v_{-i}) = \inf\{v_i \text{ such that } \Phi_i(v_i) \geq 0 \text{ and } \Phi_i(v_i) \geq \Phi_j(v_j),\ \text{ for } j \neq i\}$$

$$c_i^2(v_{-i}) = \inf\{v_i \text{ such that } \phi_i(v_i) \geq 0 \text{ and } \phi_i(v_i) \geq \phi_j(v_j),\ \text{ for } j \neq i\}$$

and

$$A_i(\bar{v}_i) = \{v_{-i} \in Y_{-i} \text{ such that } \bar{v}_i < c_i^2(v_{-i})\}.$$

The price that buyer i pays at $t = 1$ whenever he is the only buyer participating, which is given by Z_i^1, is chosen such that type $\bar{v}_i$ be indifferent between revealing his true type or waiting a period. The mechanism proposed at $t = 2$ is as follows: $M^2 = (S^2, \gamma^2)$ where $S_i^2 = [a_i, \bar{v}_i]$ and γ^2 is such that:

$$\begin{aligned}
r_i^2(v_i, v_{-i}) &= 1 \text{ if } i = I(v, g);\ v \in Y\backslash\Xi(Y) \\
&= 0 \text{ otherwise, that is for } i \neq I(v, g) \\
r_i^2(v_i, v_{-i}) &= 0;\ \text{for all } i \text{ when } v \in \Xi(Y) \\
\text{Payment} &: \\
z_i^2(v_i, v_{-i}) &= 0;\ \text{for all } i \text{ when } v \in \Xi(Y) \\
z_i^2(v_i, v_{-i}) &= c_i^2(v_{-i}) \text{ if } i = I(v, g);\ v \in Y\backslash\Xi(Y) \\
&= 0 \text{ otherwise, that is for } i \neq I(v, g)
\end{aligned}$$

Given this strategy of the seller let us consider the following strategy profile for the buyers, (along the path determined by the seller's strategy). For $v_i \in [a_i, \bar{v}_i]$ buyer i chooses a_i at $t = 1$ and claims his true type at $t = 2$. (Recall that at $t = 2$ the seller employs a direct mechanism M^2.) For $v_i \in (\bar{v}_i, b_i]$ buyer i claims v_i at $t = 1$; and $\bar{v}_i$ at $t = 2$.

The allocation rule and the payment rule implemented by this assessment are given by

$$
\begin{gathered}
p_i : V \rightarrow [0,1],\ i \in I \text{ such that} \\
p_i(v) = 0, \text{ for all } i \in I \text{ and } v \in \Xi(Y) \\
p_i(v) = \delta\ , \text{ if } i = I(v,g) \text{ and } p_j(v) = 0 \\
\text{for } j \neq I(v,g) \text{ and } v \in Y \backslash \Xi(Y) \\
p_i(v) = 1, \text{ if } i = I(v,f) \text{ and } p_j(v) = 0 \\
\text{for } j \neq I(v,f) \text{ and } v \in V \backslash Y \\
\text{for } I(v,f) \text{ defined in (19) and } I(v,g) \text{ defined in (17),} \\
\text{where } Y = [a_1, \bar{v}_1] \times ... \times [a_I, \bar{v}_I] \text{ and } \Xi^1 \subset Y.
\end{gathered}
$$

$$
\begin{aligned}
x_i(v) &= 0 \text{ if } v \in \Xi(Y) \\
x_i(v) &= \delta c_i^2(v_{-i})\ v \in Y \backslash \Xi(Y) \text{ if } i = I(v,g) \\
&= 0 \text{ otherwise} \\
x_i(v) &= c_i^1(v_{-i}) \text{ if } v_i \in (\bar{v}_i, b_i] \text{ and } v_{-i} \in V_{-i} \backslash Y_{-i} \text{ and } i = I(v,f) \\
x_i(v) &= 0 \text{ if } v_i \in (\bar{v}_i, b_i] \text{ and } v_{-i} \in V_{-i} \backslash Y_{-i} \text{ and } i \neq I(v,f) \\
x_i(v) &= \xi_i^1 \text{ if } v_i \in (\bar{v}_i, b_i] \text{ and } v_{-i} \in Y_{-i}\ .
\end{aligned}
$$

Note that indeed the allocation rule p is an element of $\mathcal{P}^*$, as we wanted to show.

Step 2: Find optimum $p \in \mathcal{P}^*$.

An element of $\mathcal{P}^*$ is indexed by $\bar{v}$. That is, by varying $\bar{v}$ we can obtain a strategy profile that implements a different element of $\mathcal{P}^*$: Vector $\bar{v}$ determines Y which determines g via $g(v) = \begin{cases} \frac{f(v)}{\int_Y f(s)ds} \text{ if } v \in Y \\ 0 \text{ otherwise} \end{cases}$, which in turn determines $\Xi(Y)$ via (16). Hence the allocation rule that raises maximal revenue for the seller can be identified by choosing $\bar{v}$ optimally. In other words the seller can obtain the revenue maximizing element of $\mathcal{P}^*$ by solving:

$$
\begin{aligned}
& \max \int_{\Xi(Y)} 0 f(v)dv + \int_{Y \backslash \Xi(Y)} \delta \Phi_{I(v,g)}(v) f(v)dv + \int_{V \backslash Y} \Phi_{I(v,f)}(v) f(v)dv \\
\bar{v}_i \quad &\in \quad [a_i, b_i].\ i \in I
\end{aligned}
$$

Following standard procedures it is easy to verify that this maximization problem is well defined (the choice set is compact, which is trivially true for this problem, and that the objective function is continuous.)[11]

Step 3: Implement p by a PBE

[11] Proof available upon request.

First we need to verify that players' strategies are best responses to the strategy of each other.

a) $i's$ strategy is a best response given σ_S and σ^B_{-i}.

In order to establish this we will use the allocation and payment rule (p,x) implemented by the assessment under consideration.

The pair (p,x) implemented by the assessment under consideration can be viewed as a direct mechanism. We will establish that it is incentive compatible and individually rational. These properties will be used to verify that σ^i_B is a best response to σ_S and to σ^{-i}_B. Note that the assessment under consideration implements an allocation rule p with the property that $\bar{p}_i$ be increasing in v_i. This property holds if σ^i_B is a best response to σ^{-i}_B and to σ_S, but it is only a necessary condition. Therefore we need to verify that for the strategy profile under consideration it holds that σ^i_B is a best response to σ_S and to σ^{-i}_B.

Establishing Feasibility of (p,x) :

Consider the expected payment for buyer i when $v_i \in [a_i, \bar{v}_i]$:

$$\begin{aligned}
\bar{x}_i(v_i) &= \int_{V_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} \\
&= \int_{V_{-i}\backslash Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} + \int_{Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i}.
\end{aligned}$$

Notice that $x_i(v_i,v_{-i}) = p_i(v_i,v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i,v_{-i})ds_i$ since when $v_{-i} \in V_{-i}\backslash Y_{-i}$ $p_i(v_i,v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i,v_{-i})ds_i = 0$ and $x_i(v_i,v_{-i}) = 0$. The same is true when $v_{-i} \in Y_{-i}$ but $i \neq I(v,g)$, that is $p_i(v_i,v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i,v_{-i})ds_i = 0 = x_i(v_i,v_{-i})$ whereas if $i = I(v,g)$ then $p_i(v_i,v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i,v_{-i})ds_i = \delta c_i^2(v_{-i}) = x_i(v_i,v_{-i})$. It follows that

$$\begin{aligned}
\bar{x}_i(v_i) &= \int_{Y-i} \left[p_i(v_i,v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i,v_{-i})ds_i \right] f_{-i}(v_{-i}) dv_{-i} \\
&= \bar{p}_i(v_i)v_i - \int_{a_i}^{v_i} \bar{p}_i(s_i)ds_i. \qquad (20)
\end{aligned}$$

Now let us consider $v_i \in (\bar{v}_i, b_i]$

$$\begin{aligned}
\bar{x}_i(v_i) &= \int_{V_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} \\
&= \int_{V_{-i}\backslash Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} + \int_{Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} \\
&= \int_{V_{-i}\backslash Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} + \int_{Y_{-i}} \xi_i^1 f_{-i}(v_{-i}) dv_{-i} \\
&= \int_{V_{-i}\backslash Y_{-i}} x_i(v_i,v_{-i}) f_{-i}(v_{-i}) dv_{-i} + \xi_i^1 F_{-i}(\bar{v}_{-i}).
\end{aligned}$$

Notice that for $v_{-i} \in V_i \backslash Y_{-i}$ we have that $x_i(v_i, v_{-i}) = 0$ if $i \neq I(v, f)$ and $x_i(v_i, v_{-i}) = c_i^1(v_{-i})$ if $i = I(v, f)$. As before notice that

$$x_i(v_i, v_{-i}) = p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i, \text{ for } v_i \in (\bar{v}_i, b_i] \text{ and } v_{-i} \in V_{-i} \backslash Y_{-i} \tag{21}$$

since if $i \neq I(v, f)$, $p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i = 0$ whereas if $i = I(v, f)$ then $p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i = c_i^1(v_{-i})$. Now for $v_{-i} \in Y_{-i}$ we have by the definition of ξ_i^1 that

$$\xi_i^1 = \frac{1}{F_{-i}(\bar{v}_{-i})} \left[\int_{A_i(\bar{v}_i)} \bar{v}_i f_{-i}(v_{-i})dv_{-i} + \int_{Y_{-i} \backslash A_i(\bar{v}_i)} \left[(1-\delta)\bar{v}_i + \delta c_i^2(v_{-i})\right] f_{-i}(v_{-i})dv_{-i} \right].$$

which as one can easily verify it equals to

$$\begin{aligned} \xi_i^1 F_{-i}(\bar{v}_{-i}) &= \int_{A_i(\bar{v}_i)} \bar{v}_i f_{-i}(v_{-i})dv_{-i} + \int_{Y_{-i} \backslash A_i(\bar{v}_i)} \left[(1-\delta)\bar{v}_i + \delta c_i^2(v_{-i})\right] f_{-i}(v_{-i})dv_{-i} \qquad (22) \\ &= \int_{Y_{-i}} \left[p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i \right] f_{-i}(v_{-i})dv_{-i} \text{ for } v_i \in (\bar{v}_i, b_i] \text{ and } v_{-i} \in Y_{-i}. \end{aligned}$$

This follows from the fact that

$$\begin{aligned} p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(s_i, v_{-i})ds_i &= v_i - \int_{\bar{v}_i}^{v_i} 1 ds_i - 0 \text{ for } v_{-i} \in Y_{-i} \text{ such that } \bar{v}_i < c_i^2(v_{-i}), \\ \text{which holds for } v_{-i} &\in A_i(\bar{v}_i); \\ &= v_i - \int_{\bar{v}_i}^{v_i} 1 ds_i - \int_{c_i^2(v_{-i})}^{\bar{v}_i} \delta ds_i \text{ for } v_{-i} \in Y_{-i} \text{ such that } \bar{v}_i \geq c_i^2(v_{-i}) \\ \text{which holds for } v_{-i} &\in Y_{-i} \backslash A_i(\bar{v}_i). \end{aligned}$$

From (21) and (22) it follows that for $v_i \in (\bar{v}_i, b_i]$

$$\bar{x}_i(v_i) = \bar{p}_i(v_i)v_i - \int_{a_i}^{v_i} \bar{p}_i(s_i)ds_i \tag{23}$$

Now from (20) and (23) we have

$$U^i_{\sigma,\mu}(\sigma^i_B(v_i), v_i) = \int_{a_i}^{v_i} \bar{p}_i(s_i)ds_i.$$

The pair (p, x) can be viewed as a direct mechanism with the following properties: $\bar{p}_i$ is increasing in v_i,

$$U^i(v_i, p, x) = \int_{a_i}^{v_i} \bar{p}_i(s_i)ds_i;$$

and $\Sigma_{i \in K} p_i(v) \leq 1$ for $v \in V$ and $p_i(v) \geq 0$. From Lemma 2 in Myerson (1981) it follows that (p, x) is incentive compatible and individually rational.

From this observation it is immediate that σ_B^i is a best response to σ_S and σ_B^{-i} : First observe that, along the path, according to σ_B^i all possible sequences of actions (possible choices at $t = 1$ and at $t = 2$ are determined by the mechanisms proposed by the seller) are chosen by some type. To see this, note that a type v_i in $[\bar{v}_i, b_i]$ claims v_i at $t = 1$ and the game ends at $t = 1$, a type $\hat{v}_i$ in $[a_i, \bar{v}_i]$ chooses a_i at $t = 1$ and $\hat{v}_i$ at $t = 2$. Suppose that type v_i has a profitable deviation from σ_B^i where he prefers to choose the actions chosen by v_i' according to σ_B^i. Then it must be the case that

$$\bar{p}_i(v_i')v_i - \bar{x}_i(v_i') > \bar{p}_i(v_i)v_i - \bar{x}_i(v_i)$$

but this is impossible since (p, x) is incentive compatible. Therefore σ_B^i is a best response to σ_S and to σ_B^{-i}.

b) We would like to establish that the buyers' strategy is a best response at the continuation game that starts at $t = 2$. Recall that given M^1, the game continues at $t = 2$ only after the history where all buyers reject M^1. Then the seller proposes M^2 which is chosen to be incentive compatible, which of course implies that each buyer can do no better than by claiming his true type. This is exactly what σ_B^i dictates to each type of each buyer. Hence for all i the buyer's continuation strategy is a best response at the continuation game that starts at $t = 2$.

c) The seller's strategy is a best response at each node: Given σ_B the mechanism employed by the seller at $t = 2$ is optimal given her posterior beliefs which are derived using Bayes' rule from the strategies of the other buyers. In order for an assessment to be a PBE the seller has to also choose M^1 optimally. By Proposition 4 this essentially reduces to choosing $\bar{v}_i$ for all i optimally. (This choice will have impact on the buyers' best response and on what the seller proposes at $t = 2$).

d)Beliefs are derived using Bayes' rule

Recall that along the path the only history when we move to $t = 2$ is whenever all buyers reject. The seller's posterior about $i's$ valuation is then given by

$$g_i(v_i) = \begin{cases} \frac{f_i(v_i)}{\int_Y f_i(s_i)ds_i} \\ 0 \end{cases},$$

and the joint posterior is given by $g(v) = g_1(v_1) \times g_2(v_2) \times ... \times g_I(v_I)$. ■

Our result demonstrates that at the optimum whenever at $t = 1$ the probability of trade with some buyer is positive, than it is equal to one. This is also the case at the commitment optimum; but it is not straightforward that this is optimal in a dynamic framework. One can think that the seller by using a mechanism at $t = 1$ that consists of lotteries, may on one hand, reduce the probability of trade at $t = 1$, but on the other hand, lead to such posterior beliefs at $t = 2$ that will allow for higher surplus extraction. That is, the seller could use $t = 1$ as an experimentation stage, that would allow her to obtain sharper information about the buyers' valuations that she could in turn use to obtain more revenue at $t = 2$. Our results show that it is not worthwhile for the seller to do so.

Assumption S, (Symmetry). The buyers are ex-ante symmetric, $f_i = f_j$ for all $i, j \in I$

Lemma 8 *Suppose that the probability density function of v_i given Y_i is given by* $g_i(v_i) = \begin{cases} \frac{f_i(v_i)}{F_i(\bar{v}_i)} \text{ if } v_i \in [a_i, \bar{v}_i] \\ 0 \text{ otherwise} \end{cases}$. *If f_i satisfies the monotone hazard rate assumption, then so does g_i.*

Proof. Note that $\frac{g_i(v_i)}{1-G_i(v_i)} = \frac{f_i(v_i)}{F_i(\bar{v}_i)-F_i(v_i)}$. Then if g_i satisfies MHR the following inequality must hold

$$f_i'(v_i)[F_i(\bar{v}) - F_i(v_i)] \geq -f_i^2(v_i). \tag{24}$$

Now if f_i satisfies MHR, we have that (4) holds. If $f_i' \geq 0$ then (24) is automatically satisfied. If $f_i' < 0$ then we have that

$$f_i'(v_i)\,[F_i(\bar{v}) - F_i(v_i)] \geq f_i'(v_i)\,[1 - F_i(v_i)] \geq -f_i^2(v_i).$$

■

From Lemma 8 we have that if the prior, f_i, satisfies MHR, then at a history where buyer i rejects M^1, so does the posterior g_i.

Corollary 1 *Assume O, MHR and that the buyers are ex-ante symmetric. Then the symmetric equilibrium of the game where the seller runs a SPA or a FPA in each period with optimally chosen reserve prices, generates maximal revenue for the seller.*

Proof. From Proposition 4 we know that given Assumption O, the seller can look for the PBE that generates the highest expected revenue, among ones that implement allocation rules in $\mathcal{P}^*$. If buyers are ex-ante symmetric then we have that[12] $\bar{v}_i = \bar{v}_j$ for all $i, j \in I$ and $i \neq j$, then the optimal allocation can be implemented by a symmetric equilibrium of the game where the seller runs a SPA or a FPA in each period with optimally chosen reserve prices.

Consider the symmetric equilibrium of a sequence of SPA with a reservation price in each period. At $t = 1$ a SPA with a reservation price will assign the object to the buyer with the highest valuation, (which due to symmetry and MHR is the buyer with the highest virtual valuation), among all buyers that submit a bid above the reservation price that the seller has posted at $t = 1$. This follows from the fact that conditional on submitting a bid above the reserve price, it is a dominant strategy for a buyer to submit a bid equal to his true valuation. Let g_i denote the posterior for $i's$ valuation, after buyer i does not submit

[12] Proof available upon request.

a bid above the reserve price at $t = 1$. From Lemma 8 it follows that it satisfies the MHR. At a second price auction trade does not take place at $t = 1$ if no-one bids above the reservation price. Given ex-ante symmetric buyers, at a symmetric equilibrium, the buyers are symmetric in the eyes of the seller at the beginning of $t = 2$ as well. At $t = 2$ a SPA will assign the object to the buyer with the highest valuation, (who due to symmetry and MHR is also the buyer with the highest virtual valuation), if his valuation is above the reservation price posted at $t = 2$. Similar arguments hold for a FPA. ■

An example is computed in the next section. In the case of ex-ante symmetric buyers sometimes the search for the optimal auction under non-commitment reduces to the search of the optimal sequence of reserve prices in an environment where the seller runs in each period a SPA or a FPA with a reserve price. For an example of optimally chosen reservation prices in SPA and FPA in a dynamic framework see McAfee and Vincent (1997).

5 How to Calculate the Optimal Auction

The question is how to choose the optimal element of $\mathcal{P}^*$. The numbers $\bar{v}_i$, $i \in I$ completely describe an element of this set. Why? These numbers determine the posterior which in turn determines the second period cutoffs as well as the ranking of posterior virtual valuations for each vector of types. In particular for all i, $\bar{v}_i$ determines the posterior given by $g_i(v_i) = \frac{f_i(v_i)}{F_i(\bar{v}_i)}$, that of course determines the buyers posterior virtual valuation, $v_i - \frac{1-G_i(v_i)}{g_i(v_i)}$, that in turn determines the cut-off that the seller will set at $t = 2$. The cut-off is given by

$$\xi_i^2(\bar{v}_i) \text{ solves } v_i - \frac{1 - G_i(v_i)}{g_i(v_i)} = 0,\ i \in I;$$

Note that Lemma **??** implies that if the prior satisfies MHR so will the posterior. At $t = 2$ the optimal M^2 will assign the object to buyer i if $v_i - \frac{1-G_i(v_i)}{g_i(v_i)} \geq 0$ and $v_i - \frac{1-G_i(v_i)}{g_i(v_i)} \geq v_j - \frac{1-G_j(v_j)}{g_j(v_j)}$ for all $j \in I$ such that $j \neq i$. Given MHR, in the case where buyers are ex-ante symmetric, when $\bar{v}_i = \bar{v}_j$ for all i and j, then $v_i - \frac{1-G_i(v_i)}{g_i(v_i)} \geq v_j - \frac{1-G_j(v_j)}{g_j(v_j)}$ reduces to

$$v_i \geq v_j. \tag{25}$$

Example 1 *2 buyers, uniform [0,1] distribution. For a pair $\bar{v}_1$ and $\bar{v}_2$ the optimal cutoffs*[13] *at $t=2$ are*

$$\begin{aligned}
\xi_1^2(\bar{v}_1) &= \frac{\bar{v}_1}{2} \text{ and } \xi_2^2(\bar{v}_2) = \frac{\bar{v}_2}{2}; \\
\textit{posteriors} &: \; g_1(v_1) = \frac{1}{\bar{v}_1} \text{ and } g_2(v_2) = \frac{1}{\bar{v}_2}; \\
\textit{posterior virtual valuations } \phi_1(v_1) &= \frac{2v_1}{\bar{v}_1} - 1 \text{ and } \phi_2(v_2) = \frac{2v_2}{\bar{v}_2} - 1; \\
\textit{if } v_1 &> \frac{\bar{v}_1}{\bar{v}_2} v_2 \text{ and } \frac{2v_1}{\bar{v}_1} - 1 \geq 0 \textit{ obtains the object at } t=2 \\
\textit{if } v_2 &> \frac{\bar{v}_2}{\bar{v}_1} v_1 \text{ and } \frac{2v_2}{\bar{v}_2} - 1 \geq 0 \textit{ obtains the object at } t=2
\end{aligned}$$

$$\begin{aligned}
& \int_0^{0.5\bar{v}_2} \left[\int_{0.5\bar{v}_1}^{\bar{v}_1} \delta(2v_1-1)dv_1 + \int_{\bar{v}_1}^{1} (2v_1-1)dv_1 \right] dv_2 + \\
& \int_{0.5\bar{v}_2}^{\bar{v}_2} \left[\begin{array}{c} \int_0^{0.5\bar{v}_1} \delta(2v_2-1)dv_1 + \int_{0.5\bar{v}_1}^{(\bar{v}_1/\bar{v}_2)v_2} \delta(2v_2-1)dv_1 \\ + \int_{(\bar{v}_1/\bar{v}_2)v_2}^{\bar{v}_2} \delta(2v_1-1)dv_1 + \int_{\bar{v}_2}^{1} (2v_1-1)dv_1 \end{array} \right] dv_2 + \\
& \int_{\bar{v}_2}^{1} \left[\int_0^{\bar{v}_1} (2v_2-1)dv_1 + \int_{\bar{v}_1}^{v_2} (2v_2-1)dv_1 + \int_{v_2}^{1} (2v_1-1)dv_1 \right] dv_2 \\
= \; & 0.58333\bar{v}_2\delta\bar{v}_1^2 - 0.75\bar{v}_2\delta\bar{v}_1 - 1.0\bar{v}_2\bar{v}_1^2 + \bar{v}_2\bar{v}_1 + 0.58333\delta\bar{v}_1\bar{v}_2^2 + 0.33333 - 0.33333\bar{v}_2^3.
\end{aligned}$$

Maximizing with respect to $\bar{v}_1$ and to $\bar{v}_2$ we obtain the following solution which turns out to be symmetric.

δ	$\bar{v}_1$	$\bar{v}_2$	*Revenue*
0	*0.5000*	*0.5000*	*0.4167*
0.2000	*0.5152*	*0.5152*	*0.4085*
0.3000	*0.5254*	*0.5254*	*0.4046*
0.4000	*0.5384*	*0.5384*	*0.4010*
0.5000	*0.5555*	*0.5555*	*0.3976*
0.7000	*0.6129*	*0.6129*	*0.3928*
0.9000	*0.7646*	*0.7646*	*0.3967*
0.9999	*0.9994*	*0.9994*	*0.4166*
1	*0.9998*	*0.9998*	*0.4166*

For this example the commitment benchmark is

$$\bar{v}_1 = \bar{v}_2 = 0.5 \textit{ and Revenue} = 0.41667$$

[13] Theses are the cut-offs describing the allocation rule and *not* reserve prices for period 1.

6 Concluding Remarks

This paper characterizes the optimal auction under non-commitment. In a two-period model, and assuming "independent private values and risk neutral buyers", we show that a revenue maximizing $PBE-$implementable allocation rule can be implemented by a PBE of the game where the seller runs a 'Myerson' auction with buyer-specific cutoffs in each period. A buyer can either claim a type above his/her cut-off or claim the lowest possible type. If no buyer claims a value above his/her cut-off, no trade takes place in the first period, and the seller runs a 'Myerson' auction in the second period with lower cut-offs. If the buyers are ex-ante symmetric, this rule can be implemented by a sequence of second or first price auctions with a reservation price in each period. The reservation price decreases overtime.

This is the first paper that studies a mechanism design problem under non-commitment in an environment where the principal faces many agents. A methodological contribution of the paper is to develop a procedure to characterize the optimal dynamic incentive schemes under non-commitment in asymmetric information environments with multiple agents, whose types are drawn from a continuum. One cannot appeal to the standard revelation principle to solve such problems. Moreover the recent extension of the revelation principle by Bester and Strausz (2001) does not apply in an environment where the principal faces many agents, see Bester and Strausz (2000)). We hope that the method presented here will prove useful for the characterization of the optimal dynamic incentive schemes under non-commitment in other asymmetric information environments. The assumption of commitment, which makes the characterization of the optimal incentive schemes a relatively straightforward task, implies that the principal will behave in a time-inconsistent manner and it is not very appealing for many applications.

7 Appendix

The Optimal Mechanism at $t = 2$ in the General Case

The analysis here follows closely Myerson (1981). More details and complete arguments can be found there. Our objective is to illustrate how one takes care of the possibility that $g_i(v_i) = 0$. Here $\phi_i(v_i, v_{-i}) = [v_i g_i(v_i) - (1 - G_i(v_i))]\, g_{-i}(v_{-i})$ plays the role of the virtual valuation in Myerson (because g_i may be zero one cannot factor it out and write $v_i - \frac{(1-G_i(v_i))}{g_i(v_i)}$). When ϕ_i is strictly increasing in v_i this is called the regular case. In the general case this is not necessarily true. What we do here is that we 'iron' the part of ϕ_i that depends on v_i. This is called J_i and it is equal to $J_i(v_i) = [v_i g_i(v_i) - (1 - G_i(v_i))]$. The 'ironed' J_i, which will give us the 'ironed' $\hat{\phi}_i$ is obtained as follows. Let

$$H_i(v_i) = \int_{a_i}^{v_i} J(s_i) ds_i \tag{26}$$

and

$$
\begin{aligned}
L_i(v_i) &= conv H_i(v_i) \\
&= \inf \left\{ \begin{array}{c} \lambda H_i(\alpha) + (1-\lambda) H_i(\beta) \text{ such that} \\ \alpha, \beta \in [a_i, b_i],\ \lambda \in [0,1] \text{ and } \lambda\alpha + (1-\lambda)\beta = v_i \end{array} \right\}.
\end{aligned}
$$

L_i is a convex function hence its derivative is increasing:

$$
\hat{J}_i(v_i) = \frac{dL_i(v_i)}{dv_i}.
$$

Now define

$$
\hat{\phi}_i(v_i, v_{-i}) = \hat{J}_i(v_i) g_{-i}(v_{-i});
$$

$$
\hat{\xi}_i(Y_i) = \inf \left\{ v_i \in Y_i,\ \hat{J}_i(v_i) = 0 \right\},\ i \in I;
$$

$$
\Xi_i(Y_i) = \left\{ v_i \in Y_i : v_i \leq \hat{\xi}_i(Y_i) \right\};
$$

$$
\Xi(Y) = \times_{i \in I} \Xi_i(Y_i).
$$

Let

$$
I(v, g) \in \arg\max_{i \in I} \hat{\phi}_i(v_i, v_{-i}). \tag{27}
$$

Consider the assignment function r given by

$$
\begin{aligned}
\text{for } v &\in \Xi(Y) \text{ set } r_i(v) = 0 \text{ for all } i \in I; \\
\text{for } v &\in Y \backslash \Xi(Y) \text{ set } r_i(v) = 1 \text{ if } i \in I(v, g) \\
& \quad\ \text{(where } i \text{ is chosen randomly in case of ties)} \\
\text{and } r_j(v) &= 0 \text{ for } j \neq I(v, g).
\end{aligned}
$$

This assignment function solves

$$
\begin{aligned}
&\max \Sigma_{i \in I} \int_{Y_i} r_i(v) \hat{\phi}_i(v) dv \\
&\text{s.t. } \bar{r}_i(v_i) \text{ increasing}
\end{aligned}
$$

This follows by the following arguments. Since by construction $\hat{J}_i$ is increasing in i, and $v_i > \tilde{v}_i$, buyer i will win with v_i whenever he wins with $\tilde{v}_i$, hence $\bar{r}_i(v_i) \geq \bar{r}_i(\tilde{v}_i)$. The given r assigns the object to the buyer with the highest $\hat{\phi}_i$, $I(v, g)$, if this is non-negative.

It remains to show that the mechanism described above solves (24) which is the actual objective function of the seller. Recall that the seller's objective function can be written as

$$\Sigma_{i\in I}\int_{Y_i} r_i(v)\phi_i(v)dv$$

which is equal to

$$\Sigma_{i\in I}\int_{Y_i} r_i(v)\hat{\phi}_i(v)dv + \Sigma_{i\in I}\int_{Y_i} r_i(v)\left[\phi_i(v)-\hat{\phi}_i(v)\right]dv. \tag{28}$$

Using integration by parts the second term of the above expression can be written as

$$\Sigma_{i\in I}\int_{Y_i} r_i(v)\left[\phi_i(v)-\hat{\phi}_i(v)\right]dv = \Sigma_{i\in I}\int_{Y_i} r_i(v)\left[J_i(v_i)-\hat{J}_i(v_i)\right]g_{-i}(v_{-i})dv \tag{29}$$

$$= \Sigma_{i\in I}\int_{Y_i} \bar{r}_i(v_i)\left[J_i(v_i)-\hat{J}_i(v_i)\right]dv \tag{30}$$

$$= \Sigma_{i\in I}\bar{r}_i(v_i)\left(H_i(v_i)-L_i(v_i)\right)|_{a_i}^{b_i} - \Sigma_{i\in I}\int_{Y_i}\left[H_i(v_i)-L_i(v_i)\right]d\bar{r}_i(v_i).$$

It can be easily seen that $H_i(a_i) = L_i(a_i)$ and $H_i(b_i) = L_i(b_i)$. Hence the first term of the above expression is zero. Substituting (29) back to (28) we obtain

$$\Sigma_{i\in I}\int_{Y_i}\bar{r}_i(v_i)\hat{J}_i(v_i)dv_i - \Sigma_{i\in I}\int_{Y_i}\left[H_i(v_i)-L_i(v_i)\right]d\bar{r}_i(v_i)$$

or

$$\int_{Y}\Sigma_{i\in I}r_i(v)\hat{\phi}_i(v)dv - \Sigma_{i\in I}\int_{Y_i}\left[H_i(v_i)-L_i(v_i)\right]d\bar{r}_i(v_i). \tag{31}$$

Assigning the object to the buyer with the highest $\hat{\phi}_i$ if it is non-negative maximizes the first term of (31) and makes the second term equal to zero. If $H_i(v_i) > L_i(v_i)$, then $\hat{J}_i$ is locally constant, therefore r_i is constant, which implies that $d\bar{r}_i(v_i) = 0$. Therefore the given mechanism maximizes $\int_Y \Sigma_{i\in I}r_i(v)\hat{\phi}_i(v)dv$ which is equivalent to maximizing $\int_Y \Sigma_{i\in I}r_i(v)\phi_i(v)dv$.

Proof of Lemma 3

Because at a PBE a buyer's strategy is a best response at each node we have that

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) \geq \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L), \tag{32}$$

holding all λ_i constant.

We now demonstrate that at a PBE the above inequality must hold with equality, that is

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) = \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) \tag{33}$$

To see this, we argue by contradiction. Suppose that

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) > \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L).$$

Let Δz_i be such that

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) - \Delta z_i = \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) \tag{34}$$

and modify M^2 as follows. Remove all actions that are not chosen by any type of buyer i and increase the expected payment associated with all actions chosen by types $v_i > v_i^H$ by a constant Δz_i, that is

$$\hat{z}_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i}) = z_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i}) + \Delta z_i, \text{ for all } v_{-i} \in Y_{-i}^i \text{ and } \lambda_{-i} \in \Lambda_{-i}(\lambda_i);$$

note that

$$E_{v-i}E_{\lambda_{-i}}\left[\hat{z}_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i})\right] = E_{v-i}E_{\lambda_{-i}}[z_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i}) + \Delta z_i] = \bar{z}_i(v_i^H) + \Delta z_i.$$

Before we move on observe that because at an equilibrium given λ_i, $\bar{r}_i$ is increasing in v_i, no type $v_i < v_i^L$ is choosing the same action at $t = 2$ as v_i^H, (or as any type $v_i \geq v_i^H$), since then it would be $\bar{r}_i(v_i) = \bar{r}_i(v_i^H) > \bar{r}_i(v_i^L)$.

Note that $\hat{M}^2$ is identical to M^2 with the modification that the payments associated with the actions chosen by types greater or equal to v_i^H are increased by Δz_i. Moreover $\hat{M}^2$ does not contain the actions of M^2 that were not chosen by any type of buyer i. We will establish that buyer i will still choose the same actions as before. We some abuse of notation we use

$$\begin{aligned} \hat{r}_i(v_i) &= E_{v-i}E_{\lambda_{-i}}\left[\hat{r}_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i})\right] \text{ and} \\ \hat{z}_i(v_i) &= E_{v-i}E_{\lambda_{-i}}\left[\hat{z}_i(v_i^H, v_{-i}, \lambda_i, \lambda_{-i})\right]. \end{aligned}$$

We now show that at the resulting mechanism, call it $\hat{M}^2$, raises higher revenue for the seller.

Step 1: For every i choosing the same action as before is a best response.

Take $v_i \in Y_i$, with $v_i \leq v_i^L$. Since at a PBE the buyer's strategy is a best response at each node we have

$$\bar{r}_i(v_i)v_i - \bar{z}_i(v_i) \geq \hat{r}_i(v_i')v_i - \bar{z}_i(v_i'), \text{ for all } v_i' \in Y_i$$

which by the definition of $\hat{M}^2$ implies

$$\bar{r}_i(v_i)v_i - \hat{z}_i(v_i) \geq \hat{r}_i(v_i')v_i - \hat{z}_i(v_i'), \text{ for all } v_i' \in Y_i \text{ s.t. } v_i' \leq v_i^L.$$

Moreover for $v_i' \geq v_i^H$ and since $\Delta z_i > 0$, it holds that

$$\bar{r}_i(v_i)v_i - \bar{z}_i(v_i) \geq \bar{r}_i(v_i')v_i - \bar{z}_i(v_i') - \Delta z_i, \text{ for all } v_i' \in Y_i \text{ , s.t. } v_i' \geq v_i^H$$

which, using the definition of $\hat{M}^2$ can be rewritten as

$$\hat{r}_i(v_i)v_i - \hat{z}_i(v_i) \geq \hat{r}_i(v_i')v_i - \hat{z}_i(v_i'), \text{ for all } v_i' \in Y_i \text{ s.t. } v_i' \geq v_i^H. \tag{35}$$

So far we have shown that if the buyer's type v_i is less or equal to v_i^L, he does not have incentive to choose an action chosen by some other type when the seller uses $\hat{M}^2$.

We now show that if $v_i = v_i^H$, type-v_i buyer does not find profitable to choose an action chosen by $v_i \neq v_i^H$. Because we are considering a PBE, buyer $i's$ strategy is a best response

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) \geq \bar{r}_i(v_i')v_i^H - \bar{z}_i(v_i'), \text{ for all } v_i' \in Y_i. \tag{36}$$

Subtracting Δz_i from both sides of (36) we obtain

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) - \Delta z_i \geq \bar{r}_i(v_i')v_i^H - \bar{z}_i(v_i') - \Delta z_i. \tag{37}$$

Using the definition of $\hat{M}^2$ (37) can be rewritten as

$$\hat{r}_i(v_i^H)v_i^H - \hat{z}_i(v_i^H) \geq \hat{r}_i(v_i')v_i^H - \hat{z}_i(v_i'), \text{ for all } v_i' \in Y_i \text{ such that } v_i' \geq v_i^H. \tag{38}$$

Now we will demonstrate that v_i^H does not have an incentive to choose an action chosen by $v_i \leq v_i^L$. Recall (34) that states that

$$\bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) - \Delta z_i = \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L).$$

Since the buyers strategy is a best response we have that

$$\bar{r}_i(v_i^L)v_i^L - \bar{z}_i(v_i^L) \geq \bar{r}_i(v_i')v_i^L - \bar{z}_i(v_i'), \text{ for all } v_i' \in Y_i; \tag{39}$$

moreover $\bar{r}_i$ is increasing in v_i which implies that $\bar{r}_i(v_i^L) \geq \bar{r}_i(v_i')$ for $v_i' \leq v_i^L$. Because $v_i^H > v_i^L$ (39) for $v_i' \leq v_i^L$ implies that

$$\bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) \geq \bar{r}_i(v_i')v_i^H - \bar{z}_i(v_i') \tag{40}$$

which together with (34), reduces to

$$\begin{aligned} \bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) - \Delta z_i &\geq \bar{r}_i(v_i')v_i^H - \bar{z}_i(v_i') \\ \text{for all } v_i' &\in Y^i \text{ such that } v_i' \leq v_i^L. \end{aligned} \tag{41}$$

Using the definition of $\hat{M}^2$ (41) can be rewritten as

$$\hat{r}_i(v_i^H)v_i^H - \hat{z}_i(v_i^H) \geq \hat{r}_i(v_i')v_i^H - \hat{z}_i(v_i'), \text{ for all } v_i' \in Y_i \text{ such that } v_i' \leq v_i^L.$$

From (38) and (41) it follows that when the seller employs $\hat{M}^2$ type v_i^H will choose the same action as before. It is straightforward to show, that since v_i^H does not have incentive to choose another action so does $v_i \geq v_i^H$. We have therefore demonstrated that when the seller employs $\hat{M}^2$ buyers will find optimal to choose the same actions as when she employs M^2.

Step 2: We also need to verify that buyer i gets at least his outside option payoff which is zero. For $v_i \leq v_i^L$ this follows from the fact that M^2 guarantees buyer i payoff zero, we therefore have

$$\bar{r}_i(v_i^L)v_i^L - \bar{z}_i(v_i^L) \geq 0. \tag{42}$$

For $v_i \geq v_i^H$ it suffices to check that this is true for v_i^H. From (42), and since $v_i^H > v_i^L$ it follows that

$$\bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) \geq 0,$$

and by (34)

$$\hat{r}_i(v_i^H)v_i^H - \hat{z}_i(v_i^H) = \bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H) - \Delta z_i = \bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) \geq 0$$

Hence given $\hat{M}^2$ for all the buyers choosing the same actions as with M^2 is a best response at $t = 2$; moreover $\hat{M}^2$ raises strictly higher revenue than M^2. The seller has a profitable deviation at $t = 2$ contradicting the fact that we are considering a PBE. Therefore (33) indeed holds. ■

Proof of Lemma 5

Consider a $v_i \in \bar{Y}_i \backslash Y_i$. Since Y_i is closed there exists an open interval around v_i, that is not in Y_i. Let (v_i^L, v_i^H) denote the largest such interval. We will establish the result by showing that $\bar{p}_i(v_i) = \bar{p}_i(v_i^L)$.

From Lemma 3 we know that after each history where buyer i rejects M^1, the mechanism that the seller employs at $t = 2$ must be such that

$$\bar{r}_i(v_i^L)v_i^H - \bar{z}_i(v_i^L) = \bar{r}_i(v_i^H)v_i^H - \bar{z}_i(v_i^H). \tag{43}$$

Let $\bar{p}_i^2$ denote the expected discounted probability, from the ex-ante point of the view, that buyer i obtains the object at $t = 2$ and $\bar{x}_i^2$ the expected discounted payment , from the ex-ante point of the view, that buyer i has to incur at $t = 2$. For every history that trade does not occur at $t = 1$ and buyer i has rejected M^1 an equality like (43) holds. By taking expectation over all such histories we obtain

$$\bar{p}_i^2(v_i^L)v_i^H - \bar{x}_i^2(v_i^L) = \bar{p}_i^2(v_i^H)v_i^H - \bar{x}_i^2(v_i^H). \tag{44}$$

If a buyer rejects M^1 he will never obtain the object at $t = 1$ no matter what his opponents do. Therefore if v_i rejects M^1 then it must hold that $\bar{p}_i(v_i) = \bar{p}_i^2(v_i)$. Since types in Y_i reject M^1 we have that

$$\bar{p}_i(v_i) = \bar{p}_i^2(v_i) \text{ and } \bar{x}_i(v_i) = \bar{x}_i^2(v_i). \tag{45}$$

Also from (44) and (45) we have that

$$\bar{p}_i(v_i^L)v_i^H - \bar{x}_i(v_i^L) = \bar{p}_i(v_i^H)v_i^H - \bar{x}_i(v_i^H). \tag{46}$$

We now demonstrate that $\bar{p}_i(v_i) = \bar{p}_i(v_i^L)$ for all $v_i \in (v_i^L, v_i^H)$.

We will argue by contradiction. Suppose that there exists $v_i \in (v_i^L, v_i^H)$ such that $\bar{p}_i(v_i) \neq \bar{p}_i(v_i^L)$. Note that since we are looking at a PBE it must be the case that

$$\bar{p}_i(v_i)v_i - \bar{x}_i(v_i) \geq \bar{p}_i(v_i^L)v_i - \bar{x}_i(v_i^L)$$

$$\left[\bar{p}_i(v_i) - \bar{p}_i(v_i^L)\right] v_i \geq \bar{x}_i(v_i) - \bar{x}_i(v_i^L).$$

Since $\bar{p}_i$ is increasing we have that $\bar{p}_i(v_i) \geq \bar{p}_i(v_i^L)$ and because $\bar{p}_i(v_i) \neq \bar{p}_i(v_i^L))$ it holds that $\bar{p}_i(v_i) > \bar{p}_i(v_i^L)$. From this observation and the fact that $v_i^H > v_i^L$, (recall that $v_i \in (v_i^L, v_i^H)$), we have that

$$\begin{aligned} \left[\bar{p}_i(v_i) - \bar{p}_i(v_i^L)\right] v_i^H &> \bar{x}_i(v_i) - \bar{x}_i(v_i^L) \text{ or} \\ \bar{p}_i(v_i)v_i^H - \bar{x}_i(v_i) &> \bar{p}_i(v_i^L)v_i^H - \bar{x}_i(v_i^L) \end{aligned}$$

or by (46)

$$\bar{p}_i(v_i)v_i^H - \bar{x}_i(v_i) > \bar{p}_i(v_i^H)v_i^H - \bar{x}_i(v_i^H),$$

which implies that v_i^H can benefit by mimicking the behavior of v_i. Contradiction. Therefore $\bar{p}_i(v_i) = \bar{p}_i(v_i^L)$ for all $v_i \in (v_i^L, v_i^H)$. ■

Proof of Lemma

Given the TBR we know that the set of types of buyer i that reject M^1 will be of the form $[a_i, \bar{v}_i]$. Let $\bar{v}_i$ denote the highest type of buyer i that rejects M^1, that is $\bar{v}_i$ is greater or equal to ξ_i^1 which, as the reader may recall, is defined by

$$\xi_i^1 \text{ solves } v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = 0;$$

We want to establish that at a revenue maximizing PBE $\bar{v}_i \geq \xi_i^1$ for all i. This inequality says that the seller never wants to trade with strictly positive probability at $t = 1$ with a buyer whose virtual valuation is strictly negative. Let us consider buyer i and suppose that for $v_i \in [a_i, \bar{v}_i]$ buyer i rejects M^1 with probability one at $t = 1$. Then the seller's posterior beliefs after the buyer rejects M^1 are given by $G_i(v_i) = \frac{F_i(v_i)}{F_i(\bar{v}_i)}$ and the corresponding virtual valuation is given by

$$v_i - \frac{1 - \frac{F_i(v_i)}{F_i(\bar{v}_i)}}{\frac{f_i(v_i)}{F_i(\bar{v}_i)}} = v_i - \frac{\frac{F_i(\bar{v}_i) - F_i(v_i)}{F_i(\bar{v}_i)}}{\frac{f_i(v_i)}{F_i(\bar{v}_i)}} = v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)},$$

Given this posterior virtual valuation we know that at $t = 2$ buyer i will never receive the object with strictly positive probability at $t = 2$ if his valuation is below ξ_i^2 where ξ_i^2 solves

$$\xi_i^2 \text{ solves } v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)} = 0. \tag{47}$$

From Lemma 8 we know that if the prior satisfies MHR so will a posterior that is a downward truncation of the prior distribution. Hence (47) has a unique solution. Moreover this cut-off can be alternative obtained as the solution of the optimal price by a monopolist who is facing a downward sloping demand $[F_i(\bar{v}_i) - F_i(\xi_i^2)]$. The monopolist problem is

$$[F_i(\bar{v}_i) - F_i(\xi_i^2)]\xi_i^2$$

The FOC necessary conditions for a maximum (which are also sufficient given MHR), are

$$\begin{aligned}
[F_i(\bar{v}_i) - F_i(\xi_i^2)] - f_i(\xi_i^2)\xi_i^2 &= 0 \text{ or since } f_i(\xi_i^2) > 0 \\
v_i - \frac{F_i(\bar{v}_i) - F_i(\xi_i^2)}{f_i(\xi_i^2)} &= 0
\end{aligned}$$

Since f_i is continuous so will F_i which ensures continuity of the monopolists objective function. From the Theorem of the Maximum we then have that ξ_i^2 is a continuous function of $\bar{v}_i$ and hence it is differentiable almost everywhere. Let us examine the optimal cut-off for buyer i ignoring the fact that there exist other buyers for the moment. Given an allocation rule in $\mathcal{P}^{PBE}$ expected revenue from buyer i is given by

$$\int_{\xi_i^2(\bar{v}_i)}^{\bar{v}_i} \delta\bar{r}_i(v_i)J_i(v_i)f_i(v_i)dv_i + \int_{\bar{v}_i}^{b} \bar{p}_i(v_i)J_i(v_i)f_i(v_i)dv_i.$$

Let us now differentiate with respect to $\bar{v}_i$: we obtain that

$$\begin{aligned}
&\delta\bar{r}_i(\bar{v}_i)J_i(\bar{v}_i)f_i(\bar{v}_i) - \delta\bar{r}_i(\xi_i^2(\bar{v}_i))J_i(\xi_i^2(\bar{v}_i))f_i(\xi_i^2(\bar{v}_i))\frac{\partial\xi_i^2(\bar{v}_i)}{\partial\bar{v}_i} - \bar{p}_i(\bar{v}_i)J_i(\bar{v}_i)f_i(\bar{v}_i) \qquad (48) \\
= \; & -\left(\bar{p}_i(\bar{v}_i) - \delta\bar{r}_i(\bar{v}_i)\right)J_i(\bar{v}_i)f_i(\bar{v}_i) - \delta\bar{r}_i(\xi_i^2(\bar{v}_i))J_i(\xi_i^2(\bar{v}_i))f_i(\xi_i^2(\bar{v}_i))\frac{\partial\xi_i^2(\bar{v}_i)}{\partial\bar{v}_i}
\end{aligned}$$

by the monotonicity of $\bar{p}_i$ we have that

$$\bar{p}_i(\bar{v}_i) \geq \delta\bar{r}_i(\bar{v}_i);$$

from Lemma 6 in Skreta 2004, we have that $\xi_i^2(\bar{v}_i)$ is increasing in $\bar{v}_i$ from which we obtain that

$$\frac{\partial\xi_i^2(\bar{v}_i)}{\partial\bar{v}_i} \geq 0.$$

From the last two observations it follows that (48) is strictly positive for $\bar{v}_i < \xi_i^1$ since if this is the case we have $J_i(\bar{v}_i) < 0$ and $J_i(\xi_i^2(\bar{v}_i)) < 0$. From these arguments alone we could conclude at the revenue maximizing PBE it cannot be the case that $\bar{v}_i < \xi_i^1$. But we have ignored the effect of a change of $\bar{v}_i$ of the probabilities of obtaining the object in the second period. A change in $\bar{v}_i$ will affect also q_j^2 for $j \in \{1, ..., I\}$ via the effect that it will have on the ranking of the virtual valuations. Recall that buyer i wins the object at $t = 2$ if his posterior virtual valuation is the highest among all ones and it is non-negative, that is we must have

$$\begin{aligned} v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)} &\geq v_j - \frac{F_j(\bar{v}_j) - F_j(v_j)}{f_j(v_j)} \text{ and} \\ v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)} &\geq 0 \end{aligned}$$

there is a special issue here - the non-negativity constraint does not depend of the whole vector of types. Now let us consider a small increase in $\bar{v}_i$, that is $\bar{v}_i + \varepsilon$, for some $\varepsilon > 0$, than the ranking will change. Let

$$\begin{aligned} &\hat{v}_i, \hat{v}_1 \text{ denote the vector of types where} \\ \hat{v}_i - \frac{F_i(\bar{v}_i) - F_i(\hat{v}_i)}{f_i(\hat{v}_i)} &= v_1 - \frac{F_1(\bar{v}_1) - F_1(\hat{v}_1)}{f_1(\hat{v}_1)} \end{aligned}$$

and i may win with $\bar{v}_i$ but may loose with $\bar{v}_i + \varepsilon$, in which case the object goes to 1. Now the vector of valuations where the inequality flips is $(\hat{v}_1, \hat{v}_i, ...)$ and the same for the remaining of the buyers. Let $r^2(v)$ denote the allocation at $t = 2$ given $\bar{v}_i$ and let $\hat{r}^2$ denote the allocation rule at $t = 2$ given cut-off $\bar{v}_i + \varepsilon$. The vector of types where these two rules differ are at points where the ranking of $i's$ virtual valuation flips and are located at points where virtual valuations are equal to each other

$$\Sigma_{i \in I} \int_Y [r_i^2(v) - \hat{r}_i^2(v)] J_i(v) f(v) dv$$

but $r_i^2(v) = \hat{r}_i^2(v)$ for all $v \in Y$ except the vectors of valuation where the ranking of virtual valuations changes all candidates for that are

$$\hat{v}_1, \hat{v}_i, v_2, v_3, ..., v_I, \text{ where } v_2 \in Y_2, .., v_I \in Y_I$$

and so forth for all buyers, but these set of vectors is of measure zero. (we have fixed $\hat{v}_1, \hat{v}_i, ...$) so when we integrate with respect to $\hat{v}_1$ and to $\hat{v}_i$ we will get zero. In short the effect of a change in $\bar{v}_i$ on the second period allocation has expected zero

$$\Sigma_{i \in I} \int_Y [r_i^2(v) - \hat{r}_i^2(v)] J_i(v) f(v) dv = 0$$

It will only affect the allocation for a vector of types of measure zero. Hence our preliminary analysis captures the effect of $\bar{v}_i$. We can hence conclude that at the revenue maximizing PBE it cannot be the case that $\bar{v}_i < \xi_i^1$.

Proof of Proposition 4

We start by repeating a few definitions introduced in the main text:

$$\Phi_i(v_i) = v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)}.$$

For each $v \in V$, $v = (v_1, v_2, ..., v_I)$, we use $I(v, f)$ to denote the buyer whose valuation corresponds to the highest Φ_i

$$I(v, f) \in \arg\max_{i \in I} \Phi_i(v_i);$$

also we use ξ_i^1 to denote the solution to $\Phi_i(v_i) = 0$. Given MHR for all $v_i \geq \xi_i^1$ we have that $\Phi_i(v_i) \geq 0$ and for all $v_i < \xi_i^1$ we have that $\Phi_i(v_i) < 0$. We designate as $\Xi_i^1 = [a_i, \xi_i^1]$ the set of $i's$ valuations such that $\Phi_i(v_i)$ is negative, and by Ξ^1 the set of vectors $v = (v_1, ..., v_I)$ where the virtual valuations of all buyers are negative, that is $\Xi^1 = \times_{i \in I}[a_i, \xi_i^1]$.We now continue with the proof.

Consider an assessment (σ, μ) that implements an allocation rule $p \in \mathcal{P}$. Let Y denote the support of the seller's posterior after all buyers reject M^1 and let M^2 denote the mechanism that the seller employs at $t = 2$ after all buyers reject. We assume that Y has strictly positive measure. Given the imposed tie-breaking rule Y is convex, and from Lemma 7 we have that $\Xi^1 \subset Y$. We show that there exist $\hat{p} \in \mathcal{P}^*$ such that $R(\hat{p}) \geq R(p)$.

Now consider an assessment $(\hat{\sigma}, \hat{\mu})$, that implements an allocation rule $\hat{p}$ such that for $v \in V \backslash Y$ $\hat{p}$ assigns the object with probability 1 to the buyer with the highest virtual valuation, that is

$$\text{for } v \in V \backslash Y \ \hat{p}_i(v) = \begin{cases} 1 \text{ if } i = I(v, f) \\ 0 \text{ otherwise} \end{cases}.$$

Let $\hat{M}^1$ denote the mechanism that the seller proposes at $t = 1$ according to $(\hat{\sigma}, \hat{\mu})$. Types in Y reject $\hat{M}^1$. At $t = 2$ after the history where all buyers have rejected $\hat{M}^1$, the seller employs M^2 that is optimal given for $v \in Y$ all buyers at $t = 1$ reject M^1 in the first period, that is for $v \in Y$ we have that

$$\hat{p}(v) = p(v) \text{ and } \hat{x}(v) = x(v). \tag{49}$$

From Proposition 2 we know that M^2 assigns the object at $t = 2$ with probability 1 to the buyer that has the highest posterior virtual valuation. Given this observation note that $\hat{p}$ is an element of $\mathcal{P}^*$. We now proceed to show that $\hat{p}$ generates higher revenue for the seller than p.

Expected revenue for the seller at an assessment that implements $\hat{p}$ is given by

$$R(\hat{p}) = \int_Y \Sigma_{i \in I} \hat{p}_i(v) \Phi_i(v_i) f(v) dv + \int_{V \backslash Y} [\Phi_{I(v,f)}(v_{I(v,f)}) + \Sigma_{\substack{i \in I \\ i \neq I(v,f)}} 0 \cdot \Phi_i(v_i)] f(v) dv. \tag{50}$$

Then because of (49)

$$R(\hat{p}) - R(p) = \int_{V \backslash Y} [\Phi_{I(v,f)}(v_{I(v,f)}) - \Sigma_{i \in I} \bar{p}_i(v) \Phi_i(v_i)] f(v) dv \geq 0 \tag{51}$$

where the last inequality follows from the fact that for all $i \in I$ and $v \in V \backslash \Xi^1$ $\Phi_{I(v,f)}(v_{I(v,f)}) \geq \Phi_i(v_i) \geq 0$ (Recall that in this case $\Xi^1 \subset Y$).

From (51) we have that

$$R(\hat{p}) - R(p) \geq 0.$$

■

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