Sequentially Optimal Mechanisms*

Vasiliki Skreta†
UCLA

January 2005

Abstract

This paper characterizes the revenue maximizing allocation mechanism in a T-period model under non-commitment. A risk neutral seller has one object to sell and faces a risk neutral buyer whose valuation is private information and drawn from an arbitrary bounded subset of the real line. The seller has all the bargaining power; she designs a mechanism to sell the object at t but cannot commit not to propose another mechanism at $t+1$ if trade does not occur at t. A mechanism consists of a game form and is endowed with a communication device (mediator). The buyer may employ mixed strategies. We show that the optimal mechanism is to post a price in each period. A methodological contribution of the paper is to develop a procedure to characterize the optimal dynamic incentive schemes under non-commitment in asymmetric information environments that is valid irrespective of the structure of the agent's type. *Keywords: mechanism design, non-commitment, bargaining under incomplete information, optimal auctions, durable good monopoly. JEL Classification Codes: C72, D44, D82.*

*This work is based on Chapter 1 of my doctoral thesis submitted to the Faculty of Arts and Sciences at the University of Pittsburgh. I am indebted to Masaki Aoyagi and Phil Reny for superb supervision, guidance and support. I am especially grateful to Kim-Sau Chung and Radu Saghin. Many thanks to Andreas Blume, Roberto Burguet, Hal Cole, Dean Corbae, Jacques Cremer, Jim Dana, Peter Eso, Philippe Jehiel, Patrick Kehoe, Erzo Luttmer, Ellen McGrattan, Mark Satterthwaite, Kathy Spier, Dimitri Vayanos, Asher Wolinsky and Charles Zheng for helpful discussions and comments. I would also like to thank the Department of Management and Strategy, KGSM, Northwestern University for its warm hospitality, the Andrew Mellon Pre-Doctoral Fellowship, the Faculty of Arts and Sciences at the University of Pittsburgh and the TMR Network Contract ERBFMRXCT980203 for financial support. A significant amount of this work was performed while I was at the University of Minnesota and the Federal Reserve Bank of Minneapolis. All remaining errors are my own.

†*Department of Economics, University of California, Los Angeles; Box 951477, Los Angeles, CA 90095-1477, USA. T 310 794 7241, F 310 825 9528, skreta@econ.ucla.edu*

1. Introduction

This paper establishes that the revenue maximizing allocation mechanism in a T-period model where the seller behaves sequentially rationally, is to post a price in each period. It also develops a methodology to derive the optimal mechanism under non-commitment in asymmetric information environments. The bargaining under incomplete information literature[1] acknowledges that bargainers behave sequentially rationally. If it is common knowledge that gains of trade exist, parties cannot credibly commit to stop negotiating at a point where no agreement is reached.[2] It examines possible outcomes of negotiations between individuals under the assumption that players make deterministic offers at each round. Often the right to make offers is assigned to one of the negotiating parties. Suppose that the uninformed one makes the offers. Would it be beneficial for her instead of making a take-it-or-leave-it offer at each round, to employ more sophisticated bargaining procedures? Would that possibility allow her to learn the other party's private information faster? What is the optimal negotiating process from the uninformed party's point of view? The aim of this paper is to answer these questions.

Our characterization extends the literature on optimal negotiation/selling mechanisms, by requiring the seller to behave sequentially rationally, and it provides a foundation for take-it-or-leave-it offers in the bargaining and the durable goods monopoly literatures.

In the optimal auction literature it is assumed that the individual choosing the rules, the mechanism designer, can commit never to propose another mechanism in the future, even in the event that the mechanism she initially chose failed to realize any of the existing gains of trade. This excludes the possibility of employing a mechanism in the future that may perform better. It requires that the seller behave in a non-credible way at $t = 2$, as it is often far-fetched to assume that the seller will indeed throw away a valuable object at $t = 2$ if it does not sell at $t = 1$.[3] Riley and Zeckhauser (1983) provide a characterization of the optimal selling procedure under commitment in an similar environment that is closest to ours and show that at the optimum the seller posts a price.[4] In this paper we drop the assumption of commitment and characterize the optimum requiring that the seller behaves optimally given the information that she has at each point.

On the other hand, the literatures on bargaining under incomplete information and on durable goods monopoly, (Stokey (1981), Bulow (1982), Gul-Sonnenschein-Wilson (1986)), acknowledge the impossibility of commitment, and require that the seller behaves sequentially rationally, but restrict her strategy to be

[1] See, for instance, Sobel and Takahashi (1983) and Fudenberg, Levine and Tirole (1985).

[2] Bargainers *do* commit to the protocol within each negotiation round. "Non-commitment" means that bargainers do not stop negotiating if they now that gains of trade exist.

[3] Real world examples about the inability of the sellers to commit can be found in McAfee and Vincent (1997).

[4] The optimal auction literature, (see the seminal contributions of Myerson (1981) and Riley and Samuelson (1981)) characterizes the optimal selling procedure under commitment when the seller faces many buyers whose valuations are private and independently distributed. For the case of a single buyer the optimal auction is a posted price.

a sequence of posted prices. In this paper we provide a foundation for posted prices, or take-it-or-leave-it offers, by considering all possible alternative institutions and showing that a seller who behaves sequentially rationally, can do no better than to simply post prices. Our result provides a foundation for posted prices in the durable-good monopoly literature. Hart and Tirole (1988), HT, analyze a similar problem in a finite-horizon framework under non-commitment and commitment and renegotiation. In the non-commitment case the seller's strategy consists of a sequence of *prices*. Our model differs from the one in HT in that we consider a continuum of types and in that we allow the seller to employ arbitrary mechanisms. McAfee and Vincent (1997) examine sequentially optimal auctions under the assumption that the seller's strategy is a sequence of *reservation prices*, and the buyers follow a stationary strategy.

We look at the following scenario. An uninformed party, the seller, owns a unit of an indivisible object and faces a buyer whose valuation is unknown to the seller. It is commonly known that the buyer's valuation is distributed according to some distribution F whose support is a measurable and bounded subset of the real line. The seller and the buyer interact for $T < \infty$ periods, (or stages), and they discount future payoffs with the same discount factor δ. The game ends as soon as trade takes place, or in the bargaining interpretation of the model, as soon as agreement in reached. At each stage the seller proposes a procedure to sell the object, which can be arbitrarily complicated. In particular, the seller proposes a game in normal form for the buyer to play. The outcome of this stage game determines the probability of trade and the payment at that stage. The seller also uses a mediator where the buyer can send messages, and in turn receive recommendations on how to play the given game that the seller proposed. We require that both the seller and the buyer behave optimally at each stage. This implies that the seller at each stage t, and after each sequence of moves where no trade has taken place up to t, chooses a game form and a mediator optimally given the information that she has obtained from her interaction with the buyer up to t. Technically, we require the buyer's and the seller's strategies together with the seller's beliefs to be a Perfect Bayesian Equilibrium, PBE, and our objective is to characterize a PBE that guarantees highest expected discounted revenue for the seller. We show that even though the seller can be employing any procedure, the optimal procedure is to simply post a price in each period. And in the bargaining interpretation of the model, our result says that if the uniformed party is choosing the negotiation procedure at each stage, she can do no better, than simply making take-it-or-leave-it offers.

Another contribution of this work is methodological. This is the first paper that provides a complete characterization of the optimal mechanism under non-commitment in an asymmetric information environment, where the agent's type is not restricted to be finite. In order to do so, we develop a solution method that is valid irrespective of whether the type space is finite or a continuum.[5] The early papers on

[5] In fact our analysis is very general, and valid for both the case of a continuum of types and finite types, since we allow the support of the distribution of the buyer's type to be any measurable bounded subset of the real line. In a $T-$ period model even if one starts with a continuum of types, because of the complexity of the strategy spaces the support of the principal's

dynamic mechanism design, (Freixas, Guesnerie and Tirole (1985), FGT, Laffont and Tirole (1988)), LT, establish that under non-commitment the principal cannot appeal to the standard revelation principle in order to characterize the optimal mechanism. This makes the characterization of the optimal contract extremely difficult.[6] For this reason FGT characterize the optimal incentive schemes among the class of *linear* incentive schemes. LT consider arbitrary schemes but examine only special classes of equilibria, without characterizing the optimum. A remarkable result is derived in a recent paper by Bester and Strausz (2001), BS, who show that when the principal faces *one* agent whose type space is *finite,* she can, without loss of generality, restrict attention to mechanisms where the message space has the same cardinality as the type space. As BS illustrate, in order to find the optimal mechanism one has to check which incentive compatibility constraints are binding. In an environment with limited commitment, constraints may be binding 'upwards' and 'downwards'. Even if one could obtain an analog of the BS result for the continuum type case, which is indeed challenging, it does not seem straightforward to generalize the procedure of checking which incentive compatibility constraints are binding. Our method is based on looking for equilibrium outcomes and is valid irrespective of the structure of the type space.

In the model employed in this paper the seller in each stage proposes a mechanisms that consists of a game form endowed with a communication device (mediator). As a consequence, the buyer at each stage is choosing costless actions, the messages that he sends to the mediator, as well as costly actions, the action chosen in the game form. To the best of our knowledge this is the first paper that allows for such general model of mechanisms in an environment where the mechanism designer behaves sequentially rationally. Now given that a mechanism in the paper is defined as a game form together with a mediator, a question that naturally arises is what the seller observes at each stage, that is whether the seller observes merely the action chosen by the buyer and whether trade took place or not, or she also observes the exchange of messages between the buyer and the mediator. We provide the characterization of the optimum under various assumptions regarding the amount of information that the seller observes at each stage. This is the first work that examines the role of costly as well as cheap information simultaneously. How transparent institutions are is intimately related with the "commitment power" of the mechanism designer: if she obtains no information throughout play, then the "commitment solution" is sequentially rational.

posterior beliefs at $t = 2$ can be arbitrarily complicated, (whereas in the case where one starts with a finite type space, the type space at the beginning of a subsequent period is again finite). And since the game that starts at $t = 2$ is isomorphic to the whole game - with the difference that it lasts one period less, we write our model directly for arbitrary type spaces.

[6] See the discussion in Laffont and Tirole (1993), Ch. 9, and Salanie (1998), Ch.6.

2. The Environment

In this section we describe the formal setup considered in this paper. We first describe players and their preferences, the timing of the game; we then define mechanisms, assessments, social choice functions, outcomes of the game, what we mean by implementation, and finally our equilibrium concept.

A risk neutral seller owns a unit of an indivisible object. Her valuation for the object is normalized to zero. She faces one risk neutral buyer whose valuation v is private information and is distributed on V according to F. The convex hull of V is an interval $[a, b]$ where $-\infty < a \leq b < \infty$. All elements of the game except the realization of the buyer's valuation are common knowledge. Time t is discrete and $t = 1, ..., T < \infty$. Both the seller and the buyer discount the future with the same discount factor, δ. The seller's goal is to maximize expected discounted revenue. The buyer aims to maximize surplus. We now describe the timing of the game.

Timing

- At the beginning of period $t = 1$ nature determines the valuation of the buyer. Subsequently the seller proposes a mechanism. The mechanism is played and if the buyer obtains the object the game ends, else we move on to period $t = 2$.
- At $t = 2$ the seller proposes a mechanism. The mechanism is played and if the buyer obtains the object the game ends, else we move on to period $t = 3$.

....

- At $t = T$ the seller proposes a mechanism. The mechanism is played and the game ends at the end of period T irrespective of whether trade takes place or not.
- At any t the buyer can obtain his outside payoff which is normalized to zero, by not participating in the mechanism proposed by the seller.

We continue by defining what we mean by the word "mechanism."

Mechanisms

A mechanism consists of a game form and a communication system, (mediator). A *game form* $G_t = (S_t, g_t)$ consists of a set of actions S_t and a mapping $g_t : S_t \to [0, 1] \times \mathbb{R}$ that maps an action $s_t \in S_t$ into outcomes. An outcome of a mechanism is a probability that the buyer obtains the object at period t, $r_t(s_t) \in [0, 1]$ and an expected payment $z_t(s_t) \in \mathbb{R}$. A pair $(r_t(s_t), z_t(s_t))$ is called a contract. A *communication system,* (mediator) consists of a set of reports that the buyer sends B_t, a set of recommendations N_t that the buyer may receive, and a mapping $c_t : B_t \rightarrow \Delta(N_t)$ that maps a report of the buyer β_t to probability measures over recommendations $n_t \in N_t$. The number $c_t(n_t \,|\beta_t)$ denotes the conditional probability that n_t would be the recommendation received by the buyer when he reports β_t.[7] A game form together with a mediator is a *mechanism*. The set of all possible mechanisms is denoted by $\mathcal{M}$.

The purpose of the mediator is to allow the informed party, (the buyer), to send payoff irrelevant, "cheap" messages to the uninformed party, (the seller).[8] The message β that the buyer submits to the mediator is cheap because his payoff at t is determined only by the action s_t. The same is true for the recommendation that the buyer receives from the mediator n_t. Even though both pieces of information β_t and n_t are "cheap" they differ in the following sense. Messages β_t have to satisfy best response constraints, whereas n_t comes out of the communication device and hence is not subject to such constraints. From the seminal contribution of the literature on strategic information transmission by Crawford and Sobel (1982), CS, we know that such "cheap" information affects equilibria because it affects the beliefs of the responder, which in our case is the seller. Through the exchange of cheap messages the mediator may expand the set of equilibrium payoffs. This completes our discussion on the role of mediators in the definition of mechanisms. Now we can move on and talk about strategies and assessments.

Assessments

An *assessment* consists of a strategy profile and a belief system. A *strategy profile* $\sigma = (\sigma_i)_{i=S,B}$, specifies a strategy for each player. In order to talk about strategies we need a couple of more pieces of notation. Let $\Delta(B_t)$ denote the set of probability measures over B_t and let $\Delta(S_t)$ denote the set of probability measures over S_t. With a slight abuse of notation we use β_t to denote a probability distribution over messages in B_t. Let I_S and I_B denote the information sets of the seller and the buyer respectively. A strategy for the seller, σ_S, is a sequence of maps from I_S to $\mathcal{M}$. A *behavioral* communication strategy of the buyer, σ_B, consists of a mapping from $V \times I_B$ to a probability distribution over reports, and a mapping from $V \times I_B \times N_t$ to a probability distribution over S_t, that is $\sigma_B^t(v, i_B^t) = \{(\beta_t, \delta_t) \text{ s.t. } \beta_t \in \Delta(B_t),\ \delta_t : N_t \rightarrow \Delta(S_t) \,|v, i\}$. We use $\delta(s \,|n_t)$ to denote the probability that the buyer chooses s given that he receives recommendation n_t. A *belief system*, μ, maps I_S to the set of probability distributions over V. Let $F(v|i_S^t)$ denote the seller's beliefs about the buyer's valuation at information set i_S^t, $t = 1, ..., T$. Sometimes when we are referring to

[7] As we will argue later our analysis can also handle the possibility that the seller submits messages into the communication device, so long as these messages are observed by the buyer.

[8] Mediators also function as coordination devices. From the literature on correlated and communication equilibrium we know that "helping" players to coordinate play, may extend the set of equilibria, and hence the set of implementable social choice functions. In our single-agent environment the role of the communication device as a way to coordinate play in each stage is indeed limited: in a one player game there is no need for coordination.

a particular information set we simplify the notation by setting $F(v|i_S^t) = F_t(v)$, $t = 1, ..., T$.[9] For $t = 1$ $F(v|i_S^1) = F(v)$, that is, the seller has correct prior beliefs. A strategy profile σ and a belief system μ is an assessment.

Given an assessment, (no need to be an equilibrium), the outcome from the *ex-ante* point of view is an *allocation rule* $p(\sigma, \mu)$ and a *payment rule* $x(\sigma, \mu)$. The rule $p(\sigma, \mu)(v)$ is the expected, discounted probability that a $v-$type buyer will obtain the object given the assessment (σ, μ) when his valuation is v

$$p(\sigma, \mu)(v) = \sum_{t=1}^{T} \left[\delta^t 1_{\{\text{trade at } t\}} |(\sigma, \mu), v\right]$$

and $x(\sigma, \mu)(v)$ is the expected, discounted payment that a $v-$type buyer will incur given (σ, μ) and it is formally defined as

$$x(\sigma, \mu)(v) = \sum_{t=1}^{T} \delta^t \left[1_{\{\text{trade at } t\}} \cdot \{\text{expected payment at } t\} |(\sigma, \mu), v\right].$$

It is possible that different strategy profiles lead to the same allocation rule and payment rules.

It will be useful to define the outcomes of assessments at a continuation game that starts at t when the seller's information set is i_t^S. At t and each history of moves where trade has not taken place up to t, the situation is isomorphic to the game as a whole, with the only difference being that the seller's beliefs are now given by $F_t(.|i_t^S)$. Then we can talk about the social choice function implemented by the assessment (σ, μ) at the continuation game that starts at t, when the seller's information set is i_t^S. Let $p^{t,i_t^S}(\sigma, \mu)$, $x^{t,i_t^S}(\sigma, \mu)$ denote the allocation and the payment rule implemented by the restriction of a strategy profile σ at a continuation game that starts at information set i_t^S. The objects $p^{t,i_t^S}(\sigma, \mu)$, $x^{t,i_t^S}(\sigma, \mu)$ are the analogs of $p(\sigma, \mu), x(\sigma, \mu)$ for the particular continuation game under consideration. As before, we will often suppress the notation (σ, μ).

In order to talk about implementation we define what we mean by social choice functions in the environment under consideration.

Social Choice Functions, Allocation Rules and Payment Rules

In the environment under consideration a social choice function specifies for each valuation of the buyer v and each period t a probability of trade $r_t(v) \in [0, 1]$ and an expected payment $z_t(v) \in \mathbb{R}$. Given a social choice function $\{r_t(v), z_t(v)\}_{t=1}^T$ we can define the corresponding *allocation rule* p and *payment rule* x by

$$\begin{aligned} p(v) &= r_1(v) + (1 - r_1(v))\delta [r_2(v) + (1 - r_2(v))\delta [.....]] \text{ and} \\ x(v) &= z_1(v) + (1 - r_1(v))\delta [z_2(v) + (1 - r_2(v))\delta [.....]] . \end{aligned}$$

[9] But the reader should keep in mind that there are many different histories that lead to no trade up to period t and for each of such history there is in general a different posterior.

Allocation rules map valuations to expected discounted probabilities of trade and payment rules map valuations to expected discounted payments. The expected payoff of the buyer and the seller respectively, from the interim point of view is given by

$$\begin{aligned} \text{Expected Payoff of type } v \text{ buyer} &= p(v)v - x(v) \text{ and} \\ \text{Expected Payoff of the seller} &= \int_a^b x(v)dF(v), \end{aligned}$$

where the integral in the expression of the seller's expected payoff comes from the fact that the seller does not know the valuation of the buyer. There are many different social choice functions $\{r_t(v), z_t(v)\}_{t=1}^T$ that lead to the same $p(v)$ and $x(v)$, and hence to the same payoffs to the buyer and the seller. All such social choice functions are equivalent for our purposes and hence when we talk about a social choice function we will simply mean their "reduced versions"given by p and x. Now that we have specified what we mean by a social choice function we can talk about implementation.

An assessment (σ, μ) implements the (reduced) social choice function p and x if for all $v \in V$ $p(\sigma, \mu)(v) = p(v)$ and $x(\sigma, \mu)(v) = x(v)$.

The set of social choice functions that we implement depends on the solution concept. In this paper we require assessments to form a Perfect Bayesian Equilibrium.

Solution Concept

A *Perfect Bayesian Equilibrium, (PBE),* is a strategy profile, σ, and a belief system, μ, that satisfy:

1. For all $v \in [a, b]$ the buyer's strategy is a best response at each i_b^t and t, given the seller's strategy.

2. Given $F_t(.|i_S^t)$ and the buyer's strategy, the seller chooses at each i_S^t and t an optimal mechanism.

3. $F_t(.|i_S^t)$ is derived from F_{t-1} given i_S^t using Bayes' rule whenever possible.

At a PBE we require strategies to dictate optimal behavior at each information set. How do the buyer's and the seller's information sets look like depends on what they observe during play. The amount of information that the seller observes determines, is some sense, her commitment power[10] and we call it *degree of transparency* of mechanisms.

Transparency of Mechanisms

Definition 1 *The degree of transparency of a mechanism is the amount of information that the seller observes at each stage.*

[10] If the seller does not observe anything, then all $BNE's$ are $PBE's$ and the "commitment solution" is sequentially rational.

We assume that the seller observes the message that the buyer submits to the mediator, the action that he chooses as well as whether trade takes place or not. In the "Technical Appendix for Sequentially Optimal Mechanisms" we establish that our result in a number of alternative assumptions regarding the degree of transparency of mechanisms. Throughout we assume that the buyer observes the mechanism that the seller proposes at each stage as well as whether trade takes place or not.

The buyer's strategy is a best response at each node, if there is no type v and no information set i_t^B where the buyer can obtain higher expected payoff by behaving differently. The buyer can deviate either by claiming a different report to the mediator or by choosing a different action. The requirement that the buyer cannot benefit by submitting a different message to the mediator implies for all $v \in V$ and $i_B^t \in I_B$ that the following inequality must hold

$$\begin{aligned} & \int_{n_t \in N} c_t(n_t \,|\beta_t) \delta(s_t \,|n_t) [r(s_t)v - z(s_t) + p^{t+1,i_{t+1}^S}(v)v - x^{t+1,i_{t+1}^S}(v)] dn_t \qquad (1) \\ \geq & \int_{n_t \in N} c_t(n_t \left| \hat{\beta}_t \right.) \delta(s_t \,|n_t) [r(s_t)v - z(s_t) + p^{t+1,i_{t+1}^S}(v)v - x^{t+1,i_{t+1}^S}(v)] dn_t, \end{aligned}$$

for all β_t that are in the support of the buyer's communications strategy at t, and all $\hat{\beta}_t \in B_t$. Myerson (1991) refers to the above constraints as adverse selection constraints. The requirement that the buyer is choosing an optimal action implies that the following inequality must hold

$$\begin{aligned} & r(s_t)v - z(s_t) + p^{t+1,i_{t+1}^S}(v)v - x^{t+1,i_{t+1}^S}(v) \qquad (2) \\ \geq \; & r(\hat{s}_t)v - z(\hat{s}_t) + p^{t+1,i_{t+1}^S}(v)v - x^{t+1,i_{t+1}^S}(v), \end{aligned}$$

for all actions in $s_t \in S_t$ that $\delta(s_t \,|n_t)$ assigns positive probability, and for all $\hat{s}_t \in S_t$. Myerson (1991) calls these moral hazard constraints: the buyer chooses the action that he prefers most. Inequalities (1) and (2) guarantee that the buyer's strategy is a best response at each information set.

Remark: It may seem that constraints (1) are redundant given that the buyer can still choose any action he likes. This would be true in a static scenario, but in our case a message submitted by the buyer to the mediator β_t influences the probability distribution of recommendations that he receives from the mediator, the $n_t's$. For the cases that the seller observes β_t or β_t and n_t, these pieces of information may affect her posterior beliefs about the buyer's valuation and p^{t+1,i_{t+1}^S} and x^{t+1,i_{t+1}^S} will be functions of β_t and n_t. Hence the set of PBE's depends on the amount of information observed by the seller since this determines her beliefs.

Our objective is to find an assessment that is a Perfect Bayesian Equilibrium, PBE, and guarantees highest expected revenue for the seller among all $PBE's$. In order to find a revenue maximizing PBE we will search for a social choice function that maximizes expected revenue among all social choice functions that are implemented by assessments that are $PBE's$. The following section describes the technical difficulties of the problem at hand, as well as our solution method.

3. Technical Difficulties & The Procedure

In this section we explain why one cannot employ the revelation principle in order to characterize a revenue maximizing PBE and we discuss two possible ways that one can follow to sidestep these difficulty.

In an environment where the mechanism designer has limited commitment we cannot employ the revelation principle to find the optimal mechanism at each stage. Information revelation is costly because the mechanism designer, the seller in this case, cannot commit not to use it and exploit the buyer in the future. The buyer of course anticipates this; and as it was realized in the earlier literature on mechanism design without commitment, (Freixas, Guesnerie and Tirole (1985), Hart and Tirole (1988), Laffont and Tirole (1988) and (1993)), with the exception of the final period, one cannot use the standard revelation principle in order to find the optimal mechanism in each stage.[11] Without the help of the revelation principle one may need to consider mechanisms with arbitrary message spaces, which is indeed what we do here, but then it becomes quite challenging even to write down the seller's optimization problem. And as we already discussed in the introduction, the version of the revelation principle in Bester and Strausz (2001), is only valid for finite types. Indeed it is difficult to translate the idea that it is enough to have one message per type in the case of a continuum of types.

There are at least two ways to proceed without the help of the revelation principle. One can look for equilibria or look for equilibrium outcomes.

Looking for equilibria is indeed very ambitious given the size of strategy spaces and the complexity of the game. Mechanisms employed at $t = 1, ..., T$ depend on the seller's posterior. Along the equilibrium path the posterior is determined by Bayes' rule from the buyer's strategy, the mechanism proposed by the seller, and the action chosen by the buyer. There can be infinitely many choices at $t = 1$ that end up in no trade since lotteries are allowed. Each of these choices leads to a different posterior and an optimal period-1 mechanism. At an equilibrium the mechanism at $t = 1$ has to be optimally chosen taking into account not only revenue at $t = 1$ but also what beliefs the seller will have after each history where there is no trade at $t = 1$, which in turn will determine the optimal mechanism for $t = 2$ and so on. And in order for someone to find the revenue maximizing PBE, one has to find all $PBEs$ of the game, then calculate the corresponding revenue, and finally compare the seller's revenue at each of these $PBEs$. A paper that follows the first approach and looks for *equilibria* is Laffont and Tirole (1988). That paper looks for perfect

[11] To see why, suppose that at period one the seller employs a direct revelation mechanism, the buyer has claimed to have valuation v, and according to this mechanism no trade takes place. If the seller behaves sequentially rationally, she will try to sell the object at $t = 2$ using a different mechanism. And in the case that the buyer has revealed his true valuation at $t = 1$, the seller has complete information at $t = 2$. She can therefore use this information to extract all the surplus from the buyer. In this situation the buyer will have an incentive to manipulate the seller's beliefs. One would expect, and it is indeed true, that he will not always reveal his valuation truthfully at the beginning of the relationship. Proposition 1 in Laffont and Tirole (1988) establishes that there is no PBE where each type of the buyer chooses a different action at $t = 1$. Hence truth-telling with probability one cannot occur at any PBE at $t = 1$.

Bayesian equilibria of a two-period regulation game under non-commitment, where the firm's type is drawn from a continuum. They provide some properties of equilibria but they do not obtain a characterization of an optimal equilibrium.[12] Indeed looking for equilibria is very difficult, even in our model which is in some dimensions simpler than LT (1988).

We are able to sidestep the difficulties that arise in the analysis of LT (1988) by looking at equilibrium *outcomes* instead of equilibria. Investigating properties of equilibrium outcomes is much simpler. In one sentence, what we do is to characterize the properties of $p(\sigma,\mu)'s$ and $x(\sigma,\mu)'s$ that are implemented by assessments that are $PBE's$ and then choose the one that the seller prefers. This idea was inspired from Riley and Zeckhauser (1983). Obviously the set of $PBE-$implementable allocation and payment rules is a subset of the BNE implementable ones. In order to characterize this subset we examine what restrictions the requirement that (σ,μ) be a PBE implies on p and x. In this way we obtain allocation rules and payment rules that satisfy *necessary* conditions of being PBE implementable. We then choose the p and x that the seller likes best among those that satisfy the necessary conditions of being PBE implementable and verify that there exists indeed a PBE that implements them. In one sense, this is in line with the approach of the revelation principle which prescribes a way to obtain all the set of BNE implementable allocation and payment rules: "just look at the social choice functions that can be implemented by truth-telling equilibria of direct revelation mechanisms." While our focus is still on equilibrium outcomes, rather than equilibria, we do not work with a "canonical family" of mechanisms. This alternative route in the case that we are interested in BNE implementable social choice functions is as simple as working with direct revelation mechanisms, but has the additional advantage that it also allows us to obtain properties of PBE-implementable social choice functions. Focusing on outcomes renters mechanism design "without commitment" a tractable problem, even if one works with mechanisms with arbitrary message spaces.

In the section that follows, we employ the procedure sketched here to formulate the seller's problem.

[12] First they show that there is no separating equilibrium, that is, there exists no equilibrium where each type of the agent chooses a different action at $t = 1$. Then they consider the situation where the uncertainty about the agents' type is very small, that is, the agent's type belongs in a very small interval. In this case the distortion of a full pooling continuation equilibrium compared to the commitment optimum, goes to zero as the length of the interval of possible types goes to zero. They also show that there exist other equilibria that have the property that the distortion compared to the commitment case goes to zero as the length of the interval of possible types goes to zero. These equilibria exhibit, what they call infinite reswitching or pooling over a large scale. Hence for small uncertainty, that is when the agent's type is almost known, these three kinds of continuation equilibria are candidates for an optimum. For the case that uncertainly is not trivial the authors do not say which equilibria may be optimal.

4. Formulation of the Problem

Here we employ the idea we sketched in the previous section in order to write down the seller's maximization problem. The seller's seeks to solve

$$\max_{p,x} \int_V x(v)dF(v)$$

subject to p, x being PBE implementable.

Let's translate the requirement that p and x be PBE implementable. First of all p has to satisfy resource constraints. Since there is only one object to be allocated, it must be the case that $p(v) \in [0,1]$ for all v. Second, at a PBE the buyer's and the seller's strategy must be a best response at each information set. For the buyer this implies at the very least, that there is no type v that can benefit by behaving as type v' does. For the seller this implies that at each information set her strategy must specify an optimal sequence of mechanisms employed in the remainder of the game. Finally from the fact that the buyer can always choose not to participate in a mechanism, we get that the buyer's expected payoff for the continuation of the game must be non-negative. It follows that if p and x are implemented by a PBE they must, at the very least, satisfy the constraints of the following Program, which we call **Program A:**

$$\max_{p,x} \int_V x(v)dF(v)$$

Subject to:

IC "incentive constraints," the buyer's strategy is such that

$$p(v)v - x(v) \geq p(v')v - x(v'); \text{ for all } v, v' \in V$$

PC "voluntary participation constraints,"

$$p(v)v - x(v) \geq 0 \text{ for all } v \in V$$

RES "resource constraints" for all $v \in V$

$$0 \leq p(v) \leq 1$$

$SRC(t, i_S^t)$ "sequential rationality constraints," for all t, $t = 2, ..., T$, and each history of no trade at t, i_S^t,the seller chooses a mechanism that maximizes revenue:

$$\max_{p^{t,i_S^t}, x^{t,i_S^t}} \int_{Y_{t,i_S^t}} x^{t,i_S^t}(v)dF_{t,i_S^t}(v) \tag{3}$$

subject to:

$IC(t, i_S^t)$: $p^{t,i_S^t}(v)v - x^{t,i_S^t}(v) \geq p^{t,i_S^t}(v')v - x^{t,i_S^t}(v')$, for all $v, v' \in Y_{t,i_S^t}$

$PC(t, i_S^t)$: $p^{t,i_S^t}(v)v - x^{t,i_S^t}(v) \geq 0$ for all $v \in Y_{t,i_S^t}$

$RES(t, i_S^t)$: $0 \leq p^{t,i_S^t}(v) \leq 1$ for all $v \in Y_{t,i_S^t}$,

where Y_{t,i_S^t} is the support of the posterior F_{t,i_S^t}.

$SRC(t+j, i_S^{t+j})$ for all $j = 1, ...T - t$ and each history of no trade at $t+j$, i_S^{t+j}, $p^{t+j,i_S^{t+j}}$, $x^{t+j,i_S^{t+j}}$ satisfy sequential rationality constraints.

Beliefs posterior beliefs F_{t,i_S^t} are derived using the buyer's strategy and Bayes' rule whenever possible.

Summarizing, if p, x are implemented by an assessment (σ, μ) that is a PBE, then conditions IC, PC, $RES(t)$ and $SRC(t)$ will be satisfied. These are hence necessary conditions for p and x to be PBE implementable. In what follows we will obtain a solution of Program A, p and x, and establish that there exists indeed an assessment that is a PBE and it implements p and x. The solution proceeds by induction. First we examine the problem when $T = 1$ and show that the optimum is implemented by posting a price. Then we move on to $T = 2$ and establish again that at an optimum the seller posts a price in each period. Finally, by induction we show that the same result holds for any $T < \infty$.

5. The solution for $T = 1$

Here we solve the problem when the game lasts for only one period. For $T = 1$ all sequential rationality constraints become irrelevant. Players' strategies must be merely mutual best responses and all $BNE's$ are $PBE's$. Then the seller's problem reduces to

$$\max_{p,x} \int_V x(v)dF(v)$$

subject to:

IC $p(v)v - x(v) \geq p(v')v - x(v')$; for all $v, v' \in V$

PC $p(v)v - x(v) \geq 0$ for all $v \in V$

RES $0 \leq p(v) \leq 1.$ for all $v \in V$.

Even though this problem is isomorphic to the problem in the classical works of Myerson (1981) or Riley and Zeckhauser (1983), their solution approach does not go through because it requires that the type space be an interval. The key step there is to rewrite revenue as a function solely of the allocation, which is the gist of the famous revenue equivalence theorem. That step relies on the type space being an interval, which is not necessarily true in our model, where we work with probability measures that have support arbitrary measurable subsets of the real line. For the problem at hand it is essential to allow for such generality in the distributions. For any $T > 1$ in order to solve for the revenue maximizing PBE we need to know what will be the optimal mechanism in the final period of the game. That problem is isomorphic to the problem when $T = 1$. And because we allow for the use of general mechanisms by the seller and mixed strategies by the buyer, posteriors maybe quite complicated indeed. For this reason we have to propose a solution that is valid for arbitrary posterior distributions and arbitrary type spaces - that are nor necessarily finite, nor convex.

The idea is to solve an artificial problem where the type space is an interval, but its solution restricted to V solves Program A. In particular, in the Proposition that follows we establish that we can obtain a solution of Program A by solving an artificial problem, Program B, where we require the constraints to hold on the whole convex hull of V which is the interval $[a, b]$.

Program B for $T = 1$:

$$\max_{p,x} \int_a^b x(v) dF(v)$$

subject to:

IC $\quad p(v)v - x(v) \geq p(v')v - x(v')$; for all $v, v' \in [a, b]$

PC $\quad p(v)v - x(v) \geq 0$ for all $v \in [a, b]$

RES $\quad 0 \leq p(v) \leq 1$. for all $v \in [a, b]$.

Proposition 1 [13] *Let p^A, and p^B denote solutions of Program A and Program B respectively, and let $R(p^A)$ and $R(p^B)$ denote the seller's expected revenue at each of these solutions. Then*

$$R(p^A) = R(p^B).$$

Proof. See Skreta (2004b). ■

Before proceeding let us offer a very brief sketch of the proof of Proposition 1. Programs A and B have the same objective function and they differ only in the constraint set. Program B has a lot more constraints than Program A therefore $R(p^A) \leq R(p^B)$. We establish that the reverse inequality is true by showing that a solution p and x of Program A appropriately extended on $[a, b]$ satisfies all the constraints of Program B. In particular we extend a solution of A, call it $\bar{p}_A$, as follows. For a type v in $[a, b] \backslash V$ we set the value of $\bar{p}_A$ equal to the value of p_A at the largest type in V less or equal to v. Since p_A is incentive compatible on V and no real options have been added, the resulting allocation rule is feasible for program B. It follows that the values of these programs must be the same.

Now that we have replaced the type space with its convex hull, we can obtain properties of feasible allocation and payment rules using standard techniques. Let $U_{\sigma,\mu}(\sigma_B(v), v) = p(v)v - x(v)$ denote the

[13] The conjecture that such a result may be available arose from discussions with Kim-Sau Chung.

buyer's expected discounted payoff when his valuation is v given the assessment (σ, μ). From Lemma 2 in Myerson (1981) we have:

Lemma 1 *If p, x are implemented by a BNE, they satisfy IC, PC and RES constraints and the following conditions must hold: for all $v \in [a, b]$ (a) $p(v)$ is increasing in v (b) $U_{\sigma,\mu}(\sigma_B(v), v) = \int_a^v p(s)ds + U_{\sigma,\mu}(\sigma_B(a), a)$ (c) $U_{\sigma,\mu}(\sigma_B(a), a) \geq 0$ and (d) $0 \leq p(v) \leq 1$.*

Using these properties and standard arguments, we can write the seller's expected revenue as a function only of the allocation rule and the payoff that accrues to the lowest type of the buyer[14]

$$\int_a^b x(v)dF(v) = \int_a^b p(v)vdF(v) - \int_a^b p(v)[1 - F(v)]dv - U_{\sigma,\mu}(\sigma_B(a), a).$$

From the analysis in Myerson (1981), we know that if p solves

$$\max_{p \in \Im} \int_a^b p(v)vdF(v) - \int_a^b p(v)[1 - F(v)]dv, \tag{4}$$

where

$$\Im = \{p : [a, b] \to [0, 1]\text{: } p \text{ is increasing}\}\,. \tag{5}$$

and $x(v) = p(v)v - \int_a^v p(s)ds$,which guarantees that $U_{\sigma,\mu}(\sigma_B(a), a) = 0$, then this is an optimal mechanism.

Our objective is to choose a function p that is increasing and such that (4) is maximized. Because (4) is linear in p it follows that the maximizer is an extreme point of the feasible set. Extreme points of the set of increasing functions from $[0, 1]$ to $[0, 1]$ are step functions that jump from zero to one. The optimal allocation rule has the bang-bang property: the seller trades with zero probability with types below a cutoff of v^*, ($p(v) = 0$ for $v \leq v^*$), whereas trades with probability 1 with types $v^* \geq v$, ($p(v) = 1$ for $v \geq v^*$). This allocation rule can be implemented by posting a price of v^*. The cutoff in Myerson (1981) is given by a point where the (ironed) virtual valuation is zero. Here the cut-off v^* is the smallest point with the property that the integral of the virtual valuation to any point to the right of it is non-negative. The Proposition that follows states this result and provides a formal definition of the cut-off v^*.

[14] If F has a density then revenue can be rewritten as

$$\begin{aligned}\int_a^b x(v)dF(v) &= \int_a^b p(v)\left[vf(v) - \int_v^b f(s)ds\right]dv - U_{\sigma,\mu}(\sigma_B(v), v) \\ &= \int_a^b p(v)\left[vf(v) - [1 - F(v)]\right]dv - U_{\sigma,\mu}(\sigma_B(v), v).\end{aligned}$$

If the density is strictly positive, then we obtain the familiar expression

$$\int_a^b p(v)\left[v - \frac{[1 - F(v)]}{f(v)}\right]f(v)dv - U_{\sigma,\mu}(\sigma_B(v), v).$$

Proposition 2 [15] *When $T = 1$ the seller maximizes revenue by posting a price v^*, where*

$$v^* \equiv \inf \left\{ v \in [a,b] \ s.t. \ \int_v^{\tilde{v}} s dF(s) - \int_v^{\tilde{v}} [1 - F(s)] ds \geq 0, \ \text{for all } \tilde{v} \in [v,b] \right\}. \tag{6}$$

The revenue maximizing allocation and payment rule are given by

$$\begin{aligned} p(v) &= 1 \ \text{if } v \geq v^* \\ &= 0 \ \text{if } v < v^* \end{aligned} \quad \text{and} \quad \begin{aligned} x(v) &= v^* \ \text{if } v \geq v^* \\ &= 0 \ \text{if } v < v^* \end{aligned} . \tag{7}$$

The proof of this result can be found in the "Technical Appendix for Sequentially Optimal Mechanisms."

This characterization is a generalization of the analysis in Myerson (1981) and Riley and Zeckhauser (1983), "commitment solution," for distributions that have arbitrary support.[16] It is also valid for cases that the posterior does not necessarily satisfy the monotone hazard rate property. Myerson (1981) contains an analysis that describes how to deal with this case for distributions that have strictly positive densities, using the well-known, by now, "ironing technique." The approach we use here is different from the ironing technique, but has the advantage that (6) allows us to obtain comparative statics results on how the price posted depends on F, something that is not that obvious with the "ironing" in Myerson (1981). When $T > 1$ the sequential rationality constraints at T imply that for all histories i_S^T, p^{T,i_S^T} and x^{T,i_S^T} solve a problem that is isomorphic to the one just solved. It will be then crucial for our analysis to be able to say how the price that the seller will post at the final period of the game depends of the posterior (see the proofs of Lemmata 2, 4, and 7.) In this section we have obtained a solution for the case that $T = 1$ that is valid for arbitrary distributions and it also allows to obtain comparative statics results on how the price that the seller posts depends on the distribution of the buyer's valuation; we now continue with the solution of the problem when $T = 2$.

6. Sequentially Optimal Mechanisms When $T = 2$

In this section we characterize sequentially optimal mechanisms when $T = 2$. As in the case where $T = 1$, we first show that it is without loss of generality to solve an artificial program where we replace the type

[15] I thank Phil Reny for suggesting parts of the proof of Proposition 2.

[16] The "commitment solution" is also the optimal $BNE-$implementable allocation and payment rule in a T period long game, T arbitrarily large, where we do not require the seller's strategy to be sequentially rational. The revenue maximizing allocation rule in a multi-stage game with commitment can be then implemented by the following assessment: the seller makes a take-it-or-leave-it offer in each period, $t = 1, ..., T$ of v^*. The buyer's strategy is as follows: for $v \geq v^*$ the buyer accepts the seller's offer at $t = 1$ and for $v < v^*$ the buyer rejects the seller's offer at $t = 1, ..., T$. It is very easy to see that players' strategies are mutual best responses hence the given assessment is a BNE.

space with its convex hull. Program B for $T = 2$ is given by

$$\max_{p,x} \int_a^b x(v)dF(v)$$

subject to:

IC $\quad p(v)v - x(v) \geq p(v')v - x(v')$; for all $v, v' \in [a, b]$

PC $\quad p(v)v - x(v) \geq 0$ for all $v \in [a, b]$

RES $\quad 0 \leq p(v) \leq 1.$ for all $v \in [a, b]$.

$SRC(t = 2, i_S^2)$ "sequential rationality constraints." For each information set of no trade at $t = 2$, i_S^2, the seller chooses a mechanism that maximizes revenue:

$$\max_{p^{2,i_S^2}, x^{2,i_S^2}} \int_{\bar{Y}_{2,i_S^2}} x^{2,i_S^2}(v)dF_{2,i_S^2}(v)$$

subject to

$IC(t = 2, i_S^2)$ $p^{2,i_S^2}(v)v - x^{2,i_S^2}(v) \geq p^{2,i_S^2}(v')v - x^{2,i_S^2}(v')$, for all $v, v' \in \bar{Y}_{2,i_S^2}$

$PC(t = 2, i_S^2)$ $p^{2,i_S^2}(v)v - x^{2,i_S^2}(v) \geq 0$ for all $v \in \bar{Y}_{2,i_S^2}$

$RES(t = 2, i_S^2)$ "resource constraints" $0 \leq p^{2,i_S^2}(v) \leq 1$ for all $v \in \bar{Y}_{2,i_S^2}$

$Beliefs$ posterior beliefs F_{2,i_S^2} are derived using the buyer's strategy and Bayes' rule whenever possible.

Program B is exactly the same as Program A but with V and Y_{T,i_S^T} replaced by their convex hulls $[a, b]$ and $\bar{Y}_{T,i_S^T}$ respectively.

Proposition 3 *The value of Program A and Program B is the same.*

Program B at $t = 2$ and i_S^2 has more constraints than the corresponding program A at this information set, hence the value of Program B will be weakly less, but from the of Proposition 1 we know that at each information set a solution of $SRC(t = 2, i_S^2)$ extended on $\bar{Y}_{2,i_S^2} = [\underline{v}, \bar{v}]$ solves Program B at $t = 2$ and i_S^2. Therefore, essentially nothing changes on the set of constraints described in $SRC(t = 2, i_S^2)$. Using exactly the same arguments as in the proof of Proposition 1, one can show that a solution p and x of Program A appropriately extended on $[a, b]$ satisfies all the constraints of Program B. Since these two programs have the same objective function and differ only in the constraint set, it follows that the values of these programs must be the same.

Then from standard arguments we have that Program B can be rewritten as

$$\max_{p,x} \int_a^b p(v)vdF(v) - \int_a^b p(v)[1 - F(v)]dv \tag{8}$$

subject to:

$$\begin{aligned}
\Im_{[a,b]} &= \{p:[a,b]\to[0,1]\text{: } p \text{ is increasing}\},\\
x(v) &= p(v)v-\int_a^v p(s)ds \text{ and}\\
p(v) &\in [0,1] \text{ for all } v\in[a,b].
\end{aligned}$$

$SRC(t=2,i_S^2)$ "sequential rationality constraints." For each history i_S^2

$$\max_{p^{2,i_S^2},x^{2,i_S^2}} \int_{\bar{Y}_{2,i_S^2}} p^{2,i_S^2}(v)vdF_{2,i_S^2}(v)-\int_{\bar{Y}_{2,i_S^2}} p^{2,i_S^2}(v)[1-F_{2,i_S^2}(v)]dv$$

subject to

$$\begin{aligned}
\Im_{\bar{Y}_{2,i_S^2}} &= \left\{p^{2,i_S^2}:\bar{Y}_{2,i_S^2}\to[0,1]\text{: } p^{2,i_S^2} \text{ is increasing}\right\},\\
x^{2,i_S^2}(v) &= p^{2,i_S^2}(v)v-\int_{\underline{v}}^v p^{2,i_S^2}(s)ds \text{ and}\\
p^{2,i_S^2}(v) &\in [0,1] \text{ for all } v\in\bar{Y}_{2,i_S^2}.
\end{aligned}$$

How does a solution of this program look like? Without the sequential rationality constraints, we know that posting a price of v^* in each period solves the seller's problem. This is optimal, *given* that the seller can commit not to try to sell the item using a different mechanism in a subsequent period and this solution does not satisfy the sequential rationality constraints. If the object remains unsold at date $t=2$ the seller knows that there exist gains from trade but they were not realized because the price she posted was above the buyer's valuation. Then posting a price of v^* at $t=2$ is not sequentially rational given that the buyer's type lies in $[0,v^*]$. If the seller behaves sequentially rationally she will try to sell the item at $t=2$ using a different mechanism that maximizes revenue from that point on, which clearly changes strategic considerations at $t=1$. Does is pay for the seller to use the first stage mechanism as an experimentation device to learn where does the buyer's valuation lie? That is, does the seller use a mechanism in the first period that allows her to infer with precision the type of the buyer, hoping that she can use her sharper estimate to extract the buyer's surplus in a subsequent period? Or is it too costly in terms of expected revenue to do so? Does the seller offer a set of lotteries at period $t=1$ or does she simply post a price? What is the most lucrative way to learn? Our characterization will provide answers to these questions.

Before we move to the solution let us take a closer look at the objective function of the problem described in (8). One may think that even if there is a second stage where the seller can use the information that she learns about the buyer, still at least for $t=1$, the seller wants to trade with probability zero with types of the buyer below v^* and with probability one with types above v^*. This reasoning is not complete since sequential rationality constraints imply that after any history of no trade at $t=1$ the seller will at

$t = 2$ post a price that is optimally chosen given some posterior. Then overall in the game we will have $p(v) = \delta(= 0 + \delta \cdot 1)$, for some types below v^*. So even if we were restricting attention to posted prices, the seller may not wish to set prices such that $p(v) = 1$ for $v \geq v^*$ because it may be too costly to have to deal with the remaining types $[a, v^*]$ at $t = 2$. "Too costly" here means that the price at $t = 2$ that is optimal given $\frac{F(v)}{F(v^*)}$, may be too low from the ex-ante point of view: remember that from the ex-ante point of view the seller would love to commit to set $p(v) = 0$ for $v \in [a, v^*]$. But then a higher price then v^* may do the job. This will indeed turn out to be the case but in order to establish it we have to show that there is nothing else that does better. Posted prices allow only for a certain class of posteriors: truncations of the original distribution. That is, possible posteriors are of the form $\frac{F(v)}{F(\bar{v})}$, for some $\bar{v} \in [a, b]$, and hence the price posted at $t = 2$ will be simply a function of $\bar{v}$. Is it then possible to have a posterior F_2 whose support has convex hull $[a, \bar{v}]$, such that $z_2(F_2) > z_2(\bar{v})$? We will show that any posterior that can arise at a PBE, where we allow the seller to employ arbitrarily complicated mechanisms, and the buyer arbitrarily complicated strategies, does not allow the seller to support higher prices at $t = 2$. This is an important step of our characterization.

Our objective is to find a revenue maximizing $PBE-$implementable allocation and payment rule. The set of feasible social choice functions depends on (i) how big is the class of mechanisms that the seller employs (ii) the generality of the buyer's strategy (iii) the degree of transparency of mechanisms. We will obtain the solution by gradually looking at more general set-ups.

Outline of the Solution

First, we restrict attention to allocation rules implemented by strategy profiles, where the seller at $t = 1$ employs simple mechanisms that separate types into two groups: "high" and "low" ones, and show that a revenue maximizing allocation rule among this class is implemented by posting a price in each period. Then we look at another simple environment where the seller employs mechanisms at $t = 1$ that are equivalent to a set of probability-payment pairs, (a set of contracts); the seller observes the contract chosen by the buyer at $t = 1$ and the buyer employs pure and "simple" strategies. Finally, we consider the general case where the mechanism consists of a game form and a mediator, and the buyer employs mixed strategies where potentially non-convex sets of types choose the same action with positive probability at $t = 1$. The degree of transparency of mechanisms in this general case is as follows. We assume that the seller observes the message that the buyer submits to the mediator, the action that he chooses as well as whether trade takes place or not. After we describe necessary conditions that allocation rules satisfy if they are implemented by strategy profiles that are $PBE's$, we then show that the seller prefers allocation rules implemented by the simple assessments examined in the first step. But then at an optimum the seller posts a price in each period. This result is robust in a number of different assumptions regarding the degree of transparency of mechanisms. Details can be found in the "Technical Appendix for Sequentially

Rational Mechanisms." In the next section we start our exploration with the characterization of a solution in the case that the seller employs mechanisms at $t = 1$ that contain two options.

6.1 Revenue Maximizing PBE among 2-Option Mechanisms

Here we look for the revenue maximizing allocation rule among the ones implemented by strategy profiles where the seller employs very simple mechanisms. We call this class of strategy profiles *two-options at* $t = 1$, *price below the optimal at* $t = 2$.

First, the seller proposes at $t = 1$ a mechanism that simply consists of a game form with two actions. Action s_r leads to contract (r, z) and action s_1 leads to contract $(1, z_1)$, there is no mediator. In this simple setup the first period mechanism reduces to a set of two contracts $M_1 = \{(r, z), (1, z_1)\}$, where $r \in [0, 1]$ and $z, z_1 \in \mathbb{R}$. Option (r, z) is targeted to "low" types and option $(1, z_1)$ is targeted to "high" valuation types. The only possibility that trade does not take place at $t = 1$ is when the buyer chooses the low probability option. The buyer's strategy is as follows: at $t = 1$ types $v \in [a, \bar{v})$ choose (r, z) and types in $(\bar{v}, b]$ choose $(1, z_1)$, where $\bar{v} \in [a, b]$.[17] Now at $t = 2$ after the history where the buyer chose (r, z) at $t = 1$ and no trade took place, the seller chooses a price $\hat{z}_2$, such that $\hat{z}_2 \leq z_2(\bar{v})$, where $z_2(\bar{v})$ would have been the optimal cut-off given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v})}$. The buyer at $t = 2$ accepts $\hat{z}_2$ for $v \in [\hat{z}_2, \bar{v})$ and rejects $\hat{z}_2$ for $v \in [a, \hat{z}_2)$. Type $\bar{v}$ is indifferent between choosing: (r, z) at $t = 1$ and $(1, \hat{z}_2)$ at $t = 2$ versus choosing $(1, z_1)$ at $t = 1$, that is $\bar{v} = \frac{z_1 - z - (1-r)\delta\hat{z}_2}{1-r-(1-r)\delta}$. Such an assessment is not necessarily a PBE since the seller at $t = 2$ may be choosing a cut-off below the optimal one. The allocation rules implemented by such an assessment is of the form

$$\begin{aligned} p(v) &= r \text{ for } v \in [a, \hat{z}_2) \\ p(v) &= r + (1-r)\delta \text{ for } v \in [\hat{z}_2, \bar{v}) \\ p(\bar{v}) &\in (r - (1-r)\delta, 1) \\ p(v) &= 1 \text{ for } v \in (\bar{v}, b]. \end{aligned} \tag{9}$$

Definition 2 *We call $\mathcal{P}_2^*$ the set of allocation rules that have the shape described in (9) for some $\bar{v} \in [a, b]$, $r \in [0, 1]$, and $\hat{z}_2 \leq z_2(\bar{v})$, where $z_2(\bar{v})$ is the optimal price at $t = 2$ given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v})}$.*

The next result states that a revenue maximizing element of $\mathcal{P}_2^*$ can be implemented by a PBE of the game where the seller posts a price in each period.

Proposition 4 *Let p^* denote a solution of $\max_{p \in \mathcal{P}_2^*} R(p)$. Then p^* can be implemented by a PBE of the game where the seller posts a price in each period.*

Proposition 4 establishes that if the seller restricts attention to period one mechanisms that contain two options: one targeted to the "low" types, (r, z), and one targeted to the "high" types, $(1, z_1)$, then at

[17] We assume that $ra - z \geq 0$ so that all types of the buyer have expected payoff at least as high as their outside option.

the optimum this kind of mechanism reduces to a posted price: the options available are $(0,0)$ and $(1, z_1)$. Before moving on to attack the problem in its most general form, we illustrate the procedure in a (one more!) simplified version of the problem.

6.2 The Revenue Maximizing PBE when Mechanisms are Sets of Contracts and Buyer employs Pure and "Simple" Strategies.

Here we characterize a revenue maximizing PBE in yet another simplified version of the game under consideration. In this simpler version $i)$ a mechanism is a set of contracts: it is simply a deterministic game form; there are no mediators, and each action s leads to a different contract (r, z), $ii)$ the seller observes the action chosen by the buyer and whether trade took place or not, $iii)$ the buyer's strategy is pure and "simple" in the sense that the set of types that choose a particular action at $t = 1$ is convex.

Fix a PBE and let (r, z) denote the contract that is chosen by the smallest type, which is a, at $t = 1$. Moreover let $[a, \bar{v}]$ denote the types that choose (r, z). (It is possible that this interval is degenerate, that is $a = \bar{v}$), and let $z_2(\bar{v})$ denote the price that the seller will post at $t = 2$ after the history that the buyer chose (r, z) at $t = 1$ and no trade took place. Since at a PBE the buyer's strategy must be a best response at each node, types above $z_2(\bar{v})$ will accept this price at $t = 2$ and types below $z_2(\bar{v})$ will reject.

Necessary Conditions: If an allocation rule is implemented by a PBE of this class it is of the form

$$\begin{aligned} p(v) &= r \text{ for } v \in [a, z_2(\bar{v})) \\ p(v) &= r + (1-r)\delta \text{ for } v \in [z_2(\bar{v}), \bar{v}) \\ r + (1-r)\delta &\leq p(v) \leq 1 \text{ for } v \in [\bar{v}, b]. \end{aligned} \tag{10}$$

33Now we will demonstrate that at a revenue maximizing PBE of this simpler game, the seller posts a price in each period. In establishing this we need an intermediate result.

Lemma 2 *Suppose that the posterior is given by $F_2(v) = \frac{F(v)}{F(\bar{v})}$, then z_2 is increasing in $\bar{v}$. Moreover if the posterior has a density, then z_2 is continuous in $\bar{v}$.*[18]

Proposition 5 *Suppose that $i)$ a mechanism is a set of contracts (r, z), $ii)$ the seller observes the contract chosen by the buyer and whether trade took place or not, $iii)$ the buyer's strategy is pure and "simple" in the sense that the set of types that choose a contract at $t = 1$ is convex. Then at a revenue maximizing PBE the seller posts a price in each period.*

Proof. We establish that each allocation of the form described in (10) is dominated in terms of expected revenue by an allocation rule in $\mathcal{P}_2^*$. Take an allocation rule among the ones given in (10), call it p. Expected

[18] Lemma 2 refers to properties of the price that the seller posts in the final period of the game and it is valid for any T not simply $T = 2$.

revenue can be then written as

$$\int_a^{z_2(\bar{v})} rvdF(v) - \int_a^{z_2(\bar{v})} r[1-F(v)]dv$$
$$+\int_{z_2(\bar{v})}^{\bar{v}} (r+(1-r)\delta)vdF(v) - \int_{z_2(\bar{v})}^{\bar{v}} (r+(1-r)\delta)[1-F(v)]dv$$
$$+\int_{\bar{v}}^{b} p(v)vdF(v) - \int_{\bar{v}}^{b} p(v)[1-F(v)]dv.$$

We construct an allocation rule, which we call $\hat{p}$, that is an element of $\mathcal{P}_2^*$ and it generates higher revenue than p. For the range $[a,\bar{v})$ we set $\hat{p}(v)=p(v)$. For types in $v\in[\bar{v},b]$, we choose $\hat{p}$ optimally imposing the constraint that the resulting allocation rule is increasing on $[a,b]$ and ignoring all sequential rationality constraints for types $v\in[\bar{v},b]$. For types $[\bar{v},b]$ it is, in some sense, as if we are solving a "commitment problem." Let

$$v^{**}\equiv\inf\left\{v\in[\bar{v},b]\text{ s.t. }\int_v^{\tilde{v}} sdF(s)-\int_v^{\tilde{v}}[1-F(s)]ds\geq 0,\text{ for all }\tilde{v}\in[v,b]\right\}, \tag{11}$$

then for the same reasons as in the proof of Proposition 2 we would like to set $\hat{p}$ equal to its lowest possible value for the types $v\in[\bar{v},v^{**})$, which it is now $p(v)=r+(1-r)\delta$, and set it equal to its largest possible value for the region where the virtual valuation is on average positive, that is $p(v)=1$. The optimal allocation is

$$\begin{aligned}
\hat{p}(v) &= r\text{ for }v\in[a,z_2(\bar{v}))\\
\hat{p}(v) &= r\text{ for }v\in[a,z_2(\bar{v}))\\
\hat{p}(v) &= r+(1-r)\delta\text{ for }v\in[\bar{v},v^{**})\\
\hat{p}(v) &= 1\text{ for }v\in[v^{**},b].
\end{aligned}$$

Now from Lemma 2 we have that $z_2(\bar{v})\leq z_2(v^{**})$, hence the resulting allocation rule is an element of $\mathcal{P}_2^*$ and it generates higher revenue for the seller than p. But from Proposition 4 we know that the revenue maximizing allocation rule is implemented by a PBE of the game where the seller posts a price in each period. ■

The characterization of a revenue maximizing PBE in the cases where we allow for arbitrarily complicated mechanisms and strategies of the buyer, will proceed in lines parallel to the ones outlined above.

6.3 The Solution with General Mechanisms & Strategies

We turn to characterize a revenue maximizing PBE allowing for arbitrary mechanisms and strategies.

Necessary Conditions at a PBE

We start by drawing the allocation rule from the smallest types. Let s denote an action that leads to (r, z), where this is the contract with the smallest "r", with the property that type a is either "choosing" (r, z) with strictly positive probability at $t = 1$, or is indifferent between doing and not doing so. The set of types that choose with positive probability s is not necessarily be convex, nor will all types choose that action with probability one.

Proposition 6 *If an allocation rule is implemented by an assessment that is a PBE it satisfies the following properties (i) increasing in v on $[a, b]$ (ii) $0 \leq p(v) \leq 1$ for $v \in [a, b]$ and (iii)*

$$\begin{gathered} p(v) = r \text{ for } v \in [a, z_2(F_2)) \\ r \leq p(z_2(F_2)) \leq r + (1-r)\delta \\ p(v) = r + (1-r)\delta \text{ for } v \in (z_2(F_2), \bar{v}) \\ r + (1-r)\delta \leq p(\bar{v}) \leq 1 \end{gathered} \tag{12}$$

for some $\bar{v} \in [a, b]$, $r \in [0, 1]$, and $z_2(F_2)$ optimal given F_2, where F_2 is a probability measure that has support a subset of the interval $[a, \bar{v}]$.

Definition 3 *We call $\mathcal{P}_2$ the set of allocation rules that satisfy the properties described in Proposition 6.*

Remark 1 *It is possible that $z_2 = a$ in which case we have that $p(a) \in [r, r + (1-r)\delta]$. It is also possible that $\bar{v} = a$ in which case $p(a) \in [r + (1-r)\delta, 1]$ or that $\bar{v} = b$ in which case $p(b) \in [r + (1-r)\delta, 1]$.*

Before proceeding let us comment on the shape of $PBE-$implementable allocation rules. First since we have not specified anything for types in $[\bar{v}, b]$, we do not know exactly the shape of the allocation rule: it depends on the seller's and the buyer's strategy. But for certain if p is implemented by an assessment that is a PBE it has to be an increasing function. Now for $v \in [a, \bar{v}]$ it is surprising to see that the shape of a $PBE-$implementable allocation rule is the same as the one that we would get in the case that *all* types in $[a, \bar{v}]$ were choosing with probability one a report to the mediator β and action s that leads to (r, z); only the location of the second period price differs. In the case where all types in $[a, \bar{v}]$ choose (β, s) with probability one, that price, call it $z_2(\bar{v})$, should have been optimal given posterior beliefs $\frac{F(v)}{F(\bar{v})}$. But in general, $z_2(F_2)$ is optimally chosen given some posterior F_2 whose support has convex hull $[a, \bar{v}]$. Of course now that we are considering general mechanisms and strategies, F_2 can be quite complicated and not just mere truncations of the original distribution.

Given the shape of $PBE-$implementable allocation rules the proof of Proposition 5 will go through as is, if it turned out that $z_2(F_2) \leq z_2(\bar{v})$. Next we establish that it is indeed the case that $z_2(F_2) \leq z_2(\bar{v})$. This is done in three steps. First we show that allowing the possibility to observe message β will not lead to beliefs that support higher prices at $t = 2$, compared to the case where the seller simply observes s. Second, we use this observation to conclude that we can without any loss view a mechanism as a game form where each action leads to a different contract. Third we show, that allowing for mixed or complicated strategies for the buyer does not lead to $z_2(F_2) > z_2(\bar{v})$ either.

Can the seller benefit from observing the cheap messages β?

First we establish that allowing the seller to observe the cheap messages β does not benefit her. One may be wondering, and quite rightly so, whether cheap messages β can affect the seller's beliefs, given that the seller observes costly information as well. Does the presence of costly information hinder the role of the "cheap" information? Suppose that the choice of action s follows two different messages that the buyer submits to the mediator. That is, sometimes the buyer is reporting message β and then choosing s, and sometimes reporting $\hat{\beta}$ and then choosing s. Let $\tilde{F}_2$ denote the seller's posterior after she observes (β, s) and let $\hat{F}_2$ denote the seller's posterior after she observes $(\hat{\beta}, s)$. Also let F_2 denote the seller's posterior after she observes *only* action s. Our objective is to compare $z_2(F_2)$ with $z_2(\hat{F}_2)$ and $z_2(\tilde{F}_2)$.

First let us examine how $z_2(\hat{F}_2)$ and $z_2(\tilde{F}_2)$ relate to each other.

Lemma 3 *Consider a PBE and let $z_2(\tilde{F}_2)$ respectively $z_2(\hat{F}_2)$ denote the prices that the seller will post after she observed β and s and $\hat{\beta}$ and s respectively. Then it must be the case that $z_2(\tilde{F}_2) = z_2(\hat{F}_2)$.*

Proof: Suppose not, and without any loss assume that $z_2(\tilde{F}_2) < z_2(\hat{F}_2)$. Then for all $v \in V$ we have that

$$[r(s) + (1 - r(s))\delta]\, v - \left[z - (1 - r(s))\delta z_2(\tilde{F}_2)\right] > [r(s) + (1 - r(s))\delta]\, v - \left[z - (1 - r(s))\delta z_2(\hat{F}_2)\right],$$

hence for all $v \in [z_2(\tilde{F}_2), b]$ the buyer strictly prefers to report β instead of $\hat{\beta}$ at least for the portion of the time that those types plan to chose s. But then when the seller sees $\hat{\beta}$ and s, she can infer that the valuation of the buyer is below $z_2(\tilde{F}_2)$, which in turn implies that a price of $z_2(\hat{F}_2) > z_2(\tilde{F}_2)$ cannot be optimal. Contradiction. Hence given some mechanism the choice of s uniquely determines the optimal price at $t = T$. ■

Now we turn to investigate the relationship of $z_2(F_2)$ with $z_2(\hat{F}_2)$ and $z_2(\tilde{F}_2)$.

Lemma 4 *"Cheap" information in β's does not lead to higher prices at $t = 2$ that is $z_2(F_2) \geq z_2(\tilde{F}_2) = z_2(\hat{F}_2)$.*

Remark 2 *Lemma 4 has been established assuming that there are only two different reports β and $\hat{\beta}$ followed by the same action s. This was done for expositional simplicity and it is without any loss.*

Mechanisms as Sets of Contracts

With the help of Lemma 4, we now establish that it is without any loss to think of mechanisms as being merely sets of contracts.

Lemma 5 *It is without any loss to view mechanisms as a set of contracts.*

Proof. From Lemma 4 it follows that it does not pay for the seller to observe β. Since here by assumption we are considering the case that the seller does not observe n, [19] than we can merely think of a mechanism as a game form. Now, our general formulation allows for mechanisms that consist of game forms where more than one action leads to the same contract. That is, we allow for game forms that contain s and $\hat{s}$ such that $r(s) = r(\hat{s})$ and $z(s) = z(\hat{s})$. Following the same reasoning as in the proof of Lemma 3, one can show that if both these actions are chosen with strictly positive probability, than it is the case that the price at $t = 2$ after the seller observes β, s is equal to the price at $t = 2$ after the seller observes $\beta, \hat{s}$. But then from Lemma 4 we know that these prices cannot be higher than the one where all types choosing either s or $\hat{s}$ choose only one of the two. From this observation it follows that it does not pay for the seller to employ game forms where more then one action leads to the same contract. We can then merely think of a game form as a set of contracts. ■

Summarizing, we have established that the possibility that the seller observes the payoff irrelevant messages $\beta's$ will not enable her to support higher prices at $t = 2$ compared to the case where she cannot. With the help of this observation, we have concluded in Lemma 5, that it is without any loss to think of a mechanism as being simply a set of contracts; the seller observes which contract was chosen and whether trade takes place or not. All this analysis was performed assuming that the buyer maybe using a mixed strategy.

Does the seller benefit from "sophisticated" strategies of the buyer?

Now we turn to investigate whether allowing for the buyer to employ mixed strategies will lead to posteriors that support higher prices at $t = 2$ compared to the case that he uses pure strategies.

This will be established in three steps. First we sketch an example in order to illustrate that is indeed possible that strictly positive measures of types are randomizing. This is not immediately obvious since from Proposition 6 we know that the shape of the allocation rules is as if all types in $[a, \bar{v}]$ are choosing (r, z) with positive probability. Then, we investigate which types in $[a, \bar{v}]$ may be choosing a contract other then (r, z) with positive probability, and finally we establish that posteriors that are consistent with equilibrium behavior when the buyer is mixing and/or non-convex sets of types are choosing (r, z), do not support higher prices at $t = 2$.

Let us first describe an example that shows that it is possible for types in $[a, \bar{v}]$ to be choosing with positive probability a contract different from (r, z).

[19] The case where the seller observes n is discussed in the section of robustness.

Example 1 *(Randomizations and Complicated Strategies of the Buyer are possible). Consider a strategy profile where the mechanism that the seller proposes at* $t = 1$ *contains three contracts* $(0, 0)$, (δ, z_δ) *and* $(1, z_1)$. *Suppose that after the history where the buyer chose* $(0, 0)$ *at* $t = 1$ *the optimal price at* $t = 2$ *is such that* $\delta z_2 = z_\delta$. *Then, all types of the buyer are indifferent between choosing* $(0, 0)$ *at* $t = 1$ *and* $(1, z_2)$ *at* $t = 2$ *and choosing* (δ, z_δ) *at* $t = 1$ *and* $(0, 0)$ *at* $t = 2$. *The reason for this is that for all* v

$$\begin{aligned} 0v - 0 + \delta(v - z_2) &= \delta v - z_\delta + (1 - \delta)(0v - 0) \text{ or} \\ \delta v - \delta z_2 &= \delta v - z_\delta. \end{aligned}$$

Then the buyer's strategy may dictate "sophisticated" mixing between $(0, 0)$ *and* (δ, z_δ) *at* $t = 1$.

The next result states which types in $[a, \bar{v}]$ maybe choosing at $t = 1$ a contract other than (r, z) with positive probability.

Lemma 6 *Suppose that a mechanism is a set of contracts, and consider a PBE where* $[a, \bar{v}]$, *denotes the convex hull of the set of types that choose contract* (r, z) *with positive probability at* $t = 1$. *Also let* z_2 *denote the cut-off that the seller will chose at* $T = 2$ *after the history that the buyer chose* (r, z) *at* $t = 1$ *and no trade took place. Then only types in* $[z_2, \bar{v}]$ *may be choosing a contract different from* (r, z) *with positive probability at* $t = 1$.

We use Lemma 6 to describe possible posteriors that can arise at a PBE. Let $m(v)$ denote the probability that type v is choosing contract (r, z). We will assume that m is a measurable function of v, with $m(v) \in [0, 1]$ for all $v \in [a, \bar{v}]$. From Lemma 6 it follows that $m(v) = 1$ for $v \in [a, z_2)$. For the cases where $a < z_2$, then it is immediate that, even if we allow for any possible randomization, the posterior at $T = 2$, after the buyer chose (r, z) at $t = 1$ is going to be of the form

$$F_2^m(v) = \begin{cases} \frac{F(v)}{F(z_2) + \int_{z_2}^{\bar{v}} m(s) dF(s)}, & v \in [a, z_2) \\ \frac{F(z_2) + \int_{z_2}^{v} m(s) dF(s)}{F(z_2) + \int_{z_2}^{\bar{v}} m(s) dF(s)}, & v \in [z_2, \bar{v}] \end{cases}, \tag{13}$$

where m is a measurable function of v, with $m(v) \in [0, 1]$ for all $v \in [a, \bar{v}]$ and where $F(z_2) + \int_{z_2}^{\bar{v}} m(s) dF(s) > 0$ since $z_2 > a$.

Now that we know how the set of possible beliefs looks like when the buyer potentially employs mixed strategies, or nonconvex sets of types choose a given action, we investigate whether such beliefs support higher prices at $t = 2$, compared to the case where all types in $[a, \bar{v}]$ choose a given contract with probability one.

Lemma 7 *Let* $z_2 > a$ *denote an optimal cut-off at* $t = 2$ *given beliefs* $F_2^m(v)$ *given by (*13*),* $z_2(\bar{v})$ *denote an optimal cut-off at* $t = 2$ *given beliefs* $F_2(v) = \frac{F(v)}{F(\bar{v})}$, *then* $z_2(\bar{v}) \geq z_2$.[20]

[20] The case that $z_2 = a$ is trivial, since it follows immediately that $z_2(\bar{v}) \geq z_2$.

The reason why this is true follows from the fact that only types above z_2 maybe be choosing a contract other then (r, z) with positive probability. To put it very roughly, there is less weight put on the higher types of $[a, \bar{v}]$. Equilibrium considerations force that only types above the second-period cut-off maybe randomizing between different contracts and for this reason the cut-off that is optimal given such posteriors will be weakly lower than the one when all types in $[a, \bar{v}]$ choose (r, z).

Lemmata 4, and 7 are valid for the last period of the game irrespective of the length of the game, that is they are valid for any T not just $T = 2$. They are key steps in our characterization: complicated strategies or mixed strategies of the buyer do not pay in the sense that the set of possible posterior beliefs induced by such strategies does not allow the seller to support higher prices at $t = 2$, compared to the case where convex set of types choose the same action with probability one. We are now ready to state our result.

Theorem 1 *Under non-commitment the seller maximizes expected revenue by posting a price in each period.*

Proof. The result can be established following the exact lines of the Proof of Proposition 5 and replacing $z_2(\bar{v})$ with $z_2(F_2)$. From Lemmata 4 and 7 we have that $z_2(F_2) \leq z_2(\bar{v})$, and from Lemma 2 we have that $z_2(\bar{v}) \leq z_2(v^{**})$. From the last two inequalities we get that $z_2(F_2) \leq z_2(v^{**})$ and therefore the resulting allocation rule is an element of $\mathcal{P}_2^*$. But Proposition 4 tells us that an optimal allocation rule out of $\mathcal{P}_2^*$ is implemented by a PBE of the game where the seller posts a price in each period. ■

Let us recap the arguments used to establish Theorem 1: we started with an allocation rule p that satisfies necessary conditions of being implemented by a $PBE-$ we have only imposed all the sequential rationality constraints for the range $[a, \bar{v})$. Then taking as given the shape of p for $[a, \bar{v})$, we optimally chose $\hat{p}(v)$ for $v \in [\bar{v}, b]$ ignoring all sequential rationality constraints and imposing only the requirement that the resulting allocation rule be monotonic on $[a, \bar{v})$. Using the observations from Lemmata 4 and 7, we established that $\hat{p}$ is an element of $\mathcal{P}_2^*$. Since we can follow these steps for each allocation rule in $\mathcal{P}_2$, and an optimal one among the allocation rules in $\mathcal{P}_2^*$ can be indeed implemented by a PBE of the game where the seller posts a price in each period, we can conclude that at an optimal PBE the seller posts a price in each period.

Intuitively the reason why simply posting a price is optimal, is that proposing a mechanism at $t = 1$ with more options has higher cost than benefit. The potential benefit from offering a mechanism with many options, is that it may allow for more possibilities for types to self-select at $t = 1$, which in turn in case that no trade takes place at period $t = 1$, provides the seller with more precise information about the seller's valuation at $t = 2$. The cost is that by providing more possibilities to self-select also increases the possibilities to masquerade as another type. It turns out that the cost of having higher types behaving as if they are lower ones, is higher than the gain obtained by having more precise information at $t = 2$.

7. Robustness: Longer Horizon & Alternative Degrees of Transparency

Our objective has been to characterize the revenue maximizing PBE in a multi-stage game where the seller chooses mechanisms at each stage. As we have discussed, the set of $PBE's$ depends (i) on the generality of mechanisms that the seller employs (ii) on the generality of the buyer's strategy and (iii) on the length of the time horizon and finally (iv) on the degree of transparency of mechanisms, which is intimately related with the "commitment" power of the seller.

We will now argue that our result is robust in a number of different directions. First our result is valid for any $T < \infty$. The main structure of the proof of the $T = 2$ case extends to longer horizons, but some of the details of the arguments become more involved. The interested reader is referred to the "Technical appendix for Sequentially Optimal Mechanisms" for the details. This document also contains three more extensions that are related to the degree of transparency of mechanisms. Here we have assumed that the seller observes the message that the buyer submits to the mediator, β, the action that he chooses s, and whether trade takes place or not.[21] We obtain the same result if we assume any of the following assumptions regarding the degree of transparency of mechanisms (i) the seller observes the cheap messages β, n, the costly action s, and whether trade takes place or not, (ii) the seller observes only the cheap messages β, n and whether trade took place or not and finally (iii) the seller observes only whether trade takes place or not.

Now regarding the generality of the strategy that the buyer maybe employing, we have not imposed any restrictions. The buyer may be employing mixed strategies and non-convex set of types maybe choosing the same messages and actions. Regarding the generality of definition of "mechanisms" we have been very general: we have assumed that a mechanism consists of a deterministic[22] game form, and a mediator that maps a report by the buyer to a recommendation. We have also examined a number of alternative scenaria regarding the degree of transparency of mechanisms.

It maybe worth comparing our definition of mechanism with the various definitions employed in the literature so far. In LT, who consider an environment closest to ours, a mechanism is, in our terminology, a set of contracts. In HT and in the literature that follows them, see for instance Rey and Salanie (1996), a mechanism consists of a set of messages for the agent and a deterministic mapping that maps reports to allocations. (These authors argue that in a set-up where the principal has no private information the principal's reports are immaterial and can be subsumed in the allocation.) In all these papers the principal observes everything. In Bester and Strausz (2001) a mechanism is a deterministic game form $G = (g, S)$: the agent chooses s which leads to an outcome $g(s)$ in some set X. Regarding the degree of

[21] It actually turned out that the message send by the buyer to the mediator is redundant, and this case then reduces to the case where the seller observes only the costly action of the buyer, s, as well as whether trade takes place or not.

[22] The mapping from actions s to contracts $(r(s), z(s))$ is deterministic, but recall that the buyer can be randomizing over $s's$.

transparency in BS (2001), the principal observes s and can base the future mechanisms, (or other currently non-contractible actions), on the choice of s. In Bester and Strausz (2003) a mechanism is again of the form $G = (g, S)$, but there g is random, that is a choice of s induces via $g(s)$ a probability distribution over S, and the principal does NOT observe s - observes only $g(s)$.[23]

Assuming that game forms are deterministic is without any loss of generality for most degrees of transparency of mechanisms. In particular, it is without loss in cases where the seller observes *(i)* β, s and whether trade took place or not, *(ii)* β, n, s and whether trade took place or not *(iii)* merely whether trade took place or not and $(iv)\beta$, n and whether trade took place or not. To see why this is the case in *(i)* and (ii), suppose that given an action s, $g(s)$ is (r, z) with probability π and $(\hat{r}, \hat{z})$ with probability $(1-\pi)$. Given that the seller observes s her beliefs about the buyer's valuation will be the same irrespective of whether the outcome of the game form is (r, z) or $(\hat{r}, \hat{z})$ - in other words the second period price will be simply a function of s, call it $z_2(s)$. Types below $z_2(s)$ reject this price at $t = 2$, hence for those types the expected outcome from choosing s is $p = \pi r + (1-\pi)\hat{r}$ and $x = \pi z + (1-\pi)\hat{z}$. Now types above $z_2(s)$ accept the price at $t = 2$ and for those types the expected outcome from choosing s is given by $p = \pi(r+(1-r)\delta)+(1-\pi)(\hat{r}+(1-\hat{r})\delta)$ and $x = \pi(z + (1-r)\delta z_2(s)) + (1-\pi)(\hat{z} + (1-\hat{r})\delta z_2(s))$, which can be also rewritten as $p = \pi r + (1-\pi)\hat{r}+$ $[1 - \pi r - (1-\pi)\hat{r}]\,\delta$ and $x = \pi z + (1-\pi)\hat{z}+$ $[1 - \pi r - (1-\pi)\hat{r}]\,\delta z_2(s)$. These outcomes can arise by a deterministic mapping that maps s to $\tilde{r} = \pi r + (1-\pi)\hat{r}$ and $\tilde{z} = \pi z + (1-\pi)\hat{z}$. Therefore in cases (i) and (ii), where the actions chosen by the buyer are observed by the seller, it is without any loss to assume that game forms are deterministic. For cases (iii) and (iv) where the seller does not observe s nor the contract, it is clear that it does not matter whether the game form is random or it is deterministic.

There are two cases that we have not addressed, but our analysis can be easily amended to address them. The first case is when game forms are random, the seller observes contracts, but does not observe s. In this case the same contract may have been chosen by types that choose different $s's$. Moreover because of the randomness, it is as if the buyer is choosing with positive probability contracts among which he is not indifferent. In this case there is, in some sense, a weak link between the information that the seller obtains and the (costly) action that is chosen by the buyer. The same weak relationship between costly actions of the buyer and the information that the seller observes exists in the scenario where best response constraints are imposed ex-ante only, (the buyer's strategy must only satisfy (1) and not necessarily (2)),[24] and the seller does not observe the message that the buyer sends to the mediator.[25] These two cases, even though superficially they appear unrelated, they can be addressed following the same steps. The analysis would proceed more or less along the lines of the section that provides a characterization of the result

[23] The fact that the principal does not observe s, [and cannot either precisely determine it from $g(s)$, since $g(s)$ is random], is termed noisy communication by those authors.

[24] In this case the mechanism designer has the power to employ mediators that enforce actions to the buyer: the buyer is only free to affect the "recommendations" he receives through the report that he submits to the mediator.

[25] The mechanims in Bester and Strausz (2003) can be viewed in this way.

for minimal amounts of information, studied in Section 1.2 in the "Technical Appendix for Sequentially Optimal Mechanisms." Preliminary work suggests that our result will go through. We leave the details for future work.

8. Commitment and Non-Commitment: Revenue Comparisons

In this section we compare the expected revenue for the seller when she employs a revenue maximizing mechanism under commitment and under non-commitment. Given commitment posting a price equal to v^*, (given by (6)), in each period is optimal. We have shown that when the seller behaves sequentially rationally the revenue maximizing mechanism is to post a price in each period. Let z_t denote the price posted at t. This sequence of prices has to be sequentially rational. The seller can replicate the situation under non-commitment in the commitment case by posting z_1 at $t = 1$ and z_2 at $t = 2$, instead of posting v^* in each period. From this observation it follows that in general

$$R_C \geq R_{NC}(\delta),$$

where R_C denotes the highest revenue that the seller can achieve under commitment and R_{NC} the highest revenue under non-commitment. When the buyer and the seller are very patient, (in this model the buyer and the seller have the same discount factor), the seller will find it beneficial to move all trade in the last period of the game. In the last period of the game she has commitment power. If $\delta = 1$ by shifting all trade at T she obtains expected revenue equal to R_C, which is the best she can hope for. It follows that when $\delta = 1$ expected revenue under commitment and under non-commitment coincide. On the other hand, for δ very small the value of the object at $t = 2$ is almost zero to the buyer no matter what his valuation is, so there is not much surplus for the seller to extract. When the seller and the buyer are very impatient the situation is almost equivalent to the full commitment case. The seller posts at $t = 1$ the revenue maximizing price as in the environment with commitment; therefore we get that $R_{NC}(0) = R_C$. From the above observations it follows that for extreme values of the discount factor the seller can achieve the same expected revenue under commitment and under non-commitment. For intermediate values of the discount factor it holds that $R_{NC} < R_C$. To get some idea about the magnitude of the difference in expected revenue we present an example.

Example 2 *Suppose that $T = 2$. Assume that the buyer's valuation is uniformly distributed on the interval [0,1]. For this environment the optimal mechanism under commitment is to post a price $v^* = 0.5$ in each period. The corresponding expected revenue is $R_C = 0.25$. Now let us look at the non-commitment case. Let $\bar{v}$ denote the valuation of the buyer who is indifferent between accepting z_1 at $t = 1$ and accepting z_2 at $t = 2$. It is given by $\bar{v} = \frac{z_1 - \delta z_2}{1-\delta}$. For the assumed prior we have that, if the buyer rejects the price offer at*

$t = 1$, then $F_2(v) = \frac{v}{\bar{v}}$. The price posted at $t = 2$ is given by $z_2 = \frac{\bar{v}}{2}$. Substituting this expression of z_2 into $\bar{v}$ we get that $z_1 = \bar{v}\,(1 - 0.5\delta)$. Given the above relationship between z_1, $\bar{v}$ and z_2 the seller will pick

$$\bar{v} \in \arg\max (1 - \bar{v})\bar{v}\,(1 - 0.5\delta) + \delta\left(\bar{v} - \frac{\bar{v}}{2}\right)\frac{\bar{v}}{2}.$$

The following table gives the solution for different values of the discount factor.

Discount Factor δ	**Price at t=1, $\mathbf{z}_1$**	**Price at t=2, $\mathbf{z}_2$**	$\bar{\mathbf{v}}$	$\mathbf{R}_{NC}$
0.0001	0.49999	0.25001	0.50002	0.24999
0.3	0.46612	0.27419	0.54839	0.23306
0.4	0.45714	0.28571	0.57143	0.22857
0.45	0.45330	0.29245	0.58491	0.22665
0.5	0.45	0.3	0.6	0.225
0.7	0.44474	0.34211	0.68422	0.22237
0.9	0.46538	0.42308	0.84615	0.23269
0.9999	0.49995	0.4999	0.9998	0.24998
1	0.5	0.5	1	0.25

9. Concluding Remarks

This paper establishes that the revenue maximizing allocation mechanism in a T-period model under non-commitment is to post a price in each period. It also develops a procedure to derive the optimal mechanism under non-commitment in asymmetric information environments. This method does not rely on the revelation principle.

Previous work has assumed that the seller's strategy is to post a price and the problem of the seller is to find what price to post. We provide a reason for the seller's choice to post a price, even though she can use infinitely many other possible institutions: posted price selling is the optimal strategy in the sense that it maximizes the seller's revenue. We hope that the methodology developed in this paper will prove useful in deriving the optimal dynamic incentive schemes under non-commitment in other asymmetric information environments.

In the future we plan to study the problem in an infinite-horizon framework, which may be a more appropriate model to study mechanism design under non-commitment. This problem is involved with issues which require careful analysis beyond the scope of this paper.

10. Appendix

Proof of Proposition 4

The seller's expected revenue given a strategy profile that implements an element of $\mathcal{P}_2^*$ is given by

$$\begin{aligned} R(p) &= \int_a^{\hat{z}_2} rsdF(s) - \int_a^{\hat{z}_2} r[1-F(s)]ds \\ &+ \int_{\hat{z}_2}^{\bar{v}} (r+(1-r)\delta)sdF(s) - \int_{\hat{z}_2}^{\bar{v}} (r+(1-r)\delta)[1-F(s)]ds \\ &+ \int_{\bar{v}}^{b} sdF(s) - \int_{\bar{v}}^{b} [1-F(s)]ds \end{aligned}$$

Differentiating with respect to r we get that

$$\begin{aligned} \frac{\partial R(p)}{\partial r} &= \int_a^{\hat{z}_2} sdF(s) - \int_a^{\hat{z}_2} [1-F(s)]ds \\ &+ \int_{\hat{z}_2}^{\bar{v}} (1-\delta)sdF(s) - \int_{\hat{z}_2}^{\bar{v}} (1-\delta)[1-F(s)]ds \end{aligned}$$

which does not depend on r. If $\frac{\partial R(p)}{\partial r} \geq 0$, then at an optimum it must be $r=1$ and if $\frac{\partial R(p)}{\partial r} < 0$ then at an optimum it must be $r=0$.

Now let us investigate the optimal value of $\hat{z}_2$. Since $z_2(\bar{v})$ is optimal given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v})}$ it follows that

$$\left(F(\bar{v}) - F(z_2(\bar{v}))\right) z_2(\bar{v}) \geq \left(F(\bar{v}) - F(\hat{z}_2)\right) \hat{z}_2. \tag{14}$$

Let $\hat{z}_1$ the price that the seller must most at $t=1$ to keep type $\bar{v}$ indifferent between $\hat{z}_1$ at $t=1$ and $z_2(\bar{v})$. It is given by $\hat{z}_1 = (1-\delta)\bar{v} + z_2(\bar{v}) \geq z_1 = (1-\delta)\bar{v} + z_2$ from which we get that

$$\left(1 - F(\bar{v})\right) \hat{z}_1 \geq \left(1 - F(\bar{v})\right) z_1. \tag{15}$$

And combining (14) and (15) we obtain that

$$\begin{aligned} & \left(1 - F(\bar{v})\right) \hat{z}_1 + \left(F(\bar{v}) - F(z_2(\bar{v}))\right) \delta z_2(\bar{v}) \\ \geq \; & \left(1 - F(\bar{v})\right) z_1 + \left(F(\bar{v}) - F(\hat{z}_2)\right) \delta \hat{z}_2, \end{aligned}$$

for all $\hat{z}_2 \leq z_2(\bar{v})$. It follows that at an optimum $\hat{z}_2 = z_2(\bar{v})$. ■

Proof of Lemma 2

We argue by contradiction. Suppose that $z_2(\hat{v}) < z_2(\bar{v})$, then $[F(\hat{v}) - F(\bar{v})]z_2(\bar{v}) > [F(\hat{v}) - F(\bar{v})]z_2(\hat{v})$. Moreover by the definition of $z_2(\bar{v})$ it follows that $[F(\bar{v}) - F(z_2(\bar{v}))]z_2(\bar{v}) \geq [F(\bar{v}) - F(z_2(\hat{v}))]z_2(\hat{v})$. Combining these two inequalities we get

$$\begin{aligned} & \frac{1}{F(\hat{v})} \left[\left(F(\hat{v}) - F(\bar{v})\right) z_2(\bar{v}) + \left(F(\bar{v}) - F(z_2(\bar{v}))\right) z_2(\bar{v}) \right] \\ > \; & \frac{1}{F(\hat{v})} \left[\left(F(\hat{v}) - F(\bar{v})\right) z_2(\hat{v}) + \left(F(\bar{v}) - F(z_2(\hat{v}))\right) z_2(\hat{v}) \right] \end{aligned}$$

contradicting the definition of $z_2(\hat{v})$. ■

Proof of Proposition 6

Consider a PBE assessment (σ, μ) and let p denote the allocation rule implemented by it. Let s denote an action that leads to (r, z), where this is the contract with the smallest "r", with the property that type a is either "choosing" (r, z) with strictly positive probability at $t = 1$, or is indifferent between doing and not doing so. Also let Y denote the set of types of the buyer that report message β and choose s at $t = 1$ with strictly positive probability, and let $[a, \bar{v}]$, with $a \leq \bar{v}$, denote its convex hull. From the solution for $T = 1$ we have that after the history that the buyer reported message β, chose action s, and no trade took place at $t = 1$, the seller will maximize revenue by posting a price in period $t = 2$. Let us call this price as z_2 and define

$$\begin{aligned} v_L &= \inf \{v \in Y \text{ s.t. } v \text{ accepts } z_2 \text{ at } 2\} \\ v_H &= \sup \{v \in Y \text{ s.t. } v \text{ accepts } z_2 \text{ at } 2\}. \end{aligned}$$

By definition types v_L and v_H either choose (r, z) at $t = 1$ and accept z_2 at $t = 2$ with positive probability or are indifferent between this sequence of actions and the actions that they are actually choosing.

First we show that for $v \in (v_L, v_H)$ we have that $p(v) = r + (1 - r)\delta$, then we establish that $z_2 = v_L$ and finally we show that for $v \in (a, v_L)$ we have that $p(v) = r$.

Step 1: For $v \in (v_L, v_H)$, where $v_L \neq v_H$ we have that $p(v) = r + (1 - r)\delta$. We will establish this result by observing that if a $v \in (v_L, v_H)$ is choosing a sequence of actions with positive probability, then it must be that the expected discounted probability $\hat{p}$ and the expected discounted payment $\hat{x}$ must be such that $\hat{p} = r + (1 - r)\delta$ and $\hat{x} = z + (1 - r)\delta z_2$. Otherwise depending on whether $\hat{p} < r + (1 - r)\delta$ or $\hat{p} \geq r + (1 - r)\delta$ either type $v - \varepsilon$ or $v + \varepsilon$ have a profitable deviation.

Step 2 : We show that the smallest type that accepts z_2 must be equal to it:$v_L = z_2$. First observe that the fact that at a PBE the buyer's strategy must be a best response to the seller's strategy implies that

$$(r + (1 - r)\delta) v_L - (z + (1 - r)\delta z_2) \geq r v_L - z.$$

We now show that this inequality must hold with equality. We argue by contradiction. Suppose not, that is

$$(r + (1 - r)\delta) v_L - (z + (1 - r)\delta z_2) > r v_L - z.$$

then the seller can increase z_2 by Δz such that

$$(r + (1 - r)\delta) v_L - (z + (1 - r)\delta z_2) - \Delta z = r v_L - z,$$

and raise higher revenue at the continuation game that starts at 2. All types $v \geq v_L$ still prefer to choose $(1, z_2)$ at $t = 2$ then to choose $(0, 0)$. Hence at a PBE we have that

$$(r + (1 - r)\delta)\, v_L - (z + (1 - r)\delta z_2) = r v_L - z, \tag{16}$$

from which it is immediate that $v_L = z_2$.

Step 3: For $v \in (a, v_L)$, where $a \neq v_L$ we have that $p(v) = r$. Again if a $v \in (a, v_L)$ is choosing a sequence of action with positive probability then it must be that the expected discounted probability $\hat{p}$ and the expected discounted payment $\hat{x}$ must be such that $\hat{p} = r$ and $\hat{x} = z$. Otherwise depending on whether $\hat{p} < r$ or $\hat{p} \geq r$ either type $v - \varepsilon$ or $v + \varepsilon$ have a profitable deviation.

From Steps 1-3 it follows that $p(v) = r + (1 - r)\delta$ for $v \in (v_L, v_H)$, and $p(v) = r$, for $v \in (a, v_L)$ where $v_L = z_2$. Hence the allocation rule is

$$\begin{array}{c} p(v) = r \text{ for } v \in [a, z_2) \\ r \leq p(z_2) \leq r + (1 - r)\delta \\ p(v) = r + (1 - r)\delta \text{ for } v \in (z_2, \bar{v}) \\ r + (1 - r)\delta \leq p(\bar{v}) \leq 1 \end{array}.$$

Note that $p(a)$ cannot be strictly less then r by the definition of (r, z), (in order for $p(a) \leq r$ it must be the case that type a is choosing a sequence of actions that implement $\hat{p} < r$, but this contradicts the definition of (r, z) which is the smallest "r" contract that type a chooses with positive probability at $t = 1$, or is indifferent between choosing or not. ■

Proof of Lemma 4

Suppose that the interval $[a, \bar{v}]$ is the convex hull of the types that choose the costly action s and suppose that a type v in $[a, \bar{v}]$ chooses the "cheap" message β with probability $\beta(v)$ and the cheap message $\hat{\beta}$ with probability $1 - \beta(v)$. We assume that both messages are chosen with positive probability, which implies that $\int_a^{\bar{v}} \beta(t) dF_2(t) > 0$ and $\int_a^{\bar{v}} (1 - \beta(t)) dF_2(t) > 0$. Thus the seller's posteriors at the beginning of the final period of the game after observing the costless message β and s, and respectively $\hat{\beta}$ and s, given by $\tilde{F}_2(s) = \frac{\int_a^s \beta(t) dF_2(t)}{\int_a^{\bar{v}} \beta(t) dF_2(t)}$ and $\hat{F}_2(s) = \frac{\int_a^s (1-\beta(t)) dF_2(t)}{\int_a^{\bar{v}} (1-\beta(t)) dF_2(t)}$, where F_2 is the seller's posterior in the case that she observes only s. From Lemma 3 we know that in order that both these payoff irrelevant messages to be employed in equilibrium it must be the case that $z_2(\tilde{F}_2) = z_2(\hat{F}_2)$. Our goal is to establish that $z_2(F_2) \geq z_2(\tilde{F}_2) = z_2(\hat{F}_2)$. Assume wlog that $z_2(F_2) < z_2(\tilde{F}_2)$.

By the definition of $z_2(F_2)$ we have that

$$\int_{z_2(F_2)}^{z_2(\tilde{F}_2)} s dF_2(s) - \int_{z_2(F_2)}^{z_2(\tilde{F}_2)} [1 - \int_a^s dF_2(t)] ds \geq 0. \tag{17}$$

But since $z_2(F_2)$ is not a maximizer for beliefs $\tilde{F}_2$ we have that[26]

$$\int_{z_2(F_2)}^{z_2(\tilde{F}_2)} s\beta(s)dF_2(s) - \int_{z_2(F_2)}^{z_2(\tilde{F}_2)} [\int_a^{\bar{v}} \beta(t)dF_2(t) - \int_a^s \beta(t)dF_2(t)]ds < 0. \tag{18}$$

Similarly we have that

$$\int_{z_2(F_2)}^{z_2(\hat{F}_2)} s(1-\beta(s))dF_2(s) - \int_{z_2(F_2)}^{z_2(\hat{F}_2)} [\int_a^{\bar{v}} (1-\beta(t))dF_2(t) - \int_a^s (1-\beta(t))dF_2(t)]ds < 0 \tag{19}$$

adding (18) and (19) and recalling that $z_2(\tilde{F}_2) = z_2(\hat{F}_2)$ we obtain that

$$\begin{aligned} & \int_{z_2(F_2)}^{z_2(\hat{F}_2)} [s(1-\beta(s)) + \beta(s)s]\, dF_2(s) - \int_{z_2(F_2)}^{z_2(\hat{F}_2)} [\int_a^{\bar{v}} [(1-\beta(t)) + \beta(t)]\, dF_2(t) - \int_a^s [(1-\beta(t)) + \beta(t)]\, dF_2(t)]ds \\ = & \int_{z_2(F_2)}^{z_2(\hat{F}_2)} sdF_2(s) - \int_{z_2(F_2)}^{z_2(\hat{F}_2)} [\int_a^{\bar{v}} dF_2(t) - \int_a^s dF_2(t)]ds \\ = & \int_{z_2(F_2)}^{z_2(\hat{F}_2)} sdF_2(s) - \int_{z_2(F_2)}^{z_2(\hat{F}_2)} [1 - \int_a^s dF_2(t)]ds < 0, \end{aligned} \tag{20}$$

Where the last equality follows from the fact that $\int_a^{\bar{v}} dF_2(t) = 1$. But (20) contradicts (17). ■

Proof of Lemma 6

We argue by contradiction. Suppose that there exists $\hat{v} \in [a, z_2)$ that is randomizing between (r, z) and some other contract at $t = 1$.[27] From Proposition 6 we know that it must be the case that for $v \in [a, z_2)$ we have that $p(v) = r$. Then if $\hat{v}$ is randomizing it must be choosing a contract $(\hat{r}, \hat{z})$ such either (a) $r = \hat{r} + (1-\hat{r})\delta$, (b) $\hat{r} = r$.

First to see that (a) is impossible note that since $\hat{v}$ is choosing $(\hat{r}, \hat{z})$ at $t = 1$ and accepts $(1, \hat{z}_2)$ at $t = 2$, then it must be the case that $\hat{z}_2 \leq \hat{v}$. If $a = \hat{z}_2$ then type a is indifferent between $(\hat{r}, \hat{z})$ and (r, z) contradicting the definition of (r, z). If on the other hand, $a < \hat{z}_2$ then for type $\hat{z}_2$ it must hold that $\hat{r}\hat{z}_2 - \hat{z} = (\hat{r} + (1-\hat{r})\delta)\, \hat{z}_2 - \hat{z} - (1-\hat{r})\delta\hat{z}_2 = r\hat{z}_2 - z$. But since $a < \hat{z}_2$ and $\hat{r} \leq r$ we have that

[26] From straightforward calculations we get the two expressions of revenue at the continuation game that starts at $t = 2$

$$\int_a^{\bar{v}} sd\tilde{F}_2(s) - \int_a^{\bar{v}} [1 - \tilde{F}_2(s)]ds = \frac{1}{\int_a^{\bar{v}} \pi(t)dF_2(t)} \left\{ \int_a^{\bar{v}} s\pi(s)dF_2(s) - \int_a^{\bar{v}} [\int_a^b \pi(t)dF_2(t) - \int_a^s \pi(t)dF_2(t)]ds \right\}.$$

and

$$\int_a^{\bar{v}} sd\hat{F}_2(s) - \int_a^{\bar{v}} [1 - \hat{F}_2(s)]ds = \frac{1}{\int_a^b (1-\pi(t))dF_2(t)} \left\{ \int_a^{\bar{v}} s(1-\pi(s))dF_2(s) - \int_a^{\bar{v}} [\int_a^b (1-\pi(t))dF_2(t) - \int_a^s (1-\pi(t))dF_2(t)]ds \right\}.$$

Note that since $\frac{1}{\int_a^{\bar{v}} \pi(t)dF_2(t)}$ and $\frac{1}{\int_a^b (1-\pi(t))dF_2(t)}$ are constants we can ignore them.

[27] The only relevant case is the case $a < z_2$ since the case $a \geq z_2$ trivially implies that all types that may be randomizing are greater or equal to z_2.

$\hat{r}a - \hat{z} > (\hat{r} + (1 - \hat{r})\delta)\, a - \hat{z} - (1 - \hat{r})\delta\hat{z}_2 = ra - z$ contradicting the fact that a is choosing (r, z) with positive probability.

Possibility (b) is not relevant either since $r = \hat{r}$ implies $z = \hat{z}$, which implies in turn that $(\hat{r}, \hat{z})$ is the same contract at (r, z). To see this we argue by contradiction. Suppose not, and wlog let $\hat{z} < z$. Depending on whether $\hat{z}_2 < z_2$ or $\hat{z}_2 \geq z_2$, there are two cases to consider. If $\hat{z}_2 < z_2$ then for all $v \in [a, b]$ we have that $\hat{r}v - \hat{z} > rv - z$ and $(\hat{r} + (1 - \hat{r})\delta)v - \hat{z} - (1 - \hat{r})\delta\hat{z}_2 > (r + (1 - r)\delta)v - z - (1 - r)\delta z_2$, but then the types that are choosing (r, z) with strictly positive probability are not best-responding, contradicting the supposition that we are looking at a PBE. Now in the case that $z_2 < \hat{z}_2$ then for all $v \in [a, b]$ we have that $\hat{r}v - \hat{z} > rv - z$ and $(\hat{r} + (1 - \hat{r})\delta)v - \hat{z} - (1 - \hat{r})\delta\hat{z}_2 < (r + (1 - r)\delta)v - z - (1 - r)\delta z_2$, which implies that all types in $[a, z_2)$ strictly prefer $(\hat{r}, \hat{z})$ over (r, z) and all types above z_2 strictly prefer (r, z) over $(\hat{r}, \hat{z})$. These observations imply that the seller posts at $t = 2$ a price $\hat{z}_2 > z_2$ given a posterior that has support a subset of $[a, z_2]$, contradicting the fact that we are looking at PBE. Hence $r = \hat{r}$ implies that $z = \hat{z}$, but then (r, z) and $(\hat{r}, \hat{z})$ are the same contract. ■

Proof of Lemma 7

Let us first define

$$\phi(v, \tilde{v}\,|F_2\,) = F(\bar{v}) \cdot \frac{1}{F(\bar{v})}\left[\int_v^{\tilde{v}} sdF(s) - \int_v^{\tilde{v}} (F(\bar{v}) - F(t))dt\right], \tag{21}$$

and

$$\phi(v, \tilde{v}\,|F_2^m\,) = \left[F(z_2) + \int_{z_2}^{\bar{v}} m(t)dF(t)\right] \cdot \left[\int_v^{\tilde{v}} sdF_2^m(s) - \int_v^{\tilde{v}} [1 - F_2^m(s)]ds\right].$$

For $v, \tilde{v} \in [a, z_2)$, we have that

$$\phi(v, \tilde{v}\,|F_2^m\,) = \left[\int_v^{\tilde{v}} tdF(t) - \int_v^{\tilde{v}} \left(F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}} m(s)dF(s) - F(t)\right) dt\right].$$

Hence for $v, \tilde{v} \in [a, z_2)$ $\phi(v, \tilde{v}\,|F_2^m\,)$ and $\phi(v, \tilde{v}\,|F_2\,)$ differ by a constant. To make $\phi(v, \tilde{v}\,|F_2\,)$ more easily comparable with $\phi(v, \tilde{v}\,|F_2^m\,)$, let us do some rewriting. By adding and subtracting $\int_v^{\tilde{v}} F(z_2)dt$ to $\phi(v, \tilde{v}\,|F_2\,)$ we get that

$$\phi(v, \tilde{v}\,|F_2\,) = \int_v^{\tilde{v}} tdF(t) - \int_v^{\tilde{v}} (F(z_2) + F(\bar{v}) - F(z_2) - F(t))\, dt.$$

Now because $m(v) \in [0, 1]$, we have that $F(z_2) + F(\bar{v}) - F(z_2) \geq F(z_2) + \int_{z_2}^{\bar{v}} m(t)dF(t)$ which implies that

$$\phi(v, \tilde{v}\,|F_T\,) \leq \hat{\phi}(v, \tilde{v}\,|F_T^m\,), \tag{22}$$

for all $v, \tilde{v} \in [a, z_2)$.

Suppose that $z_2(\bar{v}) < z_2$, now by the definition of $z_2(\bar{v})$ it follows that

$$\phi(z_2(\bar{v}), \tilde{v}\,|F_2\,) \geq 0, \text{ for all } \tilde{v} \in [z_2(\bar{v}), z_2],$$

which together with (22) implies that

$$\phi(z_2(\bar{v}), \tilde{v} \,|F_2^m) \geq 0, \text{ for all } \tilde{v} \in [z_2(\bar{v}), z_2],$$

and by the definition of z_2 we have that

$$\phi(z_2, \tilde{v} \,|F_2^m) \geq 0, \text{ for all } \tilde{v} \in [z_2, \bar{v}],$$

but the last two inequalities imply

$$\phi(z_2(\bar{v}), \tilde{v} \,|F_2^m) \geq 0, \text{ for all } \tilde{v} \in [z_2(\bar{v}), \bar{v}],$$

contradicting the definition of z_2. Hence we have shown that $z_2(\bar{v}) \geq z_2$. ■

References

[1] Aumann, R. J. and S. Hart (2003): "Long Cheap Talk," *Econometrica,* 71, 1619-1660.

[2] Bester, H., and R. Strausz (2001): "Contracting with Imperfect Commitment and the Revelation Principle: The Single Agent Case." *Econometrica,* 69, 1077-1098.

[3] Bester H. and R. Strausz (2003): "Contracting with Imperfect Commitment and Noisy Communication," mimeo.

[4] Bulow, J. (1982): "Durable Goods Monopolists," *Journal of Political Economy*, 90, 314-322.

[5] Coase, R. (1972): "Durability and Monopoly," *Journal of Law and Economics*, 15, 143-149.

[6] Crawford, V. and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431-1451.

[7] Freixas, X., R. Guesnerie, and J. Tirole (1985): "Planning under Incomplete Information and the Ratchet Effect," *Review of Economic Studies,* 52, 173-192.

[8] Fudenberg, D., D. Levine, and J. Tirole (1985): "Infinite-horizon Models of Bargaining with One-sided Incomplete Information," In *'Game Theoretic Models of Bargaining'*, edited by Alvin Roth. Cambridge University Press.

[9] Gul, F., Sonnenschein, H., and R. Wilson (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory* 39, 155-190.

[10] Hart, O., and J. Tirole (1988): "Contract Renegotiation and Coasian Dynamics," *Review of Economic Studies*, 55, 509-540.

[11] Kolmogorov, A. N. and S.V. Fomin (1970): *Introductory Real Analysis.* Dover Publications.

[12] Laffont,J.J., and J. Tirole (1988): "The Dynamics of Incentive Contracts," *Econometrica*, 56, 1153-1175.

[13] Laffont,J.J., and J. Tirole (1993): *A Theory of Incentives in Procurement and Regulation*, MIT Press.

[14] McAfee, R. P. and D. Vincent (1997): "Sequentially Optimal Auctions," *Games and Economic Behavior*, 18, 246-276.

[15] Munkres, I. (1975): *Topology: A First Course.* Prentice Hall.

[16] Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

[17] Myerson, R. (1982): "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, 10, 67-81.

[18] Myerson, R. (1986): "Multistage Games with Communication," *Econometrica*, 54, 323-358.

[19] Myerson, R. (1991): *Game Theory*, Harvard University Press, Chapter 6.

[20] Myerson, R. (1994): "Communication, Correlated Equilibria, and Incentive Compatibility," in R. J. Aumann and S. Hart (eds.), Handbook of Game Theory, with Economic Applications, Vol. 2, North Holland, 827-847.

[21] Riley, J. G. and W. F. Samuelson (1981): "Optimal Auctions," *American Economic Review, 71, 381-392.*

[22] Riley, J. and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle, When to hold Firm," *Quarterly Journal of Economics*, 98, 267-287.

[23] Rey, P. and B. Salanie (1996): "On the Value of Commitment in Contracting with Asymmetric Information," *Econometrica*, 64, 1395-1414.

[24] Royden, H. (1962): *Real Analysis*, 2nd Edition, New York: Macmillan.

[25] Salanie, B. (1997): *The Economics of Contracts*, MIT Press.

[26] Skreta, V. (2004b): "Revenue Equivalence and Mechanism Design for Arbitrary Type Spaces," mimeo UCLA.

[27] Sobel, J. and I. Takahashi (1983): "A Multi-stage Model of Bargaining," *Review of Economic Studies*, 50, 411-426.

[28] Stokey, N. (1981): "Rational Expectations and Durable Goods Pricing," *Bell Journal of Economics*, 12, 112-128.