Applications of Subsampling, Hybrid, and Size-Correction Methods

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Abstract

This paper analyzes the properties of subsampling, hybrid subsampling, and size-correction methods in two non-regular models. The latter two procedures are introduced in Andrews and Guggenberger (2009a). The models are non-regular in the sense that the test statistics of interest exhibit a discontinuity in their limit distribution as a function of a parameter in the model. The first model is a linear instrumental variables (IV) model with possibly weak IVs estimated using two-stage least squares (2SLS). In this case, the discontinuity occurs when the concentration parameter is zero. The second model is a linear regression model in which the parameter of interest may be near a boundary. In this case, the discontinuity occurs when the parameter is on the boundary.

The paper shows that in the IV model one-sided and equal-tailed two-sided subsampling tests and confidence intervals (CIs) based on the 2SLS t statistic do not have correct asymptotic size. This holds for both fully- and partially-studentized t statistics. But, subsampling procedures based on the partially-studentized t statistic can be size-corrected. On the other hand, symmetric two-sided subsampling tests and CIs are shown to have (essentially) correct asymptotic size when based on a partially-studentized t statistic. Furthermore, all types of hybrid subsampling tests and CIs are shown to have correct asymptotic size in this model. The above results are consistent with "impossibility" results of Dufour (1997) because subsampling and hybrid subsampling CIs are shown to have infinite length with positive probability.

Subsampling CIs for a parameter that may be near a lower boundary are shown to have incorrect asymptotic size for upper one-sided and equal-tailed and symmetric two-sided CIs. Again, size-correction is possible. In this model as well, all types of hybrid subsampling CIs are found to have correct asymptotic size.

Keywords: Asymptotic size, finite-sample size, hybrid test, instrumental variable, over-rejection, parameter near boundary, size correction, subsampling confidence interval, subsampling test, weak instrument.

JEL Classification Numbers: C12, C15.

1 Introduction

This paper continues the investigation initiated in Andrews and Guggenberger (2009a, 2009b, 2010) (hereafter denoted AG2, AG3, and AG1) of the properties of subsampling and subsampling-based procedures in non-regular models. We apply the results of AG1-AG3 to two models. The first model is an instrumental variables (IVs) regression model with possibly weak IVs. This is a leading example of a broad class of models in which lack of identification occurs at some point(s) in the parameter space. It is a model that has been studied extensively in the recent econometrics literature. For this reason, it is a natural model to use to assess the behavior of subsampling methods. The second example that we consider in this paper concerns a CI when the parameter of interest may be near a boundary. This example is a generalization of the example used in the introduction of AG1 to illustrate heuristically a problem with subsampling. Here we treat the example rigorously.

In the first example, for comparability to the literature, we focus on a model with a single right-hand-side (rhs) endogenous variable and consider inference concerning the parameter on this variable. It is well-known that standard two-stage least squares (2SLS) based t tests and CIs have poor size properties in this case, e.g., see Dufour (1997), Staiger and Stock (1997), and references cited therein. In particular, one-sided, symmetric two-sided, and equal-tailed two-sided fixed critical value (FCV) tests have finite-sample size of 1.0. Furthermore, these tests cannot be size-corrected by increasing the FCV.[1]

We are interested in the properties of subsampling methods in this model. We are also interested in whether the hybrid and size-correction (SC) methods introduced in AG2 can be used to obtain valid inference in this well-known non-regular model. Hence, we investigate the size properties of subsampling and hybrid tests based on the 2SLS estimator. The test results given here apply without change to CIs (because of location invariance). We also consider size-corrected versions of these methods. Alternatives in the literature to the size-corrected methods include the conditional likelihood ratio (CLR) test of Moreira (2003), the rank CLR test of Andrews and Soares (2007), and the adaptive CLR test of Cattaneo, Crump, and Jansson (2007). These tests are asymptotically similar, and hence, have good size properties. Also, their power properties have been shown to be quite good in Andrews, Moreira, and Stock (2006, 2007, 2008) and the other references above. Other tests in the literature that are robust to weak IVs include those given in Kleibergen (2002, 2005), Guggenberger and Smith (2005, 2008), and Otsu (2006). Although we have not investigated the power properties of the hybrid and SC subsampling tests considered here, we ex-

[1] The finite-sample (or exact) *size* of a test is defined to be the maximum rejection probability of the test under distributions in the null hypothesis. A test is said to have *level* α if its finite-sample size is α or less. The *asymptotic size* of a test is defined to be the limit superior of the finite-sample size of the test. The finite-sample (or exact) *size* of a confidence interval (or confidence set) is defined to be the minimum coverage probability of the confidence interval under distributions in the model. Analogously, a confidence interval is said to have *level* $1-\alpha$ if its finite-sample size is $1-\alpha$ or greater. The *asymptotic size* of a confidence interval is defined to be the limit inferior of the finite-sample size of the confidence interval. A test is called *asymptotically similar* if the limit of the null rejection probability of the test is the same under any sequence of nuisance parameters.

pect that they are inferior to those of the CLR, rank CLR, and adaptive CLR tests. Hence, we do not advocate the use of subsampling methods in the weak IV model for inference on the parameter of a single rhs endogenous variable.

However, the CLR-based tests and the other tests mentioned above do not apply to inference concerning the parameter on one endogenous variable when multiple rhs endogenous variables are present that may be weakly identified. This is a testing problem for which no asymptotically similar test is presently available. The methods analyzed in this paper are potentially useful for such inference problems. We leave this to future research.

We now summarize the results for the IV example. We show that subsampling tests and CIs do not have correct size asymptotically, but can be size-corrected. The asymptotic rejection probabilities of the subsampling tests are found to provide poor approximations to the finite-sample rejection probabilities in many cases. But, the finite-sample adjusted asymptotic rejection probabilities introduced in AG2 perform very well across all scenarios. In consequence, the adjusted size-corrected subsampling (ASC-Sub) tests perform well. For example, the nominal 5% ASC-Sub tests based on partially-studentized t statistics have finite-sample sizes of 4.4, 5.3, and 4.4% for upper one-sided, symmetric two-sided, and equal-tailed two-sided tests in a model with $n = 120$, $b = 12$, 5 IVs, and normal errors.

The hybrid test is found to have correct size asymptotically and very good size in finite samples for upper one-sided and symmetric two-sided tests—4.8 and 4.7%, respectively. For equal-tailed two-sided tests, the hybrid test has correct size asymptotically, but is conservative in finite samples. For the same parameter values as above, the nominal 5% hybrid test has finite-sample size of 2.8%.

We show that nominal $1-\alpha$ subsampling CIs have infinite length with probability $1-\alpha$ asymptotically when the model is completely unidentified and the correlation between the structural and reduced-form errors is ± 1. This holds for both fully- and partially-studentized t statistics. This result is of particular interest given Dufour's (1997) result that the 2SLS CI based on a fixed critical value, and any CI that has finite length with probability one, have a finite-sample size of zero for all sample sizes. The results given in this paper are consistent with those of Dufour (1997) and explain why size-correction of subsampling procedures is possible even in the presence of lack of identification at some parameter values.

In the second example we consider a multiple linear regression model where the regression parameter of interest θ ($\in R$) is restricted to be non-negative. We consider a studentized t statistic based on the least squares estimator of θ that is censored to be non-negative.

The results for this example are summarized as follows. Lower one-sided, symmetric two-sided, and equal-tailed two-sided subsampling CIs for θ based on the studentized t statistic do not have correct asymptotic coverage probability. In particular, these three nominal $1-\alpha$ CIs have asymptotic confidence levels of $1/2$, $1-2\alpha$, and $(1-\alpha)/2$, respectively. Hence, the lower and equal-tailed subsampling CIs perform very poorly in terms of asymptotic size. The finite-sample sizes of these tests are found to be close to their asymptotic sizes in models with $(n = 120, b = 12)$ and

($n = 240, b = 24$) and normal errors and regressors. Size-correction is possible for all three types of subsampling CIs. The SC subsampling CIs are found to have good size in finite samples, but display a relatively high degree of non-similarity. The upper one-sided subsampling CI has correct asymptotic size $1 - \alpha$.

We show that all types of FCV and hybrid CIs have correct asymptotic size—no size correction is necessary. These CIs are found to have finite-sample sizes that are fairly close to their nominal sizes. The FCV CIs exhibit the smallest degree of finite-sample non-similarity, which has CI length advantages. Hence, somewhat ironically, the best CIs in this example are FCV CIs that ignore the presence of a boundary. We caution, however, that the scope of this result is limited to CIs when a scalar parameter of interest may be near a boundary and no other parameters are.

Using results in the literature, such as Andrews (1999, 2001), the asymptotic results given here for subsampling, FCV, and hybrid CIs should generalize to a wide variety of models other than regression models in which one or more parameters may be near a boundary.

The Appendix of the paper provides necessary and sufficient conditions for size-correction (of the type considered in AG2) to be possible in the general set-up considered in AG1 and AG2.

Literature that is related to this paper include AG1 and AG2, as well as Politis and Romano (1994) and Politis, Romano, and Wolf (1999). Andrews and Guggenberger (2009c) discusses an additional example regarding the performance of subsampling methods. Somewhat related is the paper by Moreira, Porter, and Suarez (2009) on bootstrapping the CLR test in an IV regression model with possibly weak IVs.

The remainder of this paper is organized as follows. Section 2 summarizes the most relevant results in AG1 and AG2 to make the paper more self-contained. Section 3 discusses the IV regression example. Section 4 discusses the regression example in which the parameter of interest may be near a boundary. An Appendix contains the verifications of assumptions in AG1 and AG2, including proofs of the asymptotic distributions of t statistics in these examples. The Appendix also provides the necessary and sufficient conditions for size-correction to be possible.

2 Summary of AG1 and AG2

The treatment of the two examples considered in Sections 3 and 4 relies heavily on the theoretical results on the "asymptotic size" of a test given in AG1 and AG2. To make the paper more self-contained and easier to read, we summarize in this section some of the most relevant results of AG1 and AG2. We illustrate and motivate the assumptions and theoretical results in AG1 and AG2 through a simplified version of the weak IV example of Section 3. We also provide a brief discussion of the relevance of the two examples considered in Sections 3 and 4.

Asymptotic Size

We are interested in determining the "asymptotic size" of a test $H_0 : \theta = \theta_0$

defined as

$$AsySz(\theta_0) = \limsup_{n\to\infty} \sup_{\gamma\in\Gamma} P_{\theta_0,\gamma}(T_n(\theta_0) > c(1-\alpha)), \tag{2.1}$$

where $T_n(\theta_0)$ is the test statistic, $c(1-\alpha)$ denotes the critical value of the test at nominal size α, and $\gamma \in \Gamma$ denotes a nuisance parameter vector and its parameter space. Note that the $\sup_{\gamma\in\Gamma}$ is taken before the $\limsup_{n\to\infty}$. This definition reflects the fact that our interest is in the exact finite-sample size of the test $\sup_{\gamma\in\Gamma} P_{\theta_0,\gamma}(T_n(\theta_0) > c(1-\alpha))$. We use asymptotics to approximate the finite-sample size. Uniformity over $\gamma \in \Gamma$ is built into the definition of $AsySz(\theta_0)$. If only the pointwise null rejection probability of a test is controlled but the convergence is not uniform over $\gamma \in \Gamma$ then, at every sample size n, the finite-sample null rejection probability of the test might differ substantially from its nominal size for certain $\gamma = \gamma_n$. This is a problem (and the pointwise justification is not meaningful) if the lack of uniformity is "in an upward direction" and the finite-sample null rejection probability exceeds the nominal size of the test. Obviously, if $AsySz(\theta_0) > \alpha$, then the nominal level α test has asymptotic size greater than α and the test does not have correct asymptotic level.

In this paper, $T_n(\theta_0)$ denotes a t statistic (or the absolute value of a t statistic or (-1) times a t statistic).

Critical Values

We consider three types of critical values $c(1-\alpha)$. The first type of critical value is fixed (FCV), $c_{Fix}(1-\alpha)$, that is, it is non-random, and could, for example, be the $1-\alpha$ quantile of a standard normal distribution. The second type of critical value is a subsampling critical value denoted by $c_{n,b}(1-\alpha)$. To describe it, let $\{b_n : n \geq 1\}$ be a sequence of subsample sizes. As is standard, for the asymptotic results we assume that $b_n \to \infty$ and $b_n/n \to 0$ as $n \to \infty$. For brevity, we sometimes write b_n as b. The number of data subsamples of length b is $q_n = n!/((n-b)!b!)$. Let $L_{n,b}(x)$ and $c_{n,b}(1-\alpha)$ denote the empirical distribution function and $1-\alpha$ sample quantile, respectively, of subsample statistics $\{\widehat{T}_{n,b,j} : j = 1, ..., q_n\}$. They are defined by

$$\begin{aligned} L_{n,b}(x) &= q_n^{-1} \sum_{j=1}^{q_n} 1(\widehat{T}_{n,b,j} \leq x) \text{ for } x \in R \text{ and} \\ c_{n,b}(1-\alpha) &= \inf\{x \in R : L_{n,b}(x) \geq 1-\alpha\}. \end{aligned} \tag{2.2}$$

Under Assumption Sub2 of AG1, $\{\widehat{T}_{n,b,j} : j = 1, ..., q_n\}$ equals $\{T_{n,b,j}(\theta_0) : j = 1, ..., q_n\}$, where $T_{n,b,j}(\theta_0)$ are subsample statistics that are defined just as $T_n(\theta_0)$ is defined but are based on the data in the jth subsample of length b rather than the entire data set. Under Assumption Sub1 of AG1, $\{\widehat{T}_{n,b,j} : j = 1, ..., q_n\}$ equals $\{T_{n,b,j}(\widehat{\theta}_n) : j = 1, ..., q_n\}$, where $\widehat{\theta}_n$ is an estimator of θ_0. The nominal level α subsampling test rejects H_0 if

$$T_n(\theta_0) > c_{n,b}(1-\alpha). \tag{2.3}$$

Third, the critical value could be a hybrid critical value, defined in AG2 as the maximum of the subsampling and the FCV, $c^*_{n,b}(1-\alpha) = \max\{c_{n,b}(1-\alpha), c_{Fix}(1-\alpha)\}$.

Thus, the nominal level α hybrid test is defined to reject H_0 if

$$T_n(\theta_0) > c^*_{n,b}(1-\alpha). \tag{2.4}$$

We consider upper and lower one-sided and symmetric and equal-tailed two-sided tests. For example, the nominal level α equal-tailed two-sided subsampling test rejects H_0 if

$$T_n(\theta_0) > c_{n,b}(1-\alpha/2) \text{ or } T_n(\theta_0) < c_{n,b}(\alpha/2). \tag{2.5}$$

The equal-tailed hybrid test is defined analogously with $c_{n,b}(1-\alpha/2)$ and $c_{n,b}(\alpha/2)$ replaced by $\max\{c_{n,b}(1-\alpha/2), c_{Fix}(1-\alpha/2)\}$ and $\min\{c_{n,b}(\alpha/2), c_{Fix}(\alpha/2)\}$, respectively.

Example: Consider the simple model given by a structural equation and a reduced–form equation $y_1 = y_2\theta + u$, $y_2 = z\pi + v$, where $y_1, y_2, z \in R^n$ and $\theta, \pi \in R$ are unknown parameters. Assume $\{(u_i, v_i, z_i) : i \leq n\}$ are i.i.d. with distribution F, where a subscript i denotes the i-th component of a vector. To test $H_0 : \theta = \theta_0$ against a two–sided alternative say, the t statistic $T_n(\theta_0) = |n^{1/2}(\widehat{\theta}_n - \theta_0)/\widehat{\sigma}_n|$ and critical value $c_{Fix}(1-\alpha) = z_{1-\alpha/2}$ is used, where $\widehat{\theta}_n = (y_2' P_z y_2)^{-1} y_2' P_z y_1$, $\widehat{\sigma}_n = \widehat{\sigma}_u (N^{-1} y_2' P_z y_2)^{-1/2}$, $\widehat{\sigma}^2_u = (n-1)^{-1}(y_1 - y_2\widehat{\theta}_n)'(y_1 - y_2\widehat{\theta}_n)$, and $z_{1-\alpha}$ denotes the $1-\alpha$ quantile of a standard normal distribution. The nuisance parameter vector γ equals (F, π), where certain restrictions are imposed on F, such as conditional homoskedasticity, exogeneity of the instrument, and existence of second moments.

Nuisance parameters

The parameter γ is decomposed into three components: $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. The points of discontinuity of the asymptotic distribution of the test statistic of interest are determined by the first component, γ_1. The parameter space of γ_1 is Γ_1. The second component, γ_2, of γ also affects the limit distribution of the test statistic, but does not affect the distance of the parameter γ to the point of discontinuity. The parameter space of γ_2 is Γ_2. The third component, γ_3, of γ does not affect the limit distribution of the test statistic. The parameter space for γ_3 is $\Gamma_3(\gamma_1, \gamma_2)$, which generally may depend on γ_1 and γ_2. The parameter space Γ for γ satisfies

Assumption A. (i) Γ satisfies

$$\Gamma = \{(\gamma_1, \gamma_2, \gamma_3) : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \gamma_3 \in \Gamma_3(\gamma_1, \gamma_2)\} \tag{2.6}$$

and (ii) $\Gamma_1 = \prod_{m=1}^{p} \Gamma_{1,m}$, where $\Gamma_{1,m} = \lfloor \gamma^\ell_{1,m}, \gamma^u_{1,m} \rfloor$ for some $-\infty \leq \gamma^\ell_{1,m} < \gamma^u_{1,m} \leq \infty$ that satisfy $\gamma^\ell_{1,m} \leq 0 \leq \gamma^u_{1,m}$ for $m = 1, ..., p$ and $\lfloor$ denotes the left endpoint of an interval that may be open or closed at the left end. Define $\rfloor$ analogously for the right endpoint.

Assumption A imposes a finite dimensional product space structure on γ_1 and γ_2.

Example (continued): Decompose the nuisance parameter into $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_1 = |(E_F z_i^2)^{1/2}\pi/\sigma_v|$, $\gamma_2 = \rho$, and $\gamma_3 = (F, \pi)$, where $\sigma^2_v = E_F v_i^2$, $\sigma^2_u = E_F u_i^2$, and $\rho = Corr_F(u_i, v_i)$. The parameter spaces for γ_1 and γ_2 are $\Gamma_1 = R_+$ and $\Gamma_2 = [-1, 1]$. The details for the restrictions on the parameter space $\Gamma_3 = \Gamma_3(\gamma_1, \gamma_2)$

for γ_3 are given below and are such that the following CLT holds under sequences $\gamma = \gamma_n$ for which $\gamma_2 = \gamma_{2,n} \to h_2$:

$$\left(\begin{array}{c} (n^{-1}z'z)^{-1/2}n^{-1/2}z'u/\sigma_u \\ (n^{-1}z'z)^{-1/2}n^{-1/2}z'v/\sigma_v \end{array} \right) \to_d \left(\begin{array}{c} \psi_{u,h_2} \\ \psi_{v,h_2} \end{array} \right) \sim N(0, \left(\begin{array}{cc} 1 & h_2 \\ h_2 & 1 \end{array} \right)). \qquad (2.7)$$

In this example, the asymptotic distribution of the statistic $T_n(\theta_0)$ has a "discontinuity" at $\gamma_1 = 0$. Under different sequences $\gamma_1 = \gamma_{1,n}$ such that $\gamma_{1,n} \to 0$, the limit distribution of $T_n(\theta_0)$ may be different. More precisely, denote by $\gamma_{n,h}$ a sequence of nuisance parameters $\gamma = \gamma_n$ such that $n^{1/2}\gamma_1 \to h_1$ and $\gamma_2 \to h_2$ and $h = (h_1, h_2)$. It is shown below that under $\gamma_{n,h}$, the limit distribution of $T_n(\theta_0)$ depends on h_1 and h_2 and only on h_1 and h_2. As long as h_1 is finite, the sequence γ_1 converges to zero, yet the limit distribution of $T_n(\theta_0)$ does not only depend on the limit point 0 of γ_1, but depends on how precisely γ_1 converges to zero, indexed by the convergence speed $n^{1/2}$ and the localization parameter h_1. In contrast, the limit distribution of $T_n(\theta_0)$ only depends on the limit point h_2 of γ_2 but not on how γ_2 converges to h_2. In that sense, the limit distribution is discontinuous in γ_1 at 0, but continuous on Γ_2 in γ_2. The parameter γ_3 does not influence the limit distribution of $T_n(\theta_0)$ by virtue of the CLT in (2.7).

If $h_1 < \infty$, it is shown below that under $\gamma_{n,h}$

$$\left(\begin{array}{c} y_2'P_z u/(\sigma_u\sigma_v) \\ y_2'P_z y_2/\sigma_v^2 \\ \widehat{\sigma}_u^2/\sigma_u^2 \end{array} \right) \to_d \left(\begin{array}{c} \xi_{1,h} \\ \xi_{2,h} \\ \eta_{u,h}^2 \end{array} \right) = \left(\begin{array}{c} (\psi_{v,h_2} + h_1)\psi_{u,h_2} \\ (\psi_{v,h_2} + h_1)^2 \\ (1 - h_2\xi_{1,h}/\xi_{2,h})^2 + (1 - h_2^2)\xi_{1,h}^2/\xi_{2,h}^2 \end{array} \right) \qquad (2.8)$$

and thus $T_n(\theta_0) \to_d J_h$, where J_h is the distribution of $|\xi_{1,h}/(\xi_{2,h}\eta_{u,h}^2)^{1/2}|$. If $h_1 = \infty$, $T_n(\theta_0) \to_d J_h$, where in this case J_h is the distribution of the absolute value of a standard normal random variable independent of h_2.

Formalizing the additional aspects of the example, we now define the index set of h vectors for the different asymptotic null distributions of the test statistic $T_n(\theta_0)$ of interest. Let

$$\begin{aligned} H = \{h = (h_1, h_2) \in R_\infty^{p+q} : \exists \{\gamma_n = (\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}) \in \Gamma : n \geq 1\} \\ \text{such that } n^{1/2}\gamma_{n,1} \to h_1 \text{ and } \gamma_{n,2} \to h_2\}, \end{aligned} \qquad (2.9)$$

where $R_\infty = R \cup \{\pm\infty\}$.

Definition of $\{\gamma_{n,h} : n \geq 1\}$: Given $h = (h_1, h_2) \in H$, let $\{\gamma_{n,h} = (\gamma_{n,h,1}, \gamma_{n,h,2}, \gamma_{n,h,3}) : n \geq 1\}$ denote a sequence of parameters in Γ for which $n^{1/2}\gamma_{n,h,1} \to h_1$ and $\gamma_{n,h,2} \to h_2$.

In the example, $H = R_{+,\infty} \times [-1, 1]$, where $R_{+,\infty} = \{x \in R : x \geq 0\} \cup \{\infty\}$. The sequence $\{\gamma_{n,h} : n \geq 1\}$ is defined such that under $\{\gamma_{n,h} : n \geq 1\}$, the asymptotic distribution of $T_n(\theta_0)$ depends on h and only h. This is formalized in the following assumption and has already been illustrated in the above example.

Assumption B. For all $h \in H$, all sequences $\{\gamma_{n,h} : n \geq 1\}$, and some distributions J_h, $T_n(\theta_0) \to_d J_h$ under $\{\gamma_{n,h} : n \geq 1\}$.

For subsampling tests we need additional assumptions.

Assumption C. (i) $b \to \infty$ and (ii) $b/n \to 0$.

Assumption D. (i) $\{T_{n,b,j}(\theta_0) : j = 1, ..., q_n\}$ are identically distributed under any $\gamma \in \Gamma$ for all $n \geq 1$ and (ii) $T_{n,b,j}(\theta_0)$ and $T_b(\theta_0)$ have the same distribution under any $\gamma \in \Gamma$ for all $n \geq 1$.

Assumption C holds by choice of the blocksize. In the linear IV example, Assumption D holds trivially by the i.i.d. assumption on the data. For the hybrid test we need one additional assumption.

Assumption K. The asymptotic distribution J_h in Assumption B is the same (proper) distribution, call it J_∞, for all $h = (h_1, h_2) \in H$ for which $h_{1,m} = +\infty$ or $-\infty$ for all $m = 1, ..., p$, where $h_1 = (h_{1,1}, ..., h_{1,p})'$.

In the linear IV example, Assumption K holds with J_∞ equal to the distribution of a standard normal random variable. Theorem 1 in AG1 provides a formula for $AsySz$. In contrast to the formula of $AsySz$ in (2.1), the formula in the theorem can be used for explicit calculation. It shows that the "worst case" sequence of nuisance parameters, a sequence that yields the highest asymptotic null rejection probability, is of the type $\{\gamma_{n,h} : n \geq 1\}$. The formulation of the theorem is valid for one-sided and symmetric two-sided case. The equal-tailed case can be dealt with analogously. Let $R_{-,\infty} = \{x \in R : x \leq 0\} \cup \{-\infty\}$ and denote by $c_g(1-\alpha)$, for $g \in H$, the $1-\alpha$ quantile of J_g.

Theorem 1 (simplified version of AG1): Under Assumptions A-D and K, continuity of J_h, and additional weak technical assumptions, the asymptotic size of a FCV, subsampling, and hybrid test equals

$$\sup_{h \in H} [1 - J_h(c_{Fix}(1-\alpha))],$$
$$\sup_{(g,h) \in GH} [1 - J_h(c_g(1-\alpha))],$$
$$\sup_{(g,h) \in GH} [1 - J_h(\max(c_g(1-\alpha), c_{Fix}(1-\alpha)))],$$

respectively, where

$$\begin{aligned} GH = \{&(g,h) \in H \times H : g = (g_1, g_2),\ h = (h_1, h_2),\ g_2 = h_2, \text{ and for} \\ &m = 1, ..., p,\ \text{(i) } g_{1,m} = 0 \text{ if } |h_{1,m}| < \infty,\ \text{(ii) } g_{1,m} \in R_{+,\infty} \text{ if } h_{1,m} \\ &= +\infty,\ \text{and (iii) } g_{1,m} \in R_{-,\infty} \text{ if } h_{1,m} = -\infty\}. \end{aligned} \tag{2.10}$$

The set GH arises in the size formula, because when the limit distribution of the test statistic is J_h for some $h \in H$, then the probability limit of the subsampling critical value, $c_{n,b}(1-\alpha)$, is the $1-\alpha$ quantile of the limit distribution J_g, viz., $c_g(1-\alpha)$, for some $g \in H$ for which $(g,h) \in GH$.

Example (continued): We have $AsySz(\theta_0) = \sup_{h \in R_{+,\infty} \times [-1,1]} [1 - J_h(z_{1-\alpha/2})]$ for the FCV case, where for $h_1 < \infty$, J_h is the distribution of $|\xi_{1,h}/(\xi_{2,h}\eta_{u,h}^2)^{1/2}|$ and for $h_1 = \infty$, J_h is the distribution of the absolute value of a standard normal random variable. The asymptotic size can be calculated easily by simulation of J_h over a fine grid of vectors h in H.

In situations where the limit distribution of the test statistic is discontinuous in the above sense, the resulting FCV, subsampling, and hybrid procedures are size distorted in many examples we have studied. Often the test procedures can be size corrected by appropriately increasing the critical value of the test. For one-sided and symmetric two-sided tests, the size-corrected fixed critical value (SC-FCV), subsampling (SC-Sub), and hybrid (SC-Hyb) tests with nominal level α are defined to reject the null hypothesis $H_0 : \theta = \theta_0$ when

$$\begin{aligned} T_n(\theta_0) &> cv(1-\alpha), \\ T_n(\theta_0) &> c_{n,b}(1-\alpha) + \kappa(\alpha), \\ T_n(\theta_0) &> \max\{c_{n,b}(1-\alpha), c_\infty(1-\alpha) + \kappa^*(\alpha)\}, \end{aligned} \tag{2.11}$$

respectively, where

$$\begin{aligned} cv(1-\alpha) &= \sup_{h \in H} c_h(1-\alpha), \\ \kappa(\alpha) &= \sup_{(g,h) \in GH} [c_h(1-\alpha) - c_g(1-\alpha)], \\ \kappa^*(\alpha) &= \sup_{h \in H^*} c_h(1-\alpha) - c_\infty(1-\alpha), \text{ and} \\ H^* &= \{h \in H : \text{for some } (g,h) \in GH, c_g(1-\alpha) < c_h(1-\alpha)\}. \end{aligned} \tag{2.12}$$

Equal-tailed SC tests can be defined analogously.

Theorem 1 (AG2): Suppose the Assumptions of Theorem 1 in AG1 hold and additional technical conditions. Then, the SC-FCV, SC-Sub, and SC-Hyb tests satisfy $AsySz(\theta_0) = \alpha$.

The results presented above for tests hold also (with minor modifications) for CIs.

In this paper, we study the asymptotic size and the possibility of size-correction in two examples where the limit distribution of the test statistic has the discontinuity feature discussed above. The first example is inference on the structural parameter in a linear IV regression with possibly weak instruments and the second example is inference on a slope parameter in a linear regression model when another parameter is restricted to be non-negative.

The examples are instructive because, besides other findings, they illustrate that (i) subsampling based inference is not a panacea and often is (extremely) size distorted, (ii) the asymptotic size of subsampling in a given model can vary widely across lower one-sided, upper one-sided, two-sided symmetric, and two-sided equal-tailed inference, (iii) sometimes FCV methods have correct asymptotic size when subsampling methods do not and vice versa, (iv) the "simple" hybrid procedure often produces inference with correct asymptotic size, (v) the (more difficult) SC methods in AG2 can often be successfully applied (even when the hybrid method fails) but not always, and (vi) that implementation of subsampling tests based on partially- rather than fully-studentized statistics can be beneficial from a size perspective.

While (SC) subsampling or hybrid methods in these examples are probably not the preferred inference method from a power perspective they may prove to be beneficial for other reasons. For example, in current research, robustness properties of (SC) subsampling methods in the linear IV are investigated in situations where the IVs are "slightly" correlated with the structural error term. In such situations the asymptotic size of (SC) subsampling methods might prove to be less distorted than the one of inference procedures that are more competitive from a power perspective. Also, (SC) subsampling methods may prove to be the best currently available inference procedure in situations where (currently) no similar inference procedure is known, namely, in a modification to the linear IV example, where there are multiple rhs endogenous variables.

3 Instrumental Variables Regression with Possibly Weak Instruments

3.1 IV Model and Tests

The model we consider consists of a structural equation with one right-hand side endogenous variable y_2 and a reduced-form equation for y_2:

$$\begin{aligned} y_1 &= y_2\theta + X\zeta + u, \\ y_2 &= Z\pi + X\phi + v, \end{aligned} \tag{3.1}$$

where $y_1, y_2 \in R^n$ are endogenous variable vectors, $X \in R^{n\times k_1}$ for $k_1 \geq 0$ is a matrix of exogenous variables, $Z \in R^{n\times k_2}$ for $k_2 \geq 1$ is a matrix of IVs, and $(\theta, \zeta', \phi', \pi')' \in R^{1\times k_1\times k_1\times k_2}$ are unknown parameters. Let $\overline{Z} = [X{:}Z]$ and $k = k_1 + k_2$. Denote by u_i, v_i, X_i, Z_i, and $\overline{Z}_i$ the i-th rows of u, v, X, Z, and $\overline{Z}$, respectively, written as column vectors (or scalars).

The null hypothesis of interest is $H_0 : \theta = \theta_0$. The alternative hypothesis may be one-sided or two-sided. Below we consider upper and lower one-sided and symmetric and equal-tailed two-sided tests of nominal level α of the null hypothesis H_0.

We define a partially-studentized test statistic $T_n^*(\theta_0)$ as follows:

$$T_n^*(\theta_0) = \frac{n^{1/2}(\widehat{\theta}_n - \theta_0)}{\widehat{\sigma}_n},\ \widehat{\theta}_n = \frac{y_2' P_{Z^\perp} y_1}{y_2' P_{Z^\perp} y_2},\ \widehat{\sigma}_n = (n^{-1} y_2' P_{Z^\perp} y_2)^{-1/2}, \tag{3.2}$$

$Z^\perp = Z - P_X Z$, and $P_X = X(X'X)^{-1}X'$. If no X appears, $Z^\perp = Z$. Note that $T_n^*(\theta_0)$ does not employ an estimator of $\sigma_u = StdDev(u_i)$. Hence, it is only partially-studentized. The standard fully-studentized test statistic is

$$T_n^*(\theta_0)/\widehat{\sigma}_u, \text{ where } \widehat{\sigma}_u^2 = (n-1)^{-1}(y_1^\perp - y_2^\perp\widehat{\theta}_n)'(y_1^\perp - y_2^\perp\widehat{\theta}_n) \text{ and } y_m^\perp = y_m - P_X y_m \tag{3.3}$$

for $m = 1, 2$.

Standard nominal level α 2SLS tests based on a fixed critical value (FCV) employ the test statistic $T_n(\theta_0)/\widehat{\sigma}_u$, where $T_n(\theta_0) = T_n^*(\theta_0), -T_n^*(\theta_0)$, and $|T_n^*(\theta_0)|$ for upper one-sided, lower one-sided, and symmetric two-sided tests, respectively. In each case, the test rejects H_0 if

$$T_n(\theta_0)/\widehat{\sigma}_u > c_\infty(1-\alpha), \tag{3.4}$$

where $c_\infty(1-\alpha) = z_{1-\alpha}$, $z_{1-\alpha}$, and $z_{1-\alpha/2}$, respectively, and $z_{1-\alpha}$ denotes the $1-\alpha$ quantile of the standard normal distribution. Note that for the FCV tests full studentization of the test statistic is necessary for the normal critical values to be suitable (when the IVs are strong).

Next, we consider subsampling tests based on $T_n^*(\theta_0)$, rather than $T_n^*(\theta_0)/\widehat{\sigma}_u$. The rationale for using the partially-studentized t statistic, $T_n^*(\theta_0)$, is that σ_u^2 is difficult to estimate when the IVs are weak and a subsampling test does not require normalization for the scale of the error because the subsample statistics have the same error scale as the full sample statistic. It turns out that omitting the estimator of σ_u^2 improves the performance of the subsampling tests considerably. It also simplifies the asymptotic distribution of the t statistic considerably.

In the definition of $L_{n,b}(x)$ in (2.2), $\{\widehat{T}_{n,b,j} : j = 1, ..., q_n\}$ equals $(T_{n,b,j}^*(\theta_0) : j = 1, ..., q_n\}$, the latter being partially-studentized subsample t statistics that are defined just as $T_n^*(\theta_0)$ is defined but are based on the data in the jth subsample of length b. That is, $T_{n,b,j}^*(\theta_0) = b^{1/2}(\widehat{\theta}_{n,b,j} - \theta_0)/\widehat{\sigma}_{n,b,j}$, where $\widehat{\theta}_{n,b,j}$ and $\widehat{\sigma}_{n,b,j}$ are analogues of $\widehat{\theta}_n$ and $\widehat{\sigma}_n$, respectively, based on the jth subsample. Note that the subsample t statistic $T_{n,b,j}^*(\theta_0)$ is centered at the null hypothesis value θ_0, rather than the full-sample estimator $\widehat{\theta}_n$, which is often used in other examples. The reason is that the full-sample estimator is not consistent if the IVs are weak and, hence, centering at this value would yield poor performance of the subsampling test.

The nominal level α subsampling test rejects $H_0 : \theta = \theta_0$ if

$$T_n(\theta_0) > c_{n,b}(1-\alpha). \tag{3.5}$$

Next, we define a hybrid test that differs somewhat from the definition given above and in AG2 in order for the test to be a combination of the subsampling test that does not rely on an estimator of σ_u^2 and the FCV test that does. The nominal level α hybrid test is defined to reject H_0 if

$$T_n(\theta_0) > \max\{c_{n,b}(1-\alpha), \widehat{\sigma}_u c_\infty(1-\alpha)\} \tag{3.6}$$

(where $c_\infty(1-\alpha)$ is as above).

Now we consider equal-tailed two-sided tests. The equal-tailed FCV test is the same as the symmetric FCV test by symmetry of the normal distribution. The nominal level α equal-tailed two-sided subsampling test rejects H_0 if

$$T_n(\theta_0) > c_{n,b}(1-\alpha/2) \text{ or } T_n(\theta_0) < c_{n,b}(\alpha/2), \tag{3.7}$$

where $T_n(\theta_0) = T_n^*(\theta_0)$. The equal-tailed hybrid test is defined analogously with $c_{n,b}(1-\alpha/2)$ and $c_{n,b}(\alpha/2)$ replaced by $\max\{c_{n,b}(1-\alpha/2), \widehat{\sigma}_u c_\infty(1-\alpha/2)\}$ and $\min\{c_{n,b}(\alpha/2), \widehat{\sigma}_u c_\infty(\alpha/2)\}$, respectively, where $c_\infty(1-\alpha/2) = z_{1-\alpha/2}$ and $c_\infty(\alpha/2) = -z_{1-\alpha/2}$.

Upper, lower, symmetric, and equal-tailed nominal level α CIs based on the tests above are defined by

$$\begin{aligned} CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\sigma}_n c_{1-\alpha}, \infty), \\ CI_n &= (-\infty, \widehat{\theta}_n + n^{-1/2}\widehat{\sigma}_n c_{1-\alpha}], \\ CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\sigma}_n c_{1-\alpha}, \widehat{\theta}_n + n^{-1/2}\widehat{\sigma}_n c_{1-\alpha}], \text{ and} \\ CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\sigma}_n c_{1-\alpha/2}, \widehat{\theta}_n - n^{-1/2}\widehat{\sigma}_n c_{\alpha/2}], \end{aligned} \tag{3.8}$$

where for FCV, Sub, and Hyb CIs we have $c_\beta = \widehat{\sigma}_u c_\infty(\beta)$, $c_{n,b}(\beta)$, and $\max\{c_{n,b}(\beta), \widehat{\sigma}_u c_\infty(\beta)\}$, respectively, for $\beta = 1-\alpha, 1-\alpha/2$, and $\alpha/2$ except that for Hyb CIs when $\beta = \alpha/2$ we have $c_\beta = \min\{c_{n,b}(\beta), \widehat{\sigma}_u c_\infty(\beta)\}$.[2]

3.2 Assumptions and Parameter Space

We assume that $\{(u_i, v_i, X_i, Z_i) : i \le n\}$ are i.i.d. with distribution F. We define a vector of nuisance parameters $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by

$$\begin{aligned} \gamma_1 &= ||\Omega^{1/2}\pi/\sigma_v||, \ \gamma_2 = \rho, \text{ and } \gamma_3 = (F, \pi, \zeta, \phi), \text{ where} \\ \sigma_v^2 &= E_F v_i^2, \ \sigma_u^2 = E_F u_i^2, \ \rho = Corr_F(u_i, v_i), \\ \Omega &= Q_{ZZ} - Q_{ZX} Q_{XX}^{-1} Q_{XZ}, \text{ and } Q = \begin{bmatrix} Q_{XX} & Q_{XZ} \\ Q_{ZX} & Q_{ZZ} \end{bmatrix} = E_F \overline{Z}_i \overline{Z}_i'. \end{aligned} \tag{3.9}$$

We choose this specification for γ_1 and γ_2 because the asymptotic distribution of the t statistic depends only on these scalar parameters, as shown below.

The parameter spaces for γ_1 and γ_2 are $\Gamma_1 = R_+$ $(= \{x \in R : x \ge 0\})$ and $\Gamma_2 = [-1, 1]$. For given $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, the parameter space for γ_3 is

$$\begin{aligned} &\Gamma_3(\gamma_1, \gamma_2) = \\ &\{(F, \pi, \zeta, \phi) : E_F u_i^2 = \sigma_u^2, \ E_F v_i^2 = \sigma_v^2, \ E_F \overline{Z}_i \overline{Z}_i' = Q = \begin{bmatrix} Q_{XX} & Q_{XZ} \\ Q_{ZX} & Q_{ZZ} \end{bmatrix}, \ \& \\ &E_F u_i v_i/(\sigma_u \sigma_v) = \rho \text{ for some } \sigma_u^2, \sigma_v^2 > 0, \text{ some pd } Q \in R^{k \times k}, \ \& \text{ some } \pi \in R^{k_2} \\ &\text{that satisfy } ||\Omega^{1/2}\pi/\sigma_v|| = \gamma_1 \text{ for } \Omega = Q_{ZZ} - Q_{ZX} Q_{XX}^{-1} Q_{XZ}, \ \& \ \rho = \gamma_2; \\ &\zeta, \phi \in R^{k_1}; \ E_F u_i \overline{Z}_i = E_F v_i \overline{Z}_i = 0; \ E_F(u_i^2, v_i^2, u_i v_i) \overline{Z}_i \overline{Z}_i' = (\sigma_u^2, \sigma_v^2, \sigma_u \sigma_v \rho) Q; \\ &\lambda_{\min}(E_F \overline{Z}_i \overline{Z}_i') \ge \varepsilon; \ \left\| E_F \left(|u_i/\sigma_u|^{2+\delta}, \ |v_i/\sigma_v|^{2+\delta}, \ |u_i v_i/(\sigma_u \sigma_v)|^{2+\delta} \right)' \right\| \le M, \ \& \\ &\left\| E_F \left(||\overline{Z}_i u_i/\sigma_u||^{2+\delta}, \ ||\overline{Z}_i v_i/\sigma_v||^{2+\delta}, \ ||\overline{Z}_i||^{2+\delta} \right)' \right\| \le M\} \end{aligned} \tag{3.10}$$

for some constants $\varepsilon > 0$, $\delta > 0$, and $M < \infty$, where pd denotes "positive definite."

[2] Because the 2SLS estimator is location equivariant, the finite-sample distribution of the 2SLS t statistic under the null hypothesis $H_0 : \theta = \theta_0$ does not depend on θ_0. In consequence, test results for fixed θ_0 automatically hold uniformly over $\theta_0 \in R$. This implies that the test results apply immediately to CIs constructed by inverting the tests. Hence, in this example, there is no need to adjust the assumptions and definitions as in Section 6 of AG2 and Section 9 of AG3.

The tests introduced above are equivalent to analogous tests defined with $T_n^*(\theta_0)$, $T_{n,b,j}^*(\theta_0)$, and $\widehat{\sigma}_u$ replaced by

$$T_n^{**}(\theta_0) = T_n^*(\theta_0)/\sigma_u,\ T_{n,b,j}^{**}(\theta_0) = T_{n,j}^*(\theta_0)/\sigma_u,\ \text{and}\ \widehat{\sigma}_u/\sigma_u, \tag{3.11}$$

respectively. (They are "equivalent" in the sense that they generate the same critical regions.) The reason is that for all of the tests above $1/\sigma_u$ scales both the test statistic and the critical value equally, e.g., $T_n^*(\theta_0) > \widehat{\sigma}_u c_\infty(1-\alpha)$ iff $T_n^{**}(\theta_0) > (\widehat{\sigma}_u/\sigma_u)c_\infty(1-\alpha)$. We determine the asymptotic size of the tests (denoted by $AsySz(\theta_0)$) written as in (3.11) because this eliminates σ_u from the asymptotic distributions that arise and, hence, simplifies the expressions.

3.3 Asymptotic Distributions

In this section, we determine the asymptotic null distribution of the test statistic $T_n^{**}(\theta_0)$ under certain sequences of parameters. The sequences that we consider are the ones that determine the asymptotic size of the tests based on the results in AG1 and AG2. By asymptotic size, we mean the limit of the finite sample size, which is the maximum over $\gamma \in \Gamma$ of the rejection probability of the test under H_0, see (2.1) below. Not surprisingly, these sequences correspond to the sequences considered in the weak IV asymptotics of Staiger and Stock (1997).

The asymptotic distributions of the statistic $T_n^{**}(\theta_0)$ depend on a localization parameter $h = (h_1, h_2)' \in H$, where the parameter space H is

$$H = R_{+,\infty} \times [-1, 1]. \tag{3.12}$$

For $h \in H$, let $\{\gamma_{n,h} : n \geq 1\}$ denote a sequence of parameters with subvectors $\gamma_{n,h,j}$ for $j = 1, 2, 3$ defined by

$$\begin{aligned}
\gamma_{n,h,1} &= ||(\Omega_n^{1/2}\pi_n/(E_{F_n}v_i^2)^{1/2}||,\ \Omega_n = E_{F_n}Z_iZ_i' - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}E_{F_n}X_iZ_i',\\
\gamma_{n,h,2} &= Corr_{F_n}(u_i, v_i),\ n^{1/2}\gamma_{n,h,1} \to h_1,\ \gamma_{n,h,2} \to h_2,\ \text{and}\\
\gamma_{n,h,3} &= (F_n, \pi_n, \zeta_n, \phi_n) \in \Gamma_3(\gamma_{n,h,1}, \gamma_{n,h,2}).
\end{aligned} \tag{3.13}$$

As shown in the Appendix, under any sequence $\{\gamma_{n,h} : n \geq 1\}$, we have the following convergence results

$$\begin{aligned}
&\begin{pmatrix} (n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}n^{-1/2}Z^{\perp\prime}u/\sigma_u \\ (n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}n^{-1/2}Z^{\perp\prime}v/\sigma_v \end{pmatrix} \to_d \begin{pmatrix} \psi_{u,h_2} \\ \psi_{v,h_2} \end{pmatrix} \sim N(0, V_{h_2} \otimes I_{k_2})\ \text{for}\\
&V_{h_2} = \begin{bmatrix} 1 & h_2 \\ h_2 & 1 \end{bmatrix},\ n^{-1}(u'u/\sigma_u^2, v'v/\sigma_v^2, u'v/(\sigma_u\sigma_v)) \to_p (1, 1, h_2),\\
&\Omega_n^{-1}(n^{-1}Z^{\perp\prime}Z^\perp) \to_p I_{k_2},\ n^{-1}\overline{Z}'[u{:}v] \to_p 0,\ \text{and}\ (E_{F_n}X_iX_i')^{-1}(n^{-1}X'X) \to_p I_{k_1},
\end{aligned} \tag{3.14}$$

where $\psi_{u,h_2}, \psi_{v,h_2} \in R^{k_2}$ and $h_2 \in [-1, 1]$. These convergence results are very similar to the ones given in Staiger and Stock (1997).

If $h_1 < \infty$, then the IVs are weak, see (3.13). In this case, it follows from (3.14) (see the Appendix) that jointly under $\{\gamma_{n,h}\}$, we have

$$\begin{pmatrix} y_2' P_{Z^\perp} u/(\sigma_u \sigma_v) \\ y_2' P_{Z^\perp} y_2/\sigma_v^2 \end{pmatrix} \rightarrow_d \begin{pmatrix} \xi_{1,h} \\ \xi_{2,h} \end{pmatrix} = \begin{pmatrix} (\psi_{v,h_2} + h_1 s_{k_2})' \psi_{u,h_2} \\ (\psi_{v,h_2} + h_1 s_{k_2})'(\psi_{v,h_2} + h_1 s_{k_2}) \end{pmatrix}, \quad (3.15)$$

where s_{k_2} is any vector in R^{k_2} that lies on the unit sphere, i.e., $||s_{k_2}|| = 1$, (which holds because the distribution of $(\xi_{1,h}, \xi_{2,h})$ is invariant to s_{k_2}, see the Appendix). This, (3.14), and some calculations (see the Appendix) yield

$$\begin{pmatrix} T_n^{**}(\theta_0) \\ \widehat{\sigma}_u^2/\sigma_u^2 \end{pmatrix} \rightarrow_d \begin{pmatrix} \eta_h^{**} \\ \eta_{u,h}^2 \end{pmatrix} = \begin{pmatrix} \xi_{1,h}/\xi_{2,h}^{1/2} \\ (1 - h_2 \xi_{1,h}/\xi_{2,h})^2 + (1 - h_2^2)\xi_{1,h}^2/\xi_{2,h}^2 \end{pmatrix} \quad (3.16)$$

under $\{\gamma_{n,h} : n \geq 1\}$. Let J_h^{**} be the distribution of $\eta_h^{**} = \xi_{1,h}/\xi_{2,h}^{1/2}$. It depends on k_2, but not on k_1. The random variable $\eta_{u,h}$ is positive a.s. except when $h_1 = 0$ and $h_2 = \pm 1$. In the latter case, $\eta_{u,h} = 0$ a.s. because $\xi_{1,h} = \pm \xi_{2,h}$. Note that with fixed $\overline{Z}$ and normal (u, v), the distribution of $T_n^{**}(\theta_0)$ is *exactly* J_h^{**} with $h_1 = (Z'Z)^{1/2}\pi/\sigma_v$.

Next, suppose that $h_1 = \infty$. In this case, the IVs are strong, see (3.13). It is shown in the Appendix that under the null hypothesis and $\{\gamma_{n,h} : n \geq 1\}$ with $h_1 = \infty$, we have

$$T_n^{**}(\theta_0) \rightarrow_d \eta_h^{**} \sim N(0, 1), \ \widehat{\sigma}_u^2/\sigma_u^2 \rightarrow_p \eta_{u,h}^2 = 1, \quad (3.17)$$

and J_h^{**} is the standard normal distribution function.

The asymptotic distribution function J_h of $T_n(\theta_0)$ is given by $J_h = J_h^{**}$, $-J_h^{**}$, and $|J_h^{**}|$ for the upper, lower, and symmetric tests, respectively, where $-J_h^{**}$ and $|J_h^{**}|$ are the distribution functions of $-X$ and $|X|$, respectively, if $X \sim J_h^{**}$. Equations (3.16) and (3.17) imply that Assumption B of AG1 holds for $T_n(\theta_0)$ as defined above.

3.4 Asymptotic Size

For upper and lower one-sided and symmetric two-sided tests, the asymptotic sizes of the nominal level α FCV, subsampling, and hybrid tests, respectively, are

$$\begin{aligned} AsySz(\theta_0) &= \sup_{h \in H} P(\eta_h > \eta_{u,h} c_\infty(1-\alpha)), \\ AsySz(\theta_0) &= \sup_{(g,h) \in GH} [1 - J_h(c_g(1-\alpha))], \text{ and} \\ AsySz(\theta_0) &= \sup_{(g,h) \in GH} P(\eta_h > \max\{c_g(1-\alpha), \eta_{u,h} c_\infty(1-\alpha)\}), \end{aligned} \quad (3.18)$$

where $\eta_h = \eta_h^{**}$, $-\eta_h^{**}$, and $|\eta_h^{**}|$ for the upper, lower, and symmetric tests, respectively. The result for the subsampling test follows from Theorem 1(ii) of AG1. The results for the FCV and hybrid tests follow from variations of Theorem 1(i) of AG1 and Theorem 1 of AG2. The assumptions used for these results are verified in the Appendix.

For upper, lower, and symmetric FCV tests, the result in (3.18) implies that $AsySz(\theta_0) = 1$. This follows from (3.18) by considering the properties of these tests when the IVs are asymptotically unidentified, i.e., $h_1 = 0$, and the correlation between the errors h_2 is ± 1. For $h^\dagger = (0, \pm 1)'$, we have (i) $\eta_{u,h^\dagger} = 0$ a.s., which implies that $AsySz(\theta_0) \geq P(\eta_{h^\dagger} > 0)$, (ii) $\eta^{**}_{h^\dagger} = \xi_{1,h^\dagger}/\xi^{1/2}_{2,h^\dagger} = \pm\xi^{1/2}_{2,h^\dagger}$ a.s., (iii) $\eta_{h^\dagger} = \xi^{1/2}_{2,h^\dagger} > 0$, $\eta_{h^\dagger} = -\xi^{1/2}_{2,h^\dagger} < 0$, and $\eta_{h^\dagger} = \xi^{1/2}_{2,h^\dagger} > 0$ a.s. for upper, lower, and symmetric tests when $h^\dagger = (0, 1)'$, and (iv) the first two inequalities in (iii) are reversed when $h^\dagger = (0, -1)'$.

Analogously to (3.18), for nominal level α equal-tailed two-sided subsampling and hybrid tests, we have

$$\begin{aligned} AsySz(\theta_0) &= \sup_{(g,h) \in GH} [1 - J^{**}_h(c^{**}_g(1-\alpha/2)) + J^{**}_h(c^{**}_g(\alpha/2))] \text{ and} \\ AsySz(\theta_0) &= \sup_{(g,h) \in GH} [P(\eta^{**}_h > \max\{c^{**}_g(1-\alpha/2), \eta_{u,h} c_\infty(1-\alpha/2)\}) \\ &\quad + P(\eta^{**}_h < \min\{c^{**}_g(\alpha/2), \eta_{u,h} c_\infty(\alpha/2)\})], \end{aligned} \tag{3.19}$$

respectively, where $c_\infty(1-\alpha)$ is the $1-\alpha$ quantile of the standard normal distribution.

3.5 Quantile Graphs

Graphs of the quantiles $c_h(1-\alpha)$ of J_h as a function of h_1 for fixed h_2, where $h = (h_1, h_2)'$, are quite informative regarding the behavior of subsampling and FCV tests. When the test statistic $T_n(\theta_0)$ has limit distribution J_h, a test will have asymptotic null rejection probability less than or equal to α only if the probability limit of the critical value is greater than or equal to the $1-\alpha$ quantile of J_h, viz., $c_h(1-\alpha)$. Hence, for a subsampling test to have correct asymptotic size, one needs $c_g(1-\alpha) \geq c_h(1-\alpha)$ for all $(g, h) \in GH$. For example, this occurs if the graph is decreasing in h_1 for each h_2. On the other hand, if the graph is increasing in h_1 for some h_2, then the subsampling test over-rejects the null hypothesis.

Figure 1 provides graphs of the quantiles, $c_h(1-\alpha)$, when J_h equals J^{**}_h and $|J^{**}_h|$ for upper one-sided and symmetric two-sided tests, respectively, as a function of $h_1 \geq 0$ for several values of h_2. (We do not consider graphs for $-J^{**}_h$ because they are the same as those for J^{**}_h with h_2 replaced by $-h_2$.) In Figure 1(a) for J^{**}_h, for positive values of h_2, the graph slopes down and exceeds the value 1.645. Hence, these quantile graphs suggest that the upper subsampling test does not over-reject asymptotically for h_2 positive. On the other hand, for h_2 negative, the graph slopes up and lies below the value 1.645. Thus, for h_2 negative, the graphs indicate that the upper subsampling test over-rejects asymptotically. Quantitative results for these tests are provided below.

In Figure 1(b) for $|J^{**}_h|$, the quantile graphs are invariant to the sign of h_2, so only non-negative values are shown. The graphs slope up very slightly for $h_2 = 0.0$ and slope down for other values of h_2. The graphs lie above the value 1.96 and by a substantial amount when h_2 is close to one. Thus, these graphs suggest that the symmetric subsampling test over-rejects slightly. Quantitative details are given below.

For upper and symmetric FCV tests, graphs of the quantiles $c_h(1-\alpha)$ of the limit distributions of the fully-studentized test statistic $T_n(\theta_0)/\widehat{\sigma}_u$ and its absolute value

$|T_n(\theta_0)/\widehat{\sigma}_u|$, which we denote by J_h^* and $|J_h^*|$, respectively, as functions of h_1 for fixed h_2 are quite informative. As above, results for lower one-sided tests are the same as those for upper tests with h_2 replaced by $-h_2$. For an FCV test to have correct asymptotic size, one needs $c_\infty(1-\alpha) \geq c_h(1-\alpha)$ for all $h \in H$. For example, this occurs if the graph is increasing in h_1 for each h_2—the opposite of the condition given above that is sufficient for subsampling test to have correct asymptotic size.

Quantile graphs for J_h^* and $|J_h^*|$ are provided in Figure 2. The general shapes of the quantile graphs in Figure 2 are the same as in Figure 1 but the magnitude of the slopes are different. In consequence, for upper one-sided tests, for positive values of h_2, the FCV test over-rejects the null hypothesis asymptotically, whereas the subsampling test does not. On the other hand, the opposite occurs for negative values of h_2. For symmetric two-sided tests, the quantiles graphs are invariant to the sign of h_2. In this case, the graphs are decreasing in h_1 for large values of $|h_2|$. This causes severe over-rejection because of the large values of the graph at $h_1 = 0$ compared to the value at $h_1 = \infty$ when $|h_2|$ is close to one (also see Figure 3 regarding this). In fact, as calculated in the previous section, $AsySz(\theta_0) = 1$ for upper, lower, and symmetric FCV tests. On the other hand, when $h_2 = 0$, the quantile graph of the symmetric FCV test is strictly increasing in h_1. This indicates that its null rejection probability is less than α asymptotically in this part of the null hypothesis even though the test has $AsySz(\theta_0) = 1$.

The graphs in Figure 2 have considerably larger slopes than those in Figure 1 for values of $|h_2|$ close to one. This implies that subsampling tests based on $T_n(\theta_0)$ are preferred to subsampling tests based on $T_n(\theta_0)/\widehat{\sigma}_u$ because the former are less non-similar asymptotically.

3.6 Size-Corrected Tests

We now discuss size-corrected (SC) tests in the IV regression model. Subsampling tests based on the partially-studentized test statistic can be size-corrected by adding a positive constant $\kappa(\alpha)$ to the subsampling critical value $c_{n,b}(1-\alpha)$. For upper, lower, and symmetric tests, the constant $\kappa(\alpha)$ is chosen to be the smallest constant such that

$$\sup_{(g,h)\in GH} (1 - J_h((c_g(1-\alpha) + \kappa(\alpha)))) \leq \alpha. \tag{3.20}$$

Results in AG2 show that the solution is

$$\kappa(\alpha) = \sup_{(g,h)\in GH} (c_h(1-\alpha) - c_g(1-\alpha)) < \infty. \tag{3.21}$$

The test that uses the critical value $c_{n,b}(1-\alpha) + \kappa(\alpha)$ is referred to as the SC-Sub test.

The equal-tailed subsampling test is size-corrected by replacing the subsampling critical values $(c_{n,b}(1-\alpha/2), c_{n,b}(\alpha/2))$ by $(c_{n,b}(1-\alpha/2) + \kappa_{ET}(\alpha), c_{n,b}(\alpha/2) - \kappa_{ET}(\alpha))$ for a positive constant $\kappa_{ET}(\alpha)$. The constant $\kappa_{ET}(\alpha)$ is chosen to be the smallest constant such that

$$\sup_{(g,h)\in GH} [1 - J_h(c_g(1-\alpha/2) + \kappa_{ET}(\alpha)) + J_h(c_g(\alpha/2) - \kappa_{ET}(\alpha))] \leq \alpha. \tag{3.22}$$

The resulting test is referred to as the equal-tailed SC-Sub test. The SC methods discussed above are asymptotically valid by Corollary 3 of Section 5.3 of the Appendix. The results given there also show that the symmetric two-sided SC subsampling test based on the fully-studentized test statistic has correct asymptotic size.

AG2 also introduces plug-in SC methods that are preferable to the SC method described above. However, such methods are not applicable in this example because it is not possible to consistently estimate $\rho = Corr(u_i, v_i)$ when the IVs are weak, i.e., $h_1 < \infty$.

Size-correction methods for FCV tests also are considered in Section 5.3 of the Appendix. However, it is shown in Section 5.1.3 of the Appendix that it is not possible to size-correct FCV tests in this example (at least by the methods considered there). The FCV test cannot be size-corrected because $\sup_{h \in H} c_h(1-\alpha) = \infty$ for upper, symmetric, and equal-tailed tests, see Figure 3. Figure 3 provides quantile graphs of $|J_h^{**}|$ for the fully-studentized t statistic for values of h_2 very close to one. It illustrates that the .95 quantile approaches infinity as $h_2 \to 1$ for $h_1 = 0$. Note that this graph is invariant to the sign of h_2. These results are consistent with the results of Dufour (1997).

The hybrid test is found to have (essentially) asymptotic size equal to its nominal size. That is, numerical calculation of $AsySz(\theta_0)$ for the hybrid test given in (3.18) shows that it equals α up to simulation error. Hence, we do not consider size-corrected versions of the hybrid test.

3.7 Finite-Sample Adjusted Asymptotic Size

AG2 introduces finite-sample adjustments to the $AsySz(\theta_0)$ of subsampling and hybrid tests that take into account that $\delta_n = (b_n/n)^{1/2}$ is not zero in a given application of interest even though the asymptotic approximations take it to be so. Given that δ_n is observed, one can adjust the asymptotic approximation to the $AsySz(\theta_0)$ of subsampling and hybrid tests using δ_n. The adjustment consists of making the following changes in (3.18) and (3.19). One replaces g (which equals $(g_1, h_2)'$) by $(\delta_n^{1/2} h_1, h_2)$ and one replaces the supremum over $(g, h) \in GH$ by the supremum over $h = (h_1, h_2)' \in H$. See AG2 for an explanation of this adjustment.

Finite-sample adjusted SC subsampling tests are defined using the adjusted asymptotic formula. In this case, the SC factor $\kappa(\alpha)$ depends on δ_n and is written $\kappa(\delta_n, \alpha)$. Calculations in AG2 show that

$$\kappa(\delta_n, \alpha) = \sup_{h=(h_1,h_2)\in H} [c_{(h_1,h_2)}(1-\alpha) - c_{(\delta_n^{1/2} h_1, h_2)}(1-\alpha)]. \quad (3.23)$$

The test that rejects when $T_n(\theta_0) > c_{n,b}(1-\alpha) + \kappa(\delta_n, \alpha)$ is referred to as the adjusted size-corrected subsampling (ASC-Sub) test. Equal-tailed ASC-Sub tests are defined by making similar adjustments to the equal-tailed SC-Sub test defined in (3.22), see AG2 for details.

Adjusted size-corrected hybrid (ASC-Hyb) tests also can be defined, see AG2 for details. But, in the cases considered below, the hybrid test has correct finite-sample adjusted asymptotic size, so the hybrid and ASC-Hyb tests are the same.

3.8 Numerical Results for Asymptotic and Finite-Sample Size

In this section we report numerical calculations of the asymptotic, finite-sample adjusted asymptotic, and actual finite-sample sizes of the tests described above. The finite-sample results are for the case of $(n, b_n) = (120, 12)$, mean zero normal errors with correlation ρ (denoted by h_2 in the table), $k_2 = 5$ standard normal IVs Z_i that are independent of each other and the errors, $k_1 = 0$ exogenous regressors X_i, a π vector with equal elements, and without loss of generality $\sigma_u = \sigma_v = 1$ and $\theta_0 = 0$. Results for this case are reported in Table I. Results for the same case except with $k_2 = 1$ IVs are reported in Table II. In Tables I and II the subsampling test results are based on the partially-studentized t statistic while the FCV test results are based on the fully-studentized t statistic. The hybrid test results are based on a combination of the two, as described above.

To dramatically increase computational speed, the finite-sample subsampling and hybrid results are based on $q_n = 119$ subsamples of consecutive observations.[3] Hence, only a small fraction of the "120 choose 12" available subsamples are used. In cases where the subsampling and hybrid tests have correct asymptotic size, their finite-sample performance is expected to be better when all available subsamples are used than when only $q_n = 119$ are used. This should be taken into account when assessing the results of the tables.

The expressions for $AsySz(\theta_0)$ in (3.18) and (3.19) are given as suprema of functions of $(h_1, h_2) \in H$ or $((g_1, h_2), (h_1, h_2)) \in GH$. In Tables I and II, in columns 2, 7, and 9, we report the suprema of these functions over $h_1 \geq 0$ with h_2 fixed for a grid of h_2 values and nominal level $\alpha = .05$ for subsampling, FCV, and hybrid tests, respectively, under the headings Sub Asy, FCV Asy, and Hyb Asy.[4] Recall that h_1 indexes the strength of the IVs and h_2 indexes the correlation between the errors u_i and v_i (as they appear in the asymptotic distribution). The results for upper, symmetric, and equal-tailed tests are given in panels (a), (b), and (c), respectively, of Tables I and II. In columns 3 and 10, we report analogous finite-sample adjusted asymptotic values for the Sub and Hyb tests under the headings Sub Adj-Asy and Hyb Adj-Asy. In columns 4, 8, and 11, we report the actual finite-sample values for the

[3] This includes 10 "wrap-around" subsamples that contain observations at the end and beginning of the sample, for example, observations indexed by $(110, ..., 120, 1)$. The choice of $q_n = 119$ subsamples is made because this reduces rounding errors when q_n is small when computing the sample quantiles of the subsample statistics. The values ν_α that solve $\nu_\alpha/(q_n + 1) = \alpha$ for $\alpha = .025$, .95, and .975 are the integers 3, 114, and 117. In consequence, the .025, .95, and .975 sample quantiles are given by the 3rd, 114th, and 117th largest subsample statistics. See Hall (1992, p. 307) for a discussion of this choice in the context of the bootstrap.

[4] The results in Table IV are based on $20,000$ simulation repetitions. For the finite-sample results, the search over h_1 is done on the intervals $[0, 1]$, $[1, 4]$, $[4, 10]$, and $[10, 25]$ with stepsizes 0.01, 0.1, 0.5, and 1.5, respectively, as well as the single value $h_1 = 35$. For all results, the search over h_2 is done over the set $\{-1, -.99, -.95, -.9, -.8, -.7, ..., .7, .8, .9., 95, .99, 1\}$. For the asymptotic results and the calculation of the size-correction values, the search over h_1 is done on the interval $[-10, 10]$ with stepsize 0.1 and also includes the two values $h_1 = \pm 9,999,999,999$. The size-correction values $\kappa(\alpha)$ and $\kappa(\delta, \alpha)$ for $k_2 = 5$ are as follows: for upper tests, $\kappa(.05) = 2.73$ & $\kappa(.10, .05) = 1.23$; for symmetric tests, $\kappa(.05) = 0.04$ & $\kappa(.10, .05) = 0.03$; and for equal-tailed tests, $\kappa(.05) = 2.58$ & $\kappa(.10, .05) = 1.01$.

subsampling, FCV, and hybrid tests under the headings Sub $n = 120$, FCV $n = 120$, and Hyb $n = 120$, respectively. (In the finite-sample case, the values reported are the suprema of the null rejection probabilities over $\gamma_1 \geq 0$ with h_2 fixed for a grid of h_2 values, where for present purposes h_2 denotes the finite-sample correlation ρ between the errors u_i and v_i.) In columns 5 and 6, we report the finite-sample rejection probabilities of the SC-Sub and ASC-Sub tests under the headings SC-Sub $n = 120$ and ASC-Sub $n = 120$. Tables I and II do not report results for SC-Hyb or ASC-Hyb tests because these tests are (essentially) the same as the Hyb test.

We now discuss the results in Table I. Table I(a) for upper one-sided tests shows the following: (i) The Sub and FCV tests over-reject asymptotically by a substantial amount, but the Hyb test does not over-reject asymptotically. (ii) The FCV test has asymptotic size of 100%, which is consistent with results of Dufour (1997). (iii) For the FCV test, the asymptotic rejection probabilities approximate the finite-sample rejection probabilities extremely well. (iv) For the Sub test, the asymptotic rejection probabilities approximate the finite-sample rejection probabilities very poorly. They are much too large when the finite-sample rejection probabilities exceed 5%. On the other hand, the finite-sample-adjusted asymptotic rejection probabilities are quite accurate, compare columns 3 and 4. (v) The Sub test can be size-corrected. However, the nominal 5% SC-Sub test under-rejects substantially—its size is 0.1%—because the asymptotic size of the Sub test is over-stated. But, the ASC-Sub test performs very well. The nominal 5% ASC-Sub test has finite-sample size of 4.4%. (vi) The nominal 5% Hyb test has asymptotic size 5.1% and finite-sample size of 4.8%. The Hyb test is less non-similar than the SC-Sub test and both are based on the same test statistic. So, the Hyb test is the preferred test of those considered here.

Table I(b) for symmetric two-sided tests shows the following: (i) The FCV and Sub tests over-reject asymptotically, but the Hyb test does not. (ii) The FCV test has asymptotic size of 100%. (iii) The nominal 5% Sub test only over-rejects by a small amount—its adjusted asymptotic size is 5.4% and its finite-sample size is 5.7%. (iv) The SC-Sub and ASC-Sub tests provide small corrections to the Sub test. Their finite-sample sizes are 5.2% and 5.3%, respectively. (v) The nominal 5% Hyb test has asymptotic and adjusted asymptotic size of 5% and finite-sample size of 4.7%. Hence, the Hyb test performs very well. (vi) The Hyb test is less non-similar than the SC-Sub and ASC-Sub tests and is based on the same test statistic. Hence, it is the preferred test of those considered here.

Table I(c) for equal-tailed two-sided tests shows that (i)-(v) for upper tests also hold for equal-tailed tests with the finite-sample size of the 5% ASC-Sub being 4.4%. The nominal 5% Hyb test has asymptotic size 5.0% and finite-sample size of 2.8%. Hence, the Hyb test is somewhat conservative in this case. On the other hand, the Hyb test is less non-similar than the ASC-Sub test.

The same general features exhibited in Table I, which considers $k_2 = 5$, also are exhibited in the analogous Table II, which considers $k_2 = 1$. The main quantitative difference is that the magnitude of asymptotic over-rejection of the Sub test for upper and equal-tailed tests is noticeably lower for $k_2 = 1$ than $k_2 = 5$. However, the differences between $k_2 = 1$ and $k_2 = 5$ are much less for the magnitudes of "finite-sample adjusted asymptotic over-rejection" and "finite-sample over-rejection."

3.9 Can subsampling CIs Have Infinite Length?

In this section, we address the question of whether the asymptotic results for subsampling CIs in the IV regression example are consistent with the finite-sample results of Dufour (1997). Dufour (1997, Sec. 5.2) has shown that in an IV regression model with i.i.d. normal errors and a parameter space that includes $\theta \in R$, $\pi \in R^{k_2}$, and $|\rho| \leq 1$, a necessary condition for a CI to have finite-sample level φ is that the probability the CI has infinite length is $\geq \varphi$ when $\pi = 0$ (which implies that θ is unidentified).[5] Here we show that the limit of the probability that a subsampling CI equals $(-\infty, \infty)$ (and hence has infinite length) is $1-\alpha$ when $\pi = 0$ and $\rho = \pm 1$. We also present some simulation results that indicate that the nominal $1-\alpha$ subsampling CI has infinite length with probability $\geq 1-\alpha$ when $\pi = 0$ and $\rho \in [-1, 1]$. These results are consistent with those of Dufour (1997).

We now establish the first claim in the previous paragraph. Suppose θ is the true value, then

$$\begin{aligned} \widehat{\theta}_n &= \theta + \frac{y_2' P_{Z^\perp} u}{y_2' P_{Z^\perp} y_2} \text{ and} \qquad (3.24) \\ T_n^{**}(\theta_0) &= \frac{n^{1/2}(\widehat{\theta}_n - \theta_0)/\sigma_u}{(n^{-1} y_2' P_{Z^\perp} y_2)^{-1/2}} = \frac{y_2' P_{Z^\perp} u/(\sigma_v \sigma_u)}{(y_2' P_{Z^\perp} y_2/\sigma_v^2)^{1/2}} + \frac{(\theta - \theta_0)\sigma_v/\sigma_u}{(y_2' P_{Z^\perp} y_2/\sigma_v^2)^{-1/2}}. \end{aligned}$$

In addition, suppose $\pi = 0$ and $\rho = 1$. Then, $P_{Z^\perp} y_2 = P_{Z^\perp} v$ and $u = v\sigma_u/\sigma_v$ a.s. In consequence,

$$T_n^{**}(\theta_0) = (v' P_{Z^\perp} v/\sigma_v^2)^{1/2} + \frac{(\theta - \theta_0)\sigma_v/\sigma_u}{(v' P_{Z^\perp} v/\sigma_v^2)^{-1/2}} = (v' P_{Z^\perp} v/\sigma_v^2)^{1/2}(1 + (\theta - \theta_0)\sigma_v/\sigma_u). \qquad (3.25)$$

We consider a symmetric two-sided subsampling CI. Let $c_{n,b}(\theta_0, 1-\alpha)$ denote the subsampling critical value based on $|T_n^{**}(\theta_0)|$. By an argument analogous to that used to obtain (3.25), we have

$$\begin{aligned} T_{n,b,j}^{**}(\theta_0) &= (v_{n,b,j}' P_{Z_{n,b,j}^\perp} v_{n,b,j}/\sigma_v^2)^{1/2}(1 + (\theta - \theta_0)\sigma_v/\sigma_u) \ \forall j \leq q_n \text{ and} \\ c_{n,b}(\theta_0, 1-\alpha) &= c_{n,b}(\theta, 1-\alpha)(1 + (\theta - \theta_0)\sigma_v/\sigma_u), \qquad (3.26) \end{aligned}$$

where $v_{n,b,j} \in R^b$ denotes the jth subsample of size b taken from $\{v_1, ..., v_n\}$ and $Z_{n,b,j}^\perp \in R^{b \times k_2}$ is defined analogously to $Z^\perp$ but based on the jth subsample of size b taken from $\{\overline{Z}_1, ..., \overline{Z}_n\}$.

We now have

$$\begin{aligned} P_{\theta,\gamma}(CI_n = (-\infty, \infty)) &= P_{\theta,\gamma}(\theta_0 \in CI_n \text{ for all } \theta_0 \in R) \\ &= P_{\theta,\gamma}(|T_n^{**}(\theta_0)| \leq c_{n,b}(\theta_0, 1-\alpha) \text{ for all } \theta_0 \in R) \\ &= P_{\theta,\gamma}((|T_n^{**}(\theta)| \leq c_{n,b}(\theta, 1-\alpha))) \\ &\to J_h(c_h(1-\alpha)) = 1 - \alpha, \qquad (3.27) \end{aligned}$$

[5] If the parameter space for ρ bounds $|\rho|$ away from one, then this result does not hold because in this case the diameter of Ψ_0, using Dufour's notation, is finite.

where the third equality holds by (3.25) and (3.26) and the convergence holds by Lemma 6(vi) of AG1 when $(\pi, \rho) = (0, 1)$ and, hence, $h = g = (0, 1)$. Analogous results hold when $(\pi, \rho) = (0, -1)$ and with upper and lower one-sided subsampling CIs.

Next, in Table III, we present simulations of the finite-sample probability that a symmetric two-sided subsampling CI has infinite length for $n = 120$, $b = 12$, mean zero normal errors with correlation ρ, $k_2 = 5$ standard normal IVs that are independent of each other and the errors, $k_1 = 0$, a π vector with equal elements, and (without loss of generality) $\sigma_u = \sigma_v = 1$ and $\theta_0 = 0$.

The probabilities depend on $||\pi||$, which measures the strength of the IVs and equals the square root of the expectation of the concentration parameter, and ρ, which is the correlation between the structural and reduced-form errors. Results are given for the subsampling CI constructed using (a) the "partially-studentized" t statistic, which does not use an estimator of the structural error variance σ_u^2, and (b) the "fully-studentized" t statistic, which uses an estimator of σ_u^2. Table III shows that both types of CIs have very high probabilities of having infinite length when the IVs are weak, i.e., $||\pi||$ is close to zero. The probabilities for the CI based on the fully-studentized t statistic are noticeably higher than those based on the partially-studentized t statistic. This indicates that the latter CI is preferable.

We now discuss how the probabilities in Table III are calculated. A confidence interval CI_n for θ of nominal level 95% is given as the collection of θ_0 values for which the hypothesis $H_0 : \theta = \theta_0$ is not rejected at the 5% significance level. In the present context, for partially-studentized symmetric two-sided subsampling CIs, this means that

$$\begin{aligned} CI_n &= \{\theta_0 \in R : T_n(\theta_0) \leq T_{n,b,i}(\theta_0) \text{ for at least 5\% of the } i = 1, ..., q\} \\ &= \{\theta_0 \in R : |n^{1/2}(\widehat{\theta}_n - \theta_0)/\widehat{\sigma}_n| \leq |b^{1/2}(\widehat{\theta}_{n,b,i} - \theta_0)/\widehat{\sigma}_{n,b,i}| \\ &\qquad \text{for at least 5\% of the } i = 1, ..., q\}. \end{aligned} \tag{3.28}$$

We are interested in the percentage of times that CI_n is unbounded. Rather than constructing the confidence interval by testing $H_0 : \theta = \theta_0$ for each $\theta_0 \in R$, we use a simple shortcut based on the following observation:

$$\{\theta_0 \in R : |a - w\theta_0| \leq |c - d\theta_0|\} \tag{3.29}$$

is unbounded if and only if $|w| \leq |d|$. For each simulation repetition the subsampling CI is unbounded if and only if for at least 5% of the $i = 1, ..., q$ we have[6]

$$n^{1/2}/\widehat{\sigma}_n \leq b^{1/2}/\widehat{\sigma}_{n,b,i}. \tag{3.30}$$

This condition can be checked without much computational effort. The computational method for fully-studentized statistics is the same with $\widehat{\sigma}_n$ replaced by $\widehat{\sigma}_n\widehat{\sigma}_u$ (and similarly for the subsample statistic $\widehat{\sigma}_{n,b,j}$).

[6] Note that we can be imprecise about the case when this inequality holds as an equality because this is a zero probability event. If we have equality $|b| = |d|$, we have to be careful with this statement because the set $S = \{\theta_0 \in R : |a - b\theta_0| \leq |c - d\theta_0|\}$ satisfies either $s < S$ or $s > S$ for some finite number s (in the case $a \neq c$), whereas in the case $|b| < |d|$ the set S is "unbounded in both directions."

Note that the probability that the symmetric hybrid CI is unbounded equals that for the symmetric subsampling CI.

3.10 Extensions

With some work the results of this section can be extended along the lines of Staiger and Stock (1997) to (i) any k-class estimator, including the limited information maximum likelihood (LIML) estimator, (ii) non-i.i.d. observations by defining $\Gamma_3(\gamma_1, \gamma_2)$ to be such that a convergence condition of the form of Assumption M of Staiger and Stock (1997) holds for any sequence $\{\gamma_{n,h} : n \geq 1\}$ (defined below), and (iii) multiple right-hand side endogenous variables with the parameter of interest being a subvector of the endogenous variable parameter vector. AG1 studies inference concerning the exogenous variable parameters. It is shown that the asymptotic size of a t test equals 1 both for subsampling and FCV critical values.

4 CI When the Parameter of Interest May Be Near a Boundary

4.1 Model and Confidence Intervals

In this section, we consider confidence intervals for a regression parameter θ that is known to satisfy $\theta \geq 0$. We consider the linear model with dependent variable $Y_i \in R$ and regressors $X_i \in R^k$ and $Z_i \in R$:

$$Y_i = X_i'\beta + Z_i\theta + U_i \tag{4.1}$$

for $i = 1, ..., n$. We assume $\{(U_i, X_i, Z_i) : i \geq 1\}$ are i.i.d. with distribution F and satisfy $E_F U_i^2 = \sigma_U^2 > 0$ and $E_F U_i(X_i', Z_i) = 0$. We also assume conditional homoskedasticity, that is, $E_F U_i^2 (X_i', Z_i)'(X_i', Z_i) = \sigma_U^2 Q_F$, where $Q_F = E_F(X_i', Z_i)'(X_i', Z_i) > 0$. We decompose Q_F into matrices Q_{XX}, Q_{XZ}, Q_{ZX}, and Q_{ZZ} in the obvious way. We denote by $Y, Z, U \in R^n$ and $X \in R^{n \times k}$ the matrices with rows Y_i, Z_i, U_i, and X_i', respectively, for $i = 1, ..., n$.

The parameter space for θ is R_+ and that for β is R^k. Denote by $\widehat{\theta}_n$ the censored LS estimator of θ. That is,

$$\begin{aligned} \widehat{\theta}_n &= \max\{\widehat{\theta}_{LS}, 0\}, \text{ where} \\ \widehat{\theta}_{LS} &= (Z'M_X Z)^{-1} Z'M_X Y \text{ and } M_X = I - X(X'X)^{-1}X'. \end{aligned} \tag{4.2}$$

The t statistics upon which upper, lower, and symmetric CIs are based are given by $T_n(\theta_0) = T_n^*(\theta_0)$, $-T_n^*(\theta_0)$, and $|T_n^*(\theta_0)|$, respectively. By definition,

$$\begin{aligned} T_n^*(\theta_0) &= n^{1/2}(\widehat{\theta}_n - \theta_0)/\widehat{\eta}_n, \text{ where} \\ \widehat{\eta}_n &= \widehat{\sigma}_U (n^{-1} Z'M_X Z)^{-1/2}, \\ \widehat{\sigma}_U^2 &= n^{-1} \textstyle\sum_{i=1}^n \widehat{U}_i^2, \ \widehat{U}_i = Y_i - X_i'\widehat{\beta}_n - Z_i\widehat{\theta}_n, \end{aligned} \tag{4.3}$$

and $(\widehat{\beta}_n, \widehat{\theta}_n)$ are the LS estimators of (β, θ) subject to the restriction $\theta \geq 0$.

We consider FCV, subsampling, and hybrid CIs. Upper and lower one-sided and symmetric and equal-tailed two-sided CIs of nominal level $1-\alpha$ for $\alpha < 1/2$ are defined by

$$\begin{aligned} CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\eta}_n c_{1-\alpha}, \infty), \\ CI_n &= (-\infty, \widehat{\theta}_n + n^{-1/2}\widehat{\eta}_n c_{1-\alpha}], \\ CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\eta}_n c_{1-\alpha}, \widehat{\theta}_n + n^{-1/2}\widehat{\eta}_n c_{1-\alpha}], \text{ and} \\ CI_n &= [\widehat{\theta}_n - n^{-1/2}\widehat{\eta}_n c_{1-\alpha/2}, \widehat{\theta}_n - n^{-1/2}\widehat{\eta}_n c_{\alpha/2}], \end{aligned} \tag{4.4}$$

respectively, where the left endpoint is replaced by zero if it is smaller than 0, and where the critical value $c_{1-\alpha}$ is defined as follows.[7] For FCV CIs, $c_{1-\alpha} = z_{1-\alpha}$, $c_{1-\alpha} = z_{1-\alpha}$, $c_{1-\alpha} = z_{1-\alpha/2}$, and $(c_{\alpha/2}, c_{1-\alpha/2}) = (z_{\alpha/2}, z_{1-\alpha/2})$, respectively, where $z_{1-\alpha}$ denotes the $1-\alpha$ quantile of the standard normal distribution. For subsampling CIs, $c_{1-\alpha}$ equals the $1-\alpha$ sample quantile, $c_{n,b}(1-\alpha)$, of the subsample statistics $\{T_{n,b,j}(\widehat{\theta}_n) : j = 1, ..., q_n\}$.[8] By definition, for upper, lower, symmetric, and equal-tailed CIs, the subsample t statistic is $T_{n,b,j}(\widehat{\theta}_n) = T^*_{n,b,j}(\widehat{\theta}_n)$, $-T^*_{n,b,j}(\widehat{\theta}_n)$, $|T^*_{n,b,j}(\widehat{\theta}_n)|$, and $T^*_{n,b,j}(\widehat{\theta}_n)$, respectively, where $T^*_{n,b,j}(\widehat{\theta}_n) = n^{1/2}(\widehat{\theta}_{n,b,j} - \widehat{\theta}_n)/\widehat{\eta}_{n,b,j}$ and $(\widehat{\theta}_{n,b,j}, \widehat{\eta}_{n,b,j})$ are defined just as $(\widehat{\theta}_n, \widehat{\eta}_n)$ are defined but using the jth subsample in place of the full sample. For the hybrid CIs, we take $c_{1-\alpha} = \max\{c_{n,b}(1-\alpha), z_{1-\alpha}\}$ for the upper and lower one-sided CI, $c_{1-\alpha} = \max\{c_{n,b}(1-\alpha), z_{1-\alpha/2}\}$ for the symmetric two-sided

[7]Except for the left boundary censoring, these definitions are as in (6.2) of AG2 and (3.8) above with $\widehat{\sigma}_n = \widehat{\eta}_n$ and $\tau_n = n^{1/2}$.

An alternative CI for θ could be obtained from inverting a t statistic based on the (uncensored) OLS estimator $\widehat{\theta}_{LS}$ with lower endpoint censored at zero. The resulting FCV and subsampling CIs have correct asymptotic size. The resulting FCV CI has smaller length than the FCV in the paper but smaller coverage probability in finite samples and the same comparison typically holds between the resulting subsampling CI and the SC subsampling CI in the paper. To verify this claim, consider for simplicity the symmetric two-sided CI in the paper that uses the (smaller) OLS variance estimator rather than $\widehat{\sigma}^2_u$ to estimate σ^2_u. For a FCV CI, if $\widehat{\theta}_{LS} \geq 0$ then the resulting CIs are identical. If $\widehat{\theta}_{LS} < 0$, then the alternative CI has smaller length, but also has smaller coverage probability in finite samples (but both CIs have correct asymptotic confidence size). Also, with positive probability, the alternative CI is empty (which could be artificially overcome by defining it as $\{0\}$ in this case). In sum, for a FCV CI there is a length versus coverage probability trade-off in finite samples. Next, consider the subsampling version of the alternative CI. By using $\widehat{\theta}_{LS}$ rather than $\widehat{\theta}_n$, the limit distribution of the test statistic is continuous in the nuisance parameters and therefore, the alternative subsampling CI has correct asymptotic confidence size (unlike the subsampling CIs considered in the paper whose asymptotic confidence size is $1-2\alpha$). Therefore, the meaningful length comparison is between the size-corrected subsampling CI in the paper and the alternative CI. Asymptotically, the critical value of the size-corrected subsampling procedure is $c_0(1-\alpha/2)$ when h_1 is finite and $c_\infty(1-\alpha/2)$ when $h_1 = \infty$, whereas the critical value of the alternative subsampling CI is the $1-\alpha$ quantile of $|J^*_\infty|$ in all cases, that is $c_\infty(1-\alpha)$. For a symmetric two-sided CI and $\alpha = 5\%$, $c_0(1-\alpha/2) = c_0(.975) = 1.960$, $c_\infty(1-\alpha/2) = c_\infty(.975) = 2.241$, and $c_\infty(1-\alpha) = c_\infty(.95) = 1.960$. Just like in the FCV case, it follows that the size-corrected subsampling CI has length (in large sample sizes) at least as large as for the alternative CI. While both CIs have asymptotic size equal to $1-\alpha$, the alternative CI typically has smaller coverage probability in finite samples and is empty with positive probability. These findings have been verified in simulations available from the authors upon request.

[8]The asymptotic results given below also hold when the subsample statistics are $\{T_{n,b,j}(\theta_0) : j = 1, ..., n\}$.

CI, and $(c_{\alpha/2}, c_{1-\alpha/2}) = (\min\{c_{n,b}(\alpha/2),\ z_{\alpha/2}\},\ \max\{c_{n,b}(1-\alpha/2), z_{1-\alpha/2}\})$ for the equal-tailed two-sided CI.

The coverage probability of a CI defined in (4.4) when γ is the true parameter vector is

$$P_\gamma(\theta \in CI_n) = P_\gamma(T_n(\theta) \le c_{1-\alpha}) \tag{4.5}$$

for the first three CIs and $P_\gamma(\theta \in CI_n) = P_\gamma(c_{\alpha/2} \le T_n(\theta) \le c_{1-\alpha/2})$ for the equal-tailed CI. The exact and asymptotic confidence sizes of CI_n are

$$ExCS_n = \inf_{\gamma \in \Gamma} P_\gamma(\theta \in CI_n) \text{ and } AsyCS = \liminf_{n\to\infty} ExCS_n, \tag{4.6}$$

respectively.

4.2 Parameter Space

The parameter spaces for θ, η, and β are $\Theta = R_+$, $[\eta_L, \eta_U]$ for some $0 < \eta_L \le \eta_U < \infty$, and R^k, respectively. For given $\theta, \gamma_1 \ge 0$, the parameter space for the distribution F of (U_i, X_i, Z_i) is

$$\begin{aligned}\mathcal{F}(\theta,\gamma_1) = \{&F : E_F|U_i|^{2+\delta} \le M,\ E_F U_i^2 > 0,\ E_F U_i(X_i', Z_i) = 0,\ Q_F > 0,\\ &E_F U_i^2 (X_i', Z_i)'(X_i', Z_i) = E_F U_i^2 Q_F,\ \sigma_U^2 (Q_{ZZ} - Q_{ZX} Q_{XX}^{-1} Q_{XZ})^{-1} = \eta^2,\\ &\eta \in [\eta_L, \eta_U],\ \gamma_1 = \theta/\eta\}\end{aligned} \tag{4.7}$$

for some $\delta > 0$ and $0 < M < \infty$. (The condition $E_F|U_i|^{2+\delta} \le M$ in $\mathcal{F}(\theta,\gamma_1)$ guarantees that the Liapounov CLT applies under sequences $\{\gamma_{n,h}\}$ as in (4.11).) The parameter space for $\gamma = (\gamma_1, \gamma_3) = (\theta/\eta, (\theta, \beta, F))$ is

$$\begin{aligned}\Gamma = \{&\gamma = (\gamma_1,\gamma_3) = (\theta/\eta, (\theta,\beta,F)) : \gamma_1 \in R_+,\\ &\theta/\gamma_1 \in [\eta_L, \eta_U],\ \beta \in R^k,\ \&\ F \in \mathcal{F}(\theta,\gamma_1)\}.\end{aligned} \tag{4.8}$$

4.3 Asymptotic Distributions

Here we establish the asymptotic distribution of the t statistic $T_n^*(\theta_0)$ under sequences of parameter values. As in Section 3.3, the sequences that we consider are the ones that determine the asymptotic size of the CIs of interest (according to the results in AG1-AG3). Because CIs require uniformity of the coverage probability over the parameter of interest θ, we actually need to derive the asymptotic distribution of $T_n^*(\theta_{n,h})$ under certain sequences $\{\theta_{n,h} : n \ge 1\}$ defined below.

We have

$$n^{1/2}(\widehat{\theta}_n - \theta) = \max\{n^{1/2}(Z'M_X Z)^{-1} Z' M_X U,\ \ -n^{1/2}\theta\}. \tag{4.9}$$

By the law of large numbers and the CLT,

$$\begin{aligned}&n^{1/2}(Z'M_X Z)^{-1} Z' M_X U \to_d \zeta_\eta \sim N(0,\eta^2),\ \text{where}\\ &\eta^2 = \sigma_U^2 (Q_{ZZ} - Q_{ZX} Q_{XX}^{-1} Q_{XZ})^{-1},\end{aligned} \tag{4.10}$$

under F.

The asymptotic distributions of the t statistic depend on a localization parameter h with parameter space $H = R_{+,\infty}$. We consider sequences $\{\gamma_{n,h} = (\gamma_{n,h,1}, \gamma_{n,h,3}) = (\theta_{n,h}/\eta_{n,h}, (\theta_{n,h}, \beta_{n,h}, F_{n,h})) : n \geq 1\}$ of true parameters $(\theta/\eta, (\theta, \beta, F))$ that satisfy $h = \lim_{n\to\infty} n^{1/2}\theta_{n,h}/\eta_{n,h}$, $\theta_{n,h} \geq 0$, $\eta_{n,h} \in [\eta_L, \eta_U]$, $\beta_{n,h} \in R^k$, and $F_{n,h} \in \mathcal{F}(\theta_{n,h}, \gamma_{n,h,1})$ for all $n \geq 1$. (Using the notation of AG2 and AG3, no parameters $\gamma_{n,h,2}$ or h_2 appear in this example.[9]) Under a sequence $\{\gamma_{n,h} : n \geq 1\}$, the Liapounov CLT, the continuous mapping theorem (CMT), and standard asymptotic calculations imply that

$$T_n^*(\theta_{n,h}) \to_d \max\{\zeta, -h\}, \text{ where } \zeta = \zeta_\eta/\eta \sim N(0,1). \tag{4.11}$$

Define the distribution J_h^* by

$$\max\{\zeta, -h\} \sim J_h^*. \tag{4.12}$$

As defined, J_h^* is standard normal when $h = \infty$. When $h = \infty$, we also write J_∞^* for J_h^*.

For $T_n(\theta_0) = T_n^*(\theta_0)$, $-T_n^*(\theta_0)$, and $|T_n^*(\theta_0)|$, we have

$$T_n(\theta_{n,h}) \to_d J_h, \text{ where } J_h = J_h^*, \ -J_h^*, \text{ and } |J_h^*|, \tag{4.13}$$

respectively, using the CMT. (Here $-J_h^*$ and $|J_h^*|$ denote the distributions of $-S$ and $|S|$ when $S \sim J_h^*$.) The dfs of J_h^*, $-J_h^*$, and $|J_h^*|$ are given by

$$J_h^*(x) = \begin{cases} 0 & \text{for } x < -h \\ \Phi(x) & \text{for } x \geq -h \end{cases}, \quad (-J_h^*)(x) = \begin{cases} \Phi(x) & \text{for } x < h \\ 1 & \text{for } x \geq h, \end{cases}, \quad \text{and}$$

$$|J_h^*|(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 2\Phi(x) - 1 & \text{for } 0 < x < h \\ \Phi(x) & \text{for } x \geq h \end{cases}, \tag{4.14}$$

where $\Phi(x)$ is the standard normal df. A key property of J_h^* for the asymptotic properties of subsampling CIs is that J_h^* is stochastically decreasing in h and $-J_h^*$ and $|J_h^*|$ are stochastically increasing in h.

4.4 Quantile Graphs

As discussed in Section 3.5, quantile graphs are very informative concerning the behavior of FCV, subsampling, and hybrid tests and CIs. Figure 4 provides graphs of the .95 quantiles of $-J_h^*$ and $|J_h^*|$ as a function of h. The graphs have distinctive stepwise linear shapes that are increasing functions of h. This suggests that lower, symmetric, and equal-tailed nominal .95 subsampling CIs have $AsyCS$ less than the desired value .95. On the other hand, it indicates that FCV and hybrid CIs have $AsyCS$ equal to .95, but the CIs are not asymptotically similar.

[9] Strictly speaking, the foregoing definitions of $\gamma_{n,h}$ and $\mathcal{F}(\theta_{n,h}, \gamma_{n,h,1})$ do not fit into the general CI set-up given in AG2 and AG3. The reasons are that (i) these papers consider CIs for θ, where θ is a sub-vector of $\gamma_{n,h}$, whereas here θ/η is a sub-vector of γ, and (ii) these papers allow the parameter space for $\gamma_{n,h,3}$ to depend on $\gamma_{n,h,1}$, whereas here it depends on $\gamma_{n,h,1}$ and $\theta_{n,h}$. In fact, the results of AG2 and AG3 can be altered straightforwardly to accomodate these differences.

4.5 Asymptotic Size and Size-Correction

We now apply Corollary 2 of AG2 and Corollary 1 of AG3 to determine $AsyCS$ analytically for each CI. The details of these calculations are given in Section 5.2.2 of the Appendix. The assumptions of AG2 and AG3 are verified in Section 5.2.1 below.

We find that the upper one-sided FCV, subsampling, and hybrid CIs all have $AsyCS = 1 - \alpha$ for $\alpha < 1/2$, as desired. For the lower one-sided FCV and hybrid CIs, $AsyCS = 1-\alpha$ because J_h $(= -J_h^*)$ is stochastically increasing in h. For the lower one-sided subsampling CI, $AsyCS = 1/2$, again because J_h $(= -J_h^*)$ is stochastically increasing in h. For symmetric two-sided FCV and hybrid CIs, $AsyCS = 1 - \alpha$, because J_h $(= |J_h^*|)$ is stochastically increasing in h. For the symmetric two-sided subsampling CI, $AsyCS = 1 - 2\alpha$ and the CI under-covers by α.

For equal-tailed two-sided FCV and hybrid CIs, $AsyCS = 1 - \alpha$, because J_h $(= J_h^*)$ is stochastically decreasing in h. For equal-tailed two-sided subsampling CIs, $AsyCS = 1/2 - \alpha/2$ for $\alpha < 1/2$, because J_h $(= J_h^*)$ is stochastically decreasing in h.

Lower one-sided SC and ASC subsampling CIs can be constructed using the method described in AG2. A symmetric two-sided SC subsampling CI can be constructed by making a quantile adjustment. That is, to obtain a subsampling CI with $AsyCS = 1 - \alpha$ one constructs a CI with nominal level $\xi(\alpha) = \alpha/2$. Size-correction of the equal-tailed subsampling CI using the "alternative" method defined in Section 7 of the Supplement to AG2 can be applied.

4.6 Numerical Results

Tables IV and V provide asymptotic, finite-sample adjusted asymptotic, and actual finite-sample coverage probabilities for a variety of nominal .95 CIs for this example.[10] The finite-sample results are for the case of $(n, b_n) = (120, 12)$, standard normal errors, and five regressors including four independent standard normal regressors and an intercept. The vector X_i contains three standard normal regressors and the intercept and Z_i is a standard normal regressor. The finite sample results are invariant to the error variance and the parameters β and θ.

Table IV gives finite-sample results for $n = 120$ and $b = 12$, whereas Table V gives analogous results for $n = 240$ and $b = 24$. The asymptotic and adjusted asymptotic sizes of the subsampling CIs are quite close to the finite-sample sizes, see the rows labelled Min for columns 2-4. The only exception is for symmetric CIs with the smaller sample size $n = 120$. The FCV and hybrid CIs have asymptotic and adjusted asymptotic sizes that are quite similar to the finite-sample sizes in all cases.

[10] The results of Tables IV and V are based on $20,000$ simulation repetitions. For Table IV, the search over h to determine the Min is done on the interval $[0, 5]$ with stepsize 0.01 on $[0, 1]$, stepsize .1 on $[1.0, 5.0]$, plus the values 10^{-6}, $10^{-4}, 10^{-3}$, and .005. For Table V, the search over h to determine the Min is done on the interval $[0, 2]$ with stepsize 0.01 on $[0, 1]$ and stepsize .1 on $[1.0, 2.0]$ plus the value 10^{-6}. The size-correction values are as follows: for lower tests, $\kappa(.05) = 1.645$ & $\kappa(.10, .05) = 1.14$; for symmetric tests, $\kappa(.05) = 0.315$ & $\kappa(.10, .05) = 0.321$; and for equal-tailed tests, $\kappa_{ET,1}(.05) = 0$, $\kappa_{ET,2}(.05) = 1.645$, $\kappa_{ET,1}(.10, .05) = 0$, & $\kappa_{ET,2}(.10, .05) = 1.36$. Tables IV and V do not provide asymptotic coverage probabilities for each value of θ for subsampling and hybrid CIs, just minimum coverage probabilities over all values. The reason is that there is not a one-to-one transformation from $(g, h) \in GH$ to θ.

As predicted by the asymptotic results above, lower and equal-tailed subsampling CIs have very poor finite-sample size, viz., 49.7 and 48.9%, respectively, when $n = 120$ and 51.7 and 49.1% when $n = 240$. The SC subsampling CIs have good size in most cases. The main exception is for symmetric CIs with $n = 120$, in which case the size is too high, viz., 98.6%. For $n = 240$, its size is better, viz., 96.0%. The SC subsampling CIs exhibit a relatively high degree of non-similarity, which is not desirable from a CI length perspective. The ASC CIs have size that is too low for lower and equal-tailed CIs, ranging from 88.7 to 91.7%.

The FCV CI has very good finite-sample size, ranging from 94.1 to 94.9%. Its degree of finite-sample non-similarity also is quite good (i.e., small) relative to other CIs. The hybrid CI has good finite-sample size, though not quite as good as for the FCV CI. It ranges from 94.3 to 97.6%. Somewhat ironically, the best CI in this example is the naive FCV CI that ignores the boundary problem. Its asymptotic size is correct for all types of CI, lower, upper, symmetric, and equal-tailed. Although it is not asymptotically similar, its degree of non-similarity is low relative to the other CIs that have correct size.

5 Appendix

This Appendix provides supporting technical material for the IV regression example and the parameter of interest near a boundary example. It also provides necessary and sufficient conditions for the size-correction methods considered here and in AG2 to apply to a given example. In contrast, AG2 provides sufficient conditions that are stronger than the conditions given here. The weaker conditions given here are needed for some of the results in the IV regression example.

5.1 IV Regression Example

This section of the Appendix verifies assumptions for the IV regression model, provides proofs of results stated in the text concerning this model, and shows that FCV tests cannot be size corrected in this model.

5.1.1 Verification of Assumptions

In this section, we verify Assumptions t1, Sub2, A-G, and J of AG1 and Assumption K of AG2. Under these assumptions, Theorem 1 of AG1 gives the subsampling $AsySz(\theta_0)$ result in (3.18) and variations of Theorem 1 of AG2 give the FCV and hybrid $AsySz(\theta_0)$ results in (3.18) using the continuity of the joint distribution of $(\eta_h, \eta_{u,h}^2)$ whenever $h \neq (0, \pm 1)$.

Assumption t1 holds with $\tau_n = 1/2$ by the definition of $T_n^*(\theta_0)$. Assumption Sub2 holds because the subsample statistics are centered at θ_0, rather than $\widehat{\theta}_n$. Assumption A holds by definition of $\Gamma_1 = R_+$. In Assumption B, we take $r = 1/2$. Assumption B is verified by (3.16) and (3.17). Assumption C holds by choice of b_n such that $b_n \to \infty$ and $b_n/n \to 0$. Assumptions D and E hold by the i.i.d. assumption. Assumptions

F and J hold because $J_h(x)$ is strictly increasing on R for all $h \in H$. Assumption G holds because Assumption Sub2 holds. Assumption K of AG2 holds by (3.17).

For the SC subsampling tests, asymptotic validity established in Corollary 3 below requires verification of Assumption LS that $\sup_{(g,h)\in GH}(c_h(1-\alpha) - c_g(1-\alpha)) < \infty$. We do not provide a formal proof of this condition. However, numerical results indicate that this condition holds for upper, lower, and symmetric tests based on the partially-studentized t statistic and for symmetric tests based on the fully-studentized t statistic, see Figures 1-3.

5.1.2 Proofs for IV Regression Example

First we prove (3.14). The weak law of large numbers (WLLN) for independent $L^{1+\delta}$-bounded random variables for $\delta > 0$ gives

$$n^{-1}(u'u/\sigma_u^2, v'v/\sigma_v^2, u'v/(\sigma_u\sigma_v)) \to_p (1,1,h_2),\ n^{-1}Z'Z - E_{F_n}Z_iZ_i' \to_p 0,$$
$$n^{-1}X'Z - E_{F_n}X_iZ_i' \to_p 0,\ n^{-1}\overline{Z}'[u{:}v] \to_p 0, \text{ and } n^{-1}X'X - E_{F_n}X_iX_i' \to_p 0 \tag{5.1}$$

given the moment conditions in $\Gamma_3(\gamma_1,\gamma_2)$. The last result in (5.1) gives $(E_{F_n}X_iX_i')^{-1}(n^{-1}X'X) \to_p I_{k_1}$ because the conditions in $\Gamma_3(\gamma_1,\gamma_2)$ imply that $\lambda_{\min}(E_{F_n}X_iX_i') \geq \varepsilon_1$ for some $\varepsilon_1 > 0$. This and the results in (5.1) imply that

$$n^{-1}Z'X(n^{-1}X'X)^{-1}n^{-1}X'Z - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}E_{F_n}X_iZ_i' \to_p 0. \tag{5.2}$$

Combined with the second result in (5.1), this gives $n^{-1}Z^{\perp\prime}Z^{\perp} - \Omega_n \to_p 0$. In turn, this implies that $\Omega_n^{-1}(n^{-1}Z^{\perp\prime}Z^{\perp}) \to_p I_{k_2}$, as desired, because $\sup_{n\geq 1}||\Omega_n^{-1}|| \leq M_1$ for some $M_1 < \infty$ given the condition $\lambda_{\min}(E_{F_n}\overline{Z}_i\overline{Z}_i') \geq \varepsilon > 0$ of $\Gamma_3(\gamma_1,\gamma_2)$.

It remains to establish the first result of (3.14). Using the results above, it suffices to show that

$$\begin{pmatrix} \Omega_n^{-1/2}n^{-1/2}(Z' - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}X')u/\sigma_u \\ \Omega_n^{-1/2}n^{-1/2}(Z' - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}X')v/\sigma_v \end{pmatrix} \to_d \begin{pmatrix} \psi_{u,h_2} \\ \psi_{v,h_2} \end{pmatrix}, \tag{5.3}$$

where $(\psi_{u,h_2}', \psi_{v,h_2}')' \sim N(0, V_{h_2} \otimes I_{k_2})$, because $n^{-1/2}[Z{:}X]'[u/\sigma_u{:}v/\sigma_v] = O_p(1)$ by the Liapounov CLT for independent, mean zero, $L^{2+\delta}$-bounded random variables and $\sup_{n\geq 1}||\Omega_n^{-1}|| \leq M_1 < \infty$. The Liapounov CLT implies that (5.3) holds because the left-hand side quantity has mean zero, variance matrix $V_{h_2}\otimes I_{k_2}$, and is $L^{2+\delta}$-bounded by the conditions in $\Gamma_3(\gamma_1,\gamma_2)$. This concludes the proof of (3.14). $\square$

Next, we show that the distribution of $(\xi_{1,h},\xi_{2,h})$, defined in (3.15), is invariant to s_{k_2} for s_{k_2} on the unit sphere. This is used in the convergence result in (3.15). To establish invariance, let B be an orthogonal $k_2 \times k_2$ matrix. Then, for $\lambda \in R^{k_2}$,

$$(\psi_{v,h_2}+\lambda)'(\psi_{v,h_2}+\lambda) = (B\psi_{v,h_2}+B\lambda)'(B\psi_{v,h_2}+B\lambda) \approx (\psi_{v,h_2}+B\lambda)'(\psi_{v,h_2}+B\lambda), \tag{5.4}$$

where " $\approx$ " denotes equality in distribution and " $\approx$ " holds because $(B\psi_{u,h_2}, B\psi_{v,h_2}) \sim N(0, V_{h_2} \otimes I_{k_2})$. Analogously, $(\psi_{v,h_2} + \lambda)'\psi_{u,h_2} \approx (\psi_{v,h_2} + B\lambda)'\psi_{u,h_2}$. Hence, the distribution of $(\xi_{1,h}, \xi_{2,h})$ is the same for all s_{k_2} with $||s_{k_2}|| = 1$.

Equation (3.15) follows from (3.14) by the following result. If $X_n \to_d X \sim N(0, V)$ for $X \in R^{k_2}$, $h_n \to h \in R$, and $s_n \in R^{k_2}$ satisfies $||s_n|| = 1$ for all n, where h_n and s_n are nonrandom, then $(X_n + h_n s_n)'X_n \to_d (X + hs)'X$ for any $s \in R^{k_2}$ with $||s|| = 1$. This can be proved by showing that any subsequence $\{v_n\}$ of $\{n\}$ has a subsequence $\{w_n\}$ such that the claimed convergence holds with w_n in place of n. The latter is shown by taking $\{w_n\}$ to be a sequence for which $\{s_{w_n}\}$ converges to some $s \in R^{k_2}$ with $||s|| = 1$. The result of the previous paragraph shows that the limit distribution does not depend on s provided $||s|| = 1$. □

We now show that (3.16) holds when $h_1 < \infty$. The proof uses (3.14), (3.15), and the following argument. The result $T_n^{**}(\theta_0) \to_d \eta_h$ follows immediately from (3.14). To show $\widehat{\sigma}_u^2 \to_d \eta_{u,h}^2$ when $h_1 < \infty$, we write

$$\begin{aligned} y_1^{\perp} - y_2^{\perp}\widehat{\theta}_n &= (I_n - P_X)(u - y_2(\widehat{\theta}_n - \theta_0) + X\zeta_n) \\ &= (I_n - P_X)(u - (v + Z\pi_n + X\phi_n)(\widehat{\theta}_n - \theta_0)) \\ &= u - v(\widehat{\theta}_n - \theta_0) - P_X(u - v(\widehat{\theta}_n - \theta_0)) + (Z^{\perp}\pi_n/\sigma_v)(\widehat{\theta}_n - \theta_0)\sigma_v, \end{aligned} \tag{5.5}$$

where for notational simplicity here and below we do not index σ_v by n. Next, we have

$$n^{-1}||Z^{\perp}\pi_n/\sigma_v||^2 = ||\Omega_n^{1/2}\pi_n/\sigma_v||^2(1 + o_p(1)) = \gamma_{n,h,1}^2(1 + o_p(1)) \to_p 0, \tag{5.6}$$

where the first equality holds using (3.14) and the convergence holds because $n\gamma_{n,h,1}^2 \to h_1^2 < \infty$. In addition, by (3.14),

$$\begin{aligned} &(\widehat{\theta}_n - \theta_0)\sigma_v/\sigma_u \to_d \xi_{1,h}/\xi_{2,h}, \\ &n^{-1}||P_X u/\sigma_u||^2 = n^{-1}u'X(n^{-1}X'X)^{-1}n^{-1}X'u/\sigma_u^2 \to_p 0, \text{ and} \\ &n^{-1}||P_X v/\sigma_v||^2 \to_p 0. \end{aligned} \tag{5.7}$$

Using these results, we obtain

$$n^{-1}||P_X(u - v(\widehat{\theta}_n - \theta_0))/\sigma_u||^2 \to_p 0. \tag{5.8}$$

Combining (5.5)-(5.8) yields

$$\begin{aligned} &((n-1)/n)\widehat{\sigma}_u^2/\sigma_u^2 \\ = \ &n^{-1}(y_1^{\perp} - y_2^{\perp}\widehat{\theta}_n)'(y_1^{\perp} - y_2^{\perp}\widehat{\theta}_n)/\sigma_u^2 \\ = \ &n^{-1}(u - v(\widehat{\theta}_n - \theta_0))'(u - v(\widehat{\theta}_n - \theta_0))/\sigma_u^2 + o_p(1) \\ = \ &n^{-1}u'u/\sigma_u^2 - 2(n^{-1}u'v/(\sigma_u\sigma_v))(\widehat{\theta}_n - \theta_0)\sigma_v/\sigma_u \\ &\quad + (n^{-1}v'v/\sigma_v^2)(\widehat{\theta}_n - \theta_0)^2(\sigma_v/\sigma_u)^2 + o_p(1) \\ \to_d \ &1 - 2h_2\xi_{1,h}/\xi_{2,h} + (\xi_{1,h}/\xi_{2,h})^2 \\ = \ &(1 - h_2\xi_{1,h}/\xi_{2,h})^2 + (1 - h_2^2)(\xi_{1,h}/\xi_{2,h})^2, \end{aligned} \tag{5.9}$$

where the second equality uses the Cauchy-Schwarz inequality and (5.5)-(5.8) to handle the cross-terms and the convergence holds by (3.14) and (5.7). □

Next, we establish that (3.17) holds when $h_1 = \infty$. Let $a_n = n^{1/2}\gamma_{n,h,1}$. The first result of (3.17), viz., $T_n^{**}(\theta_0) \to_d N(0,1)$, follows from

$$\frac{y_2' P_{Z^\perp} y_2}{a_n^2 \sigma_v^2} \to_p 1 \text{ and } \frac{y_2' P_{Z^\perp} u}{a_n \sigma_u \sigma_v} \to_d N(0,1) \tag{5.10}$$

under $\{\gamma_{n,h}\}$ and the null hypothesis. To establish (5.10), we have

$$\begin{aligned}
&(n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}y_2}{a_n\sigma_v} \\
&= (n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}Z^\perp\pi_n}{a_n\sigma_v} + (n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}v}{a_n\sigma_v} \\
&= (n^{-1}Z^{\perp\prime}Z^\perp)^{1/2}\Omega_n^{-1/2}\frac{n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v}{||n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v||} + o_p(1) \\
&= \frac{n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v}{||n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v||} + o_p(1),
\end{aligned} \tag{5.11}$$

where the second equality uses the definitions of a_n and $\gamma_{n,h,1}$, (3.14), and $a_n \to \infty$, and the third equality uses (3.14). This yields

$$\frac{y_2' P_{Z^\perp} y_2}{a_n^2 \sigma_v^2} = 1 + o_p(1). \tag{5.12}$$

Next, we have

$$\begin{aligned}
\frac{y_2' P_{Z^\perp} u}{a_n \sigma_u \sigma_v} &= \left((n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}y_2}{a_n\sigma_v}\right)'(n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}u}{\sigma_u} \\
&= \left(\frac{n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v}{||n^{1/2}\Omega_n^{1/2}\pi_n/\sigma_v||}\right)'(n^{-1}Z^{\perp\prime}Z^\perp)^{-1/2}\frac{n^{-1/2}Z^{\perp\prime}u}{\sigma_u} + o_p(1) \\
&\to_d N(0,1),
\end{aligned} \tag{5.13}$$

where the second equality uses (5.11) and the convergence holds by (3.14) and the fact that $\chi_n \to_d N(0, I_{k_2})$ and $\lambda_n \in R^{k_2}$ with $||\lambda_n|| = 1$ imply that $\lambda_n'\chi_n \to_d N(0,1)$. Hence, (5.10) holds and the first result of (3.17) is established.

We now establish the second result of (3.17), viz., $\widehat{\sigma}_u^2/\sigma_u^2 \to_p 1$ under $\{\gamma_{n,h}\}$ and the null hypothesis when $h_1 = \infty$. Equation (5.10) implies that

$$(\widehat{\theta}_n - \theta_0)\sigma_v/\sigma_u = O_p(a_n^{-1}) = o_p(1). \tag{5.14}$$

The desired result follows from (5.5), which holds when $h_1 = \infty$, combined with the following results:

$$
\begin{aligned}
& n^{-1}||(I_n - P_X)v(\widehat{\theta}_n - \theta_0)||^2/\sigma_u^2 \le (n^{-1}v'v/\sigma_v^2)((\widehat{\theta}_n - \theta_0)\sigma_v/\sigma_u)^2 \to_p 0, \\
& n^{-1}u'P_X u/\sigma_u^2 = (n^{-1}u'X/\sigma_u)(n^{-1}X'X)^{-1}n^{-1}X'u/\sigma_u \to_p 0, \text{ and} \\
& n^{-1}||(Z^{\perp}\pi_n/\sigma_v)(\widehat{\theta}_n - \theta_0)\sigma_v||^2/\sigma_u^2 = \frac{\pi_n'(n^{-1}Z^{\perp\prime}Z^{\perp})\pi_n}{a_n^2\sigma_v^2}O_p(1) \\
= \; & \frac{\pi_n'\Omega_n\pi_n}{a_n^2\sigma_v^2}O_p(1) = O_p(n^{-1}), \qquad (5.15)
\end{aligned}
$$

where the convergence in the first and second lines holds by (3.14) and (5.14) and the first through third equalities of the third equation hold by (5.14), (3.14), and definitions of a_n and $\gamma_{n,h,1}$, respectively. $\square$

5.1.3 Size-Correction of FCV Tests

We now show that FCV tests cannot be size corrected in the IV regression model. There are two ways in which one can define an SC-FCV test. First, consider a nominal $1-\alpha$ SC-FCV test that rejects $H_0 : \theta = \theta_0$ when

$$
\begin{aligned}
& T_n(\theta_0)/\widehat{\sigma}_u > c_\infty(1-\alpha) + \overline{\kappa}(\alpha), \text{ where } \overline{\kappa}(\alpha) < \infty \text{ satisfies} \\
& \sup_{h \in H} P(\eta_h > \eta_{uh}[c_\infty(1-\alpha) + \overline{\kappa}(\alpha)]) \le \alpha. \qquad (5.16)
\end{aligned}
$$

The condition on $\overline{\kappa}(\alpha)$ is needed for $AsySz(\theta_0) \le \alpha$. For $h^* = (0,1)$, we have $\eta_{u,h^*} = 0$ a.s. and $\eta_h^{**} = \xi_{2,h^*}^{1/2} > 0$ a.s., see the discussion following (3.16). Hence, for upper and symmetric tests, $\eta_h = \xi_{2,h^*}^{1/2} > 0$ a.s. and for all $\overline{\kappa} \in R$,

$$
\sup_{h \in H} P(\eta_h > \eta_{uh}[c_\infty(1-\alpha) + \overline{\kappa}]) \ge P(\xi_{2,h^*}^{1/2} > 0) = 1. \qquad (5.17)
$$

Thus, no value $\overline{\kappa}(\alpha) < \infty$ exists that satisfies (5.16) when $\alpha < 1$. For lower one-sided tests, one needs to replace $h^* = (0,1)$ by $h^* = (0,-1)$ in the argument above to show that no value $\overline{\kappa}(\alpha)$ exists that satisfies (5.16).

On the other hand, one can define an SC-FCV test to be one for which H_0 is rejected when

$$
T_n(\theta_0) > \widehat{\sigma}_u c_\infty(1-\alpha) + \overline{\kappa}(\alpha). \qquad (5.18)
$$

For the symmetric version of this test, under $\{\gamma_{n,h}\}$ we have

$$
\begin{aligned}
& P_{\theta_0,\gamma_{n,h}}(T_n(\theta_0) > \widehat{\sigma}_u c_\infty(1-\alpha) + \overline{\kappa}(\alpha)) \\
= \; & P_{\theta_0,\gamma_{n,h}}(|T_n^{**}(\theta_0)| > (\widehat{\sigma}_u/\sigma_u)c_\infty(1-\alpha) + \overline{\kappa}(\alpha)/\sigma_u) \\
\to \; & P(\eta_h > \eta_{uh}c_\infty(1-\alpha) + \overline{\kappa}(\alpha)/\sigma_u), \qquad (5.19)
\end{aligned}
$$

where the convergence holds by (3.16). Hence, for a test with $AsySz(\theta_0) \le \alpha$, $\overline{\kappa}(\alpha)$ needs to be defined such that

$$
\sup_{h \in H, \sigma_u > 0} P(\eta_h > \eta_{uh}c_\infty(1-\alpha) + \overline{\kappa}(\alpha)/\sigma_u) \le \alpha \qquad (5.20)
$$

when the parameter space for σ_u is R_+. For $h^* = (0,1)$ and all $\overline{\kappa} \in R$, we have

$$\sup_{h \in H, \sigma_u > 0} P(\eta_h > \eta_{uh} c_\infty(1-\alpha) + \overline{\kappa}/\sigma_u) \geq \sup_{\sigma_u > 0} P(\eta_{h^*} > \overline{\kappa}/\sigma_u) = 1. \tag{5.21}$$

Hence, no value $\overline{\kappa}(\alpha)$ exists that satisfies (5.20) when the parameter space for σ_u is R_+ and $\alpha < 1$. The argument for upper and lower one-sided tests is similar.

We conclude that FCV tests cannot be size-corrected in the IV regression model. These results are consistent with the results in Dufour (1997).

5.2 Parameter of Interest Near Boundary Example

5.2.1 Verification of Assumptions

We now verify the assumptions needed to apply Corollary 2 of AG2 and Corollary 1 of AG3 in this example. Assumptions A, B, C, etc. are stated in AG2. First, consider the case of an upper one-sided CI based on $T_n(\theta_0) = T_n^*(\theta_0)$. Assumption A holds by definition of Γ. Assumption B follows from (4.11). We choose $\{b_n : n \geq 1\}$ so that Assumption C holds. Assumption D holds by the i.i.d. assumption. Assumption E holds by the general argument given in Section 3.3 of AG1. Assumption F holds because $J_h(x) = J_h^*(x)$ is strictly increasing for $x \geq 0$ and $c_h(1-\alpha) \geq 0$ for $\alpha \leq 1/2$ by (4.14). Assumption G follows by Lemma 4 in Section 7 of AG1 under Assumption Sub1 and follows trivially under Assumption Sub2. The assumptions of Lemma 4 are verified as follows. Assumption BB holds with $(a_n, d_n) = (\tau_n \eta_{n,h}^{-1}, \eta_{n,h}^{-1})$, where $\tau_n = 1/2$, V_h is the distribution of $\max\{\zeta, -h\}$, and W_h is a point mass distribution at 1 under sequences $\gamma_{n,h} = (\theta_{n,h}/\eta_{n,h}, (\theta_{n,h}, \beta_{n,h}, F_{n,h}))'$ such that $h \in R_{+,\infty}$. Assumptions DD and EE hold by the same arguments as for Assumptions D and E. Assumption HH holds because $a_n = \tau_n \eta_{n,h}^{-1} = n^{1/2} \eta_{n,h}^{-1}$ and $\eta_{n,h} \in [\eta_L, \eta_U]$. The verification of the assumptions for the lower one-sided and two-sided cases is analogous with the exceptions of Assumptions F and J. Using (4.14), one can verify that Assumption F holds for $J_h = -J_h^*$ and $J_h = |J_h^*|$ and Assumption J holds for $J_h = J_h^*$ because for all $h \in H$ either (i) $J_h(x)$ is strictly increasing at $x = c_h(1-\alpha)$ or (ii) $J_h(x)$ has a jump at $x = c_h(1-\alpha)$ with $J_h(c_h(1-\alpha)) > 1-\alpha$ and $J_h(c_h(1-\alpha)-) < 1-\alpha$ provided $\alpha \in (0,1)$.

5.2.2 Analytic Calculation of AsyCS

In this section, we use Corollary 2 of AG2 and Corollary 1 of AG3 to calculate analytically the $AsyCS$ of subsampling, FCV, and hybrid tests.

For upper one-sided CIs, $J_h(\cdot)$ $(= J_h^*(\cdot))$ is continuous at all $x > 0$ for all $h \in R_{+,\infty}$ using (4.14). Because the $1-\alpha$ quantile of J_h is positive for any $h \in H$ given $\alpha < 1/2$, the intervals for $AsyCS$ in Corollary 1 of AG3 collapse to points. By Corollary 1 of AG3, we find that the upper one-sided FCV, subsampling, and hybrid CIs all have $AsyCS = 1-\alpha$ for $\alpha < 1/2$ because the $1-\alpha$ quantile of J_h for any $h \in H$ equals $z_{1-\alpha}$ using (4.14).

For the lower one-sided FCV CI, Corollary 1 of AG3 implies that $AsyCS \in [\inf_{h \in H} J_h(z_{1-\alpha}-), \inf_{h \in H} J_h(z_{1-\alpha})]$. In this case, J_h $(= -J_h^*)$ is stochastically increasing in

h. Hence, $\inf_{h\in H} J_h(z_{1-\alpha}) = J_\infty(z_{1-\alpha}) = \Phi(z_{1-\alpha}) = 1-\alpha$ using (4.14). Thus, $AsyCS = 1-\alpha$ for the lower one-sided FCV CI. For the lower one-sided hybrid CI, we have $\max\{c_g(1-\alpha), c_\infty(1-\alpha)\} = c_\infty(1-\alpha) = z_{1-\alpha}$ for all $g \in H$ because J_h $(= -J_h^*)$ is stochastically increasing in h. Hence, by Corollary 2 of AG2 and the above result for the FCV CI, $AsyCS = 1-\alpha$ for the lower one-sided hybrid CI.

For the lower one-sided subsampling CI, Corollary 1 of AG3 implies that $AsyCS \in [\inf_{(g,h)\in GH} J_h(c_g(1-\alpha)-), \inf_{(g,h)\in GH} J_h(c_g(1-\alpha))]$. We have

$$\inf_{(g,h)\in GH} J_h(c_g(1-\alpha)) = \inf_{h\in H} J_h(c_0(1-\alpha)) = \inf_{h\in H} J_h(0) = J_\infty(0) = 1/2, \tag{5.22}$$

where the first and third equalities hold because J_h $(= -J_h^*)$ is stochastically increasing in h, the second equality holds because $J_0(x) = 1$ for all $x \geq 0$ using (4.14), and the last equality holds because $J_\infty(x) = \Phi(x)$ using (4.14). Therefore, $AsyCS = 1/2$ for the lower one-sided subsampling CI.

We now provide the results for symmetric two-sided CIs. By Corollary 1 of AG3, we have $AsyCS \in [\inf_{h\in H} J_h(z_{1-\alpha/2}-), \inf_{h\in H} J_h(z_{1-\alpha/2})]$ for the FCV CI. Because J_h $(= |J_h^*|)$ is stochastically increasing in h and using (4.14), we have $\inf_{h\in H} J_h(z_{1-\alpha/2}) = J_\infty(z_{1-\alpha/2}) = 2\Phi(z_{1-\alpha/2}) - 1 = 1-\alpha$. Hence, $AsyCS = 1-\alpha$ for the symmetric two-sided FCV CI. For the hybrid CI, we have $\max\{c_g(1-\alpha), c_\infty(1-\alpha)\} = c_\infty(1-\alpha) = z_{1-\alpha/2}$ for all $g \in H$ because J_h $(= |J_h^*|)$ is stochastically increasing in h and using (4.14). Thus, using Corollary 2 of AG2 and the above result for the FCV CI, we have $AsyCS = 1-\alpha$ for the symmetric two-sided hybrid CI.

For the symmetric two-sided subsampling CI, Corollary 1 of AG3 implies that $AsyCS \in [\inf_{(g,h)\in GH} J_h(c_g(1-\alpha)-), \inf_{(g,h)\in GH} J_h(c_g(1-\alpha))]$. We have

$$\inf_{(g,h)\in GH} J_h(c_g(1-\alpha)) = \inf_{h\in H} J_h(c_0(1-\alpha)) = \inf_{h\in H} J_h(z_{1-\alpha}) = J_\infty(z_{1-\alpha}) = 1-2\alpha, \tag{5.23}$$

where the first and third equalities hold because J_h $(= |J_h^*|)$ is stochastically increasing in h, the second equality holds because $J_0(x) = \Phi(x)$ for all $x \geq 0$ using (4.14), and the last equality holds because $J_\infty(x) = (|J_\infty^*|)(x) = 2\Phi(x) - 1$ using (4.14). Equation (5.23) holds with $c_g(1-\alpha)-$ in place of $c_g(1-\alpha)$. Hence, the (nominal $1-\alpha$) symmetric two-sided subsampling CI has $AsyCS = 1-2\alpha$ and under-covers by α. An SC subsampling CI can be constructed by taking $\xi(\alpha) = \alpha/2$.

Next, we discuss the results for the equal-tailed two-sided CIs. Here, $J_h = J_h^*$. By Corollary 2 of AG2, $AsyCS \in [1 - Max_{ET,Type}^{\ell-}(\alpha), 1 - Max_{ET,Type}^{r-}(\alpha)]$ for $Type$ equal to Fix, Sub, and Hyb for the FCV, subsampling, and hybrid CIs, respectively. For the FCV CI, $(c_{\alpha/2}, c_{1-\alpha/2}) = (z_{\alpha/2}, z_{1-\alpha/2})$ yields

$$\begin{aligned} Max_{ET,Fix}^{r-}(\alpha) &= \sup_{h\in H}[1 - J_h(z_{1-\alpha/2}) + J_h(z_{\alpha/2}-)] \\ &= \sup_{h\in H}[1 - \Phi(z_{1-\alpha/2}) + J_h(z_{\alpha/2}-)] = \alpha/2 + J_\infty(z_{\alpha/2}-) \\ &= \alpha/2 + \Phi(z_{\alpha/2}-) = \alpha, \end{aligned} \tag{5.24}$$

where the second equality holds by (4.14), the third equality holds because J_h $(= J_h^*)$ is stochastically decreasing in h, and the fourth equality holds by (4.14). Analogously, $Max_{ET,Type}^{\ell-}(\alpha) = \alpha$. It follows that $AsyCS = 1-\alpha$ for the equal-tailed FCV CI.

For the equal-tailed hybrid CI, the quantities $\max\{c_g(1-\alpha/2), c_\infty(1-\alpha/2)\}$ and $\min\{c_g(\alpha/2), c_\infty(\alpha/2)\}$ that appear in $Max^{r-}_{ET,Hyb}(\alpha)$ equal $z_{1-\alpha/2}$ and $z_{\alpha/2}$, respectively, because $c_g(1-\alpha/2) = z_{1-\alpha/2}$ for all $g \in H$ provided $\alpha \leq 1/2$ using (4.14) and $c_g(\alpha/2) \geq c_\infty(\alpha/2) = z_{\alpha/2}$ for all $g \in H$ using the fact that J_h $(= J^*_h)$ is stochastically decreasing in h and (4.14). Hence, $Max^{r-}_{ET,Hyb}(\alpha) = Max^{r-}_{ET,Fix}(\alpha)$ and likewise with ℓ in place of R. In consequence, the result that $AsyCS = 1-\alpha$ for the equal-tailed FCV CI yields the same result for the equal-tailed hybrid CI.

Lastly, for the equal-tailed subsampling CI, we have

$$\begin{aligned}
&Max^{r-}_{ET,Sub}(\alpha) = \sup_{(g,h)\in GH} [1 - J_h(z_{1-\alpha/2}) + J_h(c_g(\alpha/2)-)] \\
&= \sup_{(g,h)\in GH} [1 - \Phi(z_{1-\alpha/2}) + J_h(c_g(\alpha/2)-)] = \alpha/2 + \sup_{h\in H} J_h(0-) \\
&= \alpha/2 + J_\infty(0-) = \alpha/2 + \Phi(0) = \alpha/2 + 1/2, \qquad (5.25)
\end{aligned}$$

where the first equality holds because $c_g(1-\alpha/2) = z_{1-\alpha/2}$ for all $g \in H$ provided $\alpha \leq 1/2$ by (4.14), the second equality holds as in (5.24), the third equality holds because $c_g(\alpha/2) \leq 0$ with equality when $g = 0$ for $\alpha \leq 1/2$ using (4.14), the fourth equality holds because J_h $(= J^*_h)$ is stochastically decreasing in h, and the fifth equality holds because $J_\infty = J^*_\infty = \Phi$ using (4.14). Likewise, $Max^{\ell-}_{ET,Sub}(\alpha) = \alpha/2 + 1/2$. Therefore, $AsyCS = 1/2 - \alpha/2$ for the equal-tailed subsampling CI. Size-correction (of the type discussed in the paper) is not possible here.

5.3 Size-Correction

5.3.1 Results for Size-Corrected Tests

In this section we consider more general definitions of SC tests than given in AG2. These conditions allow us to determine necessary and sufficient conditions for the existence of SC tests of the form given in (3.2) of AG2. These conditions relax Assumption L of AG2 and are needed to cover symmetric two-sided SC subsampling tests based on the fully-studentized t statistic in the IV example.

We start by altering the definitions of $cv(1-\alpha)$, $\kappa(\alpha)$, and $\kappa^*(\alpha)$ from those given in AG2. We define the constants $cv(1-\alpha)$, $(\in R)$, $\kappa(\alpha)$ $(\in [0,\infty))$, and $\kappa^*(\alpha)$ $(\in \{-\infty\} \cup [0,\infty))$ to be the smallest values that satisfy

$$\begin{aligned}
\sup_{h\in H} [1 - J_h(cv(1-\alpha)-)] &\leq \alpha, \\
\sup_{(g,h)\in GH} (1 - J_h((c_g(1-\alpha) + \kappa(\alpha))-)) &\leq \alpha \text{ and} \\
\sup_{(g,h)\in GH} (1 - J_h(\max\{c_g(1-\alpha), c_\infty(1-\alpha) + \kappa^*(\alpha)\}-)) &\leq \alpha, \qquad (5.26)
\end{aligned}$$

respectively.[11] The constants are defined in this way because the left-hand side of

[11] If no such smallest value exists, we take some value that is arbitrarily close to the infimum of the values that satisfy (5.26). Note that under the assumptions below, there exist values that satisfy (5.26).

each inequality in (5.26) is an upper bound on the $AsySz(\theta_0)$ of each test.[12] If (5.26) holds with $cv(1-\alpha) = c_{Fix}(1-\alpha)$ (or with $\kappa(\alpha) = 0$ or $\kappa^*(\alpha) = 0$), then (i) no size correction is needed and (ii) the SC-FCV test (or SC-Sub test or SC-Hyb test, respectively) is just the uncorrected test. It is shown in the proof of Theorem 2 of AG2 that $cv(1-\alpha)$, $\kappa(\alpha)$, and $\kappa^*(\alpha)$ as defined in (3.2) of AG2 satisfy (5.26) (under the assumptions of that Theorem). Hence, the definition of these quantities via (5.26) is indeed more general than the definition given in (3.2) of AG2.

The following conditions are necessary and sufficient for the existence of constants $cv(1-\alpha)$, $\kappa(\alpha)$, and $\kappa^*(\alpha)$, respectively, that satisfy (5.26).

Assumption LF. $\sup_{h\in H} c_h(1-\alpha) < \infty$.

Assumption LS. $\sup_{(g,h)\in GH} (c_h(1-\alpha) - c_g(1-\alpha)) < \infty$.

Define

$$\begin{aligned} H^* &= \{h \in H : \text{for some } (g,h) \in GH, c_g(1-\alpha) < c_h(1-\alpha)\} \text{ and} \\ H^{**} &= \{h \in H : J_h(c_h(1-\alpha)-) < 1-\alpha \text{ and } \exists g \in H,\ (g,h) \in GH, \\ &\quad \max\{c_g(1-\alpha), \sup_{h^*\in H^*} c_{h^*}(1-\alpha)\} = c_h(1-\alpha)\}. \end{aligned} \tag{5.27}$$

Assumption LH. (i) $\sup_{h\in H^*} c_h(1-\alpha) < \infty$, and (ii) $\sup_{h\in H^{**}} c_h(1-\alpha) < \infty$.

(If H^* is empty, $\sup_{h\in H^*} c_h(1-\alpha) = -\infty$ by definition and analogously for H^{**}.)

Below we use Assumption K of AG2 for the results concerning hybrid tests. It states that the asymptotic distribution J_h of the statistic $T_n(\theta_0)$ under $\{\gamma_{n,h} : n \geq 1\}$ is the same (proper) distribution, call it J_∞, for all $h = (h_1, h_2) \in H$ for which $h_{1,m} = +\infty$ or $-\infty$ for $m = 1, ..., p$, where $h_1 = (h_{1,1}, ..., h_{1,p})'$.

Let "iff" abbreviate "if and only if."

Lemma 1 (a) *A value* $cv(1-\alpha)$ *that satisfies* (5.26) *exists iff Assumption* LF *holds.*

(b) *A value* $\kappa(\alpha)$ *that satisfies* (5.26) *exists iff Assumption* LS *holds.*

(c) *Suppose Assumption* K *holds. A value* $\kappa^*(\alpha)$ *that satisfies* (5.26) *exists iff Assumption* LH *holds.*

(Note that Lemma 1 does not provide conditions for the existence of a *smallest* value that satisfies (5.26). Rather, it provides conditions for the existence of *some* value that satisfies (5.26).)

Assumptions LF, LS, and LH are satisfied in many examples. However, they are all violated in some examples, e.g., see the consistent model selection/super-efficient example in Andrews and Guggenberger (2009c). Size correction (at least of the type considered here) is not possible in that example. In addition, in some examples, Assumption LF fails, but Assumptions LS and LH hold. This implies that the FCV test cannot be size-corrected by the method considered here, but the subsampling

[12] This holds by Theorem 1(i) of AG1 with $c_{Fix}(1-\alpha) = cv(1-\alpha)$, by Theorem 1(ii) of AG1 with $c_{n,b}(1-\alpha) + \kappa(\alpha)$ in place of $c_{n,b}(1-\alpha)$, and by Theorem 1 of AG2 with $\max\{c_{n,b}(1-\alpha), c_\infty(1-\alpha) + \kappa^*(\alpha)\}$ in place of $\max\{c_{n,b}(1-\alpha), c_\infty(1-\alpha)\}$).

and hybrid tests can be. This occurs in the IV example considered below with symmetric two-sided tests (when the test statistic $T_n^*(\theta_0)$ is defined with σ_u estimated). Furthermore, in some examples, Assumption LS fails, but Assumptions LF and LH hold. This occurs in the IV example considered below with upper one-sided tests and $H_2 = [-1, 0]$ (when the test statistic $T_n^*(\theta_0)$ is defined with σ_u estimated). The restriction of H_2 to $[-1, 0]$ in this example, which requires the correlation between the structural and reduced form errors to be non-positive, may not arise naturally in practice. But this case serves to illustrate the point that it is possible for Assumption LS to fail even when Assumption LF holds.

Assumption LH(ii) is not restrictive because H^{**} is typically empty or a small set. Sufficient conditions for Assumption LH(ii) are either of the following:

LH(ii)$'$. For all $h \in H$, $J_h(c_h(1-\alpha)-) = 1 - \alpha$.

LH(ii)$''$. For all $h \in H$, $J_h(\cdot)$ is continuous at its $(1-\alpha)$–quantile $c_h(1-\alpha)$.

Note that LH(ii)$''$ implies LH(ii)$'$. Assumptions LH(ii)$'$ and LH(ii)$''$ imply that $H^{**} = \varnothing$ and thus imply LH(ii).

We now provide conditions under which Assumptions LF, LS, and LH are equivalent. Define

Assumption L*. $\sup_{h_2 \in H_2} c_{(0,h_2)}(1-\alpha) < \infty$.

Assumption L**. $\inf_{h \in H} c_h(1-\alpha) > -\infty$.

Assumptions L* and L** often hold, but L* is violated in the symmetric two-sided IV example mentioned above and L** is violated in the upper one-sided IV example mentioned above.

Lemma 2 (a) LF & L** $\Rightarrow$ LS, (b) LF $\Rightarrow$ LH, (c) LS & L* $\Rightarrow$ LF*,* (d) LH(i) & L* $\Rightarrow$ LF, *and* (e) *if* L* *and* L** *hold, then* LF $\Leftrightarrow$ LS $\Leftrightarrow$ LH*.*

Comment. Assumption L of AG2 is equivalent to the combination of Assumptions LF and L**. Hence, the Lemma shows that Assumption L implies Assumptions LF, LS, and LH.

The following Corollary to Theorem 1 of AG1 and Theorem 1 of AG2 shows that the SC tests have asymptotic size less than or equal to their nominal level α. Assumptions A-G are stated in AG1.

Corollary 3 (a) *Suppose Assumptions* A, B, *and* LF *hold. Then, the SC-FCV test has* $AsySz(\theta_0) \leq \alpha$.

(b) *Suppose Assumptions* A, B, C-G, *and* LS *hold. Then, the SC-Sub test has* $AsySz(\theta_0) \leq \alpha$.

(c) *Suppose Assumptions* A, B, C-G, K, *and* LH *hold. Then, the SC-Hyb test has* $AsySz(\theta_0) \leq \alpha$.

Comment. Under the assumptions of the Corollary plus Assumption M(a) (respectively, M(b), M(c) of AG2, the SC-FCV (SC-Sub, SC-Hyb) test has $AsySz(\theta_0) = \alpha$.

5.3.2 Proofs for Size-Correction Results

For notational simplicity, we write $cv(1-\alpha)$ and $c_h(1-\alpha)$ as cv and c_h hereafter.
Proof of Lemma 1. For part (a), first suppose Assumption LF holds. Consider the value $cv = \sup_{h^\dagger \in H} c_{h^\dagger} + \varepsilon$ $(< \infty)$ for some $\varepsilon > 0$. We have

$$\begin{aligned} &\sup_{h\in H}[1 - J_h(cv-)] = \sup_{h\in H}[1 - J_h((\sup_{h^\dagger\in H} c_{h^\dagger} + \varepsilon)-)] \\ \leq &\sup_{h\in H}[1 - J_h((c_h + \varepsilon)-)] \leq \sup_{h\in H}[1 - J_h(c_h)] \leq \alpha, \end{aligned} \tag{5.28}$$

where the first and second inequalities hold because J_h is nondecreasing and the last inequality holds by the definition of c_h. Equation (5.28) implies that there exists a value cv such that (5.26) holds.

To prove the converse for part (a), suppose there exists a constant cv $(\in R)$ such that (5.26) holds. Then, for all $h \in H$,

$$J_h(cv) \geq J_h(cv-) \geq 1 - \alpha, \tag{5.29}$$

where the second inequality holds by (5.26). By (5.29) and the definition of c_h, $c_h \leq cv < \infty$ for all $h \in H$. Hence, Assumption LF holds.

To prove part (b), first suppose Assumption LS holds. Consider the value $\kappa(\alpha) = \sup_{(g^*,h^*)\in GH}[c_{h^*} - c_{g^*}] + \varepsilon$ $(< \infty)$ for some $\varepsilon > 0$. Then, for any $(g, h) \in GH$, $c_g + \kappa(\alpha) \geq c_h + \varepsilon$, and we have

$$\sup_{(g,h)\in GH}[1 - J_h((c_g + \kappa(\alpha))-)] \leq \sup_{(g,h)\in GH}[1 - J_h((c_h + \varepsilon)-)] \leq \sup_{h\in H}[1 - J_h(c_h)] \leq \alpha. \tag{5.30}$$

Hence, $\kappa(\alpha)$ satisfies (5.26).

To prove the converse for part (b), suppose some $\kappa(\alpha) \in [0, \infty)$ satisfies (5.26). Then, for all $(g, h) \in GH$,

$$J_h(c_g + \kappa(\alpha)) \geq J_h((c_g + \kappa(\alpha))-) \geq 1 - \alpha, \tag{5.31}$$

where the second inequality holds by (5.26). Hence, by the definition of c_h, $c_g+\kappa(\alpha) \geq c_h$. This implies that $\kappa(\alpha) \geq c_h - c_g$ for all $(g, h) \in GH$ and Assumption LS holds.

For part (c), suppose Assumption LH holds. For some $\varepsilon > 0$, define

$$\kappa^*(\alpha) = \max\{\sup_{h^*\in H^*} c_{h^*} - c_\infty, \sup_{h^{**}\in H^{**}} c_{h^{**}} - c_\infty + \varepsilon\} \tag{5.32}$$

and recall that $\sup_{h^*\in H^*} c_{h^*} = -\infty$ when $H^* = \varnothing$ and analogously for H^{**}. By Assumption LH, $\kappa^*(\alpha) < \infty$. Then,

$$\max\{c_g, c_\infty + \kappa^*(\alpha)\} = \max\{c_g, \sup_{h^*\in H^*} c_{h^*}, \sup_{h^{**}\in H^{**}} c_{h^{**}} + \varepsilon\}. \tag{5.33}$$

For all $(g, h) \in GH$, we have

$$\max\{c_g, \sup_{h^*\in H^*} c_{h^*}\} \geq c_h \tag{5.34}$$

because $\max\{c_g, \sup_{h^* \in H^*} c_{h^*}\} < c_h$ implies that $c_g < c_h$, which implies that $h \in H^*$, which implies that $\sup_{h^* \in H^*} c_{h^*} \geq c_h$, which is a contradiction.

For any $(g, h) \in GH$ with $\max\{c_g, \sup_{h^* \in H^*} c_{h^*}\} > c_h$, we have

$$1 - J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-) \leq 1 - J_h(c_h) \leq \alpha. \tag{5.35}$$

For any $(g, h) \in GH$ with $\max\{c_g, \sup_{h^* \in H^*} c_{h^*}\} = c_h$, we have

$$1 - J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-) = 1 - J_h(\max\{c_h, \sup_{h^{**} \in H^{**}} c_{h^{**}} + \varepsilon\}-) \leq \alpha, \tag{5.36}$$

where the last inequality holds by the following argument. If $c_h \geq \sup_{h^{**} \in H^{**}} c_{h^{**}} + \varepsilon$, then $J_h(c_h-) = 1-\alpha$ (because if $J_h(c_h-) < 1-\alpha$ then $h \in H^{**}$, a contradiction) and if $c_h < \sup_{h^{**} \in H^{**}} c_{h^{**}} + \varepsilon$, then $1 - J_h(\max\{c_h, \sup_{h^{**} \in H^{**}} c_{h^{**}} + \varepsilon\}-) \leq 1 - J_h(c_h) \leq \alpha$. Combining (5.35) and (5.36) gives $\sup_{(g,h) \in GH}[1 - J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-)] \leq \alpha$, as desired.

To prove the converse of part (c), suppose that some $\kappa^*(\alpha) \in [0, \infty)$ satisfies (5.26). We show that this implies Assumption LH(i) and LH(ii). For all $(g, h) \in GH$, we have

$$J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}) \geq J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-) \geq 1 - \alpha, \tag{5.37}$$

where the second inequality holds because $\kappa^*(\alpha)$ satisfies (5.26). By (5.37) and the definition of c_h, $\max\{c_g, c_\infty + \kappa^*(\alpha)\} \geq c_h$. In consequence, either (i) $c_g \geq c_h$ or (ii) $c_g < c_h$ and $c_\infty + \kappa^*(\alpha) \geq c_h$. Hence, for all $(g, h) \in GH$ for which $c_g < c_h$, we have $c_h \leq c_\infty + \kappa^*(\alpha) < \infty$. That is, $\sup_{h^* \in H^*} c_{h^*} \leq c_\infty + \kappa^*(\alpha) < \infty$ and Assumption LH(i) holds.

If Assumption LH(ii) does not hold, then by the definition of H^{**}, we can pick a sequence a $\{(g, h) \in GH\}$ such that $c_h = \max\{c_g, \sup_{h^* \in H^*} c_{h^*}\} \to \infty$ and $J_h(c_h-) < 1 - \alpha$. From this sequence pick a $c_h > \max\{c_\infty + \kappa^*(\alpha), \sup_{h^* \in H^*} c_{h^*}\}$, which can be done given that LH(i) holds. Then, $c_h = c_g > c_\infty + \kappa^*(\alpha)$ and thus $1 - J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-) = 1 - J_h(c_h-) > \alpha$ contradicting (5.26). □

Proof of Lemma 2. For part (a), LF and L** imply that $\sup_{(g,h) \in GH}(c_h - c_g) \leq \sup_{h \in H} c_h - \inf_{h \in H} c_h < \infty$, so LS holds. Part (b) holds because $(H^* \cup H^{**}) \subset H$. To prove part (c), suppose LS & L* hold and LF does not hold. Then, by LF, there is a sequence $h_n = (h_{n,1}, h_{n,2}) \in H$ such that $c_{h_n} \to \infty$, and because $K^* = \sup_{h_2 \in H_2} c_{(0,h_2)} < \infty$, we have $c_{(h_{n,1}, h_{n,2})} - c_{(0,h_{n,2})} \geq c_{(h_{n,1}, h_{n,2})} - K^* \to \infty$. Since $((0, h_{n,2}), (h_{n,1}, h_{n,2})) \in GH$, this contradicts LS.

To prove part (d), suppose LH(i) & L* hold and LF does not hold. Then, by LF, there is a sequence $h_n = (h_{n,1}, h_{n,2}) \in H$ such that $c_{h_n} \to \infty$, and because $\sup_{h_2 \in H_2} c_{(0,h_2)} < \infty$, we have $c_{(h_{n,1}, h_{n,2})} > c_{(0,h_{n,2})}$ and $h_n \in H^*$ for n sufficiently large. LH(i) is contradicted by $c_{h_n} \to \infty$ and $h_n \in H^*$ for n sufficiently large.

Part (e) follows from parts (a)-(d). □

Proof of Corollary 3. First, the SC-FCV, SC-Sub, and SC-Hyb tests are well-defined given Assumptions LF, LS, and LH, respectively, because Lemma 1 guarantees that there exist values cv, $\kappa(\alpha)$, and $\kappa^*(\alpha)$ that satisfy (5.26).

For part (a), Theorem 1(i) of AG1 applied with $c_{Fix}(1-\alpha) = cv$ implies that the SC-FCV test satisfies $AsySz(\theta_0) \le \sup_{h \in H}[1 - J_h(cv-)] \le \alpha$,where the second inequality holds by (5.26). For part (b), Theorem 1(ii) of AG1 with $c_{n,b} + \kappa(\alpha)$ in place of $c_{n,b}$ implies that the SC-Sub test satisfies $AsySz(\theta_0) \le \sup_{(g,h) \in H}[1 - J_h((c_g + \kappa(\alpha))-)] \le \alpha$,where the second inequality holds by (5.26). For part (c), Theorem 1 of AG2 with $\max\{c_{n,b}, c_\infty + \kappa^*(\alpha)\}$ in place of $\max\{c_{n,b}, c_\infty\}$ implies that the SC-Hyb test satisfies $AsySz(\theta_0) \le \sup_{(g,h) \in H}[1 - J_h(\max\{c_g, c_\infty + \kappa^*(\alpha)\}-)] \le \alpha$,where the second inequality holds by (5.26). $\square$

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TABLE I

WEAK IV EXAMPLE: MAXIMUM (OVER h_1) NULL REJECTION PROBABILITIES ($\times 100$) FOR DIFFERENT VALUES OF THE CORRELATION h_2 FOR VARIOUS NOMINAL 5% TESTS, WHERE THE PROBABILITIES ARE ASYMPTOTIC, FINITE-SAMPLE-ADJUSTED ASYMPTOTIC, AND FINITE SAMPLE FOR $n = 120$, $b = 12$, AND $k_2 = 5$

	Test:	Sub	Sub	Sub	SC-Sub	ASC-Sub	FCV	FCV	Hyb	Hyb	Hyb
h_2	Prob:	Asy	Adj-Asy	n=120	n=120	n=120	Asy	n=120	Asy	Adj-Asy	n=120
(a) Upper 1-Sided Tests											
-1.00		85.9	38.1	37.0	0.0	4.4	5.1	5.5	5.1	5.0	2.8
-.99		85.0	37.1	36.7	0.0	4.2	5.1	5.2	5.1	5.0	2.6
-.95		81.5	34.0	33.0	0.1	3.6	5.1	5.2	5.1	5.0	2.6
-.90		76.6	30.7	29.1	0.0	2.9	5.1	5.3	5.1	5.0	2.6
-.80		65.9	24.9	23.3	0.0	1.8	5.1	5.0	5.1	5.0	2.4
-.60		42.3	16.6	14.4	0.0	0.8	5.1	4.9	5.1	5.0	2.5
-.40		23.3	11.5	9.7	0.0	0.4	5.1	5.3	5.1	5.0	2.6
-.20		11.5	7.6	5.9	0.0	0.2	5.1	5.0	5.1	5.0	2.5
.00		5.5	5.3	5.2	0.0	0.1	5.1	5.4	5.1	5.0	2.5
.20		5.0	5.0	5.0	0.0	0.2	7.1	7.4	5.0	5.0	2.9
.40		5.0	5.0	5.0	0.0	0.1	18.5	19.0	5.0	5.0	4.3
.60		5.0	5.0	4.9	0.0	0.1	43.7	44.8	5.0	5.0	4.8
.80		5.0	5.0	4.5	0.0	0.0	76.9	77.9	5.0	5.0	4.5
.90		5.0	5.0	4.5	0.0	0.1	91.9	92.3	5.0	5.0	4.5
.95		5.0	5.0	4.7	0.0	0.0	97.2	97.2	5.0	5.0	4.7
.99		5.0	5.0	4.7	0.0	0.0	99.7	99.7	5.0	5.0	4.7
1.00		5.0	5.0	4.6	0.0	0.0	100	100	5.0	5.0	4.6
Max		85.9	38.1	37.0	0.1	4.4	100	100	5.1	5.0	4.8
(b) Symmetric Two-Sided Tests											
.00		5.5	5.4	5.7	5.2	5.3	5.0	5.3	5.0	5.0	3.1
.20		5.3	5.1	5.4	4.8	5.0	5.0	5.2	5.0	5.0	2.8
.40		5.2	5.0	5.0	4.5	4.6	9.6	10.0	5.0	5.0	3.5
.60		5.0	5.0	4.9	4.4	4.5	31.3	32.3	5.0	5.0	4.6
.80		5.0	5.0	4.5	4.1	4.2	68.9	70.2	5.0	5.0	4.5
.90		5.0	5.0	4.5	3.9	4.1	88.6	88.8	5.0	5.0	4.5
.95		5.0	5.0	4.7	4.2	4.3	95.9	95.9	5.0	5.0	4.7
.99		5.0	5.0	4.7	4.1	4.3	99.6	99.6	5.0	5.0	4.7
1.00		5.0	5.0	4.6	4.0	4.2	100	100	5.0	5.0	4.6
Max		5.5	5.4	5.7	5.2	5.3	100	100	5.0	5.0	4.7
(c) Equal-Tailed 2-Sided Tests											
0.0		5.5	5.4	5.7	0.0	0.3	5.0	5.3	5.0	5.0	2.2
.20		8.3	5.9	5.7	0.0	0.3	5.0	5.2	5.0	5.0	2.0
.40		16.1	7.6	6.7	0.0	0.4	9.8	10.0	5.0	5.0	2.1
.60		31.6	10.8	9.6	0.0	0.7	31.7	32.3	5.0	5.0	2.7
.80		56.6	17.1	16.4	0.0	1.7	69.2	70.2	5.0	5.0	2.6
.90		69.9	22.7	21.8	0.0	2.9	88.7	88.8	5.0	5.0	2.7
.95		76.0	26.0	25.0	0.0	3.7	95.9	95.9	5.0	5.0	2.7
.99		80.7	29.0	28.1	0.0	4.4	99.6	99.6	5.0	5.0	2.8
1.00		82.0	30.1	29.0	0.0	4.4	100	100	5.0	5.0	2.7
Max		82.0	30.1	29.0	0.0	4.4	100	100	5.0	5.0	2.8

TABLE II

WEAK IV EXAMPLE: MAXIMUM (OVER h_1) NULL REJECTION PROBABILITIES ($\times 100$) FOR DIFFERENT VALUES OF THE CORRELATION h_2 FOR VARIOUS NOMINAL 5% TESTS, WHERE THE PROBABILITIES ARE ASYMPTOTIC, FINITE-SAMPLE-ADJUSTED ASYMPTOTIC, AND FINITE SAMPLE FOR $n = 120$, $b = 12$, AND $k_2 = 1$

	Test:	Sub	Sub	Sub	SC-Sub	ASC-Sub	FCV	FCV	Hyb	Hyb	Hyb
h_2	Prob:	Asy	Adj-Asy	n=120	n=120	n=120	Asy	n=120	Asy	Adj-Asy	n=120
(a) Upper 1-Sided Tests											
-1.00		52.7	30.2	29.9	0.4	4.5	5.1	5.3	5.1	5.0	1.9
-.99		50.4	29.4	29.3	0.3	4.0	5.1	5.0	5.1	5.0	1.7
-.95		43.9	26.3	25.8	0.2	2.8	5.1	4.8	5.1	5.0	1.7
-.90		37.8	22.9	21.7	0.2	2.1	5.1	5.1	5.1	5.0	1.8
-.80		28.3	18.4	16.2	0.1	1.2	5.1	4.8	5.1	5.0	1.7
-.60		17.3	12.1	9.8	0.0	0.7	5.1	5.3	5.1	5.0	1.9
-.40		11.2	8.6	6.3	0.0	0.4	5.1	5.3	5.1	5.0	2.0
-.20		7.3	6.5	4.8	0.0	0.2	5.1	5.1	5.1	5.0	2.0
.00		5.1	5.2	4.1	0.0	0.1	5.1	5.1	5.0	5.0	1.8
.20		5.1	5.1	4.0	0.0	0.1	5.5	5.4	5.0	5.0	1.8
.40		5.0	5.0	4.1	0.0	0.1	6.5	6.5	5.0	5.0	2.0
.60		5.0	5.0	3.9	0.0	0.1	8.9	8.8	5.0	5.0	2.0
.80		5.0	5.0	3.9	0.0	0.0	18.7	19.0	5.0	5.0	3.2
.90		5.0	5.0	3.7	0.0	0.1	32.5	32.8	5.0	5.0	3.5
.95		5.0	5.0	3.6	0.0	0.1	44.8	45.7	5.0	5.0	3.6
.99		5.0	5.0	3.9	0.0	0.1	65.8	66.1	5.0	5.0	3.9
1.00		5.0	5.0	3.7	0.0	0.1	100	99.6	5.0	5.0	3.7
Max		52.7	30.2	29.9	0.4	4.5	100	99.6	5.1	5.0	3.9
(b) Symmetric Two-Sided Tests											
.00		5.0	5.0	3.7	3.7	3.7	5.0	5.2	5.0	5.0	2.7
.20		5.0	5.0	3.7	3.7	3.7	5.0	5.1	5.0	5.0	2.7
.40		5.0	5.0	4.0	4.0	4.0	5.0	5.2	5.0	5.0	2.8
.60		5.0	5.0	3.8	3.8	3.8	5.5	5.4	5.0	5.0	2.8
.80		5.0	5.0	3.8	3.8	3.8	13.0	13.1	5.0	5.0	3.0
.90		5.0	5.0	3.7	3.7	3.7	26.4	26.6	5.0	5.0	3.5
.95		5.0	5.0	3.6	3.6	3.6	39.1	40.0	5.0	5.0	3.6
.99		5.0	5.0	3.9	3.9	3.9	62.3	62.2	5.0	5.0	3.9
1.00		5.0	5.0	3.7	3.7	3.7	100	99.6	5.0	5.0	3.7
Max		5.0	5.0	4.0	4.0	4.0	100	99.6	5.0	5.0	3.9
(c) Equal-Tailed 2-Sided Tests											
0.0		5.0	5.1	4.5	0.0	0.2	5.0	5.2	5.0	5.0	1.7
.20		5.7	5.4	4.4	0.0	0.2	5.0	5.1	5.0	5.0	1.7
.40		7.9	6.6	5.2	0.0	0.2	5.0	5.2	5.0	5.0	1.8
.60		12.8	9.0	7.2	0.0	0.4	5.6	5.4	5.0	5.0	1.7
.80		23.6	13.9	12.7	0.0	0.9	13.2	13.1	5.0	5.0	1.9
.90		33.5	19.2	17.5	0.1	1.7	26.6	26.6	5.0	5.0	1.8
.95		40.4	23.6	22.3	0.1	2.5	39.2	40.0	5.0	5.0	1.9
.99		49.5	28.6	27.1	0.3	3.6	62.4	62.2	5.0	5.0	2.1
1.00		52.7	30.2	28.9	0.3	4.2	100	99.6	5.0	5.0	2.1
Max		52.7	30.2	28.9	0.3	4.2	100	99.6	5.0	5.0	2.1

TABLE III

WEAK IV EXAMPLE: FINITE-SAMPLE PROBABILITIES ($\times 100$) THAT A NOMINAL 95% SYMMETRIC TWO-SIDED SUBSAMPLING CONFIDENCE INTERVAL HAS IINFINITE LENGTH AS A FUNCTION OF THE ERROR CORRELATION ρ AND THE SQUARE ROOT OF THE EXPECTED VALUE OF THE CONCENTRATION PARAMETER $||\pi||$ FOR $n = 120$, $b = 12$, AND $k_2 = 5$

		(a) Partially-Studentized Confidence Interval								
		$\|\|\pi\|\|$								
		0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
	0.00	95.3	89.4	63.8	27.1	5.8	0.7	0.4	0.0	0.0
	0.25	95.4	89.4	63.6	26.9	5.7	0.7	0.4	0.0	0.0
ρ	0.50	95.5	89.2	63.6	27.0	6.0	0.7	0.4	0.0	0.0
	0.75	95.5	89.4	63.7	27.0	6.2	0.7	0.6	0.0	0.0
	1.00	95.5	89.0	63.9	27.2	5.9	0.6	0.5	0.0	0.0
		(b) Fully-Studentized Confidence Interval								
	0.00	99.3	98.1	89.8	65.9	34.6	13.9	5.2	2.2	1.1
	0.25	99.3	98.2	90.1	67.5	37.8	17.2	6.9	3.0	1.4
ρ	0.50	99.4	98.2	91.1	73.5	49.5	27.7	13.6	6.3	3.0
	0.75	99.4	98.3	94.3	86.3	72.7	53.5	34.2	19.7	10.3
	1.00	99.5	99.9	100.0	99.9	99.5	96.3	85.1	64.6	42.2

TABLE IV

PARAMETER OF INTEREST NEAR A BOUNDARY EXAMPLE: ASYMPTOTIC, FINITE-SAMPLE ADJUSTED ASYMPTOTIC, AND FINITE-SAMPLE COVERAGE PROBABILITIES ($\times 100$) FOR VARIOUS NOMINAL 95% CONFIDENCE INTERVALS FOR DIFFERENT VALUES OF THE PARAMETER h FOR $n = 120$ AND $b = 12$

(a) Lower 1-Sided Confidence Intervals

	Test:	Sub	Sub	Sub	SC-Sub	ASC-Sub	FCV	FCV	Hyb	Hyb
θ	Prob:	Asy	Adj-Asy	n=120	n=120	n=120	Asy	n=120	Asy	n=120
0.0			100	100	100	100	100	100		100
10^{-6}			49.8	49.7	100	100	100	100		100
.01			51.3	51.2	100	100	100	100		100
.05			56.7	56.8	100	100	100	100		100
.10			63.3	63.8	100	95.1	100	100		100
.15			69.6	69.8	97.0	90.0	100	96.7		96.7
.20			75.6	75.4	96.2	92.6	95.0	94.3		94.3
.30			84.7	83.9	98.2	96.1	95.0	94.3		94.4
.40			91.4	89.6	99.3	98.0	95.0	94.3		94.5
.60			95.0	95.1	99.8	99.5	95.0	94.3		95.9
.80			95.0	97.0	100	99.8	95.0	94.3		97.3
1.6			95.0	97.9	100	99.9	95.0	94.3		98.1
2.5			95.0	97.9	100	99.9	95.0	94.3		98.2
Min		50.0	49.8	49.7	95.6	88.7	95.0	94.3	95.0	94.3

(b) Symmetric Two-Sided Confidence Intervals

0.0			95.0	97.9	99.0	99.1	97.5	97.1		98.7
.10			95.0	97.9	99.2	99.2	97.5	97.1		98.8
.15			95.0	97.2	99.0	99.0	97.5	97.0		98.8
.20			89.9	96.7	98.6	98.7	95.0	94.5		97.8
.25			89.9	96.7	98.6	98.6	95.0	94.1		97.6
.35			89.9	97.0	98.6	98.7	95.0	94.1		97.7
.50			91.4	97.7	98.9	98.9	95.0	94.1		98.0
1.0			95.0	98.7	99.5	99.5	95.0	94.1		98.8
Min		90.0	89.9	96.7	98.6	98.6	95.0	94.1	95.0	97.6

(c) Equal-tailed Two-sided Confidence Intervals

.00			97.5	99.2	99.2	99.2	97.5	97.1		99.3
10^{-6}			47.3	48.9	99.2	99.2	97.5	97.1		99.3
.01			48.8	50.5	99.2	99.2	97.5	97.1		99.3
.05			54.4	56.6	99.2	99.2	97.5	97.1		99.3
.10			60.8	64.3	99.2	99.0	97.5	97.1		99.3
.20			73.1	76.7	97.1	98.7	95.0	94.4		96.5
.30			82.2	85.4	98.4	98.6	95.0	94.1		96.3
.40			88.9	91.0	98.9	98.8	95.0	94.1		96.3
.60			95.0	96.1	99.1	99.1	95.0	94.1		97.2
.80			95.0	97.7	99.2	99.4	95.0	94.1		98.1
1.6			95.0	98.4	99.2	99.2	95.0	94.1		98.6
2.5			95.0	98.4	99.2	99.2	95.0	94.1		98.6
Min		47.5	47.3	48.9	97.0	91.7	95.0	94.1	95.0	96.3

TABLE V

Parameter of Interest Near a Boundary Example: Asymptotic, Finite-Sample Adjusted Asymptotic, and Finite-Sample Coverage Probabilities ($\times100$) for Various Nominal 95% Confidence Intervals for Different Values of the Parameter h for $n = 240$ and $b = 24$

	Test:	Sub	Sub	Sub	SC-Sub	ASC-Sub	FCV	FCV	Hyb	Hyb
θ	Prob:	Asy	Adj-Asy	n=240	n=240	n=240	Asy	n=240	Asy	n=240
(a) Lower 1-Sided Confidence Intervals										
0.0				100	100	100		100		100
.01				51.7	100	100		100		100
.05				58.9	100	100		100		100
.10				67.1	99.0	89.4		98.9		98.9
.15				74.1	96.6	92.9		94.9		94.9
.20				80.1	98.0	95.4		94.9		94.9
.30				88.3	99.2	98.2		94.9		94.9
.40				92.8	99.8	99.1		94.9		95.2
.60				95.3	99.9	99.7		94.9		96.4
.80				95.6	100	99.7		94.9		96.7
1.6				95.6	100	99.7		94.9		96.7
Min		50.0	49.8	51.7	95.7	88.5	95.0	94.9	95.0	94.9
(b) Symmetric Two-Sided Confidence Intervals										
0.0				95.4	97.4	97.5		97.3		97.7
.10				92.4	96.6	96.7		97.3		97.7
.15				91.8	96.0	96.1		94.7		95.7
.20				92.4	96.2	96.3		94.7		95.8
.25				92.9	96.4	96.5		94.7		95.9
.35				93.8	96.9	96.9		94.7		96.0
.50				95.1	97.4	97.5		94.7		96.4
1.0				95.6	97.8	97.8		94.7		96.7
Min		90.0	89.9	91.8	96.0	96.1	95.0	94.7	95.0	95.7
(c) Equal-tailed Two-sided Confidence Intervals										
.00				97.2				97.3		98.3
.01				49.1				97.3		98.3
.05				56.6				97.3		98.3
.10				65.2				97.3		98.3
.20				78.7				94.7		95.6
.30				87.2				94.7		95.6
.40				91.6				94.7		95.7
.60				94.3				94.7		96.4
.80				94.6				94.7		96.7
1.6				94.6				94.7		96.7
Min		47.5	47.3	49.1	92.9	89.5	95.0	94.7	95.0	95.6

FIGURE 1.—Instrumental Variables Example: .95 Quantile Graphs, $c_h(.95)$, for J_h^{**} and $|J_h^{**}|$ for the Partially-Studentized t Statistic as Functions of h_1 for Several Values of the Correlation h_2 and $k_2 = 5$

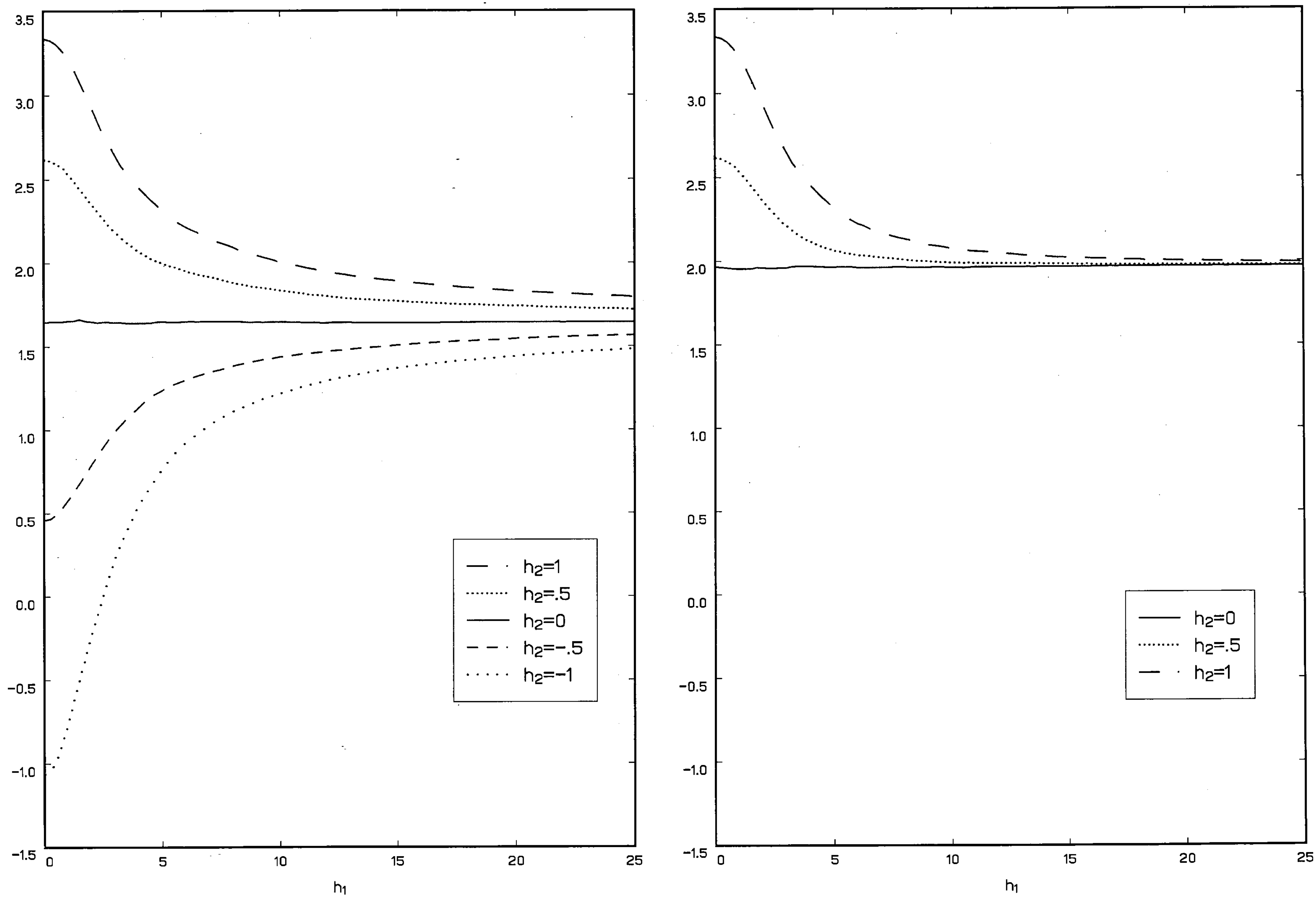

FIGURE 2.—Instrumental Variables Example: .95 Quantile Graphs, $c_h(.95)$, for J_h^* and $|J_h^*|$ for the Fully-Studentized t Statistic as Functions of h_1 for Several Values of the Correlation h_2 and $k_2 = 5$

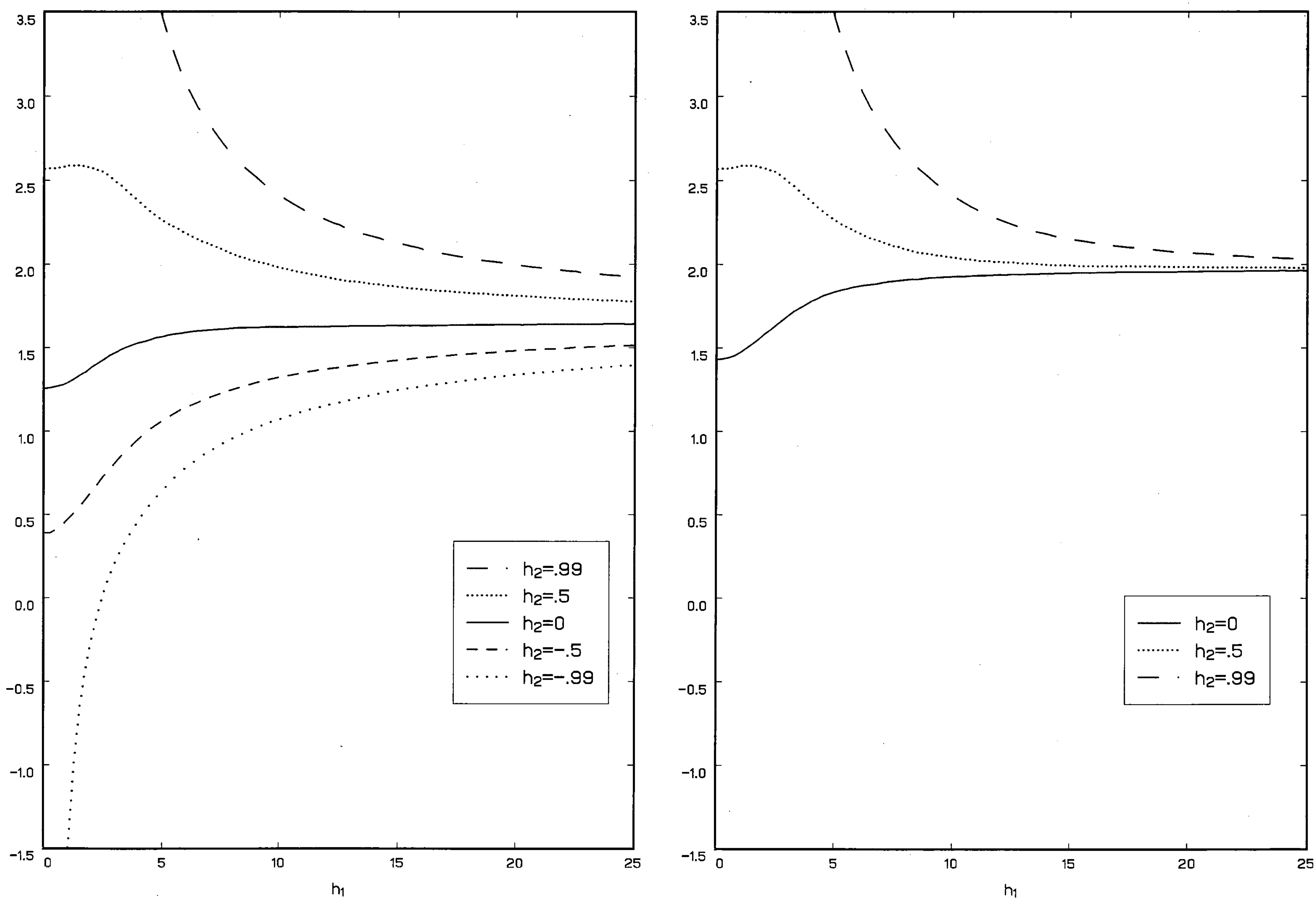

FIGURE 3.—Instrumental Variables Example: .95 Quantile Graphs, $c_h(.95)$, for $|J_h^*|$ for the Fully-Studentized t Statistic as Functions of h_1 for Several Values of the Correlation h_2 That Are Close to One and $k_2 = 5$

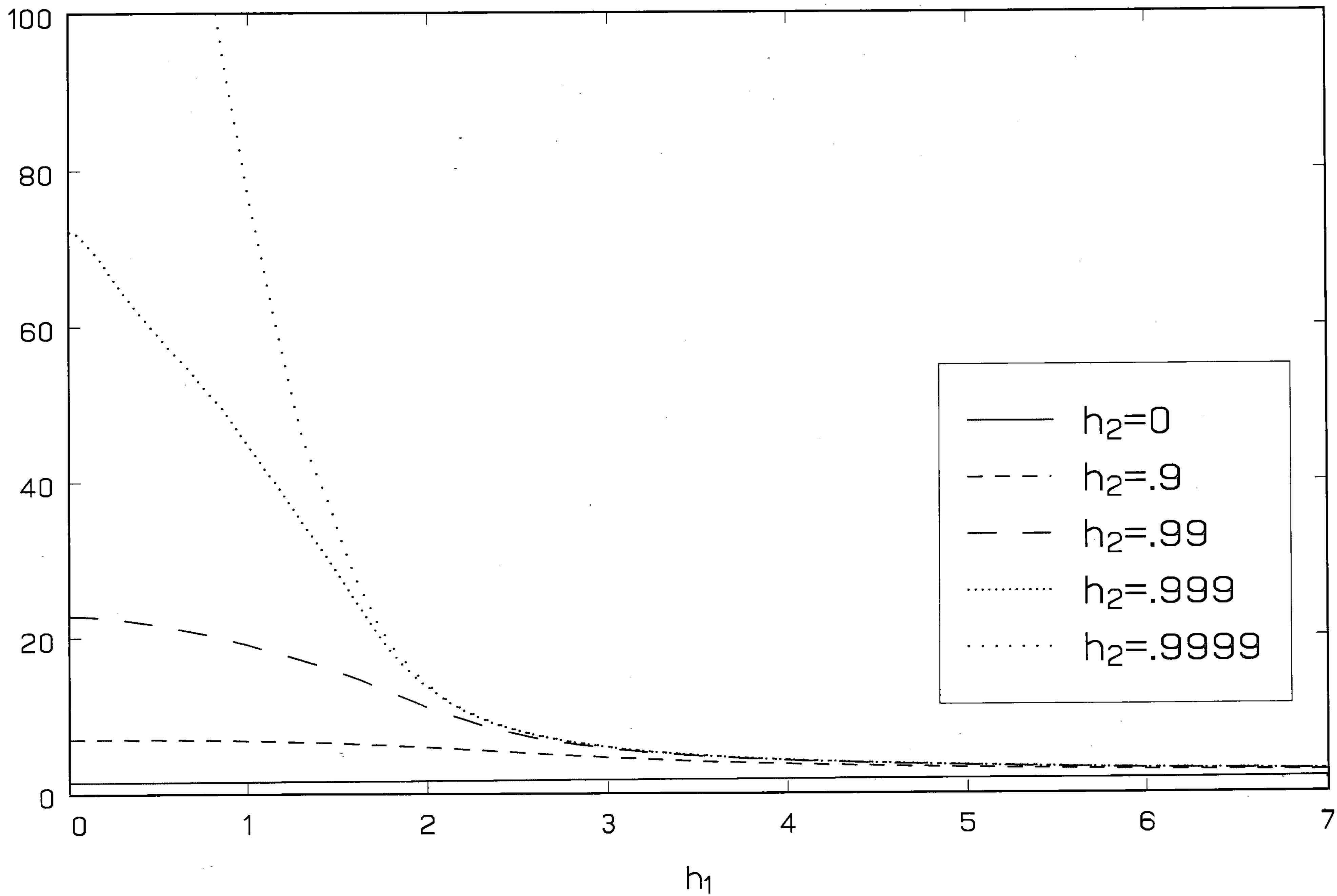

FIGURE 4.—Parameter of Interest Near Boundary Example: .95 Quantile Graphs, $c_h(.95)$, for $-J_h^*$ and $|J_h^*|$ as Functions of h