Technical Appendix for "Sequentially Optimal Mechanisms"

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Abstract

This document contains a number of omitted proofs and a more "formal" proof of the main theorem of the paper "Sequentially Optimal Mechanisms."

1. Omitted and Detailed Proofs for the Case that $T = 2$

Proof of Proposition 2

Step 1 We start by proving existence of the solution of the seller's problem when $T = 1$. The seller seeks to solve

$$\max_{p \in \Im} \int_a^b p(v) v dF(v) - \int_a^b p(v)[1 - F(v)] dv$$

where $\Im = \{p : [a, b] \to [0, 1]$, increasing$\}$.

Step 1a. (Sequential Compactness) In order to show sequential compactness of $\Im$ we will refer to the following results.

Theorem (pointwise convergence) A sequence p_n of functions from X to W converges to a function p in the topology of pointwise convergence[1] if and only if for each $s \in X (= [a, b]$ in our problem),

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[1]

Definition 1 *(Topology of pointwise convergence.) Given a point x of [0,1] and an open set U of space [0,1] let*

$$S(x, U) = \left\{ p \mid p \in [0, 1]^{[0,1]} \text{ and } p(x) \in U \right\}$$

The sets $S(x, U)$ are a subbasis for a topology on $[0,1]^{[0,1]}$ which is called the topology of pointwise convergence. The typical basis element about a function p consists of all functions g that are close to p at finitely many points.

the sequence $p_n(s)$ of points of $W(=[0,1]$ in our problem) converges to the point $p(s)$. (For a proof see Munkres "Topology: A first Course" page 281.)

Let $\{p_n\}$ be a sequence of elements of $\Im$. Then, from Helly's Selection Principle, (see Kolmogorov and Fomin p. 372), it follows that there exists $p \in \Im$ and a subsequence of $\{p_n\}$ that converges pointwise to p. From the previous Theorem it also follows that there exists $p \in \Im$ and a subsequence of $\{p_n\}$ that converges to p. Hence every sequence in $\Im$ has a convergent subsequence. It follows that $\Im$ is sequentially compact.[2]

Step 1b. (Continuity) We want to show that the objective function is continuous on $\Im$ in the topology of pointwise convergence. In order to accomplish this we will use Lebesque's Dominated Convergence Theorem.

Theorem (Lebesque's Dominated Convergence Theorem). Let g be a measurable function over a measurable set E, and suppose that $\{h_n\}$ is a sequence of measurable functions on E such that

$$|h_n(s)| \leq g(s)$$

and for almost all $s \in E$ we have $h_n(s) \to h(s)$. Then

$$\int_E h = \lim \int_E h_n.$$

(For a proof see Royden (1962) p.76.)

Take $E = [a, b]$ which is a measurable set, let

$$\begin{aligned} g &= \sup_{s \in [a,b]} |p(s)s| \\ \hat{g} &= \sup_{s \in [a,b]} |p(s)[1 - F(s)]| \, . \end{aligned}$$

Note that g, $\hat{g}$ are measurable, since they are constant functions, and g, and respectively $\hat{g}$, is an upper bound for every function

$$h(s) = p(s)s \text{ and } \hat{h}(s) = p(s)[1 - F(s)].$$

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Definition 2 *(Sequential Compactness). A topological space X is said to be sequentially compact if every infinite sequence from X has a convergent subsequence.*

Observe that h and $\hat{h}$ are measurable functions. Take $h_n(s) = p_n(s)s$ and $\hat{h}_n(s) = p_n(s)(1 - F(s))$ and apply Lebesque's Dominated Convergence Theorem with g and $\hat{g}$ defined as above:

$$\lim \int_a^b p_n(s) s dF(s) = \int_a^b p(s) s dF(s)$$

and

$$\lim \int_a^b p_n(s)[1 - F(s)]ds = \int_a^b p(s)[1 - F(s)]ds$$

hence

$$R(p_n) = \int_a^b p_n(s) s dF(s) - \int_a^b p_n(s)[1 - F(s)]ds,$$

is continuous in p_n.

Step 1c. We now demonstrate that a bounded and continuous function over a sequentially compact set has a maximum. First note that $R(p)$ is bounded by b. Let $\bar{R} = \sup_{p \in \Im} R(p)$ and let p_n be a sequence in $\Im$ such that

$$R(p_n) \geq \bar{R} - \frac{1}{n}, \; n \in \mathbb{N}.$$

Since $\Im$ is sequentially compact, every sequence has a convergent subsequence, therefore $\{p_n\}_{n \in \mathbb{N}}$, has a convergent subsequence, $\{p_{n_{(1)}}\}_{n_{(1)} \in \mathbb{N}}$, that converges to $\bar{p}$. Since R is continuous at $\bar{p}$, we have that $R(\bar{p}) = \lim_{n_{(1)} \to \infty} R(p_{n_{(1)}}) = \bar{R}$. Hence the maximum exists.

Step 2. We now proceed to show that the maximizer is of the form

$$p_z(s) = \begin{cases} 1 \text{ if } s \geq z \\ 0 \text{ if } s < z. \end{cases} \tag{1}$$

The objective function is linear in the choice variable so the maximizer will be an extreme point of the set of $\Im$. The set of extreme points of $\Im$ is

$$K = \cup_{z \in [a,b]} p_z$$

where p_z is defined in (1).

Every increasing, non-negative function p with $p(b) = 1$ can be written as a convex combination of functions as defined in (1)

$$G(v) = \int_a^b p_z(s) dp(z).$$

Let p^* be a maximizer of the problem defined in (4), in the main text. Let R^* denote the maximum value of the objective function. Then using the above representation and Fubini's theorem we have

$$\begin{aligned}
& \int_a^b p^*(v) v dF(v) - \int_a^b p^*(v)[1-F(v)]dv \\
= & \int_a^b \left\{ \int_a^b p_z(v) dp(z) \right\} v dF(v) - \int_a^b [1-F(v)] \left\{ \int_0^1 p_z(v) dp(z) \right\} dv \\
= & \int_a^b \left[\int_a^b p_z(v) v dF(v) \right] dp(z) - \int_a^b \left[\int_a^b [1-F(v)] p_z(v) dv \right] dp(z) \\
= & \int_a^b \left[\int_a^b p_z(v) v dF(v) - \int_a^b [1-F(v)] p_z(v) dv \right] dp(z) = R^*.
\end{aligned}$$

This is a convex combination of functions of the form given in (1). Hence one of these functions is a maximizer.

Step 3. Now we turn to show that the optimal cutoff is given by

$$v^* \equiv \inf \left\{ v \in [a,b] \text{ such that } \left[\int_v^{\tilde{v}} s dF(s) - \int_v^{\tilde{v}} [1-F(s)] ds \right] \geq 0, \text{ for all } \tilde{v} \in [v,b] \right\} \tag{2}$$

First note that v^* is well-defined because the set

$$\left\{ v \in [a,b] \text{ such that } \left[\int_v^{\tilde{v}} s dF(s) - \int_v^{\tilde{v}} [1-F(s)] ds \right] \geq 0, \text{ for all } \tilde{v} \in [v,b] \right\}$$

is non-empty since it contains b. Suppose that v^* does not characterize the optimal mechanism when $T = 1$. With some abuse of notation, let $R(v^*)$ denote the seller's expected revenue given an allocation rule described by (1) with cutoff v^*. If v^* is not optimal then there exists another cut-off $\tilde{v}$ such that

$$R(\tilde{v}) > R(v^*).$$

First, suppose that $\tilde{v} < v^*$. Then by the definition of v^*, there exists a $v' \in [\tilde{v}, v^*]$[3], such that

$$\int_{\tilde{v}}^{v'} s dF(s) - \int_{\tilde{v}}^{v'} [1-F(s)] ds < 0. \tag{3}$$

In this case expected revenue is given by

$$\begin{aligned}
R(\tilde{v}) = & \int_a^{\tilde{v}} 0 \cdot s dF(s) - \int_a^{\tilde{v}} 0 \cdot [1-F(s)] ds + \int_{\tilde{v}}^{v'} s dF(s) - \int_{\tilde{v}}^{v'} [1-F(s)] ds \\
& + \int_{v'}^b s dF(s) - \int_{v'}^b [1-F(s)] ds
\end{aligned}$$

[3] Actually, from the definition of v^* it follows that there exists $\hat{v} \in [v', b]$ such that $\int_{v'}^{\hat{v}} \phi(t) dt < 0$. A moment's thought will reveal that we can take $\hat{v} \leq v^*$ without any loss.

From (3) it follows that

$$\begin{aligned} R(\tilde{v}) &< \int_a^{\tilde{v}} 0 \cdot sdF(s) - \int_a^{\tilde{v}} 0 \cdot [1-F(s)]ds + \int_{\tilde{v}}^{v'} 0 \cdot sdF(s) - \int_{\tilde{v}}^{v'} 0 \cdot [1-F(s)]ds \\ &\quad + \int_{v'}^{b} sdF(s) - \int_{v'}^{b} [1-F(s)]ds = R(v'), \end{aligned}$$

clearly $\tilde{v}$ cannot be optimal.

Now suppose that $\tilde{v} > v^*$ and $R(\tilde{v}) > R(v^*)$. Then,

$$\begin{aligned} R(\tilde{v}) &= \int_a^{v^*} 0 \cdot sdF(s) - \int_a^{v^*} 0 \cdot [1-F(s)]ds + \int_{v^*}^{\tilde{v}} 0 \cdot sdF(s) - \int_{v^*}^{\tilde{v}} 0 \cdot [1-F(s)]ds \\ &\quad + \int_{\tilde{v}}^{b} sdF(s) - \int_{\tilde{v}}^{b} [1-F(s)]ds. \end{aligned}$$

From the definition of v^* it follows that $\int_{v^*}^{\tilde{v}} sdF(s) - \int_{v^*}^{\tilde{v}} [1-F(s)]ds \geq 0,$ hence

$$\begin{aligned} R(\tilde{v}) &\leq \int_a^{v^*} 0 \cdot sdF(s) - \int_a^{v^*} 0 \cdot [1-F(s)]ds + \int_{v^*}^{\tilde{v}} sdF(s) - \int_{v^*}^{\tilde{v}} [1-F(s)]ds \\ &\quad + \int_{\tilde{v}}^{b} sdF(s) - \int_{\tilde{v}}^{b} [1-F(s)]ds = R(v^*), \end{aligned}$$

contradiction. ■

Proof of Proposition 4 (Detailed)

Consider a PBE assessment (σ, μ) and let p denote the allocation rule implemented by it. Let s denote an action that leads to (r, z), where this is the contract with the smallest "r", with the property that type a is either "choosing" (r, z) with strictly positive probability at $t = 1$, or is indifferent between doing and not doing so. Also let Y denote the set of types of the buyer that report message β and choose s at $t = 1$ with strictly positive probability, and let $[a, \bar{v}]$, with $a \leq \bar{v}$, denote its convex hull. From the solution for $T = 1$ we have that after the history that the buyer reported message β, chose action s, and no trade took place at $t = 1$, the seller will maximize revenue by posting a price in period $t = 2$. Let us call this price as z_2 and define

$$\begin{aligned} v_L &= \inf \{v \in Y \text{ s.t. } v \text{ accepts } z_2 \text{ at } 2\} \\ v_H &= \sup \{v \in Y \text{ s.t. } v \text{ accepts } z_2 \text{ at } 2\}. \end{aligned}$$

By definition types v_L and v_H either choose (r, z) at $t = 1$ and accept z_2 at $t = 2$ with positive probability or are indifferent between this sequence of actions and the actions that they are actually choosing.

First we show that for $v \in (v_L, v_H)$ we have that $p(v) = r + (1-r)\delta$, then we establish that $z_2 = v_L$ and finally we show that for $v \in (a, v_L)$ we have that $p(v) = r$.

Step 1: For $v \in (v_L, v_H)$, where $v_L \neq v_H$ we have that $p(v) = r + (1-r)\delta$. Suppose not, then there exists $v \in (v_L, v_H)$ such that $p(v) \neq r + (1-r)\delta$, that is it is either a) $p(v) > r + (1-r)\delta$ or b) $p(v) < r + (1-r)\delta$. If $p(v) > r + (1-r)\delta$ then type v must be choosing with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} > r + (1-r)\delta$. At a PBE the buyer's strategy must be a best response hence it must be the case that $\hat{p}v - \hat{x} \geq (r + (1-r)\delta^{t-1})v - z - (1-r)\delta z_2$. But now since $\hat{p} > r + (1-r)\delta$ it follows that $\hat{p}v_H - \hat{x} > (r + (1-r)\delta)v_H - z - (1-r)\delta z_2$, contradicting the fact that v_H chooses (r, z) with positive probability or is indifferent between doing and not doing so. Now if $p(v) < r + (1-r)\delta$ then type v is choosing at $t = 1$ with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} < r + (1-r)\delta$ and because at a PBE the buyer's strategy is a best response then we have that $\hat{p}v - \hat{x} \geq (r + (1-r)\delta)v - z - (1-r)\delta z_2$. But now since $\hat{p} < r + (1-r)\delta$ and $v_L < v$ it follows that $\hat{p}v_L - \hat{x} > (r + (1-r)\delta)v_L - z - (1-r)\delta z_2$, contradicting the fact that v_L chooses (r, z) with positive probability or is indifferent between doing and not doing so.

Step 2 : We show that the smallest type that accepts z_2 must be equal to it:$v_L = z_2$. First observe that the fact that at a PBE the buyer's strategy must be a best response to the seller's strategy implies that

$$(r + (1-r)\delta)\, v_L - (z + (1-r)\delta z_2) \geq r v_L - z.$$

We now show that this inequality must hold with equality. We argue by contradiction. Suppose not, that is

$$(r + (1-r)\delta)\, v_L - (z + (1-r)\delta z_2) > r v_L - z.$$

then the seller can increase z_2 by Δz such that

$$(r + (1-r)\delta)\, v_L - (z + (1-r)\delta z_2) - \Delta z = r v_L - z,$$

and raise higher revenue at the continuation game that starts at 2. All types $v \geq v_L$ still prefer to choose $(1, z_2)$ at $t = 2$ then to choose $(0, 0)$. Hence at a PBE we have that

$$(r + (1-r)\delta)\, v_L - (z + (1-r)\delta z_2) = r v_L - z, \tag{4}$$

from which it is immediate that $v_L = z_2$.

Step 3: For $v \in (a, v_L)$, where $a \neq v_L$ we have that $p(v) = r$. Suppose not, then there exists $v \in (a, v_L)$ such that $p(v) \neq r$, that is it is either a) $p(v) > r$ or b) $p(v) < r$. If $p(v) > r$ then type v must be choosing with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} > r$. At a PBE the buyer's strategy must be a best response hence it must be the case that $\hat{p}v - \hat{x} \geq rv - z$. But now since $\hat{p} > r$ it follows that $\hat{p}v_L - \hat{x} > rv_L - z$, but from (4) we have that $\hat{p}v_L - \hat{x} > (r + (1-r)\delta)\, v_L - (z + (1-r)\delta z_2)$ contradicting the fact that v_L chooses (r, z) at $t = 1$ and $(1, z_2)$ at $t = 2$ with positive probability. Now if $p(v) < r$ then type v is choosing at $t = 1$ with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} < r$ and because at a PBE the buyer's strategy is a best response then we have that $\hat{p}v - \hat{x} \geq rv - z$. But now since $\hat{p} < r$ and $a < v$ it follows that $\hat{p}a - \hat{x} > ra - z$, contradicting the fact that a chooses (r, z) with positive probability or is indifferent between doing and not doing so.

From Steps 1-3 it follows that $p(v) = r + (1-r)\delta$ for $v \in (v_L, v_H)$, and $p(v) = r$, for $v \in (a, v_L)$ where $v_L = z_2$. Hence the allocation rule is

$$\begin{array}{c} p(v) = r \text{ for } v \in [a, \hat{z}_2) \\ r \leq p(\hat{z}_2) \leq r + (1-r)\delta \\ p(v) = r + (1-r)\delta \text{ for } v \in (\hat{z}_2, \bar{v}) \\ r + (1-r)\delta \leq p(\bar{v}) \leq 1 \end{array}.$$

Note that $p(a)$ cannot be strictly less then r by the definition of (r, z), (in order for $p(a) \leq r$ it must be the case that type a is choosing a sequence of actions that implement $\hat{p} < r$, but this contradicts the definition of (r, z) which is the smallest "r" contract that type a chooses with positive probability at $t = 1$). ■

An Alternative Proof of the Main Result

We start by verifying that the maximum of R over $\mathcal{P}_2^*$ and $\mathcal{P}_2$ indeed exists. In order to prove existence of a maximum we must establish that the spaces of functions $\mathcal{P}_2$ and $\mathcal{P}_2^*$ are sequentially compact and the objective function is continuous in the same topology.

Lemma A 8 The maximum of R over $\mathcal{P}_2$ and over $\mathcal{P}_2^*$ exists.

Proof. We will prove that the maximum of R over $\mathcal{P}_2$ exists. Using an identical procedure one can show that the maximum of R over $\mathcal{P}_2^*$ exists. The proof will be done for the case that the game lasts for two periods. All the arguments are valid, with more complicated notation, for the case that $T > 2$.

Continuity. Continuity of R follows from an identical argument as the one used in Step 1b, in the proof of Proposition 2.

In order to prove that the maximum exists it remains to demonstrate that $\mathcal{P}_2$ is sequentially compact in the topology of pointwise convergence.

Sequential Compactness. We will first show that every sequence $p_n \in \mathcal{P}_2$, $n \in \mathbb{N}$ has a subsequence that converges pointwise to $p \in \mathcal{P}_2$. Recall that p_n:$[a, b] \to [0, 1]$, increasing and of the form

$$\begin{array}{c} p_n(v) = r \text{ for } v \in [a, z_2^n), \\ p_n(v) = r + (1-r)\delta \text{ for } v \in [z_2^n, \bar{v}_n) \\ r + (1-r)\delta \leq p_n(v) \leq 1 \text{ for } v \in [\bar{v}_n, b] \end{array},$$

with $z_2^n \leq z_2(\bar{v}_n)$, and where $z_2(\bar{v}_n)$ is the optimal price at $t = 2$ given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_n)}$. The inequality $z_2^n \leq z_2(\bar{v}_n)$ follows from Lemmata 4 and 7 that can be found in the main text. Suppose that the limit of $\bar{v}_n$ is $\bar{v}$, (this limit exists since every bounded sequence has a convergent subsequence). From Lemma 2 we know that $z_2(\bar{v}_n)$ is increasing in $\bar{v}_n$, and hence it is continuous in $\bar{v}_n$ from the right, which guarantees that

$$\lim_{n\to\infty} z_2(\bar{v}_n) \leq z_2(\bar{v}).$$

Now for all $n \in \mathbb{N}$ we have that

$$z_2^n \leq z_2(\bar{v}_n),$$

let $\hat{z}_2 = \lim_{n\to\infty} z_2^n$, (this limit exists because z_2^n is a bounded sequence), hence taking the limit as $n \to \infty$ we get that

$$\hat{z}_2 \leq \lim_{n\to\infty} z_2(\bar{v}_n) \leq z_2(\bar{v}).$$

Let $w_1, w_2, ...$ denote the rational points of $[a, b]$. Since p_n is bounded, the sequence $\{p_n\}$ has a subsequence, $\{p_n^{(1)}\}$ that converges at point w_1. Since $\{p_n^{(1)}\}$ is also bounded, it has a subsequence $\{p_n^{(2)}\}$ converging at the point w_2 as well as the point w_1; $\{p_n^{(2)}\}$ contains a subsequence $\{p_n^{(3)}\}$ that converges at point w_3 as well as at point w_1 and w_2 and so on. The "diagonal sequence" $\{h_n\} = \{p_n^{(n)}\}$ will then converge to every rational point of $[a, b]$. The limit of this subsequence, p, is an increasing function from $[a, b]$ to $[0, 1]$. Moreover $p(s) = r$ for all the rationals in $[a, \hat{z}_2)$, and $p(s) = r + (1-r)\delta$ for all the rationals in $[\hat{z}_2, \bar{v})$. We complete the definition of p at the remaining points of $[a, b]$ by setting[4]

$$p(v) = \lim_{v\to w^-} p(w) \text{ if } v \text{ is irrational.}$$

The resulting function p is then the limit of $\{h_n\}$ at every continuity point of p, (see Kolmogorov and Fomin page 373). Since p is increasing it has at most countably many discontinuity points. Using the diagonal process we can find a subsequence of $\{h_n\}$ that converges to all the discontinuity points p, which implies that it converges pointwise everywhere to p on $[a, b]$.

[4] The notation $v \to w^-$ means that v approaches w from below.

From the above arguments it follows that $\{p_n\}_{n\in\mathbb{N}}$ has a subsequence that converges pointwise to p which is an increasing function, such that at $\hat{z}_2$ its value jumps from r to $r+(1-r)\delta$ and at $\bar{v}$ its value is $p(\bar{v}) = r+(1-r)\delta$, in other words, $p:[a,b]\to[0,1]$ is increasing and such that

$$\begin{gathered} p(v) = r \text{ for } v\in[a,\hat{z}_2), \\ p(v) = r+(1-r)\delta \text{ for } v\in[\hat{z}_2,\bar{v}) \\ r+(1-r)\delta \le p(v) \le 1 \text{ for } v\in[\bar{v},b] \\ \text{with } \hat{z}_2 \le z_2(\bar{v}). \end{gathered}$$

Therefore $p\in\mathcal{P}_2$.

From Theorem on pointwise convergence, (stated in the proof of Proposition 2), it follows that $\{p_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to p. Hence every infinite sequence in $\mathcal{P}_2$ has a convergent subsequence. Therefore, $\mathcal{P}_2$ is sequentially compact. As seen in the proof of Proposition 2, **Step 1c**, a bounded continuous function on a sequentially compact set has a maximum. ■

The following Lemma establishes that the allocation rules in $\mathcal{P}_2^*$ are extreme points of the set $\mathcal{P}_2$.

Lemma A 9. Every allocation rule in $\mathcal{P}_2$ can be approximated arbitrarily closely, in the usual metric, by a convex combination of elements of $\mathcal{P}_2^*$.

Proof. We use p and q to denote generic elements of $\mathcal{P}_2$ and $\mathcal{P}_2^*$ respectively. Every measurable function can be approximated by a step function in the usual metric generated by the norm (see for instance Royden 1962). An element of $\mathcal{P}_2$, say p, can be therefore approximated by a step function g. We now show that every step function that is arbitrarily close to an element of $\mathcal{P}_2$, can be written as a convex combination of elements of $\mathcal{P}_2^*$. Take a $p\in\mathcal{P}_2$,

$$\begin{gathered} p(v) = r \text{ for } v\in[a,\hat{z}_2), \\ p(v) = r+(1-r)\delta \text{ for } v\in[\hat{z}_2,\bar{v}) \\ r+(1-r)\delta \le p(v) \le 1 \text{ for } v\in[\bar{v},b] \\ \text{with } \hat{z}_2 \le z_2(\bar{v}), \end{gathered},$$

and a step function g, such that $|p-g|<\varepsilon$, $\varepsilon>0$ arbitrarily small. Since the restriction of p on $[a,\bar{v})$ is a step function, we can take $p(t)=g(t)$, for $t\in[a,\bar{v})$. Suppose that in the interval $[\bar{v},b]$, g has K steps. Then we can consider the division of $[\bar{v},b]$, into K subintervals, I_j, $j=1,...,K$. In each of these subintervals g takes a potentially different value g_j, where $r+(1-r)\delta\le g_j\le b$.

We now show that we can write g as a linear combination of L functions $q_1,...,q_L\in\mathcal{P}_2^*$, that is to say

$$g=\sum_{i=1}^{L}\alpha_i q_i,\ \Sigma_{i=1}^{L}\alpha_i = 1.$$

All $q_i's$ have the following characteristics

$$\begin{aligned} q_i(v) &= r,\ v \in [a, \hat{z}_2), \\ q_i(v) &= r + (1-r)\delta,\ v \in [\hat{z}_2, \hat{v}_i), \\ q_i(v) &= 1, v \in [\hat{v}_i, b], \end{aligned}$$

where $b \geq \hat{v}_i \geq \bar{v}$ and $z_2(\hat{v}_i) \geq z_2(\bar{v}) \geq \hat{z}_2$, where $\hat{z}_2$ and $z_2(\bar{v})$ are the same as in the definition of p.

The way to determine the coefficients α_i, is as follows. Suppose that for $v \in I_1$, $g_1 = g(v) = r + (1-r)\delta + \eta_1$. Then for $v \in I_1$, we have $g(v) = \Sigma_{i=2}^{L}\alpha_i q_i + \alpha_1 q_1$, where $q_i = r + (1-r)\delta$ for all $i \neq 1$ and $q_1 = 1$, $\alpha_1 = \frac{\eta_1}{1-r-(1-r)\delta}$, and of course $\Sigma_{i=1}^{L}\alpha_i = 1$. (Observe that since $q_1 = 1$ on I_1 it must be $q_1 = 1$ for $v \in I_j$, $j = 2, ..., K$.) Obviously, $\Sigma_{i=2}^{L}\alpha_i = 1 - \alpha_1 = \frac{1-r-(1-r)\delta-\eta_1}{1-r-(1-r)\delta}$. So for $v \in I_1$, we can write $g(v) = \Sigma_{i=2}^{L}\alpha_i q_i + \alpha_1 q_1 = (r + (1-r)\delta)\cdot\left(\frac{1-r-(1-r)\delta-\eta_1}{1-r-(1-r)\delta}\right) + 1\cdot\left(\frac{\eta_1}{1-r-(1-r)\delta}\right) = r + (1-r)\delta + \eta_1$. Now, suppose that for $v \in I_2$, we have $g_2 = r + (1-r)\delta + \eta_1 + \eta_2$. Then for $v \in I_2$ we can write $g(v) = \Sigma_{i=3}^{L}\alpha_i q_i + \alpha_1 q_1 + \alpha_2 q_2$, where $q_i = r + (1-r)\delta$ for all $i \neq 1, 2$ and $q_1 = 1 = q_2$, $\alpha_1 = \frac{\eta_1}{1-r-(1-r)\delta}$, $\alpha_2 = \frac{\eta_2}{1-r-(1-r)\delta}$ and $\Sigma_{i=1}^{L}\alpha_i = 1$. To verify this note that $\Sigma_{i=3}^{L}\alpha_i = 1 - \alpha_1 - \alpha_2 = \frac{1-r-(1-r)\delta-\eta_1-\eta_2}{1-r-(1-r)\delta}$. We therefore obtain, that for $v \in I_2$, $g(v) = \Sigma_{i=3}^{L}\alpha_i q_i + \alpha_1 q_1 + \alpha_2 q_2$
$= (r + (1-r)\delta)\cdot\left(\frac{1-r-(1-r)\delta-\eta_1-\eta_2}{1-r-(1-r)\delta}\right) + 1\cdot\left(\frac{\eta_1}{1-r-(1-r)\delta}\right) + 1\cdot\left(\frac{\eta_2}{1-r-(1-r)\delta}\right) = r + (1-r)\delta + \eta_1 + \eta_2$.

Continuing in a similar manner we can determine all the $\alpha_i's$. It follows that any step function that is arbitrarily close to an element of $\mathcal{P}_2$, can be written as a convex combination of elements of $\mathcal{P}_2^*$. Therefore for each $p \in \mathcal{P}_2$ there exist a g, where g is a convex combination of elements of $\mathcal{P}_2^*$, such that $|p - g| < \varepsilon$. ■

Proposition A 4 Consider a linear function $R : \mathcal{P} \rightarrow \mathbb{R}$. Suppose that there exists a set $\mathcal{P}^* \subset \mathcal{P}$, such that every element of $\mathcal{P}$ can be approximated by a convex combination of elements of $\mathcal{P}^*$. Furthermore, suppose that the maximum value of R over $\mathcal{P}$ and $\mathcal{P}^*$ exists. Then

$$\max_{p \in \mathcal{P}} R(p) = \max_{p \in \mathcal{P}^*} R(p).$$

Proof. First note that since $\mathcal{P}^* \subset \mathcal{P}$, then

$$\max_{\mathcal{P}} R \geq \max_{\mathcal{P}^*} R.$$

It is given, that every element of $\mathcal{P}$ can be arbitrarily closely approximated by a convex combination of elements of $\mathcal{P}^*$. We will use p and q to denote generic elements of $\mathcal{P}$ and $\mathcal{P}^*$ respectively, and g to denote convex combinations of elements of $\mathcal{P}^*$. Suppose that $\bar{p} \in \mathcal{P}$ is a maximizer of R, and consider a sequence

$\{g_n\}_{n\in\mathbb{N}}$ such that $g_n \to \bar{p}$. This implies that for all $\varepsilon > 0$, there exists g_n such that $|R(g_n) - R(\bar{p})| < \varepsilon$, for n large enough. From this we get that, for n large enough either $R(g_n) > R(\bar{p}) - \varepsilon$ or $R(\bar{p}) \geq R(g_n) - \varepsilon$.

Fix an n large enough. Since g_n is a convex combination of elements of $\mathcal{P}^*$, we can rewrite each element of this sequence as $g_n = \Sigma_{i=1}^{L} \alpha_i^n q_i^n$, where $q_i^n \in \mathcal{P}^*$ and $\Sigma_{i=1}^{L} \alpha_i^n = 1$. Then, because R is linear, we can write

$$R(g_n) = \Sigma_{i=1}^{L} \alpha_i^n R(q_i^n).$$

Now suppose that $R(q_i^n) < R(g_n)$ for all $i = 1, ..., L$. Then we have that $R(g_n) = \Sigma_{i=1}^{L} \alpha_i^n R(q_i^n) < R(g_n)$, but this is impossible. Hence there must exist i such that $R(q_i^n) \geq R(g_n)$. Now

$$\max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n),$$

where the first inequality follows from the fact that $q_i^n \in \mathcal{P}^*$. If $R(\bar{p}) > R(g_n)$ then

$$\max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n) > R(\bar{p}) - \varepsilon, \text{ for all } \varepsilon > 0.$$

Taking the limit as $\varepsilon \to 0$, we get that

$$\max_{\mathcal{P}^*} R(p) = R(\bar{g}) = \max_{\mathcal{P}} R(p).$$

If $R(g_n) \geq R(\bar{p})$, then from the fact that $q_i^n \in \mathcal{P}^*$ and $\mathcal{P}^* \subset \mathcal{P}$, we have

$$R(\bar{p}) \geq \max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n) \geq R(\bar{p}),$$

which again implies that all inequalities hold with equality. We therefore get

$$\max_{\mathcal{P}^*} R(p) = R(\bar{p}) = \max_{\mathcal{P}} R(p).$$

■

Now we are ready to state the main result of the paper.

Theorem A 4 Under non-commitment the seller maximizes expected revenue by posting a price in each period.

Proof. In Lemma A 8, we verified that the seller's maximization problem is well defined. From Lemma A 9 we know that an element of $\mathcal{P}_2(r, z_2(F_2), \bar{v})$ can be written as a convex combination of elements of $\mathcal{P}_2^*$. The result follows from Proposition A 4 and Lemma A 9. ■