Dynamic Global Games of Regime Change: Learning, Multiplicity and Timing of Attacks

Supplementary Material

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Abstract

This supplementary document contains a formal analysis of some of the extensions briefly discussed in Section 5 of the published version. Section A1 considers the game in which agents receive signals about the size of past attacks. Section A2 considers the game with observable shocks to the fundamentals. Section A3 considers the variant in which agents observe the shocks with a one-period lag. Section A4 considers the game with short-lived agents in which the fundamentals follow a random walk. Finally, Section A5 collects the proofs of the formal results contained in this document.

A1. Signals about past attacks

For some applications, it might be natural to assume that agents collect information—either private or public—not only about the underlying fundamentals but also about the size of past attacks. To capture this possibility, we extend the game with public news examined in Section 5.1 as follows. In every period $t \geq 2$, agents receive private and public signals about the size of the attack in the previous period. These signals are, respectively,

$$\tilde{X}_{it} = S(A_{t-1}, \tilde{\xi}_{it}) \quad \text{and} \quad \tilde{Z}_t = S(A_{t-1}, \tilde{\varepsilon}_t),$$

where $\tilde{\xi}_{it}$ is idiosyncratic noise, $\tilde{\varepsilon}_t$ is common noise, and $S : [0,1] \times \mathbb{R} \to \mathbb{R}$. To preserve Normality of the information structure, we adopt a specification similar to that in Dasgupta (2002):

$$\tilde{\xi}_{it} \sim \mathcal{N}\left(0, 1/\gamma_t^x\right), \quad \tilde{\varepsilon}_t \sim \mathcal{N}\left(0, 1/\gamma_t^z\right), \quad \text{and} \quad S\left(A, v\right) = \begin{cases} \Phi^{-1}\left(A\right) + v & \text{if } A \in (0,1), \\ v & \text{otherwise.} \end{cases}$$

The noises $\tilde{\xi}_{it}$ and $\tilde{\varepsilon}_t$ guarantee that, even if $A(\theta)$ is monotonic, the fundamentals θ never become common certainty among the agents.[1]

Since in any equilibrium of the game, agents play in period 1 as in the static benchmark, the size of attack in period 1 is given by $A_1(\theta) = \Phi(\sqrt{\beta_1}(x_1^* - \theta))$, where $x_1^* = \hat{x}_1$. This implies that in period 2, the signals the agents receive about A_1 are also additive signals about θ : $\tilde{X}_{i2} = \sqrt{\beta_1}(x_1^* - \theta) + \tilde{\xi}_{i2}$ and $\tilde{Z}_2 = \sqrt{\beta_1}(x_1^* - \theta) + \tilde{\varepsilon}_2$. The posterior beliefs about θ conditional on $(\tilde{x}^2, \tilde{z}^2, \tilde{X}_2, \tilde{Z}_2)$ are then Normal with mean $\frac{\beta_2}{\beta_2+\alpha_2}x_2 + \frac{\alpha_2}{\beta_2+\alpha_2}z_2$ and precision $\beta_2 + \alpha_2$ where

$$x_2 = \frac{\beta_1}{\beta_2}x_1 + \frac{\eta_2^x}{\beta_2}\tilde{x}_2 + \frac{\beta_1\gamma_2^x}{\beta_2}\left\{x_1^* - \frac{1}{\sqrt{\beta_1}}\tilde{X}_2\right\},$$
$$z_2 = \frac{\alpha_1}{\alpha_2}z_1 + \frac{\eta_1^z}{\alpha_2}\tilde{z}_1 + \frac{\beta_1\gamma_2^z}{\alpha_2}\left\{x_1^* - \frac{1}{\sqrt{\beta_1}}\tilde{Z}_2\right\},$$
$$\beta_2 = \beta_1 + \eta_2^x + \beta_1\gamma_2^x \quad \text{and} \quad \alpha_2 = \alpha_1 + \eta_2^z + \beta_1\gamma_2^z,$$

with x_1, z_1, β_1 and α_1 defined as in the previous sections. That is, x_2 and z_2 are sufficient statistics for $(\tilde{x}^2, \tilde{X}^2)$ and $(\tilde{z}^2, \tilde{Z}^2)$ with respect to θ. If the agents' strategies in period 2 are monotonic in $(\tilde{x}^2, \tilde{X}^2)$, then the size of attack and hence the regime outcome in that period are decreasing in θ, which in turn implies that the agents' strategies in period 2 are necessarily a threshold strategy in the statistic x_2. A similar argument applies to every $t \geq 2$: in any monotone equilibrium, the posterior beliefs about θ conditional on $(\tilde{x}^t, \tilde{z}^t, \tilde{X}^t, \tilde{Z}^t)$ are Normal with mean $\frac{\beta_t}{\beta_t+\alpha_t}x_t + \frac{\alpha_t}{\beta_t+\alpha_t}z_t$ and precision $\beta_t + \alpha_t$, where

$$x_t = \frac{\beta_{t-1}}{\beta_t}x_{t-1} + \frac{\eta_t^x}{\beta_t}\tilde{x}_t + \mathbf{1}_{t-1}\frac{\beta_{t-1}\gamma_t^x}{\beta_t}\left\{x_{t-1}^* - \frac{1}{\sqrt{\beta_{t-1}}}\tilde{X}_t\right\},$$
$$z_t = \frac{\alpha_{t-1}}{\alpha_t}z_{t-1} + \frac{\eta_t^z}{\alpha_t}\tilde{z}_t + \mathbf{1}_{t-1}\frac{\beta_{t-1}\gamma_t^z}{\alpha_t}\left\{x_{t-1}^* - \frac{1}{\sqrt{\beta_{t-1}}}\tilde{Z}_t\right\},$$
$$\beta_t = \beta_{t-1} + \eta_t^x + \mathbf{1}_{t-1}\beta_{t-1}\gamma_t^x \quad \text{and} \quad \alpha_t = \alpha_{t-1} + \eta_t^z + \mathbf{1}_{t-1}\beta_{t-1}\gamma_t^z;$$

where $\mathbf{1}_{t-1}$ is an indicator function that takes value 1 if $A_{t-1} \in (0,1)$ and 0 otherwise, and x_{t-1}^* is the threshold played in period $t-1$. It follows that the conditions in Proposition 3 continue to characterize the entire set of monotone equilibria—the only difference is that the statistics x_t and z_t are now endogenous, as defined above, and that the thresholds x_t^* and θ_t^* are now functions, not only of z^t, but also of $\tilde{Z}^t$.

The multiplicity result of Theorem 2 thus extends directly to this environment. Similarly, the structure of dynamics remains the same as in the game with public news, except for the property that an unsuccessful attack does not necessarily reduce the incentives for further attacks. This is because an unsuccessful attack now also generates new private and public signals, which in some cases may offset the impact of the knowledge that the regime survived past attacks. To see this,

[1] We assume that these signals are uninformative when $A = 0$ or $A = 1$ to avoid the possibility that agents can detect (collective) deviations. Since agents are infinitesimal, this would not affect equilibrium outcomes, but would require us to specify out-of-equilibrium beliefs.

consider the case where all signals are private $(\gamma_t^x > 0, \eta_t^x \geq 0, \gamma_t^z = \eta_t^z = 0)$, in which case the only novel effect is that an unsuccessful attack leads to an endogenous increase in β_t. A further attack is then possible only if this increase is large enough, like in the benchmark game. On the other hand, when the endogenous signal is public $(\gamma_t^z > 0 = \gamma_t^x)$, a new attack becomes possible if this signal is low enough, like in the case with exogenous public news. Signals about the size of past attacks can thus substitute for the exogenous arrival of private and public information and lead to "snow-balling effects" where new attacks become possible immediately after unsuccessful ones.

A2. Observable shocks

Consider the game with observable shocks described in Section 5.3 of the paper. The characterization of monotone equilibria was completed there. Here we prove that "essentially" all equilibria of the benchmark game $\Gamma(0)$ can be approximated by equilibria of the game with observable shocks $\Gamma(\delta)$, for δ small enough (This result was discussed at the end of Section 5.3 without proof).

As in the case with unobservable shocks (Theorem 3 in the paper), we rule out knife-edge equilibria where U is tangent to the horizontal axis. But, unlike that case, convergence is established in probability, for the equilibrium thresholds here are functions of the sequences of observable shocks.

Proposition A1 *For any $\varepsilon > 0$ and $T < \infty$, there exists $\delta(\varepsilon, T) > 0$ such that the following is true for all $\delta < \delta(\varepsilon, T)$:*

For any equilibrium $\{x_t^*, \theta_t^*\}_{t=1}^{\infty}$ of $\Gamma(0)$, for which $\theta_t^* \notin \arg\max_{\theta^*} U\left(\theta^*, \theta_{t-1}^*, \beta_t, \alpha, z\right)$ for all $t \in \{2, ..., T\}$, there exists an equilibrium $\left\{x_t^\delta\left(\omega^t\right), \theta_t^\delta\left(\omega^t\right)\right\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ such that

$$\Pr\left(\left|\theta_t^\delta\left(\omega^t\right) - \theta_t^*\right| \leq \varepsilon \ \forall t \in \{1, ..., T\} \right) \ \geq \ 1 - \varepsilon.$$

A3. Shocks observable with lag

In this section, we discuss a variant of the game with shocks in which agents observe the shocks with a one-period lag. This variant was briefly discussed at the end of Section 5.3.

The game structure is the same as in the model with fully observable shocks (Section 5.3), except that ω_t becomes known only at the end of period t. The property that the contemporaneous shock is unobservable introduces an additional source of uncertainty about the regime outcome in the current period and may even reintroduce the lower-dominance region. At the same time, the property that the shock is revealed at the end of the period ensures that the learning induced by the knowledge that the regime survived past attacks continues to take the simple and sharp form of a truncation in the support of the agents' beliefs about θ, as in the case with fully observable shocks.

Equilibrium characterization, multiplicity and dynamics. Monotone equilibria are now characterized by sequences $\{x_t^*(\omega^{t-1}), \theta_t^*(\omega^t)\}_{t=1}^{\infty}$ such that agents attack in period t if and only if $x_t \leq x_t^*(\omega^{t-1})$ and the status quo survives period t if and only if $\theta > \theta_t^*(\omega^t)$; note that strategies in period t are contingent only on ω^{t-1} since ω_t is not observed at the time agents choose whether or not to attack, but the regime outcome still depends on ω_t, since ω_t directly affects the size of attack necessary for regime change.

To compute the expected net payoff from attacking, we need to adjust the conditional probability of regime change as follows. For a given threshold rule $\bar{x}_t$, regime change occurs in period t when the fundamentals are θ if and only if $\theta + \delta\omega_t \leq \Phi\left(\sqrt{\beta_t}\,(\bar{x}_t - \theta)\right)$, or equivalently $\omega_t \leq \bar{\omega}_t^{\delta}(\theta; \bar{x}_t) \equiv \left[\Phi\left(\sqrt{\beta_t}\,(\bar{x}_t - \theta)\right) - \theta\right]/\delta$. Conditional on θ, the probability of regime change in period t is therefore given by

$$p_t^{\delta}(\theta; \bar{x}_t) \equiv \Pr\left(\omega_t \leq \bar{\omega}_t^{\delta}(\theta; \bar{x}_t)\right) = F\left(\bar{\omega}_t^{\delta}(\theta; \bar{x}_t)\right).$$

The updating of posterior beliefs, on the other hand, is the same as in the game with fully observable shocks. Let $\bar{\theta}_t(\bar{x}_t, \Omega)$ be implicitly defined by $\bar{\theta}_t + \Omega = \Phi\left(\sqrt{\beta_t}\left(\bar{x}_t - \bar{\theta}_t\right)\right)$. Next, consider any sequence of threshold rules $\{\bar{x}_t(\omega^{t-1})\}_{t=1}^{\infty}$ and define the sequence $\{\bar{\theta}_t(\omega^t)\}_{t=1}^{\infty}$ recursively by $\bar{\theta}_t(\omega^t) = \max\left\{\bar{\theta}_{t-1}(\omega^{t-1}), \bar{\theta}_t\left(\bar{x}_t(\omega^{t-1}), \delta\omega_t\right)\right\}$, with $\bar{\theta}_0 = -\infty$ and $\omega_0 = 0$. When agents follow the strategy associated with $\{\bar{x}_t(\omega^{t-1})\}_{t=1}^{\infty}$, posterior beliefs over θ in period t are again characterized by truncated normal distributions with truncation at $\bar{\theta}_{t-1}(\omega^{t-1})$.

Let then $\Psi_t\left(\theta | x, \bar{\theta}_{t-1}\right)$ denote the c.d.f. of an agent's posterior about θ conditional on having statistic x and on believing that $\theta > \bar{\theta}_{t-1}$; this is simply

$$\Psi_t\left(\theta | x, \bar{\theta}_{t-1}\right) = \begin{cases} 1 - \dfrac{\Phi\left(\sqrt{\alpha+\beta_t}\left(\frac{\beta_t}{\alpha+\beta_t}x_t + \frac{\alpha}{\alpha+\beta_t}z - \theta\right)\right)}{\Phi\left(\sqrt{\alpha+\beta_t}\left(\frac{\beta_t}{\alpha+\beta_t}x_t + \frac{\alpha}{\alpha+\beta_t}z - \bar{\theta}_{t-1}\right)\right)} & \text{if } \theta > \bar{\theta}_{t-1} \\ 0 & \text{if } \theta \leq \bar{\theta}_{t-1} \end{cases}$$

which is exactly the same as in the benchmark model. Next, let $v_t^{\delta}\left(x, \bar{x}_t, \bar{\theta}_{t-1}\right)$ denote an agent's expected net payoff from attacking in period t when he has sufficient statistic $x \in \mathbb{R}$, all other agents follow monotone strategies in that period with threshold $\bar{x}_t \in \overline{\mathbb{R}}$, and the agent believes that $\theta > \bar{\theta}_{t-1}$; this is given by

$$v_t^{\delta}\left(x, \bar{x}_t, \bar{\theta}_{t-1}\right) = \int_{-\infty}^{+\infty} F\left(\bar{\omega}_t^{\delta}(\theta; \bar{x}_t)\right) d\Psi_t\left(\theta | x, \bar{\theta}_{t-1}\right) - c.$$

Finally, define

$$V_t^{\delta}\left(\bar{x}_t, \bar{\theta}_{t-1}\right) \equiv \begin{cases} \lim_{x \to +\infty} v_t^{\delta}\left(x, \bar{x}_t, \bar{\theta}_{t-1}\right) & \text{if } \bar{x}_t = +\infty \\ v_t^{\delta}\left(\bar{x}_t, \bar{x}_t, \bar{\theta}_{t-1}\right) & \text{if } \bar{x}_t \in \mathbb{R} \\ \lim_{x \to -\infty} v_t^{\delta}\left(x, \bar{x}_t, \bar{\theta}_{t-1}\right) & \text{if } \bar{x}_t = -\infty \end{cases}.$$

V_t^{δ} is the analogue of the function U in the benchmark model: it represents the net payoff from attacking in period t for the marginal agent with threshold $\bar{x}_t$.

Since v_t^δ is continuous in x, $\bar{x}_t$ and $\bar{\theta}_{t-1}$, V_t^δ is continuous in $\bar{x}_t$ and $\bar{\theta}_{t-1}$ for all $\bar{x}_t \in \mathbb{R}$. Moreover, since v_t^δ is bounded and monotone decreasing in x, for any given $\bar{x}_t$, $V_t^\delta\left(\bar{x}_t, \bar{\theta}_{t-1}\right)$ is well-defined at $\bar{x}_t = \pm\infty$. We thus have the following equilibrium characterization.

Proposition A2 *$\{a_t(\cdot)\}_{t=1}^\infty$ is a monotone equilibrium of $\Gamma(\delta)$ if and only if there exists a sequence $\{x_t^*\left(\omega^{t-1}\right), \theta_t^*\left(\omega^t\right)\}_{t=1}^\infty$ such that:*

(i) for all t, $a_t(\cdot) = 1$ if $x_t < x_t^*\left(\omega^{t-1}\right)$ and $a_t(\cdot) = 0$ if $x_t > x_t^*\left(\omega^{t-1}\right)$.

(ii) for $t = 1$, $x_1^* \in \mathbb{R}$ solves $V_1^\delta\left(x_1^*, -\infty\right) = 0$; and $\theta_1^*(\omega_1) = \bar{\theta}_t\left(x_1^*, \delta\omega_1\right)$.

(iii) for all $t \geq 2$, either $x_t^*\left(\omega^{t-1}\right) = -\infty$ and $V_t^\delta\left(x_t^*\left(\omega^{t-1}\right), \theta_{t-1}^*\left(\omega^{t-1}\right)\right) \leq 0$, or $x_t^*\left(\omega^{t-1}\right) \in \mathbb{R}$ solves $V_t^\delta\left(x_t^*\left(\omega^{t-1}\right), \theta_{t-1}^*\left(\omega^{t-1}\right)\right) = 0$; and $\theta_t^*\left(\omega^t\right) = \max\left\{\theta_{t-1}^*\left(\omega^{t-1}\right), \bar{\theta}_t\left(x_t^*\left(\omega^{t-1}\right), \delta\omega_t\right)\right\}$.

An equilibrium always exists.

The equilibrium characterization is thus similar to that with observable shocks; one only has to adjust the agents' expected payoff from attacking to take into account the uncertainty about the regime outcome introduced by unobservable contemporaneous shocks.

As $\delta \to 0$, the impact of shocks on regime outcomes vanishes, thus ensuring a similar convergence result as the one we established in the previous section for the case with observable shocks.

Proposition A3 *For any $\varepsilon > 0$ and $T < \infty$, there exists $\delta(\varepsilon, T) > 0$ such that the following is true for all $\delta < \delta(\varepsilon, T)$:*

For any equilibrium $\{x_t^*, \theta_t^*\}_{t=1}^\infty$ of $\Gamma(0)$ for which $\theta_t^* \notin \arg\max_{\theta^*} U\left(\theta^*, \theta_{t-1}^*, \beta_t, \alpha, z\right)$ for all $t \in \{2, ..., T\}$, there exists an equilibrium $\left\{x_t^\delta\left(\omega^{t-1}\right), \theta_t^\delta\left(\omega^t\right)\right\}_{t=1}^\infty$ of $\Gamma(\delta)$ such that

$$\Pr\left(\left|\theta_t^\delta\left(\omega^t\right) - \theta_t^*\right| \leq \varepsilon \ \forall t \in \{1, ..., T\} \right) \ \geq \ 1 - \varepsilon.$$

A4. Changing fundamentals with short-lived agents

In Section 5.5 we introduced and briefly analyzed a game with short-lived agents where the "fundamentals" (summarized by the critical size of attack necessary for regime change) follow a random walk. Here we prove that Proposition 5 and Theorem 3, which we established for the case with long-lived agents and unobservable shocks, apply also to this game. To keep the analysis self-contained, we first briefly revisit the description of the game and the characterization of beliefs and payoffs that is in Section 5.5.

The game. A regime change occurs in period t if and only if $A_t \geq h_t$, where h_t follows a Gaussian random walk: $h_1 = \theta \sim N(z, 1/\alpha)$ and $h_t = h_{t-1} + \delta\omega_t$ for $t \geq 2$, with $\omega_t \sim N(0,1)$, i.i.d. across time and independent of θ. Once the status quo is abandoned, the game is over. As long as the status quo is in place, a new cohort of agents replaces the old one in each period; each cohort is of measure 1 and lives exactly one period. Agents who are born in period t must choose whether or not to attack the status quo, after receiving private signals $x_{it} = h_t + \xi_{it}$, where $\xi_{it} \sim \mathcal{N}(0, 1/\beta_t)$

is i.i.d. across agents and independent of h_s for any $s \neq t$. Payoffs are as in the benchmark model: the net payoff from attacking in period t is $1-c$ if the status quo is abandoned in that period and $-c$ otherwise, while the payoff from not attacking is zero.

Equilibrium characterization, multiplicity and dynamics. Let $\Psi_t^\delta\left(h_t, \bar{x}^{t-1}\right)$ denote the c.d.f. of the common posterior in period t about h_t, when agents in earlier cohorts attacked in periods $\tau \leq t-1$ if and only if $x_\tau < \bar{x}_\tau$. When earlier cohorts followed such strategies, the status quo survived period τ if and only if $h_\tau > \bar{\theta}_\tau\left(\bar{x}_\tau\right)$, where $\bar{\theta}_\tau\left(\bar{x}_\tau\right)$ is the solution to $\Phi\left(\sqrt{\beta_\tau}\left(\bar{x}_\tau - h_\tau\right)\right) = h_\tau$. Therefore, for $t \geq 2$, $\Psi_t^\delta\left(h_t; \bar{x}^{t-1}\right)$ is recursively defined by

$$\Psi_t^\delta\left(h_t; \bar{x}^{t-1}\right) = \frac{\int_{\bar{\theta}_{t-1}(\bar{x}_{t-1})}^{+\infty} \Phi\left(\frac{h_t - h_{t-1}}{\delta}\right) d\Psi_{t-1}^\delta\left(h_{t-1}; \bar{x}^{t-2}\right)}{1 - \Psi_{t-1}^\delta\left(\bar{\theta}_{t-1}\left(\bar{x}_{t-1}\right); \bar{x}^{t-2}\right)} \tag{A1}$$

with $\Psi_1^\delta\left(h_1\right) = \Phi\left(\sqrt{\alpha}\left(h_1 - z\right)\right)$. Next, let $\Psi_t^\delta\left(h_t | x; \bar{x}^{t-1}\right)$ denote the c.d.f. of private posterior about h_t, by Bayes' rule,

$$\Psi_t^\delta\left(h_t | x; \bar{x}^{t-1}\right) = \frac{\int_{-\infty}^{h_t} \sqrt{\beta_t}\phi\left(\sqrt{\beta_t}\left(x - h_t'\right)\right) d\Psi_t^\delta\left(h_t'; \bar{x}^{t-1}\right)}{\int_{-\infty}^{+\infty} \sqrt{\beta_t}\phi\left(\sqrt{\beta_t}\left(x - h_t'\right)\right) d\Psi_t^\delta\left(h_t'; \bar{x}^{t-1}\right)}. \tag{A2}$$

The expected net payoff from attacking in period t for an agent with signal x is thus given by $v_1^\delta\left(x; \bar{x}_1\right) = \Psi_1^\delta\left(\bar{\theta}_1\left(\bar{x}_1\right) | x\right) - c$ for $t = 1$ and

$$v_t^\delta\left(x; \bar{x}^t\right) = \Psi_t^\delta\left(\bar{\theta}_t\left(\bar{x}_t\right) | x; \bar{x}^{t-1}\right) - c$$

for $t \geq 2$. Finally, define the payoff of the marginal agent by

$$V_t^\delta\left(\bar{x}^t\right) \equiv \begin{cases} \lim_{x \to +\infty} v_t^\delta\left(x; \bar{x}^t\right) & \text{if } \bar{x}_t = +\infty \\ v_t^\delta\left(\bar{x}_t; \bar{x}^t\right) & \text{if } \bar{x}_t \in \mathbb{R} \\ \lim_{x \to -\infty} v_t^\delta\left(x; \bar{x}^t\right) & \text{if } \bar{x}_t = -\infty \end{cases} \tag{A3}$$

The following then provides the algorithm for characterizing monotone equilibria.

Proposition A4 *For any $\delta > 0$, $\{a_t(\cdot)\}_{t=1}^\infty$ is a monotone equilibrium for $\Gamma(\delta)$ if and only if there exists a sequence $\{x_t^*\}_{t=1}^\infty$ such that:*

(i) for all t, $a_t(\cdot) = 1$ if $x_t < x_t^*$ and $a_t(\cdot) = 0$ if $x_t > x_t^*$,

(ii) for $t = 1$, $x_1^* \in \mathbb{R}$ and $V_1^\delta\left(x_1^*\right) = 0$,

(iii) for any $t \geq 2$, either $x_t^* = -\infty$ and $V_t^\delta\left(x^{*t}\right) \leq 0$, or $x_t^* \in \mathbb{R}$ and $V_t^\delta\left(x^{*t}\right) = 0$.

An equilibrium exists for any $\delta > 0$.[2]

Finally, the next result establishes that essentially any equilibrium of the benchmark game can be approximated by an equilibrium of the random-walk game for δ small enough.

[2] Given a sequence of thresholds $\{x_t^*\}_{t=1}^\infty$ characterizing a monotone equilibrium, the sequence of thresholds $\{h_t^*\}_{t=1}^\infty$ characterizing the associated regime outcomes is simply given by $h_t^* = \bar{\theta}_t\left(x_t^*\right)$ for any $t \geq 1$.

Proposition A5 *For any $\varepsilon > 0$ and any $T < \infty$, there exists $\delta(\varepsilon, T) > 0$ such that the following is true for all $\delta < \delta(\varepsilon, T)$:*

For any equilibrium $\{x_t^*\}_{t=1}^{\infty}$ of $\Gamma(0)$ such that $x_t^* \notin \arg\max_x V_t^0\left(x^{*t-1}, x\right)$ for all $t \in \{2, ..., T\}$, there exists an equilibrium $\{x_t^\delta\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ such that, for all $t \leq T$, either $|x_t^* - x_t^\delta| < \varepsilon$, or $x_t^* = -\infty$ and $x_t^\delta < -1/\varepsilon$.

A5. Proofs

Proof of Proposition A1. To establish Proposition A1, we first prove the following weaker claim:

Result A1.a *For any $\varepsilon > 0$, any $T < \infty$, and any sequence $\{\theta_t^*\}_{t=1}^{T}$ that is part of an equilibrium of $\Gamma(0)$ and such that $\theta_t^* \notin \arg\max_{\theta^*} U\left(\theta^*, \theta_{t-1}^*, \beta_t, \alpha, z\right)$ for all $t \leq T$, there exists a $\hat{\delta} = \hat{\delta}\left(\varepsilon, T, \{\theta_t^*\}_{t=1}^{T}\right) > 0$ such that, whenever $\delta \leq \hat{\delta}$, there exists an equilibrium $\left\{\theta_t^\delta\left(\omega^t\right)\right\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ such that*

$$\Pr\left(\left|\theta_t^\delta\left(\omega^t\right) - \theta_t^*\right| \leq \varepsilon, \forall t \in \{1, ..., T\} \right) \geq 1 - \varepsilon. \tag{A4}$$

Given Result A1.a, the stronger result in the proposition then follows by letting $\delta(\varepsilon, T)$ be the minimum of $\hat{\delta}\left(\varepsilon, T, \{\theta_t^*\}_{t=1}^{T}\right)$ across all different sequences $\{\theta_t^*\}_{t=1}^{T}$ that can be part of an equilibrium of $\Gamma(0)$; that $\delta(\varepsilon, T) > 0$ is ensured by the fact that the set of such sequences is finite for any finite $T < \infty$.

To prove Result A1.a, we proceed in four steps, using an argument based on induction: step 1 shows that the result holds for $T = 1$; step 2 provides a sufficient condition for the result to hold for T conditional on holding for $T - 1$; steps 3 and 4 prove that this condition is satisfied both for the case where $\theta_T^* = \theta_{T-1}^*$ (step 3) and for the case where $\theta_T^* > \theta_{T-1}^*$ (step 4).

To simplify notation, let $\Omega \equiv \delta\omega$, and for any $t \geq 1$ and any $\left(\bar{\theta}_t, \bar{\theta}_{t-1}, \Omega\right)$ such that $\bar{\theta}_t \geq \bar{\theta}_{t-1}$ and $\Omega \in \left[-\bar{\theta}_t, 1 - \bar{\theta}_t\right]$, define $V_t\left(\bar{\theta}_t, \bar{\theta}_{t-1}, \Omega\right) \equiv U\left(\bar{\theta}_t + \Omega, \bar{\theta}_{t-1} + \Omega, \beta_t, \alpha, z + \Omega\right)$. Furthermore, for any $\varepsilon > 0$, $T < \infty$, and $\left\{\bar{\theta}_t\right\}_{t=1}^{T} \in \mathbb{R}^T$, let

$$B_{\varepsilon,T}\left(\left\{\bar{\theta}_t\right\}_{t=1}^{T}\right) \equiv \left\{\left\{\theta_t'\right\}_{t=1}^{T} \in \mathbb{R}^T : \left|\bar{\theta}_t - \theta_t'\right| \leq \varepsilon, \forall t = 1, ..., T\right\}.$$

Step 1. By Propositions 1 and 4, the (unique) first-period equilibrium threshold θ_1^* of $\Gamma(0)$ satisfies $V_1(\theta_1^*, -\infty, 0) = 0$, while the (also unique) first-period equilibrium threshold $\theta_1^\delta(\omega_1)$ of $\Gamma(\delta)$ satisfies $V_1\left(\theta_1^\delta(\omega_1), -\infty, \delta\omega_1\right) = 0$. Moreover, since $U\left(\bar{\theta}, -\infty, \beta_1, \alpha, z\right)$ is continuous and strictly decreasing in both $\bar{\theta}$ and z, $V_1\left(\bar{\theta}, -\infty, \Omega\right)$ is also continuous and strictly decreasing in both $\bar{\theta}$ and Ω. From the definition of V and of θ_1^*, we thus have that $V_1(\theta_1^* - \varepsilon, -\infty, 0) > 0 > V_1(\theta_1^* - \varepsilon, -\infty, \varepsilon)$. It follows that there exists $\bar{\Omega} \in (0, \varepsilon)$ such that $V_1\left(\theta_1^* - \varepsilon, -\infty, \bar{\Omega}\right) = 0$, implying that $\theta_1^\delta\left(\bar{\Omega}/\delta\right) = \theta_1^* - \varepsilon$. Likewise, $V_1(\theta_1^* + \varepsilon, -\infty, 0) < 0 < V_1(\theta_1^* + \varepsilon, -\infty, -\varepsilon)$ and hence there exists $\underline{\Omega} \in (-\varepsilon, 0)$ such that $V_1(\theta_1^* + \varepsilon, -\infty, \underline{\Omega}) = 0$, implying that $\theta_1^\delta(\underline{\Omega}/\delta) = \theta_1^* + \varepsilon$.

Since $V_1\left(\bar{\theta}_1, -\infty, \Omega\right)$ is continuous and strictly decreasing in both $\bar{\theta}_1$ and Ω, $\theta_1^\delta(\omega_1)$ is continuous and decreasing in ω_1. Hence $\theta_1^\delta(\omega_1) \in [\theta_1^* - \varepsilon, \theta_1^* + \varepsilon]$ if and only if $\omega_1 \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]$. There thus exists an equilibrium of $\Gamma(\delta)$ for which $\left|\theta_1^\delta(\omega_1) - \theta_1^*\right| \leq \varepsilon$ whenever $\omega_1 \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]$ and therefore $\Pr\left(\left|\theta_1^\delta(\omega_1) - \theta_1^*\right| \leq \varepsilon\right) = \Pr\left(\omega_1 \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]\right)$. Since $\underline{\Omega} < 0 < \bar{\Omega}$, $\Pr\left(\omega_1 \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]\right)$ is decreasing in δ and converges to 1 as $\delta \to 0$. It follows that there exists $\hat{\delta} > 0$ such that $\Pr\left(\omega_1 \in \left[\underline{\Omega}/\hat{\delta}, \bar{\Omega}/\hat{\delta}\right]\right) = 1 - \varepsilon$ and $\Pr\left(\left|\theta_1^\delta(\omega_1) - \theta_1^*\right| \leq \varepsilon\right) \geq 1 - \varepsilon$ for all $\delta \leq \hat{\delta}$, which proves the claim for $T = 1$.

Step 2. Suppose Result A1.a holds for $T-1$, with $T \geq 2$. This means that for any sequence $\{\theta_t^*\}_{t=1}^{T-1}$ that is part of an equilibrium of $\Gamma(0)$ and any $\varepsilon_1 \in (0, \varepsilon)$, there exists a $\hat{\delta}_{-1} = \hat{\delta}\left(\varepsilon_1, T-1, \{\theta_t^*\}_{t=1}^{T-1}\right)$ such that, for any $\delta \leq \hat{\delta}_{-1}$, there exists an equilibrium $\left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ such that

$$\Pr\left(\left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T-1} \in B_{\varepsilon_1, T-1}\left(\{\theta_t^*\}_{t=1}^{T-1}\right)\right) \geq 1 - \varepsilon_1. \tag{A5}$$

Now suppose further that we are able to prove that the following is true.

Result A1.b *For any $\varepsilon > 0$ and any sequence $\{\theta_t^*\}_{t=1}^{T}$ that is part of an equilibrium of $\Gamma(0)$, there exists an $\varepsilon_1 \in (0, \varepsilon)$ and a $\hat{\delta} \leq \hat{\delta}_{-1}$ such that for any $\delta \in (0, \hat{\delta})$, there exists an equilibrium of $\Gamma(\delta)$ that satisfies (A5) and such that, for any ω^{T-1} for which $\left|\theta_{T-1}^\delta(\omega^{T-1}) - \theta_{T-1}^*\right| \leq \varepsilon_1$,*

$$\Pr(\left|\theta_T^\delta(\omega^T) - \theta_T^*\right| \leq \varepsilon | \omega^{T-1}) \geq 1 - \varepsilon + \varepsilon_1.$$

If Result A1.b is true, then,

$$\Pr\left(\left|\theta_T^\delta(\omega^T) - \theta_T^*\right| \leq \varepsilon \mid \left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T-1} \in B_{\varepsilon_1, T-1}\left(\{\theta_t^*\}_{t=1}^{T-1}\right)\right) \geq 1 - \varepsilon + \varepsilon_1. \tag{A6}$$

But then,

$$\begin{aligned}
&\Pr\left(\left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T} \in B_{\varepsilon, T}\left(\{\theta_t^*\}_{t=1}^{T}\right)\right) \geq \\
&\geq \Pr\left(\left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T-1} \in B_{\varepsilon_1, T-1}\left(\{\theta_t^*\}_{t=1}^{T-1}\right) \text{ and } \left|\theta_T^\delta(\omega^T) - \theta_T^*\right| \leq \varepsilon\right) \\
&= \Pr\left(\left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T-1} \in B_{\varepsilon_1, T-1}\left(\{\theta_t^*\}_{t=1}^{T-1}\right)\right) \\
&\qquad \cdot \Pr\left(\left|\theta_T^\delta(\omega^T) - \theta_T^*\right| \leq \varepsilon \mid \left\{\theta_t^\delta(\omega^t)\right\}_{t=1}^{T-1} \in B_{\varepsilon_1, T-1}\left(\{\theta_t^*\}_{t=1}^{T-1}\right)\right) \\
&\geq (1 - \varepsilon_1)(1 - \varepsilon + \varepsilon_1) \\
&> 1 - \varepsilon,
\end{aligned}$$

implying that Result A1.a holds also for T.

To complete the proof of Result A1.a, it thus suffices to show that Result A1.b holds. We do so by proving the following:

Result A1.c *There exist scalars $\varepsilon_1 \in (0, \varepsilon)$, $\underline{\Omega} < 0 < \bar{\Omega}$, $\tilde{\delta} > 0$, and a function $\hat{\theta}_T : \mathbb{R}^2 \to \mathbb{R}$ such that the following hold:*

(i) for any $\bar{\theta}_{T-1} \in \left[\theta_{T-1}^* - \varepsilon_1, \theta_{T-1}^* + \varepsilon_1\right]$ and any $\Omega \in [\underline{\Omega}, \bar{\Omega}]$, either $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) = \bar{\theta}_{T-1} \geq -\Omega$,

or $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1}) > \bar{\theta}_{T-1}$ *and* $V_T\left(\hat{\theta}_T(\Omega, \bar{\theta}_{T-1}), \bar{\theta}_{T-1}, \Omega\right) = 0$*;*
(ii) for any $\bar{\theta}_{T-1} \in \left[\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1\right]$ *and any* $\delta < \tilde{\delta}$, $\Pr\left(\left|\hat{\theta}_T(\delta\omega_T, \bar{\theta}_{T-1}) - \theta^*_T\right| \leq \varepsilon\right) \geq 1 - \varepsilon + \varepsilon_1$.

We prove Result A1.c in the next two steps, distinguishing the case where $\theta^*_T = \theta^*_{T-1}$ (step 3) and where $\theta^*_T > \theta^*_{T-1}$ (step 4). Result A1.b then follows from Result A1.c by letting $\hat{\delta} = \min\{\tilde{\delta}, \hat{\delta}\left(\varepsilon_1, T-1, \{\theta^*_t\}_{t=1}^{T-1}\right)\}$ and $\left\{\theta^\delta_t(\omega^t)\right\}_{t=1}^{\infty}$ be the equilibrium of $\Gamma(\delta)$ whose sequence of thresholds coincides with that of the equilibrium that satisfies (A5) for $t \leq T-1$ together with $\theta^\delta_T(\omega^T) = \hat{\theta}_T\left(\delta\omega_T, \theta^\delta_T(\omega^{T-1})\right)$ for any ω^{T-1} such that $\theta^\delta_T(\omega^{T-1}) \in \left[\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1\right]$.

Step 3. Suppose that $\theta^*_T = \theta^*_{T-1}$, and pick any $\varepsilon_1 \in (0, \varepsilon)$ such that $\theta^*_{T-1} - \varepsilon_1 > 0$. Then, for any $\bar{\theta}_{T-1} \in \left[\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1\right]$, let $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1}) = \bar{\theta}_{T-1}$ if $\Omega \geq -\bar{\theta}_{T-1}$ and otherwise let $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1})$ be the highest solution to $V_T\left(\hat{\theta}_T, \bar{\theta}_{T-1}, \Omega\right) = 0$. Clearly, $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1})$ satisfies part (i) of Result A1.c for any δ. To see when part (ii) is also satisfied, note that, for any $\bar{\theta}_{T-1} \in \left[\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1\right]$,

$$\begin{aligned}\Pr\left(\left|\hat{\theta}_T(\delta\omega_T, \bar{\theta}_{T-1}) - \theta^*_T\right| \leq \varepsilon\right) &\geq \Pr\left(\hat{\theta}_T(\delta\omega_T, \bar{\theta}_{T-1}) = \bar{\theta}_{T-1}\right) \\ &= \Pr\left(\delta\omega_T \geq -\bar{\theta}_{T-1}\right) \\ &\geq \Pr\left(\omega_T \geq -\left(\theta^*_{T-1} - \varepsilon_1\right)/\delta\right).\end{aligned}$$

Since $\bar{\theta}_{T-1} \geq \theta^*_{T-1} - \varepsilon_1 > 0$, $\Pr\left(\omega_T \geq -\left(\theta^*_{T-1} - \varepsilon_1\right)/\delta\right)$ is strictly decreasing in δ and converges to 1 as $\delta \to 0$. It follows that there exists $\tilde{\delta} > 0$ such that $\Pr\left(\omega_T \geq -\left(\theta^*_{T-1} - \varepsilon_1\right)/\tilde{\delta}\right) = 1 - \varepsilon + \varepsilon_1$, implying part (ii) is satisfied for all $\delta \leq \tilde{\delta}$. Hence, Result A1.c is satisfied for the case $\theta^*_T = \theta^*_{T-1}$.

Step 4. Next assume that $\theta^*_T > \theta^*_{T-1}$, in which case θ^*_T solves $V_T\left(\theta^*_T, \theta^*_{T-1}, 0\right) = 0$. Suppose further that $V_T(\theta_T, \theta_{T-1}, \Omega)$ is strictly decreasing in θ_T in a neighborhood of $(\theta_T, \theta_{T-1}, \Omega) = \left(\theta^*_T, \theta^*_{T-1}, 0\right)$ (An analogous argument applies if V_T is strictly increasing in such a neighborhood, whereas the case that V_T is locally non-monotonic is ruled out by the non-tangency assumption). Then, by the Implicit Function Theorem, there exists $\varepsilon' \in (0, \varepsilon]$, $\underline{\Omega}' < 0 < \bar{\Omega}'$ and a function $\hat{\theta}_T : \mathbb{R}^2 \to \mathbb{R}$ such that $V_T\left(\hat{\theta}_T(\Omega, \bar{\theta}_{T-1}), \bar{\theta}_{T-1}, \Omega\right) = 0$ for any $(\Omega, \bar{\theta}_{T-1}) \in [\underline{\Omega}', \bar{\Omega}'] \times [\theta^*_{T-1} - \varepsilon', \theta^*_{T-1} + \varepsilon']$.

Clearly, the function $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1})$ satisfies part (i) of Result A1.c by construction. To see when it also satisfies part (ii), note that, by the continuity of V_T, $\hat{\theta}_T$ is also continuous, and hence there exist $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon']$, $\underline{\Omega} \in [\underline{\Omega}', 0)$, and $\bar{\Omega} \in (0, \bar{\Omega}']$, such that $\hat{\theta}_T(\Omega, \bar{\theta}_{T-1}) \in [\theta^*_T - \varepsilon_2, \theta^*_T + \varepsilon_2]$ whenever $(\Omega, \bar{\theta}_{T-1}) \in [\underline{\Omega}, \bar{\Omega}] \times [\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1]$. Since $\varepsilon_2 \leq \varepsilon' \leq \varepsilon$, it follows that, whenever $\bar{\theta}_{T-1} \in [\theta^*_{T-1} - \varepsilon_1, \theta^*_{T-1} + \varepsilon_1]$,

$$\begin{aligned}\Pr\left(\left|\hat{\theta}_T(\delta\omega_T, \bar{\theta}_{T-1}) - \theta^*_T\right| \leq \varepsilon\right) &\geq \Pr\left(\left|\hat{\theta}_T(\delta\omega_T, \bar{\theta}_{T-1}) - \theta^*_T\right| \leq \varepsilon_2\right) \geq \\ &\geq \Pr\left(\omega_T \in \left[\underline{\Omega}/\delta, \overline{\Omega}/\delta\right]\right).\end{aligned}$$

Since in turn $\Pr\left(\omega_T \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]\right)$ is decreasing in δ and converges to 1 as $\delta \to 0$, there exists $\tilde{\delta} > 0$ such that $\Pr\left(\omega_T \in \left[\underline{\Omega}/\delta, \bar{\Omega}/\delta\right]\right) \geq 1 - \varepsilon + \varepsilon_1$ for all $\delta \leq \tilde{\delta}$, which establishes part (ii) of Result A1.c. ■

Proof of Proposition A2. The result follows from exactly the same arguments as the proof of Proposition 5 in the main text, after adjusting the notation for beliefs. (Note that, unlike in the case of Proposition 5, here there is no need to prove convergence of beliefs: the belief updating induced by any given monotone strategy is identical to that in the benchmark model.) ■

Proof of Proposition A3. As in the proof of Proposition A1, it suffices to prove the weaker claim in Result A1.a. For this purpose, Step 1 below first establishes pointwise convergence of $V_t^{\delta}\left(\bar{x}_t, \bar{\theta}_{t-1}\right)$ to $V_t^{0}\left(\bar{x}_t, \bar{\theta}_{t-1}\right) \equiv U\left(\bar{\theta}_t\left(\bar{x}_t\right), \bar{\theta}_{t-1}, \beta_t, \alpha, z\right)$ as $\delta \to 0$, where $\bar{\theta}_t\left(\bar{x}_t\right) \equiv \bar{\theta}_t\left(\bar{x}_t, 0\right)$. Steps 2-5 then use this property to prove the result with an induction argument similar to the one in the proof of Proposition A1.

Step 1. The proof that V_t^{δ} converges pointwise to V_t^{0} as $\delta \to 0$ is similar to Step 1 in the proof of Theorem 3 in the paper; it is actually simplified by the fact that the equilibrium updating of beliefs here is identical to that in the benchmark model and hence follows directly from the convergence of regime outcomes. Indeed, for any $t \geq 1$ any $\bar{x}_t \in \overline{\mathbb{R}}$ and any $\theta \neq \bar{\theta}_t\left(\bar{x}_t\right)$,

$$\lim_{\delta \to 0} p_t^{\delta}\left(\theta; \bar{x}_t\right) = p_t^{0}\left(\theta; \bar{x}_t\right) \equiv \begin{cases} 1 & \text{if } \theta \leq \bar{\theta}_t\left(\bar{x}_t\right), \\ 0 & \text{if } \theta > \bar{\theta}_t\left(\bar{x}_t\right). \end{cases}$$

This immediately implies that, for any t, any $\bar{x}_t \in \mathbb{R}$, and any $\bar{\theta}_{t-1} \in \overline{\mathbb{R}}$,

$$\begin{aligned} \lim_{\delta \to 0} V_t^{\delta}\left(\bar{x}_t, \bar{\theta}_{t-1}\right) &= \lim_{\delta \to 0} \int_{-\infty}^{+\infty} p_t^{\delta}\left(\theta; \bar{x}_t\right) d\Psi_t\left(\theta | \bar{x}_t, \bar{\theta}_{t-1}\right) - c \\ &= \Psi_t\left(\bar{\theta}_t\left(\bar{x}_t\right) | \bar{x}_t, \bar{\theta}_{t-1}\right) - c \\ &= U\left(\bar{\theta}_t\left(\bar{x}_t\right), \bar{\theta}_{t-1}, \beta_t, \alpha, z\right) \equiv V_t^{0}\left(\bar{x}_t, \bar{\theta}_{t-1}\right). \end{aligned}$$

Step 2. Here we show that Result A1.a holds for $T = 1$. Fix $\varepsilon > 0$. In period 1, the game in which shocks are observable with a lag is isomorphic to the game in which shocks are never observable. Therefore, for any $\eta > 0$, step 2 of Theorem 3 in the paper implies immediately that there exists $\delta'(\eta) > 0$, such that for all $\delta \leq \delta'(\eta)$ there exists an equilibrium $\{x_t^{\delta}\left(\omega^{t-1}\right), \theta_t^{\delta}\left(\omega^{t}\right)\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ such that $\left|x_1^{\delta} - x_1^{*}\right| \leq \eta$. Moreover, since $\theta_1^{\delta}\left(\omega_1\right) = \bar{\theta}_1\left(x_1^{\delta}, \delta\omega_1\right)$, define $\bar{\Omega}\left(\bar{x}_1\right)$ and $\underline{\Omega}\left(\bar{x}_1\right)$ implicitly by $\bar{\theta}_1\left(\bar{x}_1, \bar{\Omega}\right) = \theta_1^{*} - \varepsilon$ and $\bar{\theta}_1\left(\bar{x}_1, \underline{\Omega}\right) = \theta_1^{*} + \varepsilon$. Therefore, $\Pr\left(\theta_1^{\delta}\left(\omega_1\right) \in \left[\theta_1^{*} - \varepsilon, \theta_1^{*} + \varepsilon\right]\right) = \Pr\left(\omega_1 \in \left[\underline{\Omega}\left(x_1^{\delta}\right)/\delta, \bar{\Omega}\left(x_1^{\delta}\right)/\delta\right]\right)$. Clearly, $\bar{\Omega}\left(x_1^{*}\right) > 0 > \underline{\Omega}\left(x_1^{*}\right)$, and by continuity, there exists $\eta_1 \in (0, \varepsilon]$ such that $\bar{\Omega}\left(x_1^{\delta}\right) > 0 > \underline{\Omega}\left(x_1^{\delta}\right)$ for any $x_1^{\delta} \in \left[x_1^{*} - \eta_1, x_1^{*} + \eta_1\right]$. Since $\Pr\left(\omega_1 \in \left[\underline{\Omega}\left(x_1^{\delta}\right)/\delta, \bar{\Omega}\left(x_1^{\delta}\right)/\delta\right]\right)$ is decreasing in δ and converges to 1 as $\delta \to 0$, there exists $\delta'' > 0$ such that $\Pr\left(\omega_1 \in \left[\underline{\Omega}\left(x_1^{\delta}\right)/\delta, \bar{\Omega}\left(x_1^{\delta}\right)/\delta\right]\right) \geq 1 - \varepsilon$ for all $\delta \leq \delta''$. We conclude that Result A1.a holds for $T = 1$ with $\hat{\delta}(\varepsilon, 1) = \min\{\delta'', \delta'(\eta_1)\}$.

Step 3. Along the same lines as in step 2 in the proof of Proposition A1, we now establish a sufficient condition for Result A1.a to hold for T periods when it holds for $T-1$ periods. In particular, fix an $\varepsilon > 0$, an $\varepsilon_1 \in (0, \varepsilon)$, a $T \geq 2$, and a sequence $\{\theta_t^{*}\}_{t=1}^{T}$ that is part of an equilibrium of $\Gamma(0)$, and suppose that there exists a $\hat{\delta}_{-1} = \hat{\delta}\left(\varepsilon_1, T-1, \{\theta_t^{*}\}_{t=1}^{T-1}\right) > 0$ such that, whenever $\delta \leq \hat{\delta}_{-1}$, there exists an equilibrium $\left\{x_t^\delta\left(\omega^{t-1}\right), \theta_t^\delta\left(\omega^t\right)\right\}_{t=1}^{\infty}$ of $\Gamma(\delta)$ that satisfies the result for $T-1$ and ε_1. Suppose further that we are able to prove the following:

Result A2.c *There exist scalars $\varepsilon_1 \in (0, \varepsilon)$, $\tilde{\delta} > 0$, such that, for any $\delta < \tilde{\delta}$, there exists a function $\hat{x}_T : \mathbb{R} \rightarrow \mathbb{R}$ that satisfied the following:*
(i) for any $\bar{\theta}_{T-1} \in \left[\theta_{T-1}^* - \varepsilon_1, \theta_{T-1}^* + \varepsilon_1\right]$, either $\hat{x}_T\left(\bar{\theta}_{T-1}\right) = -\infty$ and $V\left(-\infty, \bar{\theta}_{T-1}\right) \leq 0$, or $\hat{x}_T\left(\bar{\theta}_{T-1}\right) > -\infty$ and $V\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \bar{\theta}_{T-1}\right) = 0$;
(ii) for any $\bar{\theta}_{T-1} \in \left[\theta_{T-1}^* - \varepsilon_1, \theta_{T-1}^* + \varepsilon_1\right]$, $\Pr\left(\left|\hat{\theta}_T\left(\delta\omega_T, \bar{\theta}_{T-1}\right) - \theta_T^*\right| \leq \varepsilon\right) \geq 1 - \varepsilon + \varepsilon_1$, where $\hat{\theta}_T$ is defined by $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) \equiv \max\left\{\bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right), \bar{\theta}_{T-1}\right\}$.

Then, for any $\delta < \min\{\hat{\delta}_{-1}, \tilde{\delta}\}$, there exists an equilibrium of $\Gamma(\delta)$ that satisfies the result for $T-1$ and ε_1 and for which $x_T^\delta\left(\omega^{T-1}\right) = \hat{x}_T\left(\theta_{T-1}^\delta\left(\omega^{T-1}\right)\right)$ and $\theta_T^\delta\left(\omega^T\right) = \hat{\theta}_T\left(\delta\omega_T, \theta_{T-1}^\delta\left(\omega^{T-1}\right)\right)$ when ω^{T-1} is such that $\theta_{T-1}^\delta\left(\omega^{T-1}\right) \in \left[\theta_{T-1}^* - \varepsilon_1, \theta_{T-1}^* + \varepsilon_1\right]$. But then, by the same argument as in Step 2 of the proof of Proposition A1, this equilibrium satisfies

$$\Pr\left(\left\{\theta_t^\delta\left(\omega^t\right)\right\}_{t=1}^T \in B_{\varepsilon,T}\left(\{\theta_t^*\}_{t=1}^T\right)\right) \geq 1 - \varepsilon,$$

proving that the result holds for T with $\hat{\delta} = \min\{\hat{\delta}_{-1}, \tilde{\delta}\}$. In the next two steps, we thus prove Result A2.c, distinguishing again between the case where $\theta_T^* = \theta_{T-1}^*$ (Step 4) and the case where $\theta_T^* > \theta_{T-1}^*$ (Step 5).

Step 4. Suppose that $\theta_T^* = \theta_{T-1}^*$, and fix $\varepsilon > 0$. For all x,

$$\begin{aligned} v_T^\delta\left(x, -\infty, \bar{\theta}_{t-1}\right) &= \int_{-\infty}^{+\infty} F\left(\bar{\omega}_T^\delta\left(\theta; -\infty\right)\right) d\Psi_T\left(\theta | x, \bar{\theta}_{T-1}\right) - c \\ &= \int_{-\infty}^{+\infty} F\left(-\theta/\delta\right) d\Psi_T\left(\theta | x, \bar{\theta}_{T-1}\right) - c \\ &\leq F\left(-\bar{\theta}_{T-1}/\delta\right) - c, \end{aligned}$$

and therefore $V_T^\delta\left(-\infty, \bar{\theta}_{T-1}\right) \leq F\left(-\bar{\theta}_{T-1}/\delta\right) - c$. Now, select $\varepsilon_1 \in (0, \varepsilon)$ and $\tilde{\delta}_1 > 0$ such that $\theta_{T-1}^* > \varepsilon_1$ and $F(-\left(\theta_{T-1}^* - \varepsilon_1\right)/\tilde{\delta}_1) - c \leq 0$. Whenever $\delta \leq \tilde{\delta}_1$ and $\left|\bar{\theta}_{T-1} - \theta_{T-1}^*\right| \leq \varepsilon_1$,

$$V_T^\delta\left(-\infty, \bar{\theta}_{T-1}\right) \leq F\left(-\bar{\theta}_{T-1}/\delta\right) - c \leq F(-\left(\theta_{T-1}^* - \varepsilon_1\right)/\tilde{\delta}_1) - c \leq 0.$$

Therefore, whenever $\delta \leq \tilde{\delta}_1$, $\hat{x}_T\left(\bar{\theta}_{T-1}\right) = -\infty$ satisfies part *(i)* of Result A2.c, in which case $\bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right) = -\Omega$ and hence $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) = \max\left\{-\Omega, \bar{\theta}_{T-1}\right\}$. To check that part *(ii)* is also satisfied, notice that $\Omega > -\bar{\theta}_{T-1}$ implies $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) = \bar{\theta}_{T-1}$, and hence $\left|\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) - \theta_T^*\right| \leq \varepsilon_1 < \varepsilon$. Therefore,

$$\begin{aligned} \Pr\left(\left|\hat{\theta}_T\left(\delta\omega_T, \bar{\theta}_{T-1}\right) - \theta_T^*\right| < \varepsilon\right) &\geq \Pr\left(\delta\omega_T > -\bar{\theta}_{T-1}\right) \\ &= 1 - F\left(-\bar{\theta}_{T-1}/\delta\right) \geq 1 - F\left(-\left(\theta_{T-1}^* - \varepsilon_1\right)/\delta\right) \end{aligned}$$

for any $\bar{\theta}_{T-1} \in \left[\theta_{T-1}^* - \varepsilon_1, \theta_{T-1}^* + \varepsilon_1\right]$. Since $\theta_{T-1}^* > \varepsilon_1$, there exists $\tilde{\delta}_2 > 0$, such that $1 - F\left(-\left(\theta_{T-1}^* - \varepsilon_1\right)/\delta\right) \geq 1 - \varepsilon + \varepsilon_1$ for all $\delta \leq \tilde{\delta}_2$. Hence, Result A2.c is satisfied whenever $\delta \leq \tilde{\delta} \equiv \min\{\tilde{\delta}_1, \tilde{\delta}_2\}$.

Step 5. Suppose now that $\theta_T^* > \theta_{T-1}^*$, and fix $\varepsilon > 0$ and $\varepsilon' \in (0, \varepsilon]$ such that $\theta_{T-1}^* + \varepsilon' < \theta_T^* - \varepsilon'$. Suppose further that V_t^0 is locally decreasing in $\bar{x}_T$ at $\bar{x}_T = x_T^*$ and fix $\eta_1 > 0$ such that $V_t^0\left(x_T^* - \eta, \theta_{T-1}^*\right) > 0 > V_t^0\left(x_T^* + \eta, \theta_{T-1}^*\right)$ for all $\eta \leq \eta_1$ (An analogous argument applies if V_T^0 is locally increasing, while tangency is ruled out by assumption).

>From the pointwise convergence of V_T^δ to V_t^0, for any $\eta \in (0, \eta_1]$, there exists $\delta_1(\eta) > 0$, such that, whenever $\delta \leq \delta_1(\eta)$, $V_T^\delta\left(x_T^* - \eta, \theta_{T-1}^*\right) > 0 > V_T^\delta\left(x_T^* + \eta, \theta_{T-1}^*\right)$. By continuity with respect to θ_{T-1}^*, there also exists $\varepsilon_1(\eta) \in (0, \varepsilon')$, such that

$$V_T^\delta\left(x_T^* - \eta, \bar{\theta}_{T-1}\right) > \frac{1}{2} V_T^\delta\left(x_T^* - \eta, \theta_{T-1}^*\right) > 0 > \frac{1}{2} V_T^\delta\left(x_T^* + \eta, \theta_{T-1}^*\right) > V_T^\delta\left(x_T^* + \eta, \bar{\theta}_{T-1}\right)$$

for all $\bar{\theta}_{T-1}$ such that $\left|\bar{\theta}_{T-1} - \theta_{T-1}^*\right| \leq \varepsilon_1(\eta)$ and all $\delta \leq \delta_1(\eta)$. Therefore, whenever $\delta \leq \delta_1(\eta)$, there exists $\hat{x}_T\left(\bar{\theta}_{T-1}\right) \in [x_T^* - \eta, x_T^* + \eta]$ such that $V_T^\delta\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \bar{\theta}_{T-1}\right) = 0$, in which case part (i) of Result A2.c is satisfied for $\varepsilon_1(\eta)$ and $\tilde{\delta} \leq \delta_1(\eta)$, for any $\eta \leq \eta_1$.

To check when part (ii) is also satisfied, note that $\bar{\theta}_T(\bar{x}_T, \Omega) \in [\theta_T^* - \varepsilon', \theta_T^* + \varepsilon']$ if and only if $\Omega \in [\Omega_T(\theta_T^* + \varepsilon', \bar{x}_T), \Omega_T(\theta_T^* - \varepsilon', \bar{x}_T)]$, where $\Omega_T(\theta, \bar{x}) \equiv \Phi\left(\sqrt{\beta_T}(\theta - \bar{x})\right) - \theta$. Whenever $\bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right) \in [\theta_T^* - \varepsilon', \theta_T^* + \varepsilon']$, $\bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right) > \bar{\theta}_{T-1}$ and therefore $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) = \max\left\{\bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right), \bar{\theta}_{T-1}\right\} = \bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right)$. Moreover, $\bar{\theta}_T(\bar{x}_T, \Omega) \in [\theta_T^* - \varepsilon, \theta_T^* + \varepsilon]$ for all $\bar{x}_T \in [x_T^* - \eta, x_T^* + \eta]$, and therefore $\hat{\theta}_T\left(\Omega, \bar{\theta}_{T-1}\right) = \bar{\theta}_T\left(\hat{x}_T\left(\bar{\theta}_{T-1}\right), \Omega\right) \in [\theta_T^* - \varepsilon, \theta_T^* + \varepsilon]$, whenever $\Omega \in [\Omega_T(\theta_T^* + \varepsilon, x_T^* + \eta), \Omega_T(\theta_T^* - \varepsilon, x_T^* - \eta)]$. We conclude that

$$\begin{aligned}\Pr\left(\left|\hat{\theta}_T\left(\delta\omega_T, \bar{\theta}_{T-1}\right) - \theta_T^*\right| \leq \varepsilon\right) &\geq \Pr\left(\left|\hat{\theta}_T\left(\delta\omega_T, \bar{\theta}_{T-1}\right) - \theta_T^*\right| \leq \varepsilon'\right) \\ &\geq \Pr\left(\delta\omega_T \in \left[\Omega_T\left(\theta_T^* + \varepsilon', x_T^* + \eta\right), \Omega_T\left(\theta_T^* - \varepsilon', x_T^* - \eta\right)\right]\right)\end{aligned}$$

for all $\bar{\theta}_{T-1}$ such that $\left|\bar{\theta}_{T-1} - \theta_{T-1}^*\right| \leq \varepsilon_1(\eta)$. Since $\Omega_T(\theta_T^*, x_T^*) = 0$ and $\Omega_T(\theta_T^* - \varepsilon, x_T^*) > 0 > \Omega_T(\theta_T^* + \varepsilon, x_T^*)$, we have $\Omega_T(\theta_T^* - \varepsilon, x_T^* - \eta_2) > 0 > \Omega_T(\theta_T^* + \varepsilon, x_T^* + \eta_2)$ for some $\eta_2 \in (0, \eta_1]$, and there exists $\delta_2(\eta_2) > 0$ such that, for all $\delta \leq \delta_2(\eta_2)$,

$$\Pr\left(\omega_T \in [\Omega_T(\theta_T^* + \varepsilon, x_T^* + \eta_2)/\delta, \Omega_T(\theta_T^* - \varepsilon, x_T^* - \eta_2)/\delta]\right) \geq 1 - \varepsilon + \varepsilon_1(\eta).$$

Therefore, part (ii) of Result A2.c is satisfied with $\varepsilon_1 = \varepsilon_1(\eta_2)$ and $\tilde{\delta} = \min\{\delta_1(\eta_2), \delta_2(\eta_2)\}$. ■

Proof of Proposition A4. Since first-period beliefs are identical to those in the benchmark game, $V_1^\delta(\bar{x}_1) = V_1^0(\bar{x}_1)$ for all $\bar{x}_1 \in \mathbb{R}$, and therefore $x_1^* = \hat{x}_1$ and $h_1^* = \hat{\theta}_1$, where $(\hat{x}_1, \hat{\theta}_1)$ denote the first-period equilibrium thresholds of the benchmark game. The rest of the proof then follows from the same arguments as the proof of Proposition 5 and Lemma A2. In particular, to see that $V_t^\delta\left(\bar{x}^{t-1}, +\infty\right) = -c < 0$ for all $\bar{x}^{t-1} \in \overline{\mathbb{R}}^{t-1}$ (which rules out equilibria in which $x_t^* = +\infty$), notice that for $\bar{x}_t = +\infty$, and for any $x > 1$,

$$\Psi_t^\delta\left(\bar{h}_t(\bar{x}_t)|x, \bar{x}^{t-1}\right) = \Psi_t^\delta\left(1|x, \bar{x}^{t-1}\right) \leq \frac{\Psi_t^\delta\left(1, \bar{x}^{t-1}\right)}{\Psi_t^\delta\left(1, \bar{x}^{t-1}\right) + \int_1^{+\infty} \frac{\phi\left(\sqrt{\beta_t}(x - h_t)\right)}{\phi\left(\sqrt{\beta_t}(x - 1)\right)} d\Psi_t^\delta\left(h_t|\bar{x}^{t-1}\right)};$$

as $x \to \infty$, $\frac{\phi\left(\sqrt{\beta_t}(x-h_t)\right)}{\phi\left(\sqrt{\beta_t}(x-1)\right)} \to \infty$ whenever $h_t > 1$, and therefore $\lim_{x\to\infty} \Psi_t^\delta \left(1|x, \bar{x}^{t-1}\right) = 0$. ■

Proof of Proposition A5. Below we establish that, as $\delta \to 0$, beliefs and hence payoffs in the game with short-lived agents converge pointwise to those in the benchmark model. Given the converge of payoffs, the result then follows from the same arguments as in Steps 2-4 in the proof of Theorem 3.

Pointwise convergence of posteriors and payoffs. Consider first beliefs. Let $\Psi_t^0 \left(h_t; \bar{x}^{t-1}\right)$ denote the period-t common posterior about h_t in the benchmark model, and $\Psi_t^\delta \left(h_t; \bar{x}^{t-1}\right)$ the period-t common posterior about h_t in the game with changing fundamentals and short-lived agents. The former are simply given by the truncated Normals,

$$\Psi_t^0 \left(h_t; \bar{x}^{t-1}\right) = 1 - \frac{\Phi\left(\sqrt{\alpha}\left(z - h_t\right)\right)}{\Phi\left(\sqrt{\alpha}(z - \bar{\theta}_t\left(\bar{x}^{t-1}\right))\right)},$$

while the latter are defined by (A1) (Recall that $\bar{\theta}_t\left(\bar{x}^{t-1}\right) \equiv \min\{\theta : \theta \geq \Phi\left(\sqrt{\beta_\tau}\left(\bar{x}_\tau - \theta\right)\right) \forall \tau \leq t\} = \max_{\tau \leq t} \left\{\bar{\theta}_\tau\left(\bar{x}_\tau\right)\right\}$). By Bayes' rule, the corresponding private posteriors satisfy

$$\Psi_t^0 \left(h_t | x; \bar{x}^{t-1}\right) = \frac{\int_{-\infty}^{h_t} \sqrt{\beta_t}\phi\left(\sqrt{\beta_t}\left(x - h'\right)\right) d\Psi_t^0\left(h'; \bar{x}^{t-1}\right)}{\int_{-\infty}^{+\infty} \sqrt{\beta_t}\phi\left(\sqrt{\beta_t}\left(x - h'\right)\right) d\Psi_t^0\left(h'; \bar{x}^{t-1}\right)}$$

for the benchmark model, and similarly (replacing Ψ_t^0 with Ψ_t^δ) for the game with for the game with changing fundamentals (Clearly, the above definitions and conditions apply to $t \geq 2$; similar ones hold for $t = 1$).

To prove pointwise convergence of private posteriors, it thus suffices to prove pointwise convergence of the common posteriors. We establish this by induction. Clearly, since period 1 is identical in the two games,

$$\Psi_1^\delta\left(h_1\right) = 1 - \Phi\left(\sqrt{\alpha}\left(z - h_1\right)\right) = \Psi_1^0\left(h_1\right)$$

for any h_1. Next, consider any $t \geq 2$ and suppose that pointwise convergence holds at $t-1$. By the induction hypothesis,

$$\lim_{\delta\to 0} \Psi_{t-1}^\delta\left(h_t; \bar{x}^{t-2}\right) = \Psi_{t-1}^0\left(h_t; \bar{x}^{t-2}\right) = \begin{cases} 0 & \text{if } h_t \leq \bar{\theta}_{t-2}(\bar{x}^{t-2}) \\ 1 - \dfrac{\Phi\left(\sqrt{\alpha}\left(z - h_t\right)\right)}{\Phi\left(\sqrt{\alpha}\left(z - \bar{\theta}_{t-2}(\bar{x}^{t-2})\right)\right)} > 0 & \text{if } h_t > \bar{\theta}_{t-2}(\bar{x}^{t-2}) \end{cases}$$

for all h_t and $\bar{x}^{t-2}$. Using the above together the fact that $\lim_{\delta\to 0}\Phi\left(\left(h_t - h_{t-1}\right)/\delta\right) = 1$ whenever $h_{t-1} < h_t$ and $\lim_{\delta\to 0}\Phi\left(\left(h_t - h_{t-1}\right)/\delta\right) = 0$ whenever $h_{t-1} > h_t$, condition (A1) gives

$$\begin{aligned} \lim_{\delta\to 0} \Psi_t^\delta\left(h_t; \bar{x}^{t-1}\right) &= \begin{cases} 0 & \text{if } h_t \leq \bar{\theta}_{t-1}(\bar{x}_{t-1}) \\ \dfrac{\int_{\bar{\theta}_{t-1}(\bar{x}_{t-1})}^{h_t} d\Psi_{t-1}^0\left(h_{t-1}; \bar{x}^{t-2}\right)}{1 - \Psi_{t-1}^0\left(\bar{\theta}_{t-1}\left(\bar{x}_{t-1}\right); \bar{x}^{t-2}\right)} & \text{if } h_t > \bar{\theta}_{t-1}(\bar{x}_{t-1}) \end{cases} \\ &= \Psi_t^0\left(h_t, \bar{x}^{t-1}\right), \end{aligned}$$

for all h_t and $\bar{x}^{t-1}$, which proves the pointwise converge of posteriors in period t.

Next, consider payoffs. In the benchmark model, first-period payoffs satisfy

$$V_1^0(\bar{x}_1) = U\left(\bar{\theta}_1(\bar{x}_1), -\infty, \beta_1, \alpha, z\right) = \Psi_1^0\left(\bar{\theta}_1(\bar{x}_1)|\bar{x}_1\right) - c \quad \forall \bar{x}_1 \in \mathbb{R},$$

whereas for any $t \geq 2$,

$$V_t^0\left(\bar{x}^{t-1}, \bar{x}_t\right) = U\left(\bar{\theta}_t(\bar{x}_t), \bar{\theta}_{t-1}\left(\bar{x}^{t-1}\right), \beta_t, \alpha, z\right) = \Psi_t^0\left(\bar{\theta}_t(\bar{x}_t)|\bar{x}_t; \bar{x}^{t-1}\right) - c \quad \forall \bar{x}_t \in \mathbb{R}, \bar{x}^{t-1} \in \overline{\mathbb{R}}^{t-1}.$$

In the game with changing fundamentals, first-period beliefs are identical to those in the benchmark game, and therefore

$$V_1^\delta(\bar{x}_1) = \Psi_1^\delta\left(\bar{\theta}_1(\bar{x}_1)|\bar{x}_1\right) - c = \Psi_1^0\left(\bar{\theta}_1(\bar{x}_1)|\bar{x}_1\right) - c = V_1^0(\bar{x}_1) \quad \forall \bar{x}_t \in \mathbb{R}.$$

For $t \geq 2$, payoffs in the game with changing fundamentals satisfy

$$V_t^\delta\left(\bar{x}^{t-1}, \bar{x}_t\right) = \Psi_t^\delta\left(\bar{\theta}_t(\bar{x}_t)|\bar{x}_t; \bar{x}^{t-1}\right) - c \quad \forall \bar{x}_t \in \mathbb{R},\ \bar{x}^{t-1} \in \overline{\mathbb{R}}^{t-1},\ t \geq 2.$$

The pointwise convergence of beliefs thus implies that

$$\lim_{\delta \to 0} V_t^\delta\left(\bar{x}^{t-1}, \bar{x}_t\right) = \lim_{\delta \to 0} \Psi_t^\delta\left(\bar{\theta}_t(\bar{x}_t)|\bar{x}_t; \bar{x}^{t-1}\right) - c = V_t^0\left(\bar{x}^{t-1}, \bar{x}_t\right) \quad \forall \bar{x}_t \in \mathbb{R},\ \bar{x}^{t-1} \in \overline{\mathbb{R}}^{t-1},\ t \geq 2.$$

Note that convergence of beliefs and payoffs may fail at $\bar{x}_t = -\infty$, but, as in the case of Theorem 3, this does not affect the result. ■