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Games and Economic Behavior 62 (2008) 533–557

GAMES and Economic Behavior

www.elsevier.com/locate/geb

Revisiting games of incomplete information with analogy-based expectations

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Received 14 September 2006

Available online 20 September 2007

Abstract

This paper studies the effects of analogy-based expectations in static two-player games of incomplete information. Players are assumed to be boundedly rational in the way they forecast their opponent's state-contingent strategy: they bundle states into analogy classes and play best-responses to their opponent's average strategy in those analogy classes. We provide general properties of analogy-based expectation equilibria and apply the model to a variety of well known games. We characterize conditions on the analogy partitions for successful coordination in coordination games under incomplete information [Rubinstein, A., 1989. The electronic mail game: Strategic behavior under 'almost common knowledge'. Amer. Econ. Rev. 79, 385–391], we show how analogy grouping of the receiver may facilitate information transmission in Crawford and Sobel's cheap talk games [Crawford, V.P., Sobel, J., 1982. Strategic information transmission. Econometrica 50, 1431–1451], and we show how analogy grouping may give rise to betting in zero-sum betting games such as those studied to illustrate the no trade theorem.

JEL classification: C72; D82

Keywords: Analogy expectation; Bayesian games; Bounded rationality; Coordination; Incomplete information; Betting; Strategic information transmission

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0899-8256/$ – see front matter
doi:10.1016/j.geb.2007.06.006

1. Introduction

In games involving many states, it is implausible to assume that players understand the strategy of their opponent state by state. In this paper, we take the view that players understand only the average behavior of their opponent over bundles of states. Players are then characterized by how finely they understand the strategy of their opponent together with their information and payoff structure.

The class of games that we consider are two-player games of incomplete information. Our only departure from standard approaches lies in the modeling of players' expectations about their opponent' strategy. Players bundle states into analogy classes and play best-responses to their opponent's average strategy in those analogy classes. More precisely, given the prior probabilities of states of the world and the strategy σ_j of player j in every state, one can infer for every analogy class α_i of player $i \neq j$ the average behavior of player j in α_i. Call $\overline{\sigma}_j$ the strategy of player j that specifies that in any state belonging to α_i player j plays according to the average behavior in α_i. In equilibrium, σ_i is assumed to be a best-response to $\overline{\sigma}_j$ given player i's information partition for $i, j = 1, 2$ and $j \neq i$, where $\overline{\sigma}_j$ is interpreted as i's perception of j's strategy.

The equilibrium that is so obtained is the analog for static games of incomplete information of the solution concept considered in Jehiel (2005) for extensive form games. It is called the analogy-based expectation equilibrium, and it is interpreted in Section 3 as the limiting outcome of a learning process involving populations of player $i = 1, 2$ who would get a coarse feedback about the past behavior of players in population $j \neq i$ and no feedback on their own past performance until they exit the system.[1]

One of our main objectives is to apply the approach to a number of classic games such as coordination games (with noisy signals), strategic information transmission games and betting games. For each application, we illustrate the working of the approach, and how the obtained outcomes differ from those obtained with the standard approach. We consider various analogy groupings throughout the applications.

In one application, the e-mail game (Rubinstein, 1989), many states correspond to the same underlying payoff structure (whether the enemy is prepared or not). Our leading result for this application considers the payoff-relevant analogy partition which groups states according to the underlying payoff structure (as opposed to the fine details of the state which include the higher order beliefs of the two players). That is, our equilibrium in the payoff-relevant analogy case assumes that the only feedback transmitted from one generation of players to the next (in the underlying dynamic learning model sustaining the approach) is the average probability of attack conditional on whether or not the enemy was prepared (and not conditional on how many messages were sent back and forth). Such a view on feedback agrees with the recent literature on robust implementation (Bergemann and Morris, 2005) which points out that it may be hard to have access to (and thus have feedback about) others' beliefs and higher order beliefs.

In Crawford and Sobel's (1982) information transmission game, there is a natural notion of proximity between states as payoffs vary continuously with the state and states are linearly ordered. Our analogy partition in this application is the interval analogy partition that groups states into intervals so as to reflect the view that the feedback aggregates the communication strategy of the senders over nearby states.

[1] The latter assumption is automatically satisfied whenever each individual player plays only once.

Finally, we apply the approach to zero-sum betting games and consider the analogy partition in which states are bundled according to whether betting induces a loss or a gain to the opponent so as to reflect the idea that in some cases whether the opponent could have avoided a loss by not betting or could have made a gain by betting are more salient pieces of information on which getting feedback is easier.

Before considering the applications, we provide a general framework that allows for arbitrary analogy partitions (Section 2). In Section 3 we discuss the learning interpretation of the model and discuss a few examples of analogy partitions. In Section 4 we briefly characterize general properties of analogy-based expectation equilibria. Existence in finite Bayesian games is established (Proposition 1) and the contrast with other solution concepts is illustrated through a series of examples.

Sections 5, 6 and 7 are devoted to applications. In Section 5 we show that successful coordination may arise in Rubinstein's (1989) electronic mail game when players use the payoff-relevant analogy partition. More generally, we analyze the various forms of analogy grouping that permit successful coordination. Relatedly, we show that "global games" arguments (Carlson and van Damme, 1993), used to select a unique equilibrium (the risk dominant one), do not extend to situations in which players use a coarse analogy partitioning (bundling all states into one analogy class). In Section 6 we consider strategic information transmission games (Crawford and Sobel, 1982) for which it is shown that a coarse (but not too coarse) analogy grouping of the receiver may facilitate information transmission even though the coarseness of the analogy grouping makes the receiver's inferences harder. Finally, in Section 7 we show how equilibrium betting may arise in zero-sum settings with common priors where the no trade theorem would apply with the standard approach.

Section 8 concludes. All proofs are relegated to the appendix. The results of standard and alternative bounded rationality approaches are compared with ours in the corresponding sections when appropriate.[2]

2. Model

For ease of exposition, we confine ourselves to two-person games. Unless explicitly specified, the sets of actions and states of the world are assumed finite. Players are denoted by 1 and 2, and we adopt the convention that $i, j \in \{1, 2\}$ and $i \neq j$. An *information structure* is denoted by

$$\mathcal{I} = \langle \Omega, p, \mathcal{P}_1, \mathcal{P}_2 \rangle,$$

where Ω is a set of states of the world, p is a strictly positive common prior probability distribution on Ω, and $\mathcal{P}_i$ is the partition of states of the world representing the information of player i. We write $P_i(\omega)$ for the element of $\mathcal{P}_i$ containing ω. This set, called the information set of player i at ω, is the set of states which player i thinks possible.

A *Bayesian game* is denoted by

$$G = \langle \mathcal{I}, A_1, A_2, u_1, u_2 \rangle,$$

where $\mathcal{I}$ is an information structure, A_i is the set of actions available to player i, and $u_i : A \times \Omega \to \mathbb{R}$ is player i's state dependent utility function, where $A = A_1 \times A_2$. A strategy of player i in G is a $\mathcal{P}_i$-measurable function $\sigma_i : \Omega \to \Delta(A_i)$.

[2] A presentation of how other approaches to bounded rationality relate to the analogy-based expectation equilibrium in extensive form games can be found in Jehiel (2005, Section 7).

A pair of strategies $\sigma = (\sigma_1, \sigma_2)$ is a *Nash equilibrium* (NE) of G if for each player i, each state ω and each action $a_i^* \in \operatorname{supp}[\sigma_i(\omega)]$ (i.e., such that the probability of playing a_i^* at ω, $\sigma_i(a_i^* \mid \omega)$, is strictly positive),

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big) \sum_{a_j \in A_j} \sigma_j(a_j \mid \omega') u_i(a_i, a_j; \omega').$$

Following Jehiel (2005), we depart from Nash equilibrium by assuming that players form expectations about their opponent's behavior by grouping states into analogy classes and by averaging behavior in each analogy class.

Definition 1. An *analogy system* is a pair $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_i$ is the *analogy partition* of Ω for player i. The element of $\mathcal{A}_i$ containing ω is denoted by $\alpha_i(\omega)$ and is called the *analogy class* for player i at ω.

A *strategic environment* is summarized by $(G, \mathcal{A})$. Each player i forms an expectation about the average behavior of player j in each analogy class α_i of $\mathcal{A}_i$. In equilibrium this expectation coincides with the effective average behavior of j over the states $\omega \in \alpha_i$, and player i plays a best-response against the induced strategy of player j that specifies the same average behavior in all states $\omega \in \alpha_i$ for every α_i. Formally, given the strategy σ_j of player j, the *strategy of player j perceived by player i* (given $\mathcal{A}_i$) is defined by the function $\bar{\sigma}_j : \Omega \to \Delta(A_j)$ such that for all $\omega \in \Omega$,

$$\bar{\sigma}_j(\omega) = \frac{\sum_{\omega' \in \alpha_i(\omega)} p(\omega') \sigma_j(\omega')}{\sum_{\omega' \in \alpha_i(\omega)} p(\omega')} = \sum_{\omega' \in \Omega} p\big(\omega' \mid \alpha_i(\omega)\big) \sigma_j(\omega'). \tag{1}$$

Definition 2. A pair of strategies $\sigma = (\sigma_1, \sigma_2)$ is an *analogy-based expectation equilibrium* (ABEE) of the strategic environment $(G, \mathcal{A})$ if for all $i \in \{1, 2\}$, $\omega \in \Omega$ and $a_i^* \in \operatorname{supp}[\sigma_i(\omega)]$,

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big) \sum_{a_j \in A_j} \bar{\sigma}_j(a_j \mid \omega') u_i(a_i, a_j; \omega'),$$

where $\bar{\sigma}_j(\omega)$ is given by (1).

3. Motivation and interpretation

In this section we address the issues of how to interpret the solution concept, what features of the game players are assumed to be aware of, and how to think of the analogy partitions considered by the players.

Our premise is that players are aware of the prior, the action space and the structure of their payoffs and of their information. They are not (necessarily) aware of the information nor of the payoff structure of their opponent.

We think of each player i's perception $\bar{\sigma}_j$ about player j's strategy as the outcome of a learning process (assumed to be converging) involving populations of players 1 and 2. Specifically, assume that populations of players 1 and 2 are repeatedly matched in an anonymous way to play game G and assume each individual player plays only once.[3] Along the learning process, players of population i receive *feedback* about the average behavior of players of population j in

[3] This is similar to the learning framework considered in recurring games (Jackson and Kalai, 1997).

the various analogy classes α_i, but no feedback about the past performances of other players in populations i and j.[4] They play a best-response to this feedback, which in turn generates new data for the feedback in the next round.

Clearly, if the behaviors of players in populations 1 and 2 stabilize in the above learning model, it should correspond to an ABEE.[5] This is because the feedback generated by the strategy σ_j of players j (assumed to be stable) leads players of population i to play a best-response to $\bar{\sigma}_j$ (because $\bar{\sigma}_j$ corresponds to the real average behavior of player j as derived from σ_j in every considered analogy class of player i).

In our learning scenario, we assumed that players played only once. We could alternatively assume that players play several times, but then we should add that players receive no feedback about their past performance nor about what their matched partners did until they exit the system (so as to avoid that players make further inferences).[6]

In line with this learning interpretation, observe that the ABEE can be viewed as a selection of self-confirming equilibrium in which the signals received by players i after each round correspond to the average plays of players j over α_i (see, for example, Battigalli, 1987 or Dekel et al., 2004, for a general presentation of self-confirming equilibria for arbitrary signal structures). As it turns out, the structure of our signals led us to consider a selection of self-confirming equilibrium (which the general signal structure of Battigalli, 1987 or Dekel et al., 2004 would not permit): the selection comes from the fact that in ABEE player i is assumed to play a best-response to the *simplest* (as opposed to *some*) theory about player j's behavior that is consistent with i's observation (of j's play).[7]

The above learning model takes as exogenous the analogy classes α_i. One interpretation is that, along the learning process, only statistics about the average past play of players j in α_i are available to players i. Another interpretation is that more data are *a priori* available, but statistics about average play in α_i are more *salient* or *accessible* to subjects (see Higgins, 1996, for an account of accessibility in psychology), and somehow players' behaviors are best explained as if only these statistics were available.

In line with the above interpretation, Huck et al. (2007) have conducted an experiment in which players played two normal form games with an equal frequency (in our language, G consisted of two possible states having the same prior probability). The feedback provided to subjects between the various rounds concerned the behavior of the players over the two games. By playing on the accessibility of the feedback, the system stabilized to different outcomes, which were well predicted by the analogy approach.[8]

From an applied perspective, a question of primary importance is about which statistics about opponents' attitudes are likely to be more accessible as a function of the context. This obviously

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[4] Players could be assumed to receive additional feedback about the frequency of the various states as realized in past matches in which case we could dispense with the assumption that players are aware of the prior.

[5] Whether or not behaviors stabilize should be the subject of future work.

[6] See Esponda (2007) for a model that mixes observations about payoffs and opponents' behaviors in the context of adverse selection problems. Observe that when players observe their opponents' actions, one should have that $\mathcal{A}_i$ is finer than $\mathcal{P}_i$ as player i can map player j's action with his own private information.

[7] To the extent that player i has no knowledge at all about the payoff and information structure of player j, assuming player j's play is constant over states in α_i seems pretty natural for player i.

[8] When behaviors were accessible game by game this corresponded to the fine analogy grouping. When they were accessible only in aggregate over the two games, this corresponded to the coarse analogy grouping. The games were chosen so that the equilibrium predictions are markedly different according to the analogy grouping.

depends on the application, and to some extent this is an empirical question.[9] In this paper, we analyze from a theoretical viewpoint the implications of different analogy groupings. It is our hope that the theoretical insights obtained in our paper will help make progress in the broader understanding of how agents process data in interactive learning processes. Specifically, we consider the following families of analogy partitions.

(i) *The coarsest analogy partition.* $\mathcal{A}_i = \{\Omega\}$. In this situation player i only perceives the average behavior of player j across all states of the world. Hence, at equilibrium he plays a best-response to the belief that player j behaves in the same way in all states of the world. Such an analogy system is suited in situations in which only aggregate behaviors are made available or accessible to agents at the learning stage. This fits in well with the coarse accessibility treatment of the experiment conducted in Huck et al. (2007).

(ii) *The private information analogy partition.* $\mathcal{A}_i = \mathcal{P}_i$. In this situation player i believes in each state ω that player j behaves in the same way in all states player i conceives as possible at ω. This fits in well in situations in which agents construct themselves their statistics and somehow believe erroneously that all players share their own information structure.

(iii) *The payoff-relevant analogy partition.* $\mathcal{A}_i = \mathcal{F}$, where $\mathcal{F}$ is the partition of Ω generated by the fundamentals.[10] In this situation, player i believes that his opponent's strategy is only correlated to the fundamentals. This analogy system is particularly suited if there are many states associated with the same payoff configuration, and payoff-irrelevant information such as beliefs of any order is hard to convey to others. This will be our benchmark case in the study of the e-mail game.

In some applications, the structure of states and/or payoffs makes some analogy groupings more focal. For example,

(iv) When there is a natural notion of proximity of states (for example because the corresponding payoffs are nearby), grouping states in terms of neighborhood seems natural. In Crawford and Sobel's game, states are linearly ordered, and the Euclidean distance defines a relevant notion of proximity. We then consider the grouping of states into intervals and the corresponding interpretation of feedback is that receivers get to know the communication strategy of the sender only coarsely by range of states which are nearby.[11]

(v) In our study of betting games, we will consider the *two-type error analogy system* in which states are partitioned according to whether a player would have made a gain by betting or would have avoided a loss by not betting. Presumably, statistics about the frequency of the two types of betting errors are more salient (thus they are easier to remember, communicate and process) than statistics about the detailed behaviors in each state of the economy.

[9] Our framework does not allow the feedback transmitted to future generations to be a function of the action profile. Such an extension would require further generalization of the framework. (Note that such a dependence naturally arises in the context of extensive form games, as considered in Jehiel, 2005.)

[10] That is, if we denote by $F(\omega)$ the element of $\mathcal{F}$ containing ω, then $F(\omega) = F(\omega')$ if and only if $u_i(a;\omega) = u_i(a;\omega')$ for all $i \in \{1,2\}$ and $a \in A$.

[11] Think, for example, in the context of financial advisers of feedbacks taking the form of what the adviser previously recommended to buy as a function of how high the stock market approximately was.

It should be noted that an ABEE in the private information analogy partition (see (ii) above) is equivalent to Eyster and Rabin's (2005) fully cursed equilibrium. Observe that Eyster and Rabin's analysis (and most interesting insights) focuses on the concept of partially cursed equilibrium in which players' expectations are convex combinations of the expectations arising from the private information analogy partition and those arising under perfect rationality. By allowing for arbitrary analogy partitions, we consider a different approach to partial sophistication than that in Eyster and Rabin (which is measured by the weight assigned to the rational expectations). We believe that an advantage of our approach to partial sophistication is that it can receive an easier learning interpretation.[12]

4. Preliminaries

We first provide a basic existence result for finite games.

Proposition 1. *Every finite*[13] *strategic environment* $(G, \mathcal{A})$ *has at least one analogy-based expectation equilibrium.*

We further observe that:

(1) If $\mathcal{A}_i$ is finer[14] than $\mathcal{P}_j$ for all $i \in \{1, 2\}$, then the set of analogy-based expectation equilibria of $(G, \mathcal{A})$ coincides with the set of Nash equilibria of G. This is because the strategy of player j is $\mathcal{P}_j$-measurable and thus $\bar{\sigma}_j(\omega) = \sigma_j(\omega)$ for all $j \in \{1, 2\}$ and $\omega \in \Omega$.

(2) A pooling strategy profile is a simple strategy profile that satisfies $\sigma_i(a_i \mid \omega) = \sigma_i(a_i \mid \omega')$ for all $i \in \{1, 2\}$, $a_i \in A_i$ and $\omega, \omega' \in \Omega$. Clearly, in any strategic environment, the set of pooling analogy-based expectation equilibria coincide with the set of pooling Nash equilibria since perceived strategies coincide with actual pooling strategies.

4.1. Can an ABEE be viewed as a Nash equilibrium of a modified Bayesian game?

By definition, an ABEE of a finite strategic environment $(G, \mathcal{A})$ is equivalent to a (mixed-strategy) NE of the normal form game $\langle \{1, 2\}, (S_i)_i, (V_i)_i \rangle$, where S_i is the set of $\mathcal{P}_i$-measurable functions $s_i : \Omega \to A_i$ and,[15]

$$V_i(s_i, s_j) \equiv \sum_{\omega \in \Omega} p(\omega) \sum_{\omega' \in \alpha_i(\omega)} p\big(\omega' \mid \alpha_i(\omega)\big) u_i\big(s_i(\omega), s_j(\omega'); \omega\big). \tag{2}$$

But a question arises as to whether it is possible to view the ABEE as an NE of the original game with different information partitions and/or different payoff functions that depend solely on the state and the action profile in the state. In the next example we first show that an analogy-based expectation equilibrium cannot be seen, in general, as a Nash equilibrium of the initial Bayesian game with different information partitions.

[12] See Fudenberg (2006) for further discussion of Eyster–Rabin's partially cursed equilibrium.

[13] A strategic environment is said finite if the set of states of the world, Ω, and the set of action profiles, A, are finite.

[14] When we say "finer (coarser) than" we mean "finer (coarser) than or equal to."

[15] We are grateful to the associate editor for suggesting this transformation. A similar transformation would not work for the extensive form games considered in Jehiel (2005) because the weights attached to the various nodes would be a function of the strategy profile itself.

ω_1	L	M	R	R'
U	$(5,2)$	$(0,2)$	$(2,4)$	$(0,0)$
D	$(4,3)$	$(3,0)$	$(1,0)$	$(2,0)$

ω_2	L	M	R	R'
U	$(3,0)$	$(4,2)$	$(2,0)$	$(1,1)$
D	$(0,2)$	$(5,2)$	$(0,0)$	$(2,4)$

Fig. 1. Game of Example 1 (Jehiel, 2005).

Example 1. The game of Fig. 1, due to Jehiel (2005), where $p(\omega_1) = p(\omega_2)$ and $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}\}$, has a unique NE (obtained by iterative elimination of strictly dominated strategies) in which player 1 plays U at ω_1 and D at ω_2, and player 2 plays R at ω_1 and R' at ω_2.

If player 1 uses the coarsest analogy partition and player 2 uses the finest one, then the previous strategy profile is not an ABEE. It can be shown that the unique ABEE is the strategy profile $((D, L \mid \omega_1); (U, M \mid \omega_2))$.[16] Since both players choose a different action in each strategic form game of Fig. 1, their behavior cannot be replicated by introducing incomplete information about which strategic form game is being played.

Example 1 can also be used to illustrate that an ABEE cannot be interpreted either as a Nash equilibrium with subjective prior in which player 1 would erroneously believe that player 2's information partition is $\mathcal{P}_2 = \{\omega_1, \omega_2\}$ (and player 2 is playing a best-response to 1's strategy). (See Jehiel, 2005.)

Another way to view the ABEE as a NE of a modified game would be to allow for an appropriate transformation of players' payoffs. Yet, the following example shows that there may be no payoff transformation $\tilde{u}_1, \tilde{u}_2$ such that the set of ABEE of $(G, \mathcal{A})$ where $G = \langle \mathcal{I}, A_1, A_2, u_1, u_2 \rangle$ coincides with the set of NE of $\langle \mathcal{I}, A_1, A_2, \tilde{u}_1, \tilde{u}_2 \rangle$.

Example 2. Consider the strategic environment

ω_1	L	R
U	$(2,2)$	$(0,0)$
D	$(0,0)$	$(1,1)$

ω_2	L	R
U	$(1,1)$	$(0,0)$
D	$(0,0)$	$(2,2)$

where $p(\omega_1) = p(\omega_2)$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\{\omega_1, \omega_2\}\}$ and $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}\}$. There are three ABEE in pure strategies: $((U, L \mid \omega_1); (D, R \mid \omega_2))$, $((U, L \mid \omega_1); (U, L \mid \omega_2))$ and $((D, R \mid \omega_1); (D, R \mid \omega_2))$. Observe that the NE $((D, R \mid \omega_1); (U, L \mid \omega_2))$ is not an ABEE. Yet, in a setup in which players know the state, if $((U, L \mid \omega_1); (U, L \mid \omega_2))$ and $((D, R \mid \omega_1); (D, R \mid \omega_2))$ are NE, it should be that $((U, L \mid \omega_1); (D, R \mid \omega_2))$ and $((D, R \mid \omega_1); (U, L \mid \omega_2))$ are both NE. This shows that no payoff transformation permits the replication of the set of ABEE.

In some special cases, namely in the private information analogy partition case ($\mathcal{A}_i = \mathcal{P}_i$), an ABEE corresponds to a NE of a virtual game in which player i's payoff at state ω is taken to be the average payoff over the states of the information set $P_i(\omega)$. This is an observation made in Eyster and Rabin (2005). We extend it to the case in which $\mathcal{A}_i$ is finer than $\mathcal{P}_i$ for both players $i = 1, 2$:

Proposition 2. *If $\mathcal{A}_i$ is finer than $\mathcal{P}_i$ for all i, then the strategy profile σ is an analogy-based expectation equilibrium of the strategic environment $(G, \mathcal{A})$ if and only if σ is a Nash equilibrium*

[16] Notice that player 1 is better off (compared to the NE outcome) albeit his imperfect perception of player 2's strategy.

of the "virtual" Bayesian game $\overline{G}^{\mathcal{A}} = \langle \mathcal{I}, A_1, A_2, \bar{u}_1^{\mathcal{A}}, \bar{u}_2^{\mathcal{A}} \rangle$*, where for all* $i \in \{1,2\}$*,* $a \in A$ *and* $\omega \in \Omega$*,* $\bar{u}_i^{\mathcal{A}}(a;\omega) = \sum_{\omega' \in \Omega} p(\omega' \mid \alpha_i(\omega)) u_i(a;\omega')$*.*

4.2. Links to other solution concepts

As it turns out, in an ABEE, player i's strategy may be strictly dominated given the information partition of player j. This is so because player i's perceived strategy of player j, i.e. $\overline{\sigma}_j$, need not be measurable w.r.t. $\mathcal{P}_j$ (so that $\overline{\sigma}_j$ would not be a feasible strategy for player j).[17] This is only possible if i's analogy partition $\mathcal{A}_i$ is neither coarser nor finer than $\mathcal{P}_j$. This is illustrated in the following example.

Example 3 *(ABEE with a Strictly Dominated Strategy).* Consider the strategic environment of Fig. 2, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and $\mathcal{A}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. Whatever his analogy partition, player 2 always plays action a_2 at $\{\omega_1\}$ and b_2 at $\{\omega_2, \omega_3\}$, so given $\mathcal{A}_1$, player 1's perception of player 2's strategy is[18]: $\bar{\sigma}_2(\omega_1) = \bar{\sigma}_2(\omega_2) = (1/2, 1/2)$ and $\overline{\sigma}_2(\omega_3) = (0,1)$. Player 1's best response at $\{\omega_1\}$ is clearly a_1. At $\{\omega_2, \omega_3\}$, action b_1 gives him a payoff of 2, whereas action a_1 gives him a perceived expected payoff of $(1/2)[(1/2)3 + (1/2)0] + (1/2)[3] = 9/4 > 2$, so he plays action a_1 which is strictly dominated at $\{\omega_2, \omega_3\}$ in the Bayesian game. (Given the measurability of σ_2 w.r.t. $\mathcal{P}_2$, at $\{\omega_2, \omega_3\}$ playing a_1 (resp. b_1) yields $\frac{3}{2}$ (resp. 2) to player 1.)

In the special case in which states of the world are payoff irrelevant it is possible to relate an ABEE to a correlated equilibrium (Aumann, 1974), i.e., to a NE of the original game with private, payoff-irrelevant and possibly correlated signals. To see this, observe that when $P_i(\omega) = \{\omega\}$ and $u_i(a;\omega) = u_i(a)$ for all $i \in \{1,2\}$, $a \in A$ and $\omega \in \Omega$, σ is an ABEE if for all $i \in \{1,2\}$, $\omega \in \Omega$ and $a_i^* \in \operatorname{supp}[\sigma_i(\omega)]$,

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\omega' \in \Omega} p\big(\omega' \mid \alpha_i(\omega)\big) \sum_{a_j \in A_j} \sigma_j(a_j \mid \omega') u_i(a_i, a_j).$$

The correlated equilibrium is usually presented as a Nash–Bayes equilibrium of a game with incomplete information. As the condition shown above reveals, it may alternatively be viewed as an ABEE of a game with complete information but coarse analogy partitions (where a player's analogy partition coincides with his information partition in the usual approach).

Thus, for any correlated equilibrium distribution $\mu \in \Delta(A)$ of a strategic form game $\langle A_1, A_2, u_1, u_2 \rangle$ there exists a finite state space (Ω, p), an analogy system $\mathcal{A}$, and an ABEE of the strategic environment $(\langle \Omega, p, \mathcal{P}_1, \mathcal{P}_2, A_1, A_2, u_1, u_2 \rangle, \mathcal{A})$ with $P_i(\omega) = \{\omega\}$ and $u_i(a;\omega) =$

ω_1	a_2	b_2
a_1	(3, 3)	(3, 2)
b_1	(2, 3)	(2, 2)

ω_2	a_2	b_2
a_1	(3, 2)	(0, 3)
b_1	(2, 2)	(2, 3)

ω_3	a_2	b_2
a_1	(0, 2)	(3, 3)
b_1	(2, 2)	(2, 3)

Fig. 2. Game with a strictly dominated ABEE strategy.

[17] Remember that in our motivation for ABEE (see Section 3) we have assumed that player i need not know *a priori* the information partition of j. So in i's mind, he is not playing a dominated strategy (but a best-response to his theory about j).

[18] A vector $(x, 1-x)$ denotes the behavioral strategy of player 2 that assigns probability x to a_2 and $1-x$ to b_2.

	a	b
a	$(0, 1)$	$(1, 0)$
b	$(1, 0)$	$(0, 1)$

Fig. 3. ABEE vs. correlated equilibrium.

$u_i(a)$ for all $i \in \{1, 2\}$, $a \in A$ and $\omega \in \Omega$ that generates the outcome distribution μ. A "converse" but only partial link is established in the following proposition, with no restriction on players' information partitions.

Proposition 3. *Let $(G = \langle \Omega, p, \mathcal{P}_1, \mathcal{P}_2, A_1, A_2, u_1, u_2 \rangle, \mathcal{A})$ be a strategic environment satisfying $u_i(a; \omega) = u_i(a)$ for all $i \in \{1, 2\}$, $a \in A$ and $\omega \in \Omega$. If $\mu \in \Delta(A)$ is an outcome distribution generated by an ABEE of $(G, \mathcal{A})$ such that σ_i is measurable w.r.t. $\mathcal{A}_i$ for all i, then μ is a correlated equilibrium distribution of the strategic form game $\langle A_1, A_2, u_1, u_2 \rangle$.*

A key condition is that σ_i be measurable with respect to $\mathcal{A}_i$. When not met, an ABEE, even of a complete information strategic environment with payoff irrelevant states, may not induce a correlated equilibrium distribution. To see this, consider the strategic environment based on the strategic form game of Fig. 3, with $p(\omega_1) = p(\omega_2)$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\{\omega_1, \omega_2\}\}$ and $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}\}$. It is readily verified that the strategy profile $\sigma_1(\omega_1) = \sigma_2(\omega_1) = a$ and $\sigma_1(\omega_2) = \sigma_2(\omega_2) = b$ is an ABEE. Yet, player 1 gets 0, which cannot be rationalized in any correlated equilibrium distribution since player 1's maxmin payoff is $1/2$.[19]

5. Coordination under "almost common knowledge"

In this section we consider Rubinstein's (1989) electronic mail game, and analyze the possibility of coordinated attack for various analogy partitions. We also briefly discuss Carlson and van Damme's (1993) global game.

In the electronic mail game, the payoff matrices in each state of Nature are as in Fig. 4.[20] Referring to the original version of the coordinated attack problem (Gray, 1978; Halpern and Moses, 1990), action D corresponds to "Don't Attack" and action A corresponds to "Attack." The payoff-relevant state G_a corresponds to "The enemy is prepared" and G_b corresponds to "The enemy is unprepared." The information structure generated by the ε-noisy communication protocol where only player 1 is initially informed about the actual matrix game, sends a message to player 2 if the game is G_b and then each player sends back and forth a receipt confirmation, is represented by Fig. 5, where the state space is $\{G_a, (G_b, 0), (G_b, 1), (G_b, 2), \ldots\}$. Player 1's information sets are represented by dotted boxes and player 2's information sets by solid boxes. We assume that $1 - p > 1/2$, $\varepsilon, \delta > 0$ and $L > M > 0$, where $1 - p$ is the probability that state G_a occurs.

[19] Since the payoffs are the same in each state, this example further shows that expanding the state space, modifying the information, *and* transforming the payoffs by averaging the payoff functions across states cannot help to interpret an ABEE as an NE of a modified Bayesian game.

[20] As is often done in the literature (e.g., Morris and Shin, 1997), to simplify the exposition we have slightly modified Rubinstein's (1989) original game so that there is a strictly dominant strategy in one state of the world. Here, we consider payoffs $(-\delta, -\delta)$ instead of $(0, 0)$ for the action profile (A, A) in game G_a. This allows us to exclude the weakly dominated equilibrium strategy profile in which both players always play A, and so to get a unique NE.

G_a	D	A
D	(M, M)	$(0, -L)$
A	$(-L, 0)$	$(-\delta, -\delta)$

G_b	D	A
D	$(0, 0)$	$(0, -L)$
A	$(-L, 0)$	(M, M)

Fig. 4. The electronic mail game.

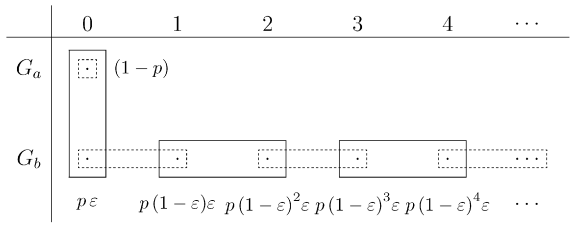

Fig. 5. Information structure in the Email game. Player 1's information sets are represented by dotted boxes and player 2's information sets by solid boxes.

Iterative elimination of strictly dominated strategies yields the unique NE: each player plays D in all states of the world.[21] Hence, even if the game G_b is mutually known up to, say, one million levels, it is never an equilibrium to play the Pareto dominant action profile (A, A). Some authors have explained how coordinated attack may be possible, but they all modify the communication protocol and thus the information structure of the Bayesian game (Morris and Shin, 1997; Urbano and Vila, 2003). Here, we will show how the analogy-based expectation approach may explain coordinated attack while keeping the same information structure of the game.

We first note that the NE is always an ABEE whatever the analogy system because the NE strategy profile is pooling. In the rest of this section we investigate whether other equilibria may emerge depending on the analogy partitions. We first consider the payoff-relevant analogy system $(A_1 = A_2 = \{\{G_a\}, \{(G_b, 0), (G_b, 1), (G_b, 2), \ldots\}\})$ in which each player i gets only to know the average probability of attack of player j conditional on the underlying stage game being G_a or G_b. We observe that coordinated attack is possible for ε not too large (i.e., when the communication protocol is sufficiently reliable).

Proposition 4. *With the payoff-relevant analogy system coordinated attack is possible at an analogy-based expectation equilibrium if and only if $\varepsilon \leqslant \frac{M}{M+L}$. In such an equilibrium players attack whenever they know G_b. Coordination failure occurs at $(G_b, 0)$ and attack is successfully coordinated in state (G_b, t) for all $t \geqslant 1$.*

The intuition for the result is as follows. Conditional on G_b being the game played, player 1 attacks always and player 2 attacks on average with probability $1 - \varepsilon$ (he does not attack only in $(G_b, 0)$ and the weight of $(G_b, 0)$ relative to other states (G_b, t), $t \geqslant 1$, is ε). Given their analogy-based expectations, players attack whenever they know G_b (this is so even for player 1 because $(1 - \varepsilon)M - \varepsilon L > 0$).

In the standard setting the ex ante expected payoff of both players is $M(1 - p)$. In the payoff-relevant analogy setting, the ex ante expected payoff of both players of the equilibrium shown

[21] Player 1 necessarily plays D at his singleton information set $\{G_a\}$, which implies that player 2 also plays D at $\{G_a, (G_b, 0)\}$. At all other information sets both players know that the game played is G_b. Each player plays A only if his opponent plays A with probability at least $\frac{L}{M+L} > 1/2$. Since D is played by player 2 at $(G_b, 0)$ and since $\Pr(G_b, 0) > \Pr(G_b, 1) > \Pr(G_b, 2) > \cdots$, this action is diffused in all other states.

in Proposition 4 is larger; it is equal to $M(1 - p\varepsilon) - Lp\varepsilon$ for player 1 and to $M(1 - p\varepsilon)$ for player 2. It is interesting to note that this is exactly the ex ante expected payoff achieved at one of the NE (that exists also under the same condition $\varepsilon \leqslant \frac{M}{M+L}$) with the simple communication protocol, where only one message is sent so that the information structure is simply $\mathcal{P}_1 = \{\{G_a\}, \{(G_b, 0), (G_b, 1)\}\}$ and $\mathcal{P}_2 = \{\{G_a, (G_b, 0)\}, \{(G_b, 1)\}\}$.[22]

The result of Proposition 4 holds whenever player 1 uses the payoff-relevant analogy partition no matter what the analogy partition of player 2 is. When player 1 is rational but player 2 uses the payoff-relevant analogy partition, we have:

Proposition 5. *When player* 1 *is rational and player* 2 *uses the payoff-relevant analogy partition, coordinated attack is possible at an analogy-based expectation equilibrium if and only if* $\varepsilon(2 - \varepsilon) \leqslant \frac{M}{M+L}$. *In such an equilibrium player* 1 *attacks in all information sets except* $\{G_a\}, \{(G_b, 0), (G_b, 1)\}$ (*where he plays D*); *player* 2 *attacks in all information sets except* $\{G_a, (G_b, 0)\}$ (*where he plays D*).

In the equilibrium of Proposition 5, player 2 attacks whenever he knows the underlying game is G_b because his belief is that player 1 attacks on average with probability $1 - [\varepsilon + (1 - \varepsilon)\varepsilon] = 1 - \varepsilon(2 - \varepsilon)$ in such an event (player 1 does not attack in $(G_b, 0)$, $(G_b, 1)$ and attacks in all other states (G_b, k), $k > 1$).

The next proposition shows that with the coarsest or the private information analogy system coordinated attack is impossible.

Proposition 6. *If at least one player uses the coarsest analogy partition* (*i.e.*, $\mathcal{A}_i = \{\Omega\}$ *for some* $i \in \{1, 2\}$) *or if both use the private information analogy partition, then the unique analogy-based expectation equilibrium consists for both players never to attack.*

While the intuition for the private information analogy partition follows the usual type of infection argument, the result for the coarsest analogy partition is of a different (simpler) nature. For example, if agent 2 uses the coarsest analogy partition, he must expect (in equilibrium) that player 1 does not attack with a probability no smaller than $1 - p$ (because at G_a, player 1 plays D and the weight of G_a relative to other states is $1 - p$). Since $1 - p > 1/2$ (and $L > M$) player 2's best-response is to never attack whatever his information, which in turn leads player 1 not to attack as well.

5.1. "Finest" analogy system allowing coordinated attack

Now we are looking for some "finest" analogy partitions such that a coordinated attack is possible after a finite number of communication periods. Assume that player 1 is "rational" so that $\mathcal{A}_1$ is finer than $\mathcal{P}_2$, and assume that player 2's analogy partition has the form

[22] Or alternatively, messages are sent back and forth as in the original formulation, but player 1 does not remember whether he received any message, and player 2 remembers only whether he receives a message but not how many messages he received. That is, $\mathcal{P}_1 = \{\{G_a\}, \{(G_b, 0), (G_b, 1), \ldots\}$ and $\mathcal{P}_2 = \{\{G_a, (G_b, 0)\}, \{(G_b, 1), \ldots\}$.

$$\mathcal{A}_2 = \Big\{\{G_a\}, \{(G_b,0),(G_b,1)\}, \ldots, \{(G_b,2l-2),(G_b,2l-1)\}, \underbrace{\{(G_b,2l),(G_b,2l+1),\ldots,(G_b,2l+1+2k^*)\}}_{\alpha_2^*}, \ldots\Big\},$$

where the last dots can be any partitioning of the remaining states. By standard arguments we obtain that player 1 plays D up to $\omega = (G_b, 2l+1)$ and player 2 plays D up to $\omega = (G_b, 2l)$. We are looking for necessary and sufficient conditions for both players to attack at an ABEE in all other information sets. If this is the case, player 1 plays optimally because he assigns probability one to player 2 choosing action A. In this situation the average strategy of player 1 in the analogy class α_2^* is

$$\begin{aligned}\bar{\sigma}_1^*(D) &= \sum_{\omega\in\alpha_2^*} p(\omega \mid \alpha_2^*)\sigma_1(\omega)(D) = \frac{p\varepsilon(1-\varepsilon)^{2l} + p\varepsilon(1-\varepsilon)^{2l+1}}{p\varepsilon\sum_{t=2l}^{2l+2k^*+1}(1-\varepsilon)^t}\\ &= \frac{2-\varepsilon}{\sum_{t=0}^{2k^*+1}(1-\varepsilon)^t} = \frac{(2-\varepsilon)\varepsilon}{1-(1-\varepsilon)^{2k^*+2}}.\end{aligned}$$

Notice that $\lim_{\varepsilon\to 0}\bar{\sigma}_1^*(D) = \frac{1}{k^*+1}$. Player 2 plays A in his information sets included in α_2^* if $\bar{\sigma}_1^*(D) \leqslant \frac{M}{M+L}$, i.e.,

$$k^* \geqslant \frac{\ln\frac{M(1-\varepsilon)^2 - L(2-\varepsilon)\varepsilon}{M}}{2\ln(1-\varepsilon)} - 1 \to \frac{L}{M} \quad (\varepsilon \to 0).$$

Since $L/M > 1$ a necessary condition is $k^* \geqslant 2$ (at least 3 information sets should be pooled together by player 2).

5.2. Global games

We have seen that at an ABEE a Pareto dominant equilibrium may be played even when it is risk dominated and when coordination is not common knowledge. In the following lines we check whether this is also possible in Carlson and van Damme's (1993) global games framework, which is similar in spirit to the coordinated attack problem. Consider the game of Fig. 6, taken from the introduction of Carlson and van Damme (1993). Each player has a safe action (action a) which yields x independently of the other player's action, and a risky action (action b) which gives a payoff of 4 if the other player also plays b and 0 otherwise. The payoff-relevant state (state of fundamentals) x is assumed to be the realization of a random variable X distributed according to a density $h(\cdot)$ which is strictly positive, continuously differentiable and bounded on $[\underline{x}, \bar{x}]$, with $\underline{x} < 0$ and $\bar{x} > 4$.

Each player i receives a private signal x_i corresponding to a random variable $X_i = X + E_i$, where E_i, E_j are distributed uniformly on $[-\varepsilon, \varepsilon]$ independently of each other and of X.[23] In

	a	b
a	(x, x)	$(x, 0)$
b	$(0, x)$	$(4, 4)$

Fig. 6. Carlson and van Damme's (1993) global game example.

[23] Hence, a state of the world is a tuple $\omega = (x, e_i, e_j) \in [\underline{x}, \bar{x}] \times [-\varepsilon, \varepsilon]^2$ and player i's information set at ω is given by $P_i(\omega = (x, e_i, e_j)) = \{\omega' = (x', e_i', e_j')\colon x' + e_i' = x + e_i\}$.

the standard rationality paradigm, Carlson and van Damme (1993) have shown that for ε small enough, there exists a unique dominant solvable equilibrium outcome. In this equilibrium player i plays b (resp. a) if $x_i < x^*(\varepsilon)$ (resp. $x_i > x^*(\varepsilon)$), and $x^*(\varepsilon) \to x^*$ as $\varepsilon \to 0$, where $x^* = 2$. Thus, in the limit as ε tends to 0, the risk dominant equilibrium of the corresponding game of complete information game is played. Observe that these results (uniqueness of the equilibrium and link to the risk dominance property) hold true independently of the distribution $h(\cdot)$ of X.[24]

Suppose that players use the coarsest analogy partitions. It is readily verified that the equilibrium shown in Carlson and van Damme is no longer an ABEE in general. To illustrate the point, assume that X is uniformly distributed on $[\underline{x}, \bar{x}]$. Then, when $\varepsilon \to 0$, the perceived strategies are

$$\bar{\sigma}_j = \frac{2-\underline{x}}{\bar{x}-\underline{x}} b + \frac{\bar{x}-2}{\bar{x}-\underline{x}} a, \quad j = 1, 2.$$

So, for player i with signal x_i choosing a yields $E(X \mid x_i) = x_i$ and choosing b is perceived to yield $4\frac{2-\underline{x}}{\bar{x}-\underline{x}}$. Best-response to $\bar{\sigma}_j$ will lead player i to choose b if $x_i < 4\frac{2-\underline{x}}{\bar{x}-\underline{x}}$ and a if the reverse inequality holds. Thus, unless $\bar{x} + \underline{x} = 4$, the NE of the original game is not an ABEE.

Since the best-response to any perceived strategy $\bar{\sigma}_j$ is a threshold strategy, all ABEE are of the following form when $\varepsilon \to 0$: player i chooses b (resp. a) if $x_i < x_i^*$ (resp. $x_i > x_i^*$), where $4H(x_j^*) = x_i^*$ and $H(\cdot)$ denotes the cumulative of $h(\cdot)$. By the theorem of intermediate values this equation always has a symmetric solution in the open interval $(0, 4)$ because $4H(0) - 0 > 0$ and $4H(4) - 4 < 0$, so there is always a symmetric ABEE in pure strategies. For example, in the case of a uniform distribution of X, i.e., $h \equiv 1/(\bar{x} - \underline{x})$, there exists a unique ABEE characterized by $x_i^* = x_j^* = x^* = \frac{4\underline{x}}{4-\bar{x}+\underline{x}} \in [0, 4]$. So, when $\bar{x} + \underline{x} > 4$ (i.e., $x^* < 2$) there is an extra bias toward the risk-dominant equilibrium and when $\bar{x} + \underline{x} < 4$ (i.e., $x^* > 2$) there is a bias toward the Pareto-dominant equilibrium.

Notice that, whatever the distribution of X, all equilibria are symmetric because $4H(x_j^*) = x_i^*$ and $4H(x_i^*) = x_j^*$ implies $x_i^* = x_j^* = x^*$ since $H(\cdot)$ is increasing. Consequently, as in the standard global game model, when the noise vanishes coordination failure never occurs. Nevertheless, there may exist several symmetric ABEE because there may be several solutions to $4H(x^\#) = x^\#$. By the theorem of intermediate values, a necessary and sufficient condition for having several (at least three) ABEE is that there exists $0 < y_1 < y_2 < 4$ such that $y_1 - 4H(y_1) > 0$ and $y_2 - 4H(y_2) < 0$. This condition is satisfied if the distribution of X is centered and sufficiently concentrated around 2.

Thus, with the coarsest analogy system, we lose the connection to risk dominance, we lose the property of uniqueness, yet we keep the property of threshold equilibria and we exclude the possibility of coordination failure.

Finally, notice that with the private information analogy system, in the limit as ε tends to 0, we find the same insight as in the standard rationality paradigm. That is, the unique equilibrium is a threshold equilibrium and the risk-dominant equilibrium is selected irrespective of the distribution of X. This can be seen directly by writing the associated virtual Bayesian game characterized in Proposition 2, which is the same as the game of Fig. 6 except that player i's payoff of type x_i when he chooses action a is equal to $E(X \mid X_i = x_i)$ instead of x, $i = 1, 2$.

[24] When X is uniformly distributed on $[\underline{x}, \bar{x}]$ the result is easily obtained by noting that the (iterated) dominance region of action a is $(\bar{x}^*(\varepsilon), \bar{x}]$ with $\bar{x}^*(\varepsilon) \leqslant 2$ and the (iterated) dominance region of action b is $[\underline{x}, \underline{x}^*(\varepsilon))$ with $\underline{x}^*(\varepsilon) \geqslant 2$, so $\underline{x}^*(\varepsilon) = \bar{x}^*(\varepsilon) = x^*(\varepsilon) = 2$ (hence, in that case the result holds even when ε does not tend to zero). For general priors the proof is done in the same way by noting that $\Pr[X_j \leqslant X_i \mid X_i]$ and $\Pr[X \leqslant X_i \mid X_i]$ tend to $1/2$ when $\varepsilon \to 0$ (when the noise vanishes it is as if the prior distribution of X were uniform). See Carlson and van Damme (1993) for more details.

6. Strategic information transmission

In this section we consider a class of sender-receiver cheap talk games studied by Crawford and Sobel (1982). The set of states of the world is the set of types of the sender (player 1), $T = [0, 1]$. They are distributed according to a smooth probability distribution with density $f(t)$ and cumulative $F(t)$ assumed to be differentiable. After learning his type, the sender chooses a costless message $m \in M = [0, 1]$. Upon receiving this message, the receiver (player 2) chooses a payoff-relevant action $a \in A = [0, 1]$. The payoffs of the sender and the receiver are respectively:

$$u_S(a; t) = -\left[a - (t + b)\right]^2,$$
$$u_R(a; t) = -[a - t]^2,$$

where $b > 0$. Hence, for all $b > 0$, when t increases both players prefer a higher action, but the best action for the sender, $t + b$, is always strictly higher than the best action for the receiver, t. The smaller b, the more congruent the preferences of the sender and the receiver.

The strategies of the sender and the receiver are respectively denoted by $\sigma_S : T \to \Delta(M)$ and $\sigma_R : M \to A$. Crawford and Sobel (1982) have shown that every NE outcome of this signaling game can be generated by an L-step partitional communication strategy of the form $\sigma_S(t) = m_l$ if $t \in [x_{l-1}, x_l)$, $l = 1, \ldots, L$, where $0 = x_0 < x_1 < \cdots < x_{L-1} < x_L = 1$ and $m_k \neq m_l$ $\forall k \neq l$. When types are uniformly distributed, such an equilibrium exists under the condition $b < \frac{1}{2L(L-1)}$. In particular, when $b \geqslant 1/4$ no information can be transmitted in equilibrium.

In this section we revisit Crawford and Sobel's game within the analogy setup. Our main observation is that even though the receiver's action is never ex ante biased (Proposition 7) and there is a bound on the quality of the inference that the receiver can make from the sender's message (Propositions 8 and 9) there may be more information transmission in the analogy setup than in the standard rationality paradigm (Proposition 10).

Specifically, assume that the receiver bundles states into analogy classes according to an interval analogy partition of the following form:

$$\mathcal{A} = \Big\{ \underbrace{[0, y_1)}_{\alpha_1}, \underbrace{[y_1, y_2)}_{\alpha_2}, \ldots, \underbrace{[y_{k-1}, y_k)}_{\alpha_k}, \ldots, \underbrace{[y_{n-1}, 1]}_{\alpha_n} \Big\}.$$

From the viewpoint of the learning story outlined in Section 3, this amounts to assuming the feedback about Senders' strategies observed by Receivers takes the form of an aggregate communication strategy for every state $s \in [y_{k-1}, y_k)$ for $k = 1, \ldots, n$.

Given this analogy partition, the strategy of the sender perceived by the receiver is, for $t \in [y_{k-1}, y_k)$,

$$\bar{\sigma}_S(t) = \int_{y_{k-1}}^{y_k} \frac{\sigma_S(s)}{F(y_k) - F(y_{k-1})} f(s)\, \mathrm{d}s.$$

By Bayes' rule we get the belief $\mu(t \mid m) = \bar{\sigma}_S(m \mid t) f(t) / \int_0^1 \sigma_S(m \mid s) f(s)\,\mathrm{d}s$ and the best-response of the receiver to $\bar{\sigma}_S$, for all $m \in \operatorname{supp}[\sigma_S]$ is[25]:

$$\sigma_R(m) = \sum_{k=1}^{n} \Pr\big[t \in \alpha_k \bigm| t \in \sigma_S^{-1}(m)\big] E(t \mid \alpha_k). \tag{3}$$

In short, the best-response requires for the receiver to choose a convex combination of the average value of t in each class α_k, $E(t \mid \alpha_k)$, where the weight of $E(t \mid \alpha_k)$ is determined by the likelihood that t lies in α_k given that message m was sent. Note that the receiver chooses the rational action $\sigma_R(m_l) = E(t \mid t \in [x_{l-1}, x_l))$ only if his analogy partition is finer than the partition generated by the sender's strategy, for example when $x_k = y_k$ for all k.

Given the form of the preferences, it is readily verified that as in the standard rationality paradigm case, any ABEE must take a partitional form described by $\sigma_S(t) = m_l$ if $t \in [x_{l-1}, x_l)$, $l = 1, \ldots, L$, where $0 = x_0 < x_1 < \cdots < x_{L-1} < x_L = 1$.[26] In addition, we show in the following proposition that the outcome of an ABEE is never ex ante biased, which follows from the equilibrium requirement that the receiver correctly perceives the *average* strategy of the sender. This should be contrasted with Kartik et al. (2007) who exhibit biased equilibrium outcomes in a variant of Crawford and Sobel's (1982) model where the receiver is naive (i.e., blindly plays the recommended action) with an exogenous and strictly positive probability.[27]

Proposition 7. *At an analogy-based expectation equilibrium the action taken by the receiver is never ex ante biased, i.e. the action made by the receiver is $E(t)$ on average.*

An important insight in Crawford and Sobel's analysis is that the number of meaningful messages sent in equilibrium is bounded whenever $b > 0$. We will show that this is also the case in our setting and that there is an upper bound on the number of messages that applies uniformly over all possible interval analogy partitions. This is to be contrasted with Blume et al. (2007) who show that an arbitrarily large number of meaningful messages can be sent in a noisy variant of the cheap talk game in which the receiver does not always observe the correct message sent by the sender.

We first observe that the number of analogy classes used by the receiver gives an upper bound on the informativeness of an equilibrium. This is essentially because if two different messages were sent with probability 1 in states lying in the same analogy class, then the receiver would react in the same way to these two messages, thereby making it equivalent to just one message. Formally, we refer to an ABEE (σ_S, σ_R) as an *L-step ABEE* if σ_S is an L-step communication strategy and $\sigma_R(m) \neq \sigma_R(m')$ for all $m, m' \in \operatorname{supp}[\sigma_S]$, $m \neq m'$. We have:

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25 Indeed, we have: $\sigma_R(m) = E_\mu(t \mid m) = \int_0^1 \mu(t \mid m) t\,\mathrm{d}t = \sum_{k=1}^n \int_{y_{k-1}}^{y_k} \frac{\Pr[\sigma_S^{-1}(m) \mid \alpha_k] f(t)}{\Pr[\sigma_S^{-1}(m)]} t\,\mathrm{d}t = \sum_{k=1}^n \frac{\Pr[\sigma_S^{-1}(m) \mid \alpha_k]}{\Pr[\sigma_S^{-1}(m)]} \int_{y_{k-1}}^{y_k} t f(t)\,\mathrm{d}t = \sum_{k=1}^n \frac{\Pr[\sigma_S^{-1}(m) \cap \alpha_k]}{\Pr[\sigma_S^{-1}(m)]} E(t \mid \alpha_k)$. To complete the characterization of the best-response we may assume that a message that is not played (i.e., not in $\operatorname{supp}[\sigma_S]$) is identified with one of the messages in $\operatorname{supp}[\sigma_S]$ with the corresponding best-response. With such an assumption, it is enough for equilibrium characterization to restrict the message space to the space of messages played in equilibrium.

26 This follows by looking at the best-response of the sender and noting that there is a one to one correspondence between the message sent and the choice of action by the receiver.

27 Ex ante biased equilibrium actions are also obtained by Morgan and Stocken (2003) in another variant of the model in which there are two types of senders (with aligned incentives or not).

Proposition 8. *If the receiver uses an interval analogy partition with n analogy classes then there exists no L-step analogy-based expectation equilibrium with $L > 2n - 1$.*

A simple corollary is that when the receiver uses the coarsest analogy partition ($n = 1$),[28] the unique ABEE is pooling: the receiver chooses action $\sigma_R(m) = E(t)$ for all m.

From Proposition 8, it might seem possible that the number L of messages grows without bound as the number n of analogy classes increases. This is however ruled out by the next proposition. A uniform bound applies essentially because either analogy classes are small and the Crawford and Sobel's bound applies or analogy classes are big but then few meaningful messages can be sent in states lying in such analogy classes. Formally, we let $\lceil \frac{1}{b} \rceil$ be the smallest integer that is no smaller than $\frac{1}{b}$. We have:

Proposition 9. *No matter what the interval analogy partition is, there exists no L-step analogy-based expectation equilibrium with $L > 6\lceil \frac{1}{b} \rceil$.*

Even though the number of messages is bounded as in Crawford and Sobel's analysis, our next and main observation is that sometimes more messages can be sent with the analogy approach than with the standard approach. More precisely, we observe that for some analogy partitions there may exist a partially revealing ABEE even when the unique NE does not allow for information transmission. Furthermore, this ABEE may Pareto dominate (ex ante) all Nash equilibria.

Proposition 10. *Assume that types are uniformly distributed and the receiver has an analogy partition with only $n = 2$ analogy classes, $\mathcal{A} = \{[0, y), [y, 1]\}$, with $y \geqslant 1/2$. If $b < \frac{1+y}{4}$ then there exists a 2-step ABEE*

$$\sigma_S(t) = \begin{cases} m_1 & \text{if } t < x^*(b, y), \\ m_2 & \text{if } t \geqslant x^*(b, y), \end{cases}$$

where

$$x^*(b, y) = \frac{2 - 2b + y - \sqrt{4b(2 + b) - 4by + y^2}}{4} < y.$$

Such an ABEE may Pareto dominate (ex ante) all Nash equilibria (the pooling as well as the 2-step NE when it exists).[29]

Since a 2-step NE exists in the uniform case if and only if $b < 1/4 < \frac{1+y}{4}$, the proposition above shows that a 2-step ABEE exists when a 2-step NE exists and, more importantly, that a 2-step ABEE may exist even when all NE are pooling, i.e., whenever $b \in (1/4, (1 + y)/4)$. In light of Propositions 8 and 9 it might be surprising that a coarse analogy partition might permit more information transmission than in the standard approach. The reason is as follows. In the standard

[28] Notice that the coarsest analogy partition also coincides with the receiver's private information analogy partition, and the finest partition coincides with the payoff-relevant partition.

[29] Notice that, as in Crawford and Sobel (1982), equilibrium outcomes cannot be compared according to interim efficiency since different sender's types prefer different equilibria (e.g., the sender of type $t = 1/2 - b$ always prefers the pooling equilibrium). Krishna and Morgan (2004) also show that it is possible to Pareto improve, ex ante, the NE found in Crawford and Sobel (1982), by allowing multiple rounds of communication and jointly controlled lotteries. However, to get information transmission for $b > 1/4$ they have to consider non-monotonic equilibria, whereas the ABEE characterized here is monotonic (i.e., the action taken by the receiver is a non-decreasing function of the sender's type).

approach, if the receiver expects the sender to follow the strategy $\sigma_S(\cdot)$ and the state is $x^*(b, y)$ the sender strictly prefers to send message m_2 thereby inducing action $(1 + x^*)/2$ rather than message m_1 that would result in action $x^*/2$ ($x^* + b$ is closer to $(1 + x^*)/2$ than to $x^*/2$ when $b > 1/4$). The analogy grouping of the receiver changes that conclusion. Because x^* is smaller than y, when message m_1 is being sent, the action of the receiver is biased upwards relative to the standard case (when m_1 is sent the action is now $y/2$ instead of $x^*/2$), and consequently when message m_2 is being sent the action of the receiver is biased downwards (this is because by Proposition 7 the action taken by the receiver is ex ante unbiased as in the standard case). These biases make message m_1 (resp. m_2) relatively more (resp. less) attractive than in the standard case, which in turn permits to sustain a 2-step ABEE.

7. Zero-sum betting

We consider the following framework: A gamble (zero-sum bet) $x : \Omega \to \mathbb{R}$ is proposed to the players. They simultaneously decide to bet (action B) or to drop out (action D). If both players bet and the state is ω player 1 gets $x_1(\omega) = x(\omega)$ and player 2 gets $x_2(\omega) = -x(\omega)$. Betting costs ε to a player regardless of whether the other player bets or not (ε is assumed to be arbitrarily small). Thus, the payoff of player i at ω is

$$u_i(a_1, a_2; \omega) = \begin{cases} x_i(\omega) - \varepsilon & \text{if } a_1 = a_2 = B, \\ -\varepsilon & \text{if } a_i = B \text{ and } a_j = D, \\ 0 & \text{if } a_i = D. \end{cases}$$

The framework is the same as that of Sebenius and Geanakoplos (1983) and Perets and Sonsino (1999) who assume that $\varepsilon = 0$ and that players bet only if they expect strictly positive payoffs.[30] In the standard rationality paradigm, we get the celebrated no trade theorem (Milgrom and Stockey, 1982): In all zero-sum betting games with $\varepsilon > 0$ the unique NE consists for both players to drop out from betting in all states of the world.

For general analogy systems though some betting is possible. In particular, define a naive (or non-strategic) betting strategy as follows:

Definition 3. The strategy of player i is a *naive betting strategy*[31] if

$$\begin{aligned} &\sigma_i(B \mid \omega) = 1 \quad \text{if } E\big[x_i \mid P_i(\omega)\big] > 0, \\ &\sigma_i(D \mid \omega) = 1 \quad \text{if } E\big[x_i \mid P_i(\omega)\big] \leqslant 0. \end{aligned}$$

Obviously, when players use the coarsest analogy partition, the naive betting strategy profile is an ABEE for $\varepsilon = 0$.[32] More generally, keeping $\mathcal{A}_i$ coarser than $\mathcal{P}_i$ and adding a small $\varepsilon > 0$ cost to betting, naive betting remains an equilibrium when the perceived chances of profitable bets are bounded away from 0.

[30] Assuming that ε is sufficiently small but strictly positive is essentially equivalent to assuming that $\varepsilon = 0$ and that each player bets only if he expects strictly positive payoff.

[31] It is referred to as naive because it is defined independently of player j's strategy; so such a betting requires no knowledge of the opponent's strategy.

[32] It is readily verified that the naive strategy profile is the only ABEE that satisfies the trembling-hand refinement (imposing that any action must be played with some arbitrarily small probability).

Proposition 11. *Assume that $\mathcal{A}_i$ is coarser than $\mathcal{P}_i$ and let $Y_i = \{\omega \in \Omega\colon\ E[x_i \mid P_i(\omega)] > 0\}$ for all $i \in \{1,2\}$.*[33]

(i) *If $\varepsilon = 0$ then the naive strategy profile is an analogy-based expectation equilibrium.*
(ii) *If for all $i \in \{1,2\}$ and $\omega \in Y_i$ we have $\alpha_i(\omega) \cap Y_j \neq \emptyset$, then there exists $\varepsilon > 0$ such that the naive strategy profile is an analogy-based expectation equilibrium.*

The next question of interest is whether there may be situations in which there is some betting and yet players do not follow the naive betting strategy. The answer is affirmative, as will be shown in Example 4. In this example, we introduce an analogy system referred to as the "two-type error" analogy system, in which a player bundles states according to whether or not he wins money in case of mutual bet. It is called the two-type error analogy system because a player bases his decision on the frequency of type I and type II errors of his opponent corresponding to the frequency with which the opponent bets whereas he should not and the frequency with which the opponent does not bet whereas he should have.

Example 4. There are five states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ which all have the same prior probability. Players' information partitions are given by

$$\mathcal{P}_1 = \big\{\{\omega_1, \omega_4, \omega_5\}, \{\omega_2, \omega_3\}\big\},$$
$$\mathcal{P}_2 = \big\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}\big\}.$$

Player 1's payoffs are given by

$$x(\omega) = \begin{cases} 9 & \text{for } \omega = \omega_1, \\ -8 & \text{for } \omega = \omega_2, \\ 7 & \text{for } \omega = \omega_3, \\ -9 & \text{for } \omega = \omega_4, \\ 1 & \text{for } \omega = \omega_5. \end{cases}$$

The two-type error analogy system is thus defined by $\mathcal{A}_1 = \mathcal{A}_2 = \{\{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4\}\}$. Naive betting would require that player 1 bets in $\{\omega_1, \omega_4, \omega_5\}$ and that player 2 bets in $\{\omega_3, \omega_4, \omega_5\}$. It is easily seen that this is not an ABEE (player 1 deviates in $\{\omega_2, \omega_3\}$). But, the following strategy profile is an ABEE, as we now check: player 1 bets always (hence he does not follow the naive betting strategy) and player 2 follows the naive betting strategy (i.e., he bets in $\{\omega_3, \omega_4, \omega_5\}$). Given that player 1 always bets, the naive betting strategy is optimal for player 2. Also,

$$\bar{\sigma}_2(\omega) = \begin{cases} \frac{2}{3}B + \frac{1}{3}D & \text{for } \omega \in \{\omega_1, \omega_3, \omega_5\}, \\ \frac{1}{2}B + \frac{1}{2}D & \text{for } \omega \in \{\omega_2, \omega_4\}. \end{cases}$$

Thus (for ε small enough), at $\{\omega_1, \omega_4, \omega_5\}$ player 1 finds it optimal to bet because $\frac{2}{3}x(\omega_1) + \frac{1}{2}x(\omega_4) + \frac{2}{3}x(\omega_5) > 0$ and at $\{\omega_2, \omega_3\}$ he finds it optimal to bet because $\frac{1}{2}x(\omega_2) + \frac{2}{3}x(\omega_3) > 0$.

[33] Y_i is the set of states in which the naive strategy of player i is to bet with probability one.

8. Conclusion

This paper has revisited a number of classic games with incomplete information relaxing the cognitive skills of the players about their ability to know or learn the strategy of their opponent. It was shown how deeply the strategic interaction is affected by such a change in the cognitive abilities of the players. Of course, the qualitative nature of the effect depends on how the players bundle states of the world into analogy classes. Analyzing which analogy groupings are effectively used in each application would require further work. From an experimental viewpoint, the methodology developed in Huck et al. (2007) can be used for that purpose.

Acknowledgments

We are grateful to Ehud Kalai, the associate editor and an anonymous referee for helpful comments and suggestions. We also thank seminar participants at Jena (Max Planck Institute), Paris Game Theory Seminar (IHP), Strasbourg (BETA), Tilburg, Cergy-Pontoise, Jerusalem (IAS), Alicante, the Second World Congress of the Game Theory Society, the 2004 European Summer Symposium in Economic Theory, and the First European Conference on Cognitive Economics.

Appendix A. Proofs

Proof of Proposition 1. The mapping (referred to as the P mapping) that maps the strategy profile (σ_i, σ_j) into the perceived strategy profile $(\bar{\sigma}_i, \bar{\sigma}_j)$ (see Definition 2) is continuous. The best-response mapping (referred to as BR) that maps the expectation profile $(\bar{\sigma}_i, \bar{\sigma}_j)$ into the set of strategy profiles (σ_i, σ_j) such that σ_i is a best-response to $\bar{\sigma}_j$ is upper hemicontinuous. It follows from Kakutani's (1941) theorem that there is a fixed point of the compound mapping $(\sigma_i, \sigma_j) \rightarrow_P (\bar{\sigma}_i, \bar{\sigma}_j) \rightarrow_{BR} (\sigma_i, \sigma_j)$. Such a fixed point is an ABEE. $\square$

Proof of Proposition 2. By definition, σ is a NE of $\bar{G}^{\mathcal{A}}$ if for each player i, each state ω and each action $a_i^* \in \operatorname{supp}[\sigma_i(\omega)]$,

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{a_j \in A_j} \sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big)\sigma_j(a_j \mid \omega') \sum_{\omega'' \in \Omega} p\big(\omega'' \mid \alpha_i(\omega')\big)u_i(a; \omega''). \tag{4}$$

We have

$$\begin{aligned}
&\sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big)\sigma_j(a_j \mid \omega') \sum_{\omega'' \in \Omega} p\big(\omega'' \mid \alpha_i(\omega')\big)u_i(a; \omega'') \\
&\quad = \frac{1}{p(P_i(\omega))} \sum_{\omega' \in P_i(\omega)} p(\omega')\sigma_j(a_j \mid \omega') \frac{1}{p(\alpha_i(\omega'))} \sum_{\omega'' \in \alpha_i(\omega')} p(\omega'')u_i(a; \omega'').
\end{aligned}$$

Since $\mathcal{A}_i$ is finer than $\mathcal{P}_i$, the last expression can be rewritten, with $P_i(\omega) = \bigcup_k \alpha^k$ and $\alpha^k \in \mathcal{A}_i$ for all k,

$$\begin{aligned}
&\frac{1}{p(P_i(\omega))} \sum_k \sum_{\omega' \in \alpha^k} p(\omega')\sigma_j(a_j \mid \omega') \frac{1}{p(\alpha^k)} \sum_{\omega'' \in \alpha^k} p(\omega'')u_i(a; \omega'') \\
&\quad = \frac{1}{p(P_i(\omega))} \sum_k \sum_{\omega'' \in \alpha^k} p(\omega'')u_i(a; \omega'') \sum_{\omega' \in \alpha^k} p(\omega' \mid \alpha^k)\sigma_j(a_j \mid \omega')
\end{aligned}$$

$$= \frac{1}{p(P_i(\omega))} \sum_k \sum_{\omega'' \in \alpha^k} p(\omega'') u_i(a; \omega'') \overline{\sigma}_j(a_j \mid \omega'')$$

$$= \frac{1}{p(P_i(\omega))} \sum_{\omega' \in P_i(\omega)} p(\omega') u_i(a; \omega') \overline{\sigma}_j(a_j \mid \omega')$$

$$= \sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big) \overline{\sigma}_j(a_j \mid \omega') u_i(a; \omega').$$

Thus, Eq. (4) is equivalent to $a_i^* \in \arg\max_{a_i \in A_i} \sum_{\omega' \in \Omega} p(\omega' \mid P_i(\omega)) \sum_{a_j \in A_j} \overline{\sigma}_j(a_j \mid \omega') u_i(a; \omega')$, which means that σ is an ABEE of $(G, \mathcal{A})$. □

Proof of Proposition 3. Let $\mathcal{M}_i = \mathcal{P}_i \wedge \mathcal{A}_i$ be the *meet* (finest common coarsening) of player i's information and analogy partitions, and let σ be an ABEE of $(G, \mathcal{A})$. By assumption, σ_i is measurable w.r.t. $\mathcal{A}_i$, and since it is measurable w.r.t. $\mathcal{P}_i$ it is also measurable w.r.t. $\mathcal{M}_i$. So, σ_i will remain a best response to $\overline{\sigma}_j$ if we replace i's information partition $\mathcal{P}_i$ by $\mathcal{M}_i$. In this new strategic environment $\mathcal{A}_i$ is finer than i's information partition, $\mathcal{M}_i$, so by Proposition 2 σ is a NE of the virtual Bayesian game $\langle \Omega, p, \mathcal{M}_1, \mathcal{M}_2, A_1, A_2, \bar{u}_1^{\mathcal{A}}, \bar{u}_2^{\mathcal{A}} \rangle$. But, since by assumption the payoff functions are independent of the state of the world, σ is a NE of $\langle \Omega, p, \mathcal{M}_1, \mathcal{M}_2, A_1, A_2, u_1, u_2 \rangle$. By the definition of a correlated equilibrium, this NE generates a correlated equilibrium distribution of $\langle A_1, A_2, u_1, u_2 \rangle$. □

Proof of Proposition 4. Player 1 plays D at G_a, so player 2 plays D at $\{G_a, (G_b, 0)\}$. In all other information sets players know G_b and have the same beliefs about the other player's strategy. Hence, if it is optimal to attack in some of these information sets then it is at all. So, an ABEE with coordinated attack exists iff all players always attack in these information sets. In that case, perceived strategies are $\overline{\sigma}_1(G_a) = \overline{\sigma}_2(G_a) = (1, 0)$, $\overline{\sigma}_1(G_b) = (0, 1)$ and $\overline{\sigma}_2(G_b) = (\varepsilon, 1 - \varepsilon)$. Player 1's perceived expected payoff when he attacks is $(1 - \varepsilon)M + \varepsilon(-L)$, which is positive (larger than when he plays D) if $\varepsilon \leqslant \frac{M}{M+L}$. Player 2's payoff when he attacks is M, so it is always optimal. □

Proof of Proposition 5. This is a corollary of the analysis of the "finest" analogy system allowing coordinated attack. □

Proof of Proposition 6. First consider the case $\mathcal{A}_2 = \{\Omega\}$. Since player 1 does not attack at G_a, the strategy of player 1 perceived by player 2 is such that $\overline{\sigma}_1(D) \geqslant 1 - p$. Hence, if player 2 attacks his perceived expected payoff is less than $pM - (1 - p)L < 0$ in all states of the world, so he never attacks. Consequently, player 1 never attacks whatever his analogy partition. Now, consider the case $\mathcal{A}_1 = \{\Omega\}$. Player 1 still plays his dominant strategy (D) at $\{G_a\}$. Consider player 2 at $\{G_a, (G_b, 0)\}$ and let $q_1 = \overline{\sigma}_1(G_a)(D)$ and $q_2 = \overline{\sigma}_1(G_b, 0)(D)$ be the average probability that player 1 chooses action D in the analogy classes used by player 2 at G_a and $(G_b, 0)$ respectively. Since $1 - p > 1/2$ and since player 1 plays D at G_a we necessarily have $q_1 > 1/2$. If player 2 chooses D, his perceived expected payoff is

$$\frac{1 - p}{1 - p + p\varepsilon} q_1 M > \frac{1 - p}{1 - p + p\varepsilon} \frac{1}{2} M.$$

If player 2 chooses A, his perceived expected payoff is

$$\frac{1-p}{1-p+p\varepsilon}\bigl(-q_1L-(1-q_1)\delta\bigr)+\frac{p\varepsilon}{1-p+p\varepsilon}\bigl(-q_2L+(1-q_2)M\bigr)$$
$$<-\frac{1-p}{1-p+p\varepsilon}\frac{1}{2}M+\frac{p\varepsilon}{1-p+p\varepsilon}M.$$

Hence, a necessary condition for A to be optimal is

$$-\frac{1-p}{1-p+p\varepsilon}\frac{1}{2}M+\frac{p\varepsilon}{1-p+p\varepsilon}M\geqslant\frac{1-p}{1-p+p\varepsilon}\frac{1}{2}M$$
$$\Leftrightarrow\quad -\frac{1-p}{2}M+p\varepsilon M\geqslant\frac{1-p}{2}M\quad\Leftrightarrow\quad p\varepsilon\geqslant 1-p,$$

which is impossible since $1-p>p$. Thus, we have $\bar{\sigma}_2(D)>1/2$, which implies that player 1 plays D in all states, so player 2 also always plays D.

In the private information analogy system case, the argument is almost the same as in the standard framework. Player 1 plays D at his information set $\{G_a\}$ because it is a strictly dominant strategy. Consider now player 2 at his information set $\{G_a,(G_b,0)\}$. If player 1 plays D at $\{(G_b,0),(G_b,1)\}$ then his strategy perceived by player 2 at $\{G_a,(G_b,0)\}$ is $\bar{\sigma}_1=(1,0)$, so player 2 plays D. If player 1 plays A at $\{(G_b,0),(G_b,1)\}$ then his strategy perceived by player 2 at $\{G_a,(G_b,0)\}$ is $\bar{\sigma}_1=(\frac{1-p}{1-p+p\varepsilon},\frac{p\varepsilon}{1-p+p\varepsilon})$. Player 2 gets a positive perceived expected payoff if he plays D. If he plays A then his perceived expected payoff is

$$\frac{1-p}{1-p+p\varepsilon}\left(\frac{1-p}{1-p+p\varepsilon}(-L)+\frac{p\varepsilon}{1-p+p\varepsilon}(-\delta)\right)$$
$$+\frac{p\varepsilon}{1-p+p\varepsilon}\left(\frac{1-p}{1-p+p\varepsilon}(-L)+M\frac{p\varepsilon}{1-p+p\varepsilon}\right)<0,$$

so player 2 necessarily plays D at his information set $\{G_a,(G_b,0)\}$. Consider now player 1 at his information set $\{(G_b,0),(G_b,1)\}$. The strategy of player 2 perceived by player 1 at this information set puts at least probability $\frac{p\varepsilon}{p\varepsilon+p(1-\varepsilon)\varepsilon}=\frac{1}{2-\varepsilon}>1/2$ to action D, so player 1's best response is D. The same reasoning applies inductively to all other players' information sets, so both players always play D. □

Proof of Proposition 7. We show that the average action taken by the receiver is the ex ante expected sender's type. This average action is

$$\int_0^1\sigma_R\bigl(\sigma_S(t)\bigr)\,\mathrm{d}t=\int_0^1\sigma_R(m)\Pr\bigl[\sigma_S^{-1}(m)\bigr]\,\mathrm{d}m=\int_0^1\sum_{k=1}^{n}\Pr\bigl[\sigma_S^{-1}(m)\cap\alpha_k\bigr]E(t\mid\alpha_k)\,\mathrm{d}m$$
$$=\sum_{k=1}^{n}E(t\mid\alpha_k)\Pr(\alpha_k)=E(t),$$

where the second equality follows from Eq. (3). □

Proof of Proposition 8. Consider an L-step partitional communication strategy of the form $\sigma_S(t)=m_l$ if $t\in[x_{l-1},x_l)$, $l=1,\dots,L$, where $0=x_0<x_1<\cdots<x_{L-1}<x_L=1$ and $m_k\neq m_l$ $\forall k\neq l$. We will show that $L\leqslant 2n-1$. For this, it suffices to prove the three following

assertions: (i) $y_1 < x_2$; (ii) $x_{L-2} < y_{n-1}$; (iii) if $y_{k-1} \leqslant x_{l-1} < x_l \leqslant y_k$ then $x_{l-2} < y_{k-1}$ and $x_{l+1} > y_k$. If (i) were not true then we would have $\sigma_R(m_1) = \sigma_R(m_2) = E(t \mid \alpha_1)$, a contradiction with an L-step ABEE. Similarly, if (ii) were not true then $\sigma_R(m_{L-1}) = \sigma_R(m_L) = E(t \mid \alpha_n)$, a contradiction. Finally, to prove (iii) assume that $y_{k-1} \leqslant x_{l-1} < x_l \leqslant y_k$ but $x_{l-2} \geqslant y_{k-1}$ or $x_{l+1} \leqslant y_k$. In the former case we get $y_{k-1} \leqslant x_{l-2} < x_{l-1} < x_l \leqslant y_k$ so $\sigma_R(m_{l-1}) = \sigma_R(m_l) = E(t \mid \alpha_k)$ and in the latter case we get $y_{k-1} \leqslant x_{l-1} < x_l < x_{l+1} \leqslant y_k$ so $\sigma_R(m_l) = \sigma_R(m_{l+1}) = E(t \mid \alpha_k)$, a contradiction. □

Proof of Proposition 9. Consider an L-step partitional communication strategy as in the proof of Proposition 8. We first observe the following additional assertion: (iv) If $y_k \leqslant x_l < x_{l+1} < x_{l+2} \leqslant y_m$ then $y_m - y_k > b$. This is because $\sigma_R(m_{l+1})$ and $\sigma_R(m_{l+2})$ both belong to $[y_k, y_m]$ (see the best response (3) of the receiver), and $\sigma_R(m_{l+1}) \leqslant x_{l+1} + b \leqslant \sigma_R(m_{l+2})$ by the sender's equilibrium condition, so $y_m - y_k > \sigma_R(m_{l+2}) - x_{l+1} \geqslant b$. Next, we give a bound on the number of messages sent in each interval of size b:

Assume $y_3 < b$. Take the largest k such that $y_k < b$. By assertion (iv), the largest l such that $x_l \leqslant y_k$ is at most 1. So, by assertion (iii) in the proof of Proposition 8, the largest l such that $x_l \leqslant y_{k+1}$ is at most 3. Hence, the largest l such that $x_l \leqslant b$ is at most 3.

Assume $y_3 \geqslant b$. By assertion (iii) the largest l such that $x_l \leqslant b$ is at most 5.

We deduce that for every y_3 the largest l such that $x_l \leqslant b$ is at most 5. We repeat inductively this argument for states lying in $[mb, (m+1)b]$ to show that $L \leqslant 6\lceil \frac{1}{b} \rceil$. □

Proof of Proposition 10. From Eq. (3), the best response of the receiver is

$$\sigma_R(m) = \begin{cases} \frac{y}{2} & \text{if } m = m_1, \\ \frac{1-xy}{2(1-x)} & \text{if } m = m_2. \end{cases}$$

The sender of type t sends the message $m \in M$ such that the response $\sigma_R(m)$ is the closest as possible to $t + b$.[34] Hence, he sends

$$\begin{cases} m_1 & \text{if } t + b < c, \\ m_2 & \text{if } t + b > c, \end{cases}$$

where $c = \frac{1+y-2xy}{4(1-x)}$ is the midpoint of $[\sigma_R(m_1), \sigma_R(m_2)]$. So, to have an equilibrium we must have $x = \frac{1+y-2xy}{4(1-x)} - b$, which gives the solution of the proposition. It can be checked that x^* is always decreasing with b and that

$$x^*(b, y) = \begin{cases} 1/2 & \text{if } b = 0, \\ 0 & \text{if } b = (1+y)/4, \end{cases}$$

so we always have $x^* < y$. We thus have a 2-step ABEE whenever $x^* > 0$, i.e., $b < \frac{1+y}{4}$. To see that such an ABEE may Pareto dominate (ex ante) all Nash equilibria let $y = 1/2$ and $b = 1/4$. Since $b < (1+y)/4 = 3/8$ the unique NE is pooling (the receiver always chooses $\sigma_R(m) = 1/2$) whereas there exists a 2-step ABEE with $x^* = 1/2 - \sqrt{2}/4 \simeq 0.146$, which leads to action $\sigma_R(m_1) = 1/4$ when $t < x^*$ and $\sigma_R(m_2) = 5/4 - 1/\sqrt{2} \simeq 0.543$ when $t \geqslant x^*$. It can be computed that the expected payoff EU_R of the receiver is $-16/192$ at the pooling equilibrium and $(-7+$

[34] To avoid deviations to off-the-equilibrium messages let, e.g., $\sigma_R(m) = y/2$ for all $m \neq m_2$. That is, any unplayed message is interpreted as m_1.

$3\sqrt{2})/48 \simeq -11.029/192$ at the 2-step ABEE, so this ABEE Pareto dominates the (unique) NE (the expected payoff of the sender is always $EU_R - b^2$). Finally, to illustrate that the 2-step ABEE may Pareto dominates all NE even when there also exists a 2-step NE, let $b = 1/8$. In that case, there is a 2-step NE, characterized by $x^* = 1/4$, $\sigma_R(m_1) = 1/8$, $\sigma_R(m_2) = 5/8$, and $EU_R = -7/192$, but since $b > 1/12$ there is no L-step NE for $L \geqslant 3$.[35] The 2-step ABEE is characterized by $x^* = (9 - \sqrt{17})/16 \simeq 0.305$, $\sigma_R(m_1) = 1/4$, $\sigma_R(m_2) = 1/4 + 4/(7 + \sqrt{17}) \simeq 0.61$, and $EU_R = (-19 + 3\sqrt{17})/192 \simeq -6.63/192$, so it Pareto dominates all NE (the pooling equilibrium and the 2-step NE). $\square$

Proof of Proposition 11. (i) If $\varepsilon = 0$ then player i's payoff at some state ω if he plays D is null and if he plays B it is equal to

$$\sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big)\bar{\sigma}_j(B \mid \omega')x_i(\omega) = \bar{\sigma}_j(B \mid \omega) \sum_{\omega' \in \Omega} p\big(\omega' \mid P_i(\omega)\big)x_i(\omega),$$

because $\bar{\sigma}_j$ is $\mathcal{P}_i$-measurable ($\mathcal{A}_i$ is coarser than $\mathcal{P}_i$). It is therefore (perceived to be) optimal for player i to bet if $E[x_i \mid P_i(\omega)] > 0$ and to drop out otherwise, which means that the naive strategy is always a (perceived) best response and thus the naive strategy profile is an ABEE.

(ii) Let σ be the naive strategy profile, i.e., $\sigma_i(B \mid \omega) = 1$ if $\omega \in Y_i$ and $\sigma_i(B \mid \omega) = 0$ if $\omega \notin Y_i$. If $\omega \notin Y_i$ then the perceived expected payoff of player i if he bets is equal to $\bar{\sigma}_j(B \mid \omega)E[x_i \mid P_i(\omega)] - \varepsilon \leqslant 0$, so to play D at ω is (perceived as) optimal for player i. If $\omega \in Y_i$, then $\alpha_i(\omega) \cap Y_j \neq \emptyset$ implies that $\bar{\sigma}_j(B \mid \omega) > 0$. Since $E[x_i \mid P_i(\omega)] > 0$ this implies that there exists $\varepsilon > 0$ such that the perceived expected payoff of player i if he bets is positive, so to play B at ω is (perceived as) optimal for player i. $\square$

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[35] Remember Crawford and Sobel's (1982) condition $b < 1/(2L(L-1))$ for an L-step NE to exist.

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