Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions*

Ed Hopkins†
Edinburgh School of Economics
University of Edinburgh
Edinburgh EH8 9JY, UK

Tatiana Kornienko‡
Department of Economics
University of Stirling
Stirling FK9 4LA, UK

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Abstract

This paper analyses comparative statics for first price auctions and all pay auctions with independent private values. In all pay auctions, bidders with low values will respond to a stochastically higher (in the sense of likelihood ratio dominance) distribution of types by playing less aggressively while high value bidders bid more. In the first price auction, a similar change results in all types playing more aggressively. Furthermore, we show that a decrease in dispersion of values, in the sense of a refinement of second order stochastic dominance, although also associated with an increase in competitiveness, may in addition result in less aggressive play by bidders with high values in both auction forms. We also find similar considerations in an oligopoly game with incomplete information: stochastically lower costs can lead to higher prices.

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†E.Hopkins@ed.ac.uk, http://homepages.ed.ac.uk/hopkinse
‡tatiana.kornienko@stir.ac.uk, http://staff.stir.ac.uk/tatiana.kornienko/

1 Introduction

Stochastic dominance relationships are essential to comparative statics in games of incomplete information, such as auctions. However, even a strong ordering of two random variables - first order stochastic dominance - can be insufficient to ensure unambiguous comparisons in some auctions (see, for example, Maskin and Riley, 2000a, footnote 14). As a consequence, several strengthenings of first order stochastic dominance have been introduced, including the monotone likelihood ratio order used for a wide class of examples (Athey, 2002) and the monotone probability ratio order (also known as conditional stochastic dominance or the reverse hazard rate order) used in auctions (Lebrun, 1998; Maskin and Riley, 2000a). These orderings of distributions allow comparative statics in games of incomplete information involving changes in distributions of general rather than specific functional form.

In this paper, we focus on the first price and the all pay auctions.[1] We assume independent private values but allow for risk aversion. For these auctions, we extend the comparative statics analysis based on likelihood ratio orders in two ways. First, we show that the standard first price auction and the all pay formats have qualitatively different comparative statics predictions. Specifically, there is a difference in response by low value bidders to a change in the distribution of types in the sense of a strong refinement of first order stochastic dominance. In the first price auction, even low types are motivated, so that a stochastically higher distribution of types leads to more aggressive bidding by all. On the contrary, in all pay auctions low-value bidders are discouraged, so that in the more competitive environment they compete less hard. Further, we show that the same considerations exist in oligopoly in that a stochastically lower distribution of costs can lead some firms to charge higher prices.

Second, while being powerful analytical tools, monotone orderings are very restrictive, ruling out many interesting cases. Being refinements on first order stochastic dominance, they offer no predictions for changes in the distributions that satisfy second order but not first order dominance. Informally speaking, this involves transformations leading to valuations (or signals, etc.) being "less dispersed" but not necessarily "higher" than before. There has been little work on the comparative statics arising from a change in distributions in terms of dispersion. For example, in auctions incomplete information simply means that there is some uncertainty about the values held by other bidders. What happens if there is a decrease in the level of uncertainty? With this question in mind, we employ a refinement of second order stochastic dominance based on the unimodality of the likelihood ratio, introduced by Ramos, Ollero and Sordo (2000). Intuitively, one would expect that such decrease in dispersion of types would lead to uniformly more aggressive play. We show that in first price auctions, a reduction in dispersion in the sense of this ordering prompts most types to bid more

[1] The established convention is to refer to the winner pays first price auction as the "first price auction", and the all pay first price auction as the "all pay auction". We use this terminology as it is more familiar to most readers, even though it is not completely satisfactory as there also exists a second price all pay auction (see Krishna and Morgan (1997)).

aggressively, but the highest types may bid less, resulting in a possible "cross" of bidding functions. Along similar lines, in all pay auctions there can be a "double-cross" of bidding functions with both low and high types bidding less.

For all pay auction, the reduction in bids caused by an increase in competition can be explained in the following way. Think of a runner about to compete in a race who learns that some of the slower competitors will be replaced with faster ones. Clearly, there would be increased competition for first place amongst fast runners. However, with a faster field of competitors, slower runners would see an even lower prospect of winning and reduce costly effort and run slower.[2] Analogously, in the all pay auction, those with high values will bid more in a more competitive environment, while low value bidders will bid less. In contrast, in a standard first price auction, low value bidders do not face the same disincentive effect. This is simply because one only pays if one wins, thus the cost of raising one's bid is offset by the lower probability of having to pay it.

Existing work on comparative statics for auctions by Lebrun (1998) and Maskin and Riley (2000a), and, for a wider class of examples, by Athey (2002) has concentrated on the conditions under which a stochastically higher distribution of valuations should lead to uniformly more aggressive bidding. Here, we extend this type of result in two ways. First, we show that such a monotone shift in types is not in fact sufficient for monotone comparative statics in all pay auctions. Further, there are plausible models of oligopoly where a stochastically lower distribution of costs will lead some firms to charge higher prices. This is not to say, however, that there are no meaningful results. Rather for all pay auctions, we make precise predictions about which classes of agents will adopt higher strategies and which lower. To our knowledge, this is the first work to identify the discouragement effect under which weak competitors optimally respond to greater competition by competing less hard.

Second, we allow for a different type of change in the distribution of types which potentially has a number of applications. What happens if the distribution of types becomes less dispersed? For a private value auction, this would mean that the group of bidders becomes more homogenous. The obvious hypothesis is that bidding will be more competitive. This hypothesis that more precise information should lead to uniformly more aggressive bidding and higher selling prices has been investigated in the context of common values by Kagel and Levin (1986) and in subsequent literature for specific functional forms of preferences and distributions of signals. More recently, Goeree and Offerman (2003) investigated the effects of more precise information on the competitive bidding in a framework that nests both private and common value cases. Yet, the major drawback of this literature is that providing agents with "more precise information" has been frequently analyzed by considering two uniform distributions with different support. While being analytically convenient, this assumption is restrictive. We show that the unimodal ratio orderings could serve as an alternative technique allowing to analyze more general pairs of distributions. We show that in general a reduction in

[2] See Fershtman and Gneezy (2005) for some experimental evidence that increased competition can lead weak competitors to quit.

dispersion does not lead to uniformly more aggressive bidding, even under quite strong regularity conditions. That is, there are plausible circumstances in which more precise information will induce some agents to bid less. Again, we characterise which agents these will be, even though, unfortunately, there is little one can say about the average bid (and thus about revenue comparisons).

It is worth remembering that measures of stochastic dominance are not confined to the economics of information. Since the famous work of Atkinson (1970) they have also been important in the literature on social welfare and the comparisons of income distributions (see Lambert (1989) for a survey). However, the ordering more commonly used in this literature is (generalized) Lorenz dominance, even though it is equivalent to second order stochastic dominance (Thistle, 1989), and, thus, both measures can be interpreted in terms of inequality. More recently, income inequality and games of incomplete information have been considered together (Hopkins and Kornienko, 2004; Samuelson, 2004; Hoppe, Moldovanu and Sela, 2005) in the context of strategic social interaction, where the question has been whether increasing equality leads to greater social competition. It is hoped that this paper will be of some interest to researchers in both fields as well as in their intersection.

2 Ordering Distributions in Terms of Dispersion

Ordering distributions in terms of stochastic dominance is now a common tool in the economics of information. However, the concentration up to now has been on first order stochastic dominance and its refinements. Clearly, those working on income distributions, since the seminal work by Atkinson (1970), have had greater interest in second order stochastic dominance (equivalent to the generalized Lorenz order - see Thistle (1989)), which allows ordering of distributions in terms of dispersion or inequality. In this section, we outline a refinement of second order stochastic dominance, which, though introduced in the context of the analysis of income distributions, we will then go on to use in comparative statics.

In what follows, we consider two distinct non-negative variables X and Y with finite means μ_X and μ_Y respectively, having distribution functions F and G, respectively, with F and G both having support $[\underline{z}, \bar{z}]$ with $0 \leq \underline{z} < \bar{z}$. Assume that F and G are twice continuously differentiable and the densities f and g are strictly positive on the corresponding supports. We employ the following definition of unimodality.[3]

Definition 1 *A function $f(z)$ is unimodal around $\hat{z}$ if $f(z)$ is strictly increasing for $z < \hat{z}$ and $f(z)$ is strictly decreasing for $z > \hat{z}$.*

[3] This is a slight strengthening of standard definitions of unimodality - for example, by Dharmadhikari and Joag-Dev (1988, Chapter 1) and by An (1998). In the first source, a function $f(z)$ is unimodal if $\int_{\underline{z}}^{z} f(t)dt$ is convex on $(\underline{z}, \hat{z})$ and concave on $(\hat{z}, \bar{z})$. In the second, the function $f(z)$ has to satisfy the following: for all $\delta > 0$, the set $D_\delta = \{z \in \Omega : f(z) \geq \delta\}$ is a convex set in Ω.

The following order of distributions was first introduced by Ramos, Ollero and Sordo (2000).

Definition 2 *Two distributions F, G satisfy the Unimodal Likelihood Ratio (ULR) order and we write $F \succ_{ULR} G$ if the likelihood ratio $L(z) = f(z)/g(z)$ is unimodal and $E[X] \geq E[Y]$.* [4]

Ramos, Ollero and Sordo (2000) showed that this order implies second order stochastic dominance (equivalently generalised Lorenz dominance).

Proposition 1 *[Ramos, Ollero and Sordo (2000), Theorems 2.1 and 2.2.] If $F \succ_{ULR} G$ and $E(X) \geq E(Y)$, then $F(z)$ and $G(z)$ cross exactly once at some $\tilde{z}$ on $(\underline{z}, \bar{z})$ and F second-order stochastically dominates G.*

If the two distributions have the same means, then the ULR order implies that G would be a mean preserving spread of F. In simple terms, if $F \succ_{ULR} G$ then distribution F is either stochastically higher than G or, if it is not stochastically higher, then it is less dispersed.[5] Consider a simple example. Suppose $G(z)$ is a uniform distribution so that its density $g(z)$ is a constant, then $L(z)$ will be unimodal if $f(z)$ is unimodal, that is, it is less dispersed than $g(z)$. This example is illustrated in Figure 1. It is well-known (see, for example, Dharmadhikari and Joag-Dev (1988)) that all logconcave functions are unimodal.[6] Thus, as Ramos, Ollero and Sordo (2000) showed, if $\log L(z)$ is concave and $\mu_X \geq \mu_Y$, then $F \succ_{ULR} G$.

From our definition of unimodality, there is a unique value of z which we denote $\hat{z}_L$ which maximizes the likelihood ratio $L(z)$, with $\hat{z}_L \leq \bar{z}$. If the mode of the ratio is located at the upper bound, that is, $\hat{z}_L = \bar{z}$, we arrive at a monotone order as a special case.

Definition 3 *The two distributions F, G satisfy the Monotone Likelihood Ratio (MLR) order and we write $F \succ_{MLR} G$, if the ratio of their densities $L(z)$ is strictly increasing.*

Milgrom (1981) introduced the MLR order to the economics of information. More recently, Athey (2002) employs the MLR order to obtain monotone comparative statics in games of incomplete information. As Milgrom (1981) points out, many well known families of distributions - for example, the normal and the exponential - satisfy the MLR order. A similar set of families of distributions satisfy ULR order. One can easily verify that, for example, if F and G are both normal or both lognormal, with $\mu_X \geq \mu_Y$ and with F having strictly lower standard deviation then $F \succ_{ULR} G$.

[4] Note that the condition on the means rules out the possibility that the mode is at the lower bound which would imply that Y first order dominates X.

[5] The ULR order implies second order stochastic dominance. Remember that if a random variable X second order stochastically dominates random variable Y, any risk averse decision maker will prefer X as, by second order stochastic dominance, either it offers a higher return and/or it is less risky.

[6] For review of logconcave and logconvex functions see An (1998).

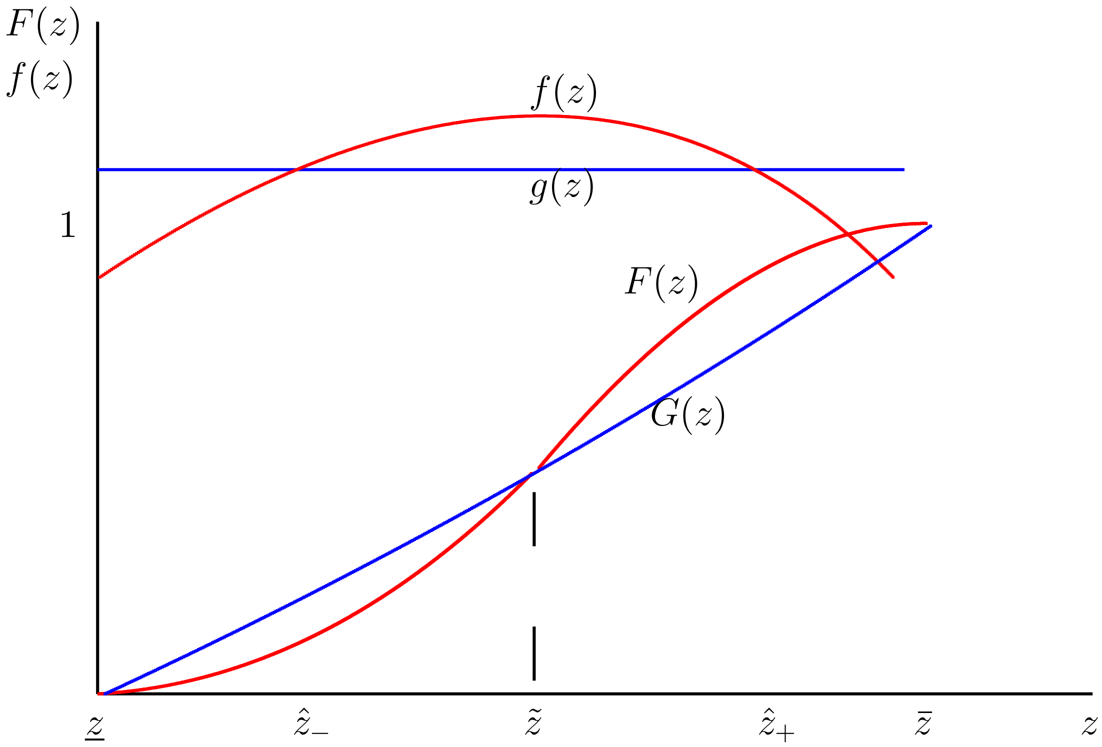

Figure 1: An Example where Distribution F ULR Dominates G

It is well-known that the MLR order implies first order stochastic dominance and other refinements of first order stochastic dominance, such as the hazard rate order and the reverse hazard rate order, see, for example, Krishna (2002, Appendix B). Similar relationships can be shown for the ULR order. Define the probability ratio and the survival ratio as respectively,

$$P(z) = \frac{F(z)}{G(z)}, \ Q(z) = \frac{1 - F(z)}{1 - G(z)}. \tag{1}$$

Proposition 2 *[Metzger and Rüschendorf (1991, Theorems 2.3 and 2.3 (c))] If $L(z)$ is unimodal with maximum at $\hat{z}_L$ then $P(z)$ is unimodal with a maximum at $\hat{z}_P \geq \hat{z}_L$ and $Q(z)$ is unimodal with a maximum at $\hat{z}_Q \leq \hat{z}_L$.*

The ratio $\sigma(z) = f(z)/F(z)$ is known as the "reverse hazard rate" in the statistics literature (see, for example, Shaked and Shanthikumar (1994)). Note that if the probability ratio of two distributions is strictly increasing then the two reverse hazard ratios are ordered, or

$$P'(z) > (<) 0 \Rightarrow \sigma_F(z) = \frac{f(z)}{F(z)} > (<) \frac{g(z)}{G(z)} = \sigma_G(z). \tag{2}$$

There is a similar relation between $Q(z)$ and the hazard ratio, which for a distribution function $F(z)$ is defined as $\lambda(z) = f(z)/(1-F(z))$. That is, it is the ratio of the density to the survival function, $1 - F(z)$. Note that

$$Q'(z) > (<)0 \Rightarrow \lambda_F(z) = \frac{f(z)}{1-F(z)} < (>) \frac{g(z)}{1-G(z)} = \lambda_G(z). \tag{3}$$

Therefore, combined with Proposition 2, these relations lead to the following corollary, which will prove useful for comparative statics.

Corollary 1 *Suppose $F \succ_{ULR} G$ then (i) $\sigma_F(z) > \sigma_G(z)$ almost everywhere on $(\underline{z}, \hat{z}_P)$, and $\sigma_F(z) < \sigma_G(z)$ almost everywhere on $(\hat{z}_P, \bar{z})$; (ii) $\lambda_F(z) < \lambda_G(z)$ almost everywhere on $(\underline{z}, \hat{z}_Q)$, and $\lambda_F(z) > \lambda_G(z)$ almost everywhere on $(\hat{z}_Q, \bar{z})$.*

Corollary 2 *Suppose $F \succ_{MLR} G$ then (i) $P'(z) > 0$ almost everywhere on $(\underline{z}, \bar{z})$, i.e. $\sigma_F(z) > \sigma_G(z)$ almost everywhere on the entire interval; (i) $Q'(z) > 0$ almost everywhere on $(\underline{z}, \bar{z})$, i.e. $\lambda_F(z) < \lambda_G(z)$ almost everywhere on the entire interval.*

3 The First Price Auction

We start with the standard (winner pays) first price auction with independent private values. We compare auctions that take place under two different distributions of valuations. We show (Proposition 4) that, when the distributions are ordered according to the ULR order, bids are higher under the less dispersed distribution except perhaps at high values. But this result also implies (Corollary 3) that when the distributions are ordered according to the MLR order, which is a special case of the ULR order and implies that the dominant distribution is first order stochastically higher, there is uniformly higher bidding under the dominant distribution.

There are $n \geq 2$ bidders each with a private value z independently drawn from a common distribution $F(z)$, which is twice differentiable with strictly positive density on its support $[\underline{z}, \bar{z}]$. Each agent makes a bid x which can be any non-negative real number. Strategies will therefore be of the form $x(z)$, a mapping from value to action. We go on to consider the effects of changes in the distribution $F(z)$ on the symmetric equilibrium strategy.

If an agent with value z wins with bid x, she is awarded the object for sale and gains a payoff $U(z-x)$, otherwise her payoff is zero. Only the winner makes a payment to the seller. We assume that $U(\cdot)$ is twice continuously differentiable with $U' > 0$ and $U'' \leq 0$ and that $U(0) = 0$. Suppose all agents adopt the same strictly increasing differentiable strategy $x(z)$, then the expected utility V of an agent of type z who bids $x(\hat{z})$, that is, as if she had type $\hat{z}$ will be

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})U(z - x(\hat{z})) \tag{4}$$

Differentiating with respect to $\hat{z}$, and setting $\hat{z}$ to z we have the following first order conditions

$$-U'(z-x)F^{n-1}(z)x'(z)+(n-1)f(z)F^{n-2}(z)U(z-x)=0 \tag{5}$$

Rearranging, we obtain the following differential equation

$$x'(z)=(n-1)\frac{f(z)}{F(z)}\frac{U(z-x)}{U'(z-x)}=(n-1)\sigma(z)\psi(x,z), \tag{6}$$

where $\sigma(z)$ is the reverse hazard rate function and $\psi(x,z)=U(z-x)/U'(z-x)$. Solution to this differential equation together with corresponding boundary condition will constitute symmetric equilibria for the auction.

Proposition 3 *[Maskin and Riley (2000b, 2003)] The unique solution to the differential equation (6), with initial conditions $x(\underline{z})=\underline{z}$ represents a symmetric equilibrium for the first price auction that is unique on $(\underline{z},\bar{z}]$.*

It turns out that the boundary conditions are important for comparative statics. Here, the lowest-value player bids right the way up to her value.[7] As we will see later, the equilibrium behaviour of the lowest type is much more aggressive in the first price auction than in all pay auction. Moreover, low value bidders in the first price auction will respond to more competitive environments by bidding more, while in the all pay they will bid less.

Specifically, we examine the effect of a more competitive distribution of types in the sense of the ULR order, introduced in the previous section, on equilibrium strategies. Remember that the ULR order implies that the dominant distribution is either higher or less dispersed than the dominated. We show that, given a distribution F that is higher in the ULR order than another distribution G, there will be more aggressive bidding by most types under the distribution F. Specifically, the bidding function under the higher distribution $x_F(z)$ will cross the other bidding function $x_G(z)$ at most once, and if this crossing does take place, it must do so at a high value.

Proposition 4 *Suppose there are two distributions F,G such that we have $F(z)\succ_{ULR} G(z)$. Let $x_F(z)$ and $x_G(z)$ be the corresponding solutions to the differential equation (6). Then, in the first price auction, $x_F(z)>x_G(z)$ on $(\underline{z},\hat{z}_P)$ where $\hat{z}_P$ is the maximum on $[\underline{z},\bar{z}]$ of the probability ratio $P(z)=F(z)/G(z)$. Further, if $\hat{z}_P<\bar{z}$, then $x_F(z)$ can cross $x_G(z)$ once and from above on $(\hat{z}_P,\bar{z})$.*

[7] Actually, a more precise boundary condition for the first price auction is that $\lim_{z\downarrow\underline{z}}x(z)=\underline{z}$. Strictly speaking, the bidder of type $\underline{z}$ is indifferent between all bids on the interval $[0,\underline{z}]$ (thus our claim that the equilibrium specified is unique only on $(\underline{z},\bar{z}]$). The boundary condition $x(\underline{z})=\underline{z}$ is used for reasons of simplicity.

Proof: First, by Proposition 2, as $F(z) \succ_{ULR} G(z)$ the probability ratio $P(z) = F(z)/G(z)$ is unimodal and, thus has a unique maximum which we label $\hat{z}_P$. Now, by Corollary 1, we have $\sigma_F(z) = f(z)/F(z) > g(z)/G(z) = \sigma_G(z)$ almost everywhere on $(\underline{z}, \hat{z}_P)$. Let us first show that if $x_F(z)$ and $x_G(z)$ do cross on $(\underline{z}, \hat{z}_P)$, then $x_F(z)$ crosses $x_G(z)$ from below. This easy to see from the equation (6), as it implies that at any point $z_\times$ such that $x_F(z_\times) = x_G(z_\times)$, we have that

$$\frac{x'_F(z_\times)}{x'_G(z_\times)} = \frac{\sigma_F(z_\times)}{\sigma_G(z_\times)} \tag{7}$$

That is, as $\sigma_F(z_\times) > \sigma_G(z_\times)$ for any $z_\times \in (\underline{z}, \hat{z}_P)$ we have $x'_F(z_\times) > x'_G(z_\times)$. This implies that there is at most a single crossing of $x_F(z)$ and $x_G(z)$ on $(\underline{z}, \hat{z}_P)$.

Consequently, there are three possible cases. First, $x_F(z) > x_G(z)$ on $(\underline{z}, \hat{z}_P)$. Second, $x_F(z) < x_G(z)$ on $(\underline{z}, \hat{z}_P)$. Third, $x_F(z) < x_G(z)$ on $(\underline{z}, z_1)$ for $z_1 < \hat{z}_P$ where z_1 is the unique crossing point of the two solutions. Note that the second and third possibilities both imply that $x_F(z) < x_G(z)$ on $(\underline{z}, \underline{z} + \epsilon)$ for some $\epsilon > 0$. In this case, as $x_F(z) < x_G(z)$ and, given our assumptions on the utility function U, it is easy to establish that ψ is decreasing in x, so that $\psi(x_F(z), z) > \psi(x_G(z), z)$ on $(\underline{z}, \underline{z}+\epsilon)$. Note that also $\sigma_F(z) > \sigma_G(z)$ on $(\underline{z}, \hat{z}_P)$. Together this implies that $x'_F(z) < x'_G(z)$ for all $z < \underline{z}+\epsilon$, which given $x_F(\underline{z}) = x_G(\underline{z})$, implies that $x_F(z) > x_G(z)$ on $(\underline{z}, \underline{z}+\epsilon)$, which is a contradiction. Thus, if $\sigma_F(z) > \sigma_G(z)$ on $(\underline{z}, \hat{z}_P)$, only the first case is possible, that is, $x_F(z) > x_G(z)$ on $(\underline{z}, \hat{z}_P)$.

Now, $F(z) \succ_{ULR} G(z)$ implies that $\sigma_F(z) < \sigma_G(z)$ on $(\hat{z}_P, \bar{z})$. Examining equation (7), it is clear that there is at most one crossing of $x_F(z)$ and $x_G(z)$ on $(\hat{z}_P, \bar{z})$ and $x_F(z)$ must cross $x_G(z)$ from above - if at all. ∎

Some intuition behind the failure of monotonicity disclosed in Proposition 4 can be explained in the following way. As the distribution of types becomes more compressed, the marginal return to raising one's bid rises as it becomes easier to surpass rivals whose values are now more closely packed, inducing more aggressive bidding. Specifically, in the first price auction, the effect of the distribution of valuations enters through the reverse hazard ratio $f(z)/F(z)$. Under Corollary 1, we know that $f(z)/F(z) > g(z)/G(z)$ for $z < \hat{z}_P$. That is, as long as $P(z)$ (as depicted in Figure 2) is rising, then incentives are higher under the less dispersed distribution F. However, for $z > \hat{z}_P$, the inequality is reversed. This is possible because in a less dispersed distribution, there are fewer high values, thus the density $f(z)$ may be quite low for high values of z (see, for example, Figure 1) and so $f(z)/F(z)$ can be lower than $g(z)/G(z)$. Thus, the bidding functions may cross for some z above $\hat{z}_P$. We now give an example of where the bidding functions do not cross, and one where they do.

Example 1 *Consider a n-bidder first price private value auction. Let $F(z)$ be $3z^2 - 2z^3$ and $G(z) = z$ both on $[0, 1]$ and both having expected values of $1/2$. Then, as G is a uniform distribution and F has a unimodal density (similar to Figure 1), we have $F \succ_{ULR} G$. If there are two risk-neutral bidders, the equilibrium bidding functions are*

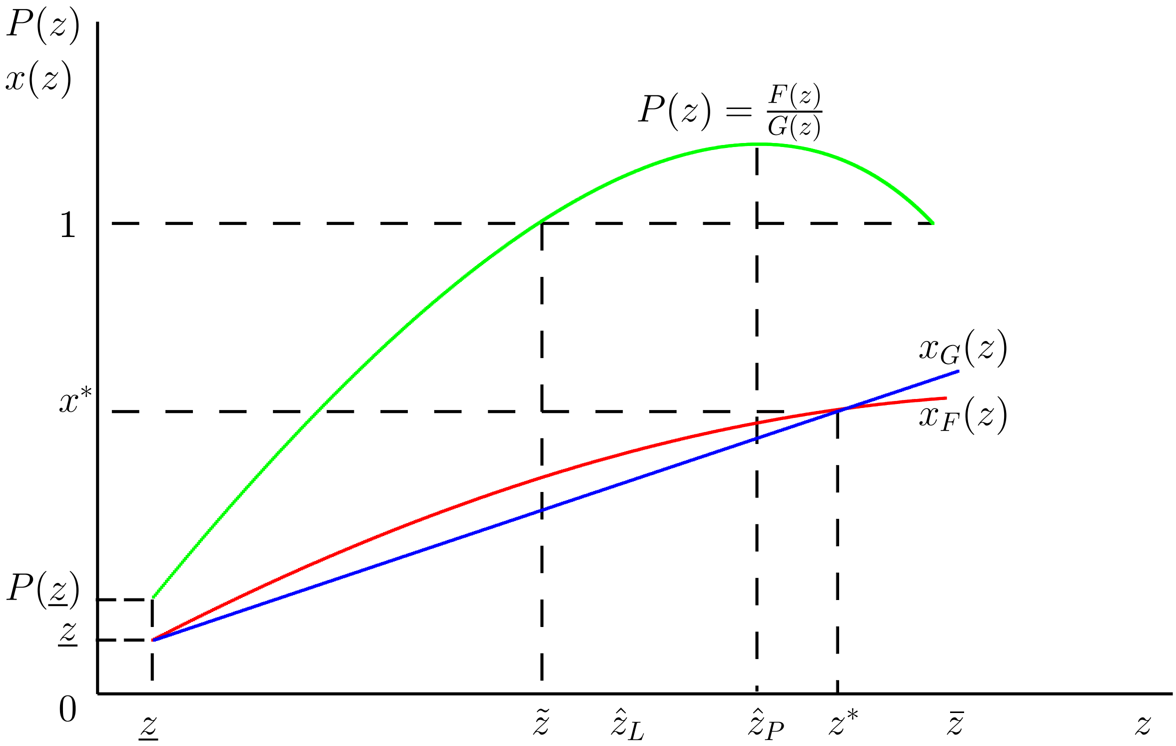

Figure 2: Comparative Statics for a First Price Auction

$x_F(z) = z(3z-4)/(4z-6)$ *and* $x_G(z) = z/2$ *for* $n = 2$*. These do not cross on* $(0, 1)$*, but they meet at the boundaries. However, for three risk neutral bidders, or equivalently if* $n = 2$ *but* $U(\cdot) = \sqrt{(\cdot)}$*, the solutions are* $x_F(z) = 2z(126 - 175z + 60z^2)/(35(3 - 2z)^2)$ *and* $x_G(z) = 2z/3$*. These cross much as in Figure 2. Note that* $\hat{z}_P$*, the maximum of* $P(z) = F(z)/G(z)$*, is equal to 0.75 and the solutions cross at approximately 0.93.*

If the maximum of the likelihood ratio is at the upper bound, i.e. $\hat{z}_L = \bar{z}$, then the ratio is monotone and by Proposition 2 the maximum of the probability ratio $P(z)$, the point $\hat{z}_P$, will be at the upper bound $\bar{z}$. Since Proposition 4 establishes that any crossing must take place to the right of $\hat{z}_P$, it follows that the solutions will not cross. That is, a stochastically higher distribution of values, in the sense of the MLR order, leads to uniformly higher bidding.

Corollary 3 *Suppose* $x_F(z)$ *and* $x_G(z)$ *are the equilibrium bidding functions for distributions* $F(z)$ *and* $G(z)$*, respectively. If* $F(z) \succ_{MLR} G(z)$*, then* $x_F(z) > x_G(z)$ *almost everywhere.*

This result is well known and is a special case of the results of Lebrun (1998) and Maskin and Riley (2001a) and Athey (2002). We reproduce it here solely as a contrast to the corresponding result for all pay auctions, given as Corollary 4 below.

4 The All Pay Auction

We have just seen that in first price auctions a stochastically higher distribution of values implies a strong competitive response. Specifically, under the MLR order, in the standard first price auction, bidding is higher almost everywhere under the stochastically higher distribution. As we will see in this section, the response to more competitive environments in the all pay auctions is markedly different from that in the first price auction. Specifically, low value bidders will respond to a distribution of values that is stochastically higher and/or more compressed by bidding less.

Again, we consider an auction with $n \geq 2$ bidders, each with a private value z independently drawn from a common twice differentiable distribution $F(z)$, with strictly positive density on its support $[\underline{z}, \bar{z}]$. If an agent with value z wins with bid x, she gains the object for sale and her payoff is $U(z-x)$. Importantly, we now assume that if she loses her payoff is $U(-x)$. That is, all bidders must pay their bid, whether they win or lose. Again, we assume that $U(\cdot)$ is twice continuously differentiable with $U' > 0$ and $U'' \leq 0$ and that $U(0) = 0$. Suppose all agents adopt the same strictly increasing differentiable strategy $x(z)$, then the expected utility of an agent of type z who bids $x(\hat{z})$, that is, as if she had type $\hat{z}$ will be

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})U(z - x(\hat{z})) + \left(1 - F^{n-1}(\hat{z})\right) U(-x(\hat{z})) \tag{8}$$

Differentiating with respect to $\hat{z}$, and setting $\hat{z}$ to z we have the following first order conditions

$$-x'(z)(U'(z-x)F^{n-1}(z)+U'(-x)(1-F^{n-1}(z)))+(n-1)f(z)F^{n-2}(z)(U(z-x)-U(-x)) = 0 \tag{9}$$

Rearranging, we obtain the following differential equation

$$\begin{aligned} x'(z) &= (n-1)f(z)F^{n-2}(z)\frac{U(z-x)-U(-x)}{U'(z-x)F^{n-1}(z)+U'(-x)(1-F^{n-1}(z))} \\ &= h(z)\gamma(F(z), z, x) \end{aligned} \tag{10}$$

where $h(z) = (n-1)f(z)F^{n-2}(z)$ is a density of the order statistic $F^{n-1}(z)$ for the highest of $n-1$ draws and $\gamma(\cdot)$ abbreviates the quotient term in the above equation. A solution to this differential equation together with corresponding boundary condition will constitute symmetric equilibria for the auction.

Proposition 5 *[Amann and Leninger (1996)] The unique solution to the differential equation (10), with initial conditions $x(\underline{z}) = 0$ represents a symmetric equilibrium for the all pay auction that is unique on $(\underline{z}, \bar{z}]$.*

Notice that the boundary condition in the all pay auction is quite different from the one in the first price auction. In particular, the lowest-value bidder in the all pay auction bids nothing, while in the first price auction she pays everything. That is, the equilibrium behaviour of the lowest type is much less aggressive in the all pay auction. Similarly, low value bidders in the all pay auction will respond to a more competitive environment by bidding even less. More precisely, we show that the bidding function under the lower distribution $x_G(z)$ is higher for low values but the two distribution functions must cross over. Finally, just as for the first price auction, under a less dispersed distribution there can be lower bidding at high values of z.

Proposition 6 *Suppose there are two distributions of values $F(z)$ and $G(z)$ with the same support $[\underline{z}, \bar{z}]$ such that $F(z) \succ_{ULR} G(z)$ and let $x_F(z)$ and $x_G(z)$ be the corresponding equilibrium bidding functions for the all pay auction given by (10). First, $x_F(z) < x_G(z)$ on $(\underline{z}, \hat{z}_-]$ where $\hat{z}_-$ is the first crossing point of $f(z)$ and $g(z)$. Second, $x_F(z)$ and $x_G(z)$ cross at least once on $(\hat{z}_-, \tilde{z})$ where $\tilde{z}$ is the crossing of $F(z)$ and $G(z)$. Third, either $x_F(z) > x_g(z)$ on $[\tilde{z}, \hat{z}_+]$, where $\hat{z}_+$ is the second crossing point of $f(z)$ and $g(z)$ or $x_F(z) > x_g(z)$ on $[\tilde{z}, \bar{z}]$ if there is no such second crossing of $f(z)$ and $g(z)$. Last, however, $x_F(z)$ and $x_G(z)$ may cross again on $(\hat{z}_+, \bar{z}]$ if $\hat{z}_+$ exists.*

Proof: First, the ULR order implies that $F(z) < G(z)$ on $(\underline{z}, \tilde{z})$. Then, from the unimodality of the ratio $f(z)/g(z)$ it must be that $f(z) < g(z)$ on some interval $(\underline{z}, \hat{z}_-)$ with necessarily $\hat{z}_- < \tilde{z}$ (see, for example, Figure 1). Then, if $F(z)$ does not (first order) stochastically dominate $G(z)$ there will be a second crossing of $f(z)$ and $g(z)$, denoted $\hat{z}_+$ on the interval $(\tilde{z}, \bar{z})$ (again see Figure 1). This implies that $h_F(z) < h_G(z)$ on $(\underline{z}, \hat{z}_-)$ (where both $f(z) < g(z)$ and $F(z) < G(z)$), while $h_F(z) > h_G(z)$ on $(\tilde{z}, \hat{z}_+)$ (where, instead, both $f(z) > g(z)$ and $F(z) > G(z)$).

Then, note from the differential equation (10) that $\partial\gamma(F, z, x)/\partial F > 0$ if $U'' < 0$ and is zero if $U'' = 0$. So, for any point of crossing in the interval $(\underline{z}, \hat{z}_-)$, we have $x'_F(z) < x'_G(z)$. So there can be only one such crossing. So either $x_G(z) > x_F(z)$ as claimed or $x_F(z) > x_G(z)$ on an interval $(\underline{z}, \breve{z})$ for some $\breve{z} > \underline{z}$. But then as $\partial\gamma(F, z, x)/\partial x < 0$ and $\partial\gamma(F, z, x)/\partial F \geq 0$, it would follow that $x'_F(z) < x'_G(z)$ on $(\underline{z}, \breve{z})$. Given $x_F(\underline{z}) = x_G(\underline{z}) = 0$, we have a contradiction. So it must be that $x_G(z) > x_F(z)$ on $(\underline{z}, \hat{z}_-)$.

Second, suppose now that $x_G(z) > x_F(z)$ on all of $(\underline{z}, \tilde{z})$, that is, there is no crossing. In equilibrium, a bidder facing distribution $F(z)$ with value z expects utility $V_F(z) = F(z)^{n-1}U(z - x(z)) + (1 - F(z)^{n-1})U(-x(z))$. Now, by the envelope theorem, we have $V'_F(z) = U'(z - x(z))F^{n-1}(z)$. Thus since $F(z) < G(z)$ on $(\underline{z}, \tilde{z})$ and $x_F(z) < x_G(z)$ we have $V'_F(z) < V'_G(z)$ so $V_F(\tilde{z}) < V_G(\tilde{z})$. But as

$$V_F(\tilde{z}) = F^{n-1}(\tilde{z})U(\tilde{z} - x_F(\tilde{z})) + (1 - F^{n-1}(\tilde{z}))U(-x_F(\tilde{z}))$$

we have $V_F(\tilde{z}) \geq V_G(\tilde{z})$ as $F(\tilde{z}) = G(\tilde{z})$ and $x_F(\tilde{z}) \leq x_G(\tilde{z})$. So, we have a contradiction and, thus in fact, $x_F(z)$ and $x_G(z)$ must cross at least once on $(\hat{z}_-, \tilde{z})$.[8]

Third, on the interval $(\tilde{z}, \hat{z}_+)$, we have both $f(z) > g(z)$ and $F(z) > G(z)$ and thus $h_F(z) > h_G(z)$. Thus, at any point of crossing of $x_F(z)$ and $x_G(z)$ on the interval, it

[8] Under the stronger assumption of risk neutrality, there will be exactly one crossing of $x_F(z)$ and $x_G(z)$ on the interval $[\underline{z}, \tilde{z}]$.

must be that $x'_F(z) > x'_F(z)$, so in fact, there can be no crossing. Finally, on the interval $[\hat{z}_+, \bar{z})$, crossing of $x_F(z)$ and $x_G(z)$ is possible where $h_F(z) < h_G(z)$. ∎

The above result is that low value bidders will bid less under the higher and/or more compressed distribution $F(z)$, but that the bidding function must cross over between $\hat{z}_-$ and the point where the two distribution functions cross $\tilde{z}$ (see Figure 3). The proposition also identifies the possibility of a further crossing that, if it exists, must take place to the right of $\hat{z}_+$. Such a crossing is illustrated in Figure 3. A specific example of these comparative statics is the following.

Example 2 *Consider a n-bidder all pay private value auction. As in Example 1, let $F(z)$ be $3z^2 - 2z^3$ and $G(z) = z$ both on $[0, 1]$. The density functions for these distributions cross at $\hat{z}_- = 0.211$ and $\hat{z}_+ = 0.789$. If there are two risk-neutral bidders, the bidding functions are $x_F(z) = (4 - 3z)z^3/2$ and $x_G(z) = z^2/2$. These are equal at 0, 1/3 and 1. That is, there is only one crossing and as per Proposition 6 it is to the right of $\hat{z}_-$. However, if there are three risk-neutral bidders then the bidding functions are $x_F(z) = 2z^5(126 - 175z + 60z^2)/35$ and $x_G(z) = 2z^3/3$. These are equal at 0, 0.438 and 0.979 so there are two crossings as in Figure 3, with the second to the right of $\hat{z}_+$ as predicted. That is, both low value and high value bidders bid less under distribution $F(z)$.*

Again the case where $F(z) \succ_{MLR} G(z)$ is a special case of the above result, where there is no second crossing of $f(z)$ and $g(z)$ and so $\hat{z}_+$ does not exist. Then, we have no crossing at high values and high value bidders bid more under the higher distribution. However, this does not affect the result about crossing at low values. So, it is the case that even when the distribution $F(z)$ is very strongly stochastically higher than $G(z)$, the higher distribution induces lower bids from those with low values. Note the contrast with the corresponding result for first price auctions, Corollary 3, where all types bid more.

Corollary 4 *Suppose $x_F(z)$ and $x_G(z)$ are the equilibrium bidding functions for distributions $F(z)$ and $G(z)$, respectively. If $F(z) \succ_{MLR} G(z)$, then $x_F(z) < x_G(z)$ on $(\underline{z}, \hat{z}_-]$, where $\hat{z}_-$ is now the unique crossing point of $f(z)$ and $g(z)$, but $x_F(z)$ and $x_G(z)$ cross so that $x_F(\bar{z}) > x_G(\bar{z})$.*

The mathematics behind this difference in behaviour for bidders with low values in first price and all pay auctions can be seen by comparing the first order conditions for the two formats. For simplicity, we assume risk neutrality and the first order condition for the first price auction (5) becomes

$$-F^{n-1}(z)x'(z) + h(z)(z - x) = 0 \tag{11}$$

and the equivalent for the all pay (9) is

$$-x'(z) + h(z)z = 0. \tag{12}$$

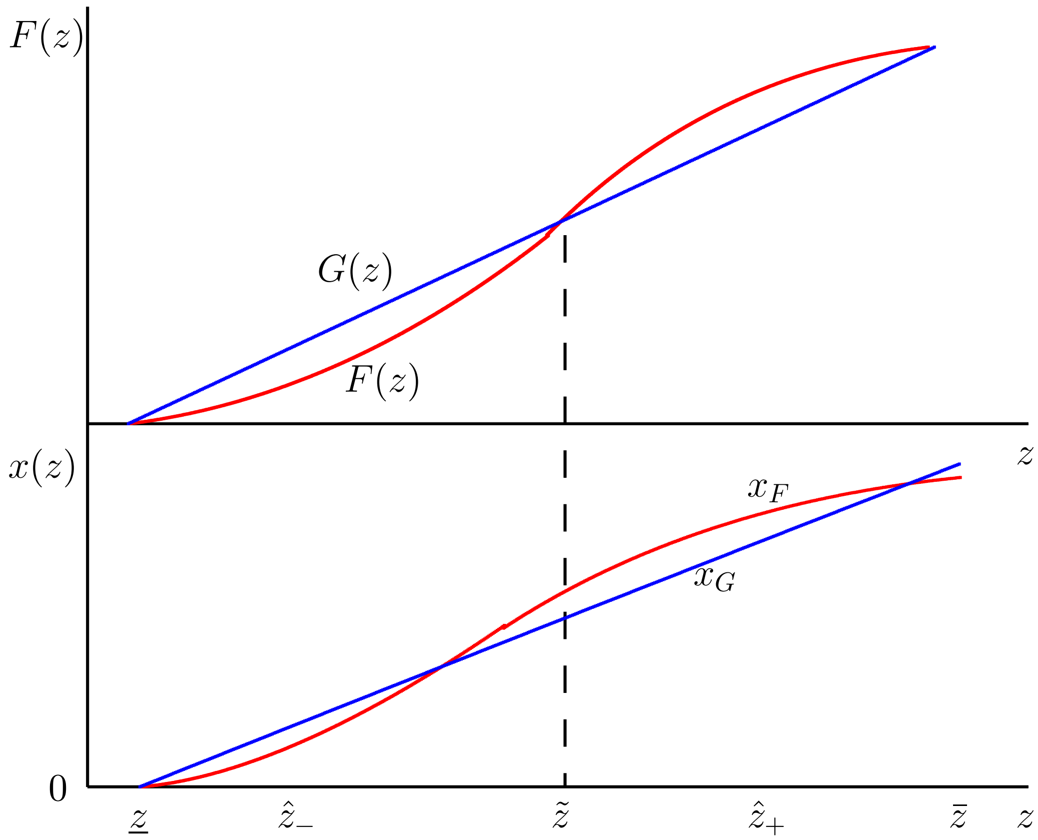

Figure 3: Comparative Statics for an All Pay Auction.

In both cases, the first term gives the marginal cost and the second the marginal benefit of raising one's bid. The marginal benefit in both cases depends on the density $h(z) = (n-1)f(z)F^{n-2}(z)$ which gives the marginal probability of winning. This density is lower for low values under distribution $F(z)$ than under $G(z)$ (for example, on the interval $(\underline{z}, \hat{z}_-)$ in Figure 1, it holds that both $f(z) < g(z)$ and $F(z) < G(z)$), giving low value bidders in the all pay auction a clearly lower incentive to compete. Simply put, under the higher distribution low values are less likely, so that a bidder with a low value is unlikely to overtake any other bidders by raising her own bid.

However, in the first price auction, the marginal cost of raising one's bid (the first term in (11)) is also lower in the stochastically higher distribution as the marginal cost of a bid here depends on the probability of winning $F^{n-1}(z)$, which is lower under F than under G for $z < \tilde{z}$. After all, in the first price auction, one only pays if one wins. Thus the change would seem to have an ambiguous effect. But if one takes the first order conditions (11) and divides by F^{n-1}, one obtains $-x'(z) + (n-1)\sigma(z)(z-x) = 0$. That is, the two effects can be combined into the reverse hazard ratio $\sigma(z)$ and we know from the results of Section 2 that this ratio will be higher on $(\underline{z}, \hat{z}_P)$ given the ULR order, or everywhere given the MLR order. Thus, in the first price auction, low value bidders respond to a higher or less dispersed distribution by bidding more (Proposition 4).

Finally, similar considerations apply to the type of contests analysed by Moldavanu and Sela (2001) that are closely related to all pay auctions. The main difference is that rather than having different values for the object for sale, competitors who differ in the cost of production compete for prizes that have a common value. Here we look at the simple case where n agents compete for a single prize with fixed common value W. The prize is awarded to the agent with the highest output. Each agent pays a cost cx to produce output x. Let $c = 1 - z$, where z is the agent's type which is an independent draw from $F(z)$ with support $[\underline{z}, \bar{z}]$ with $\bar{z} < 1$. Thus, an agent choosing $x(\hat{z})$ when all others adopt the strictly increasing strategy $x(z)$ will obtain an expected utility

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})U(W - (1-z)x(\hat{z})) + (1 - F^{n-1}(\hat{z}))U(-(1-z)x(\hat{z})).$$

A symmetric equilibrium in increasing strategies will therefore be a solution to the differential equation

$$\begin{aligned} x'(z) &= h(z)\frac{(U(W-(1-z)x) - U(-(1-z)x))}{(1-z)\left(F^{n-1}(z)U'(W-(1-z)x) + (1-F^{n-1}(z))U'(-(1-z)x)\right)} = \\ &= h(z)\xi(F(z), z, x) \end{aligned} \tag{13}$$

where $\xi(\cdot)$ abbreviates the quotient component of the equation. Again we have the all pay boundary condition of $x(\underline{z}) = 0$. Notice that the signs of the partial derivatives of the function $\xi(\cdot)$ in (13) are the same of the function $\gamma(\cdot)$ in (10), and thus it is easy to establish a result similar to that of Proposition 6, where low types will respond to a more competitive environment by bidding less.

5 An Example from Price Competition in Oligopoly

There are similar considerations for procurement auctions and oligopoly games. That is, there are plausible models of oligopoly where a reduction in costs leads some sellers to charge higher prices.

Suppose n firms each have constant marginal cost c but the exact level of that cost is private information. Each is an independent draw from a distribution $F(c)$ with a continuous positive density on $[\underline{c}, \bar{c}]$. The firms compete on price in a simultaneous move game. We assume there is a finite maximum price $\bar{p}$ that consumers are willing to pay. The standard model of price competition is of course the Betrand model. As Bertrand competition is formally similar to a procurement auction (as noted by Spulber (1995)), it easy to show that, under the MLR order, it would generate monotone comparative statics.

However, we consider another example, as given by Bagwell and Wolinsky (2002) who consider an incomplete information version of the Varian (1980) model of price dispersion. There are $n \geq 2$ firms that compete on price to sell to N consumers. Each consumer seeks to buy one unit of the good, if the price does not exceed a common reservation price $\bar{p}$. A proportion q of consumers are uninformed and purchase from a randomly chosen seller. The other $1-q$ only buy from the lowest priced firm. In the version of Bagwell and Wolinsky, each firm has private information about its marginal cost c, which is an independent draw from the distribution $F(c)$ which has support $[\underline{c}, \bar{c}]$. Expected profits for a firm with costs c from charging a price $p(\hat{c})$ when the other sellers use the symmetric strategy $p(c)$ are thus

$$V(p(\hat{c}), c, c_{-i}) = \frac{N}{n}(p(\hat{c}) - c)\left(q + n(1-q)(1-F(\hat{c}))^{n-1}\right).$$

This gives rise to the differential equation

$$p'(c) = (p-c)\left(\frac{(n-1)f(c)(1-F(c))^{n-2}}{A + (1-F(c))^{n-1}}\right), \quad p(\bar{c}) = \bar{p}. \tag{14}$$

where $A = \frac{q}{n(1-q)}$. The boundary condition is that highest-cost player chooses the highest possible price. In all pay auctions the weakest type bids zero and never wins, here she names the reservation price and never sells to the informed customers.

To explore the effect of general increase in costs on firms' pricing behavior, let us define the ratio

$$\Theta(c) = \frac{A + (1-F(c))^{n-1}}{A + (1-G(c))^{n-1}} \tag{15}$$

We first show that the monotone likelihood ratio order implies the unimodality of $\Theta(c)$.

Lemma 1 *If $F(c) \succ_{MLR} G(c)$, then $\Theta(c)$ has a unique maximum at some $\hat{c} \in (\underline{c}, \bar{c})$.*

Proof: Define $R_F(c) = (1-F(c))^{n-1}$ and $R_G(c) = (1-G(c))^{n-1}$ and use $r_i(c) = R_i'(c), i = F, G$. Note that $\Theta(\underline{c}) = \Theta(\bar{c}) = 1$ and that as F stochastically dominates G, we have $F(c) \leq G(c)$ and $\Theta(c) \geq 1$ for $c \in (\underline{c}, \bar{c})$. It is easily checked that at any point where $\Theta'(c) = 0$, then $\Theta(c) = r_F(c)/r_G(c)$. Note that since

$$\frac{r_F(c)}{r_G(c)} = \frac{f(c)}{g(c)}\left(\frac{1-F(c)}{1-G(c)}\right)^{n-2} = L(c)Q(c)^{n-2}$$

the ratio $r_F(c)/r_G(c)$ is strictly increasing on $(\underline{c}, \bar{c})$ if both $L(c) = \frac{f(c)}{g(c)}$ and $Q(c) = \frac{1-F(c)}{1-G(c)}$ are increasing on that interval. According to Corollary 2, these last two conditions are satisfied if $F(c) \succ_{MLR} G(c)$. Finally, since $r_F(c)/r_G(c)$ is increasing there can be only one turning point for $\Theta(c)$, that is, $\Theta(c)$ is unimodal with a unique maximum at some $\hat{c}$. ∎

We now employ unimodality of $\Theta(c)$ to show that in this oligopoly model, just like in the all pay auction, the MLR order is not sufficient for a uniform increase in prices.

Proposition 7 *Suppose $p_F(c)$ and $p_G(c)$ are the equilibrium pricing functions arising from the differential equation (14) for distributions $F(c)$ and $G(c)$, respectively. If $F(c) \succ_{MLR} G(c)$, then there exist a point $c_\times$ such that $p_F(c) > p_G(c)$ on $[\underline{c}, c_\times)$ and $p_F(c) < p_G(c)$ on $(c_\times, \bar{c})$ and $c_\times < \hat{c}$ where $\hat{c}$ is the maximum of $\Theta(c)$.*

Proof: Again, define $R_F(c) = (1 - F(c))^{n-1}$ and $R_G(c) = (1 - G(c))^{n-1}$ and use $r_i(c) = R_i'(c), i = F, G$. Furthermore, for $i = F, G$ define $\theta_i(c) = \frac{-r_i(c)}{R_i(c)+A}$, so that $p_i'(c) = (p_i - c)\theta_i(c)$.

According to the above Lemma, if $F(z) \succ_{MLR} G(z)$ then $\Theta(c)$ is unimodal with a unique maximum at some $\hat{c}$. It is easy to notice that $\theta_F(c) < (>)\theta_G(c)$ whenever $\Theta'(c) > (<)0$. Thus, if $p_F(c)$ and $p_G(c)$ cross, then at the point of crossing $c_\times$ it must be that $p_F'(c_\times) < p_G'(c_\times)$ (i.e. p_F cross from above) if $c_\times < \hat{c}$, and $p_F'(c_\times) > p_G'(c_\times)$ (i.e. p_F cross from below) if $c_\times > \hat{c}$. Thus, there could be only one (if any) possible crossing per interval.

Now, since $\theta_F(c) > \theta_G(c)$ on $(\hat{c}, \bar{c})$, then clearly $p_F'(c) > p_G'(c)$ on that interval. This, together with the boundary condition $p_F(\bar{c}) = p_G(\bar{c})$, implies that $p_F(c) < p_G(c)$ on $(\tilde{c}, \bar{c})$ for some $\tilde{c} > \hat{c}$. Furthermore, one can rule out a possibility of $p_F(c) > p_G(c)$ on $(\hat{c}, \tilde{c})$ since that would imply that p_F would cross p_G from above, which is a contradiction to the above conditions on the points of crossing. Thus, $p_F(c) < p_G(c)$ on $(\hat{c}, \bar{c})$.

We will now show that there is a unique crossing of $p_F(c)$ and $p_G(c)$ on $(\underline{c}, \hat{c})$. Suppose to the contrary, that there is no crossing of $p_F(c)$ and $p_G(c)$ on $(\underline{c}, \hat{c})$. But that would imply that $p_F(\underline{c}) \leq p_G(\underline{c})$. Note that in equilibrium, for $i = F, G$, a seller of cost c expects $V_i(c) = (p - c)(q + n(1 - q)R_i(c))$. Thus, if there is no crossing, we would have $V_F(\underline{c}) = (p_F(\underline{c}) - \underline{c})(q + n(1 - q)) \leq (p_G(\underline{c}) - \underline{c})(q + n(1 - q)) = V_G(\underline{c})$. To show that this cannot happen, note that, by the envelope theorem, $V_i'(c) = -n(1 - q)(A + R_i(c))$. MLR order implies that $F(c) < G(c)$ on $(\underline{c}, \bar{c})$, and thus $R_F(c) > R_G(c)$ on the entire interval. This, in turn, implies that $V_F'(c) < V_G'(c)$ on $(\underline{c}, \bar{c})$. Thus, given that $V_F(\bar{c}) = V_G(\bar{c}) = (\bar{p} - \bar{c})q$, it must be that $V_F(\underline{c}) > V_G(\underline{c})$, a contradiction. ∎

Note that here $F(c) \succ_{MLR} G(c)$ implies that $G(c)$ involves costs being generally lower. We would expect a decrease in costs to make the market uniformly more competitive. However, here a stochastically lower distribution of costs $G(c)$ induces the high cost firms (firms with costs greater than $\hat{c}$) to charge higher prices. Only low cost firms respond with lower prices. The reason for this is the presence of the uninformed consumers, who ensure that all firms have a minimum demand of qN/n, and make this game less competitive than standard Bertrand. With a lower distribution of costs, a firm at any given level of costs will be less likely to win the competition to name the lowest price and attract the informed consumers. If one's costs are high, the chances of winning can be so low, that it may be better to give up the chase. Compare this with Bertrand competition (or equivalently this model with $q = 0$). There, charging a high price ensures only zero profits. The relative lack of competitiveness for high cost firms in the Varian model enables them to respond in a relaxed way to greater competition.

6 Conclusions

In this paper, we investigate two new types of comparative statics in first price and all pay auctions, both of which give rise to non-monotone results. First, we show that in all pay auctions even a stochastically higher distribution of types does not lead to uniformly more aggressive play. We find similar results for an oligopoly game with incomplete information: a stochastically lower distribution of costs will lead to higher prices being charged by some sellers. Second, we show that refinements of second order stochastic dominance are suitable for comparative statics, but are not in general sufficient for monotonicity.

In this paper, we surveyed some stochastic orderings used to rank distributions in terms of dispersion. We also applied them to comparative statics analysis. We hope that they will find further similar applications. First, there has been a recent interest in the effect of changes in inequality in the degree of social competition (Samuelson (2004); Hopkins and Kornienko (2004); Hoppe, Moldovanu and Sela (2005)). Second, we do not investigate asymmetric auctions in this paper. However, the orderings in terms of dispersion used here could also be useful, for example, in determining the effects of one player having more precise information than other bidders. This further application of stochastic orders will be the subject of future research.

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