§10. Averaged Reduced MHD Equations on Magnetic Coordinates

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A reduced set of MHD equations for the poloidal flux $\Psi$, the stream function $\Phi$ and the plasma pressure $P$ is derived on the flux coordinates $(\rho, \theta, \zeta)$ based on the averaging method, which is given by

$$\frac{\partial \Psi}{\partial t} = -\left(\frac{\sqrt{g}}{\chi'_0} \mathbf{B} \cdot \nabla\right)\zeta \Phi + \eta \langle J \zeta \rangle \langle \zeta \rangle,$$  (1)

$$\frac{\langle \rho_m \sqrt{g} \rangle}{\chi'_0} \frac{d}{dt} \langle \sqrt{g} U \zeta \rangle = \frac{\sqrt{g}}{\chi'_0} \mathbf{B} \cdot \nabla \zeta + \langle \Omega \rangle, \langle P \rangle,$$  (2)

$$\frac{dP}{dt} = 0,$$  (3)

where $\langle \rangle$ means the average with respect to $\zeta$. $\eta$, $\chi'_0$ and $\sqrt{g}$ denote the resistivity, the $\rho$-derivative of the equilibrium toroidal magnetic flux and the Jacobian of the coordinates, respectively, and the relations of

$$\langle \zeta \rangle = \frac{\langle \sqrt{g} \rangle}{\chi'_0} \frac{df}{dt} = \frac{\partial f}{\partial \zeta} - [\Phi, f],$$  (4)

$$\langle \sqrt{g} \mathbf{B} \cdot \nabla \rangle f = \frac{\partial f}{\partial \zeta} + [\Psi, f],$$  (5)

$$[f, g] = \frac{1}{\chi'_0} \left( \frac{\partial f}{\partial \rho} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \rho} \right),$$  (6)

$$\langle \sqrt{g} U \zeta \rangle = -\frac{\partial \langle v_\theta \rangle}{\partial \rho} + \partial \langle v_\rho \rangle \zeta \theta,$$  (7)

$$\langle v_\theta \rangle = \frac{1}{\chi'_0} \frac{\partial \Phi}{\partial \theta}, \quad \langle v_\rho \rangle = -\frac{1}{\chi'_0} \frac{\partial \Phi}{\partial \rho},$$  (8)

$$\langle \sqrt{g} J \zeta \rangle = \frac{\partial \langle B_\theta \rangle}{\partial \rho} - \frac{\partial \langle B_\rho \rangle}{\partial \theta},$$  (9)

$$\langle \sqrt{g} B_\theta \rangle = -\frac{1}{\chi'_0} \frac{\partial \Psi}{\partial \theta}, \quad \langle \sqrt{g} B_\rho \rangle = \frac{1}{\chi'_0} \frac{\partial \Psi}{\partial \rho},$$  (10)

and the metric properties are used.

By setting $\partial / \partial t = \Phi = 0$, the averaged equilibrium equation,

$$\frac{d \Psi}{d\rho} \langle \sqrt{g} J \zeta \rangle + \frac{d P}{d\rho} \langle \sqrt{g} \rangle \frac{d \zeta}{d\rho} = -\frac{dF}{d\rho}$$  (11)

is obtained, which any 3D equilibrium satisfies.

The reduced equations satisfies the energy conservation relation given by

$$\frac{\partial}{\partial t} (K + M + U) = -\eta \int \langle \sqrt{g} J \zeta \rangle \frac{\langle J \zeta \rangle}{\langle \sqrt{g} \rangle} d\tau,$$  (12)

where

$$K = \frac{1}{2} \int \langle \rho_m \rangle \zeta \langle \psi \rangle \zeta \langle \psi \rangle d\tau,$$  (13)

$$M = \frac{1}{2} \int \langle B_\theta \rangle \zeta \langle B_\rho \rangle d\tau$$  (14)

and

$$U = -\int P d\tau$$  (15)

are the kinetic, the magnetic and the internal energy, respectively.

The linearized equations show the Hermitian property in the energy integral corresponding to the energy principle in the following way,

$$-\omega^2 \int \frac{1}{\chi'_0} \left\{ \langle \rho_m \rangle \zeta \frac{\partial \Phi^*}{\partial \Phi} + \langle \rho_p \rangle \zeta \frac{\partial \Phi^*}{\partial \Phi} \right\} d\tau$$

$$= \int \left\{ G_{\Phi \Phi} \frac{\partial \Phi^*}{\partial \rho} + G_{\Phi \rho} \frac{\partial \Phi^*}{\partial \theta} \right\} d\tau$$

where the perturbed quantities are assumed to have the time dependence of $\exp(i\omega t)$ and $G_{ij} \equiv \langle g_{ij} \rangle$ is used.