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Extreme Adverse Selection, Competitive Pricing, and Market Breakdown*

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Abstract: Extreme adverse selection arises when private information has unbounded support, and market breakdown occurs when no trade is the only equilibrium outcome. We study extreme adverse selection via the limit behavior of a financial market as the support of private information converges to an unbounded support. A necessary and sufficient condition for market breakdown is obtained. If the condition fails, then there exists competitive market behavior that converges to positive levels of trade whenever it is first best to have trade. When the condition fails, no feasible (competitive or not) market behavior converges to positive levels of trade.

Keywords: Adverse selection, market breakdown, separation, competitive pricing.

***JEL* Classification Numbers:** D40, D82, D83, G12, G14.

1 Introduction

The presence of adverse selection can cause severe inefficiencies. This is most starkly illustrated by Akerlof's (1970) famous example where adverse selection leads to market breakdown (i.e., no trade is the only possible equilibrium outcome). Despite the prominence of Akerlof's example, there is little work identifying the exact circumstances under which market breakdown must occur. We do so in a simple model of trade in a financial market in which an informed trader possess private information about the payoff of a risky asset.

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We are interested in two issues. The first is the role of competitive pricing in market breakdown. In a financial market context it has been argued (see Glosten and Milgrom (1985), Glosten (1989), Leach and Madhavan (1993), and Glosten (1994)) that competitive pricing leads to market breakdown when the adverse selection problem is sufficiently severe, whereas the ability of a monopolist to cross-subsidize across different trades keeps the market open under identical circumstances.

The second issue is the identification of environments in which the adverse selection problem is sufficiently severe that market breakdown occurs under all feasible market structures.[1] Models of adverse selection in financial markets that focus on the issue of competitive market breakdown typically follow Kyle's (1985) seminal contribution in assuming that the type of the informed trader is normally distributed.[2] Because of the normality assumption, these models feature what we term *extreme adverse selection*, namely a type space for the informed trader with unbounded support. In the environments we consider, extreme adverse selection is necessary, but not sufficient, for market breakdown to be unavoidable. In particular, we will show that while market breakdown does *not* arise when the private information is normally distributed, it does arise for other distributions with unbounded support.

We investigate these issues in a model similar to Glosten (1989). There is a single informed, risk-averse strategic trader (with constant absolute risk aversion preferences) and risk neutral market makers. The informed trader can act either as a buyer or as a seller; there are no restrictions on order sizes. Glosten (1989) models a two-dimension adverse selection problem in which the informed trader has private information about the expected payoff of the risky asset as well as about his endowment and both of these random variables are normally distributed. Following Biais, Martimort, and Rochet (2000), we make no parametric distribution assumptions and conduct the formal analysis in a reduced form model where the informed trader's private information is summarized by a one-dimensional type (we discuss this in more detail in remark 2.1).

Rather than studying the equilibria of a particular market-microstructure model, we analyze *feasible* trading schedules and *competitive* trading schedules (where a trading schedule specifies a quantity for each type of the informed trader). A trading schedule is *feasible* if it is optimal for the informed trader to follow that schedule given some price schedule (specifying a price for every possible quantity), and if it yields nonnegative expected profits to the market makers under this

[1] Glosten (1994) also considers this issue. See remark 3.2 for a discussion of the relationship between his and our work.

[2] Examples include Glosten (1989), Bhattacharya and Spiegel (1991), Spiegel and Subrahmanyam (1992), and Bhattacharya, Reny, and Spiegel (1995)

price schedule. In a model with a bounded type space this is equivalent to the requirement that the trading schedule can be implemented as part of an incentive compatible, interim individual rational allocation. Considering feasible trading schedules thus allows us to delineate the circumstances under which every market structure satisfying these constraints must result in the no-trade outcome, i.e., cases in which market breakdown is unavoidable.

The idea behind our definition of a competitive trading schedule is natural and straightforward. Under the price schedule supporting a *competitive* trading schedule, market makers should earn zero expected profits conditional on any particular traded quantity. (For quantities that are not traded by any type of the informed trader, we require this condition to hold for some common beliefs of the market makers consistent with the underlying distribution.) Requiring pricing to be competitive in this sense eliminates the possibility of cross-subsidization across different trades, allowing us to investigate whether such cross-subsidization is helpful in avoiding market breakdown.[3]

We follow Hellwig (1992) in studying extreme adverse selection as the limit case of a sequence of markets in which bounded supports of the distribution of the informed trader's information become arbitrary large.[4] Like Hellwig, we view the assumption of an unbounded support as an idealization of the adverse selection problem caused by large but bounded supports. Under this view, it is important to understand when a prediction of market breakdown in a model with unbounded support holds approximately in a model with large support. Moreover, it is *never* the case that the trading schedule specifying no-trade for all types of the informed trader is a feasible trading schedule in a market with extreme adverse selection (remark 4.1). Working directly with an unbounded type space would force an identification of market breakdown with the non-existence of a feasible or competitive trading schedule. The more natural identification is with the property that the trading schedule specifying no trade for all types is the only feasible or competitive

[3] A trading schedule is competitive in our sense if and only if it is a sequential equilibrium outcome of a signaling game in which the informed trader chooses a quantity of the risky asset to trade and the market makers then compete a la Bertrand to take the other side of the trade (see Kreps (1990, Section 17.3) for an extended discussion in the context of Spence (1973)-job market signaling). The analysis of Gale and Hellwig (2004), which studies a general equilibrium model of an insurance market with adverse selection, provides an alternative "micro-foundation" for our definition of competitive trading schedules.

Note that our use of the term "competitive" solely refers to a property of the equilibrium price schedule. A rather different approach to modeling competition in markets with adverse selection builds on Prescott and Townsend (1984a,b); see Bisin and Gottardi (2006) for a recent contribution along these lines.

[4] Market breakdown is not an issue in our model, i.e., there exist not only feasible but also competitive trading schedules resulting in positive levels of trade (for almost all types), unless adverse selection is extreme.

trading schedule.

We thus say that *market breakdown occurs* if, in the limit, trade does not occur. Conversely, the *market stays open* if trade occurs in the limit, for almost all values of the private information of the informed trader. Our main results

- establish a necessary and sufficient condition, which we term the *market breakdown* condition, under which market breakdown must occur for every sequence of feasible trading schedules;

- show that exactly the same condition is necessary and sufficient for market breakdown to occur for every sequence of competitive trading schedules, thus implying that competitive pricing is not a source of market breakdown; and

- show that if the market breakdown condition fails there exists a sequence of competitive trading schedule such that the market stays open, i.e., not only is it possible to avoid market breakdown but a market with competitive pricing can provide liquidity for almost all types of the informed trader.

The market breakdown condition is a condition on the mean excess function (also known as the mean residual life function) of the distribution of the informed trader's type in the market with extreme adverse selection. We show that this condition fails for thin-tailed distributions (such as the normal) but is satisfied for a class of fat-tailed distributions. In particular, for such fat-tailed distributions extreme adverse selection may cause market breakdown in the strong sense that market breakdown must occur for all sequence of feasible trading schedules.

Our conclusion that competitive pricing does not cause market breakdown differs from that in Glosten (1989), the most closely related paper, because Glosten focuses on trading schedules that are not only competitive, but also separating.[5] In a separating trading schedule the type of the informed trader is revealed to the market makers, eliminating not only the possibility of cross-subsidization across different trade sizes (as our definition of a competitive trading schedule does), but also eliminating cross-subsidization across different types of the informed trader. Due to the distortions required by separation, the competitive separating trading schedule (which is unique for standard reasons) is interim inefficient in the set of competitive trading schedules when the informed trader's private information has

[5]The literature extending Glosten's result (i.e. Bhattacharya and Spiegel (1991), Spiegel and Subrahmanyam (1992), and Bhattacharya, Reny, and Spiegel (1995)), also focuses on separating trading schedules.

bounded support (no matter how large or small).[6] If the market breakdown condition holds, the inefficiency from the additional constraint of separation disappears in the limit as every competitive trading schedule converges to the no-trade outcome. However, the condition determining whether a competitive separating trading schedule must converge to the no-trade outcome (identified in Hellwig (1992) and corresponding to the market breakdown condition in Glosten (1989)) is independent of the shape of the limit distribution. It will thus hold even when the market breakdown condition fails (as it does for the normal distribution studied by Glosten), so that the market can stay open. While it can be argued that competitive pricing should embody restrictions additional to the ones we impose in our definition of a competitive trading schedule,[7] we find it more productive to identify the inefficiency associated with the requirement of separation as a source of market breakdown.

2 The Model

2.1 Information Structure and Preferences

We consider a market for a risky asset in which market makers provide liquidity to an informed trader who, depending on his private information, may wish to buy or sell the risky asset. The informed trader's private information is summarized by his type $\theta \in \mathbb{R}$. We denote the distribution function of θ by F. We assume F is symmetric,[8] and denote its support by $[-\tau, \tau]$ with $\tau > 0$. A trade is given by a pair $(x, m) \in \mathbb{R}^2$, where x specifies the number of units of the risky asset traded and m is the corresponding payment. A purchase by the informed trader is indicated by $x > 0$ and a sale by $x < 0$; a payment from the informed trader to the markets makers is indicated by $m > 0$ and an amount received by the informed trader from

[6]We do not provide a formal proof of this assertion as it is not central to our formal analysis. While the strategic aspects of the model are quite different, this can be shown using a construction reminiscent of Riley's (1979) argument that separating outcomes are unstable in his setting.

[7]If the market is modeled as signaling game, refinements in the spirit of those proposed by Kohlberg and Mertens (1986) and Cho and Kreps (1987) imply separation and thus justify Glosten's conclusion that competitive pricing may imply market breakdown. See Gale (1992, 1996) for a related Walrasian approach to competition in markets with adverse selection yielding similar conclusions. While Kohlberg and Mertens's (1986) strategic stability has an abstract continuity motivation, the "intuitive" motivations for some of its implications seem less persuasive (Mailath, Okuno-Fujiwara, and Postlewaite, 1993). See also Laffont and Maskin (1990) who consider a financial market signalling model (more akin to the model in Leland and Pyle (1977) than to the one we consider) and argue that separating trading schedules will not be observed when they are interim inefficient in the set of sequential equilibria (as is always the case in our model).

[8]Most of our analysis carries over to the asymmetric case. See section 6 for further discussion.

the market makers by $m < 0$. Conditional on θ, the informed trader's preferences over trades are described by the utility function $u(x,\theta) - m$, where

$$u(x,\theta) = b\theta x - \frac{1}{2}rx^2, \tag{1}$$

$b > 1$, and $r > 0$. Market makers are risk neutral and maximize expected trading profits. It suffices for our purposes to consider aggregate trading profits. When market makers engage in a trade (x,m) with an informed trader of type θ, these are given by $m - v(x,\theta)$ where

$$v(x,\theta) = \theta x. \tag{2}$$

For later reference, we denote the surplus resulting from type θ trading quantity x of the risky asset by

$$s(x,\theta) = u(x,\theta) - v(x,\theta) = (b-1)\theta x - \frac{1}{2}rx^2 \tag{3}$$

and note that surplus is maximized by the trading quantity

$$q^{FB}(\theta) = \frac{b-1}{r}\theta \tag{4}$$

with resulting (first best) surplus

$$s^{FB}(\theta) = \frac{(b-1)^2}{2r}\theta^2. \tag{5}$$

Remark 2.1 Our parameterization of a market environment with an informed trader follows Hellwig (1992) and is closely related to the model in Glosten (1989) and Biais, Martimort, and Rochet (2000). It is the reduced form of a model in which the value of the risky asset is $v = t + \varepsilon$, the informed trader privately observes t and his endowment ω of the risky asset, and (t,ω,ε) are zero-mean random variables. Denote the strictly positive variances of these random variables by $\sigma_t^2, \sigma_\omega^2$, and σ_ε^2. Assuming, in addition, that the random variable ε is normally distributed and that the informed trader's preferences are CARA with risk aversion parameter $\gamma > 0$, the informed trader's preferences have the representation (1) (see Biais, Martimort, and Rochet (2000) for a derivation), where

$$\begin{aligned} \theta &\equiv \frac{t - r\omega}{b}, \\ r &\equiv \gamma\sigma_\varepsilon^2 > 0, \\ b &\equiv \frac{\sigma_t^2 + r^2\sigma_\omega^2}{\sigma_t^2} > 1. \end{aligned}$$

Restricting attention to allocations that are measurable with respect to θ, the market makers' preferences are then given by (2), provided that the additional condition

$$\theta = E[v|t - r\omega] \tag{6}$$

is satisfied.[9] In Glosten (1989) condition (6) is implied by the assumption that (t, ω) are normally distributed. Note that the normality of (t, ω) implies that the distribution of θ is normal. It is essential that our analysis not require such a restriction on the distribution of θ, and it does not. For every symmetric distribution of θ possessing a density decreasing in $|\theta|$, we can construct zero mean random variables (t, ω) such that (6) holds for any given values $b > 1$, $r > 0$. Hence, for any such choice of parameters, our model can indeed be interpreted as a reduced form of the model described above. The construction of such random variables is described in appendix A.

♦

2.2 Feasible and Competitive Trading Schedules

We are interested in studying the effects of changes in the distribution F and in particular its support $[-\tau, \tau]$. While the *market environment* is described by the triple (u, v, F), since u and v are fixed, we often abuse language by referring to either the distribution F or its support parameter τ as the environment, and similarly only make the dependence on F or τ explicit.

Throughout the following analysis we focus on implementable trading schedules. A *τ-trading schedule* is a function $q : [-\tau, \tau] \to \mathbb{R}$, specifying a trading quantity of the risky asset for every type of the informed trader. A *price schedule* is a function $p : \mathbb{R} \to \mathbb{R}$, specifying a price per unit of the risky asset. A price schedule p *implements* a τ-trading schedule q if

$$q(\theta) \in \operatorname*{argmax}_{x \in \mathbb{R}} u(x, \theta) - p(x)x, \quad \forall \theta \in [-\tau, \tau]. \tag{7}$$

A trading schedule q is *implementable* if there exists a price schedule implementing it. Due to the revelation and taxation principles, any trading schedule that can be implemented by some direct or indirect mechanism can be implemented by some price schedule.

In the following definition (and throughout) we use $E[\,\cdot\,]$ to denote the expectation with respect to F.

[9] The first best allocation *of the reduced form model*, as given by (4), maximizes the surplus under the constraint that allocations are measurable with respect to θ. This is not the same as the first best allocation in terms of the underlying model, which would provide complete insurance for the informed agent's endowment shock ω.

Definition 2.1 *Let p be a price schedule implementing the τ-trading schedule q and let F be a distribution with support $[-\tau, \tau]$. Then the price and trading schedule pair (p, q) is F-*feasible *if*

$$E[p(q(\theta))q(\theta) - v(q(\theta), \theta)] \geq 0.$$

A trading schedule is *F-feasible* (or simply *feasible* if F is obvious) if (p, q) is F-feasible for price schedule p implementing q.

Feasibility of a price and trading schedule pair requires that the trading schedule arises as the outcome of a trading process in which market makers obtain non-negative expected trading profits under the implementing price schedule. We emphasize that our notion of feasibility incorporates the incentive constraints embodied in (7). Feasibility should be interpreted as feasible in the presence of private information.

Definition 2.2 *Let p be a price schedule implementing the τ-trading schedule q and let F be a distribution with support $[-\tau, \tau]$. Then, the price and trading schedule pair (p, q) is F-*zero-profit *if*

$$p(x) = E[\theta \mid q(\theta) = x], \quad \forall x \in q([-\tau, \tau]).$$

*The pair (p, q) is F-*competitive *if in addition the* sequentiality *condition,*

$$p(x) \in [-\tau, \tau], \quad \forall x \in \mathbb{R},$$

is satisfied.

A trading schedule is *F-zero-profit* (or simply *zero-profit* if F is obvious) if (p, q) is F-zero-profit for some implementing price schedule p. A corresponding convention applies for F-competitive trading schedules. Note that if (p, q) is F-zero-profit then it is F-feasible. In particular, every competitive trading schedule is feasible.

Markets in which a monopolist specialist posts a price schedule (see Glosten (1989) for such a model) or (a finite number of) market makers post competing price schedules (Biais, Martimort, and Rochet, 2000) result in feasible, but not competitive, trading schedules in which the market makers obtain strictly positive expected trading profits. Even though market makers obtain zero expected profits, Glosten's (1994) model of a discriminatory limit order market does not result in a competitive trading schedule in the sense of definition 2.2, due to cross-subsidization across different trade sizes. The zero-profit condition in definition 2.2 requires market makers to earn zero profits *conditional on the traded quantity* of the risky asset. Imposing the zero-profit condition conditional on the traded quantity is in line with the models of competitive market making in Kyle (1985), Glosten

(1989), and Rochet and Vila (1994) and allows us to investigate whether, as suggested in Glosten (1994), cross-subsidization plays an important role in avoiding the possibility of market breakdown.

The second condition appearing in definition 2.2 is akin to a Kreps and Wilson (1982)-sequentiality requirement. It insists that for all possible quantities, the price schedule specify a price consistent with zero profits, reflecting competition between market makers with some common belief over the possible types of the informed trader who might have chosen such a quantity. As we noted in footnote 3, the analogy to sequentiality is precise when trade is modeled as a signaling game. We thus refer to this condition as the sequentiality condition. Our main results hold without the sequentiality condition; imposing it on competitive schedules merely strengthens our conclusion that competition is not a source of market breakdown.

2.3 Two Preliminary Lemmas

As the informed trader's preferences satisfy the single-crossing property, implementability imposes significant structure on the schedule:

Lemma 2.1 (Rochet (1987)) *A trading schedule q is implementable if and only if it is increasing, i.e., $\theta \leq \theta' \Rightarrow q(\theta) \leq q(\theta')$.*

Given a τ-trading schedule q implemented by a price schedule p, let $R : [-\tau, \tau] \to \mathbb{R}$ be the associated rent function given by

$$R(\theta) = b\theta q(\theta) - \frac{1}{2} r q(\theta)^2 - p(q(\theta))q(\theta).$$

Lemma 2.2 (Milgrom and Segal (2002)) *If q is a τ-trading schedule implemented by p, then*

$$R(\theta) - R(\theta') = \int_{\theta'}^{\theta} u_\theta(q(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta} = b \int_{\theta'}^{\theta} q(\tilde{\theta}) d\tilde{\theta} \tag{8}$$

for all $\theta, \theta' \in [-\tau, \tau]$.

2.4 Sequences

To capture environments where the adverse selection problem caused by the informed trader's private information is extreme, we consider sequences of market environments in which the supports of the distribution function of the informed trader's type converge to the real line. As argued by Hellwig (1992), considering type distributions with unbounded support as a limiting case provides useful insights in the structure of models with large (but bounded) support of the type distribution, while avoiding technical difficulties in models with unbounded type spaces (see remark 4.1)

Fix a *limit distribution* F^* with support $\mathbb{R}$ and say that a distribution function F is the *τ-truncation* of F^* if

$$F(\theta) = \begin{cases} 1, & \text{if } \theta > \tau, \\ \dfrac{F^*(\theta) - F^*(-\tau)}{F^*(\tau) - F^*(-\tau)}, & \text{if } \theta \in [-\tau, \tau], \\ 0, & \text{if } \theta < -\tau. \end{cases}$$

Given a sequence $\{\tau_n\}$, the τ_n-truncation of F^* is denoted by F_n. We are interested in sequences of market environments characterized by $\tau_n \to \infty$ as $n \to \infty$ (we often take $n \to \infty$ as understood). Observe that $\{F_n\}$ converges weakly to F^*.[10]

We assume F^* is symmetric and absolutely continuous, and for all $\theta \neq 0$, possesses a strictly positive and twice continuously differentiable density f^*.[11] Note that for any $\tau > 0$ these properties are inherited by any τ-truncation of F^*. We also assume F^* has a finite variance σ^2, ensuring that the expected first-best surplus (recall (5)) under the distribution F^* is finite.

Definition 2.3 *The sequence $\{(\tau_n, q_n, p_n)\}$ is* consistent *if $\tau_n \to \infty$ and for all n, q_n is a τ_n-trading schedule and p_n is a price schedule implementing q_n.*

A consistent sequence $\{(\tau_n, q_n, p_n)\}$ is feasible *(respectively,* competitive *or* zero-profit*) if (p_n, q_n) is F_n-feasible (resp., F_n-competitive or F_n-zero-profit) for all n.*

Our analysis focuses on the limit behavior of feasible and competitive sequences; in particular we identify circumstances under which all feasible (and thus all competitive) sequences converge to a closed market, as well as circumstances under which there exist competitive (and thus feasible) sequences converging to an open market in the sense of the following definition.

Definition 2.4 *A consistent sequence* converges to a closed market *if for all $\theta \in \mathbb{R}$,*

$$\lim_{n \to \infty} q_n(\theta) = 0.$$

[10] We work with truncations to simplify notation. Our analysis applies essentially unchanged to sequences of symmetric distributions with bounded supports $\{F_n\}$ converging weakly to F^*, provided each F_n also possesses a strictly positive and twice continuously differentiable density f_n for all $\theta \neq 0$ and $\sup_n \int |\theta|^\alpha \, dF_n(\theta) < \infty$ for some $\alpha > 2$, so that the relevant moments converge (Chung, 1974, Theorem 4.5.2).

[11] We do not insist on the existence of a density at zero to ensure that our analysis covers distribution functions F^* obtained by symmetrizing smooth distribution functions with support $(0, \infty)$.

A consistent sequence converges to an open market *if for all* $\theta \neq 0$, $\lim_{n\to\infty} q_n(\theta)$ *exists and*

$$\lim_{n\to\infty} q_n(\theta) \neq 0.$$

Convergence to a closed market captures the idea of a total market breakdown due to adverse selection as suggested by Akerlof (1970). Our notion of convergence to an open market is quite strong: With the possible exception of the zero type (for whom there are no gains from trade and who thus trades zero in the first-best trading schedule), *every* type trades a nonzero quantity in the limit of the sequence of trading schedules. In particular, this is significantly stronger than not converging to a closed market, which would merely require the existence of some type for which $q_n(\theta)$ does not converge to zero.

One reason for our interest in the notion of convergence to an open market is Glosten's (1994) suggestion that a necessary ingredient for a market structure to avoid market breakdown (when it can be avoided) is a "small-trade spread." Under such a spread, all types in a neighborhood of the zero type do not trade, precluding convergence to an open market. In contrast, our results show that a small-trade spread is not necessary to avoid market breakdown, since there exist competitive sequences converging to an open market (unless all feasible sequences converge to a closed market).

3 The Market-Breakdown Condition

3.1 The Main Results

To develop some intuition for our main results, consider a trading schedule in the market environment F, in which all types $\tilde{\theta} \geq \theta$ trade the quantity $x > 0$ at the corresponding competitive price

$$p(x) = E[\tilde{\theta} \mid \tilde{\theta} \geq \theta].$$

The informed trader of type θ earns nonnegative payoffs from trading x under this trading schedule if and only if

$$(b\theta - p(x))x - \frac{1}{2}rx^2 \geq 0.$$

For small x, this inequality implies (we denote partial derivatives by subscripts)

$$(b-1)\theta = s_x(0,\theta) \geq E[\tilde{\theta} - \theta \mid \tilde{\theta} \geq \theta] \equiv e(\theta), \tag{9}$$

where $e : [0,\tau] \to \mathbb{R}$ is the *mean excess function* (or *mean residual life function*). The mean excess $e(\theta)$ provides a measure for the severity of the adverse selection problem facing the market makers engaging in a trade with a type θ, since it is the difference between their expected opportunity cost of selling to all types higher than θ (i.e. $E[\tilde{\theta} \mid \tilde{\theta} \geq \theta]$ and selling just to type θ. Comparing the mean excess with the change in surplus at $x = 0$ from a marginal trade with type θ yields (9).

A sufficient condition for the convergence of a competitive sequence to an open market is a strict inequality in (9) with $e(\theta)$ replaced by

$$e^*(\theta) \equiv E^*[\tilde{\theta} - \theta \mid \tilde{\theta} \geq \theta],$$

the mean excess function for the limit distribution.

Theorem 3.1 *If there exists a type $\theta > 0$ such that*

$$(b-1)\theta > e^*(\theta), \tag{10}$$

then for every sequence $\{\tau_n\}$ with $\tau_n \to \infty$, there exists an associated competitive sequence converging to an open market.

If the hypothesis in theorem 3.1 fails, then not only does every competitive sequence converge to a closed market but more generally every *feasible* sequence converges to a closed market.

Theorem 3.2 *If the* market breakdown condition

$$(b-1)\theta \leq e^*(\theta), \qquad \forall \theta > 0 \tag{11}$$

holds, then every feasible sequence converges to a closed market.

Phrased differently: every *feasible* sequence converges to a closed market if and only if every *competitive* sequence does so, and the market breakdown condition (11) is a necessary and sufficient condition for this to occur. In other words, competition *per se* cannot lead to market failure. Furthermore, theorem 3.1 shows that if market failure can be avoided, almost all types can trade.

We prove theorem 3.1 in section 4, where we explicitly construct the associated competitive sequence for any sequence of truncations $\tau_n \to \infty$. In section 5, we prove theorem 3.2. In the remainder of this section we provide an alternative interpretation for the market breakdown condition (11) and discuss circumstances under which it will or will not hold.

Remark 3.1 The results in Hellwig (1992) establish the existence of a competitive sequence converging to an open market for $b > 2$ (see lemma 4.3 below). As theorem 3.2 holds for any $b > 1$, it is a corollary of these two results that (10) holds for any F^* (satisfying the assumptions introduced in section 2.4) and $b > 2$.

♦

3.2 Interpretation of the Market Breakdown Condition

Because we have assumed F^* to be symmetric, the market breakdown condition (11) is equivalent to[12]

$$E^*[\tilde{\theta} \mid \tilde{\theta} \leq \theta] \leq b\theta \leq E^*[\tilde{\theta} \mid \tilde{\theta} \geq \theta], \qquad \forall \theta > 0. \tag{12}$$

We can interpret this condition in terms of the informed traders and market makers marginal willingness-to-pay for the risky asset. Observing that

$$u_x(x,\theta) = b\theta - rx, \quad v_x(x,\theta) = \theta,$$

we can rewrite (12) as

$$E^*[v_x(0,\tilde{\theta}) \mid \tilde{\theta} \leq \theta] \leq u_x(0,\theta) \leq E^*[v_x(0,\tilde{\theta}) \mid \tilde{\theta} \geq \theta], \quad \forall \theta, \tag{13}$$

That is, market breakdown will occur whenever the informed traders marginal willingness-to-pay lies between the market makers' marginal willingness-to-pay conditional on trading with all types smaller than θ, and the market makers' marginal willingness-to-pay conditional on trading with all types larger than θ (where the marginal willingness-to-pay is evaluated at the endowment point). The first half of this condition precludes the possibility of engaging in profitable trades in which the informed trader sells the risky asset; the second half (which is the one corresponding to (11)) trades in which the informed trader buys the risky asset. In either case it is enough to consider marginal trades because - due to the concavity of the traders' utility functions - marginal trades are easier to support than large trades.

Remark 3.2 While our analysis is quite different in scope and focus from that in Glosten (1994),[13] his paper contains a result related to our theorem 3.2. In particular, proposition 5 in Glosten (1994) provides conditions under which any price schedule satisfying a regularity condition (see corollary 1 in Glosten (1994) and the subsequent discussion) implementing a non-zero trading schedule results in expected losses for the market makers, thus violating our feasibility condition. Applying Glosten's conditions to the limit market environment F^* of our model,[14] yields, after translating to our notation

$$E^*[\tilde{\theta} \mid u_x(0,\tilde{\theta}) \leq p] < p < E^*[\tilde{\theta} \mid u_x(0,\tilde{\theta}) \geq p], \quad \forall p \in \mathbb{R}.$$

[12] Without the symmetry assumption, (12) is the appropriate market breakdown condition.

[13] Glosten considers more general market environments than we do, but for most of his (somewhat informal) analysis considers a particular trading mechanism, namely an "open limit order book." Biais, Martimort, and Rochet (2000) provide a more rigorous analysis of some aspects of Glosten's analysis, but do not consider the issue of market breakdown.

[14] Glosten's analysis applies to a fixed market environment whereas we consider the limit behavior of consistent sequences.

Because $u_x(0,\tilde{\theta})$ has full range, we can substitute $u_x(0,\theta)$ for p in the above inequality, and then, as $u_x(0,\tilde{\theta})$ is strictly increasing in $\tilde{\theta}$, the inequality is equivalent to

$$E^*[v_x(0,\tilde{\theta}) \mid \tilde{\theta} \leq \theta] < u_x(0,\theta) < E^*[v_x(0,\tilde{\theta}) \mid \tilde{\theta} \geq \theta], \quad \forall \theta \in \mathbb{R},$$

yielding a more stringent condition than (13).

Note that Glosten (1994) does not provide a counterpart to our theorem 3.1 and suggests that competitive pricing is a source of market breakdown.

♦

3.3 When will the Market Breakdown Condition be Satisfied?

In our model, market breakdown is not an issue for distributions with finite support $[-\tau,\tau]$ (cf. Lemma 4.2 below). This is appropriately reflected by the bounded support counterpart to condition (11):

$$s_x(0,\theta) \leq e(\theta), \quad \forall \theta \in [0,\tau]. \tag{14}$$

As the right side vanishes as θ approaches the upper bound τ and the left-hand-side is clearly strictly positive for all $\theta > 0$, this condition is never satisfied.[15] While this observation is part of our motivation for studying extreme adverse selection (i.e., considering the limit as $\tau \to \infty$), it also raises the question of whether there are limit distributions F^* for which (11) holds, i.e. whether extreme adverse selection may indeed cause market breakdown.[16]

Note that every limiting distribution F^* such that $e^*(\theta)/\theta$ is a decreasing function satisfying

$$\lim_{\theta\to\infty} \frac{e^*(\theta)}{\theta} \geq (b-1) \tag{15}$$

will satisfy (11). For $b < 2$ it is easy to see that there are limit distributions satisfying these requirements. We content ourselves with providing an example using the

[15]There are models (such as Akerlof (1970)) with nonextreme adverse selection (bounded support) displaying market breakdown. The bounded-support market-breakdown condition (14) applies in these cases. In Akerlof (1970), for example, there is market breakdown only when there are no gains from trade for the worst type. But this implies that the (marginal) gains from trade $s_x(0,\theta)$ disappear. Thus, the left, as well the right, hand-side of (14) converge to zero as the type approaches the worst type.

[16]This question is of particular interest, as (to the best of our knowledge) all previous examples of market breakdown in a financial market context (see Glosten and Milgrom (1985), Leach and Madhavan (1993), Glosten (1994)) assume *risk neutral* informed traders, precluding the existence of gains from trade between the informed trader and the market makers.

Pareto distribution (the proof is in appendix C):[17]

Theorem 3.3 *Suppose $b < 2$. There exists F^* for which the market breakdown condition holds.*

Conversely, if the limit appearing in (15) is well-defined, condition (15) is clearly necessary for market breakdown. The following result builds on this observation to obtain a more explicit necessary condition for the occurrence of market breakdown. The result requires that the *proportional hazard rate* of the distribution function F^*,

$$g^*(\theta) \equiv \frac{\theta f^*(\theta)}{1 - F^*(\theta)},$$

have a well-defined limit as $\theta \to \infty$. This mild regularity condition ensures that $\lim_{\theta\to\infty} e^*(\theta)/\theta$ exists. The distributions commonly studied in economics satisfy this property.[18]

Theorem 3.4 *Suppose the limit of $g^*(\theta)$ as $\theta \to \infty$ (which may be infinite) exists and suppose for $k \geq 2$, the k^{th} moment of F^* is finite. Then for all $b \geq k/(k-1)$ the market breakdown condition (11) fails.*

When the regularity condition on the proportional hazard rate holds, theorem 3.4 implies, in particular, that for any given $b > 1$ the market breakdown condition fails if all moments of F^* exist.[19] Hence, a necessary condition for the market breakdown condition (11) is that F^* has fat tails (i.e., not all moments exist). In addition, as we have assumed F^* has a finite variance, it follows from theorem 3.4 that market breakdown can only occur if $b < 2$ holds, demonstrating the necessity of this condition in the statement of theorem 3.3 (recall from remark 3.1 that the weaker necessary condition $b \leq 2$ already follows from Hellwig (1992) and does not require the regularity condition).

Remark 3.3 Suppose, as for many commonly studied distributions, the density f^*

[17] The distribution F^* used in the proof of Theorem 3.3 not only satisfies the assumptions introduced in section 2.4, but has a density decreasing in $|\theta|$. Consequently, the argument proving Theorem 3.3 in conjunction with the construction in appendix A implies that market breakdown is not an artefact of our reduced form, but occurs in the underlying model described in remark 2.1.

[18] It is satisfied by any distribution F^* with a truncation from below possessing an increasing proportional hazard rate. See van den Berg (1994) for an extensive discussion of distributions possessing an increasing proportional hazard rate.

[19] In terms of the underlying model described in remark 2.1, if follows that if all moments of (the limit distributions) of t and ω exist (e.g., in Glosten's (1989) model in which these variables are normally distributed), the market breakdown condition fails.

of F^* is log-concave on $\mathbb{R}_+$.[20] By An (1998, proposition 1), $e^*(\theta)$ is decreasing in θ, providing a simple proof that the market breakdown condition will *not* hold for sufficiently large θ in this case. As log-concavity of the density implies that the hazard rate (and thus the proportional hazard rate) is increasing and that all moments of F^* exist (An, 1998, corollary 1), this result is a special case of theorem 3.4.

♦

4 Competitive Trading Schedules and Open Markets

In this section, we prove theorem 3.1. After conducting some preliminary analysis in section 4.1, we analyze separating competitive schedules in section 4.2. These schedules converge to open markets for $b > 2$. When $b \leq 2$, the distortions required by separation result in the separating competitive schedules converging to a closed market (even when the market breakdown condition fails). Reducing these distortions requires pooling some types, and as an intermediate step we analyze tail-pooling schedules in section 4.3. We prove theorem 3.1 in section 4.4 using *semi-pooling* competitive trading schedules. In these schedules, the distortions implied by separation are ameliorated by pooling the right set of types of informed traders.

4.1 Symmetric Trading Schedules

In every zero-profit trading schedule, the type zero informed trader does not trade, implying (from lemma 2.1) that positive types are buyers ($q(\theta) \geq 0$) and negative types are sellers ($q(\theta) \leq 0$).

Lemma 4.1 *Every zero-profit trading schedule q satisfies $q(0) = 0$ and $q(\theta)\theta \geq 0$ for all $\theta \in [-\tau, \tau]$.*

Proof. Let q be a zero-profit trading schedule implemented by the price schedule p. To simplify notation, let $x_0 = q(0)$ and $p_0 = p(x_0)$. Suppose $x_0 > 0$. Because $x = 0$ is a feasible choice for the informed trader it follows from (7) that $u(x_0, 0) - p_0 x_0 = -\frac{1}{2} r x_0^2 - p_0 x_0 \geq 0$. Consequently, we have $p_0 < 0$. The zero profit condition then implies that there exists $\theta \leq p_0$ such that $q(\theta) = x_0$. But $u(x_0, \theta) - p_0 x_0 = (\theta - p_0)x_0 - \frac{1}{2} r x_0^2 < 0$, contradicting (7). An analogous argument when $q(0) < 0$ also

[20] See Bagnoli and Bergstrom (2005) for a list of parametric families of distribution functions with log-concave densities. Note that f^* will be log-concave on $\mathbb{R}_+$ whenever it is obtained by symmetrizing the log-concave density function of a distribution with support $\mathbb{R}_+$

leads to a contradiction, implying $q(0) = 0$. Lemma 2.1 now implies $q(\theta)\theta \geq 0$ for all $\theta \in [-\tau, \tau]$. ■

Lemma 4.1 simplifies the analysis of zero-profit trading schedules, since positive types and negative types can be studied independently: Negative types sell the asset and positive types buy the asset. It is immediate that no type has an incentive to choose a quantity specified for a type of a different sign. Furthermore, because the model is symmetric, we can restrict attention to symmetric zero-profit trading schedules, where a τ-trading schedule q is *symmetric* if $q(-\theta) = -q(\theta)$ for all $\theta \in [0, \tau]$. We will do so throughout the remainder of this section. In particular, whenever convenient we specify trading schedules only for positive types (and implementing price schedules only for positive quantities) with the extension to negative types (and negative quantities) then given by symmetry.

4.2 Separating Trading Schedules

A one-to-one trading schedule q is said to be *separating*. From lemma 2.1, every separating trading schedule is strictly increasing. Note that a separating τ-trading schedule is zero-profit if and only if $p(q(\theta)) = \theta$ for all $\theta \in [-\tau, \tau]$.

The notion of a separating zero-profit trading schedule is closely related to Hellwig's (1992) notion of a Spencian outcome and plays an important role in our subsequent analysis. A slight modification of the arguments in Mailath (1987) yields the following lemma (see appendix D for the proof).

Lemma 4.2 *Let $\tau > 0$ and $\bar{x} \in (0, q^{FB}(\tau)]$. There exists a unique symmetric separating zero-profit τ-trading schedule $q : [-\tau, \tau] \to \mathbb{R}$ satisfying $q(\tau) = \bar{x}$. Furthermore, a separating zero-profit trading schedule is competitive if and only if it is symmetric and satisfies $q(\tau) = q^{FB}(\tau)$. Hence, there is a unique separating competitive τ-trading schedule for every $\tau > 0$.*

Figures 1 and 2 illustrate the separating competitive trading schedule. As usual, in a separating trading schedule, imposing the sequentiality condition determines the behavior of the "worst" types. Among positive types the worst belief the market makers can hold is $\theta = \tau$, while among negative types the worst belief is $\theta = -\tau$. Since each type receives his or her type as the price in a separating price schedule, the worst types cannot be disciplined in a separating competitive trading schedule and so choose their "first-best" quantity, $q^{FB}(\theta)$. Due to the incentive constraints, the quantities for all types in the intervals $(-\tau, 0)$ and $(0, \tau)$ are distorted from their first best level towards zero. For a given support of the type distribution, the degree of distortion is determined by the trade-off between the incentive to mislead the market and the increased cost of lowered diversification, i.e. the parameters b and

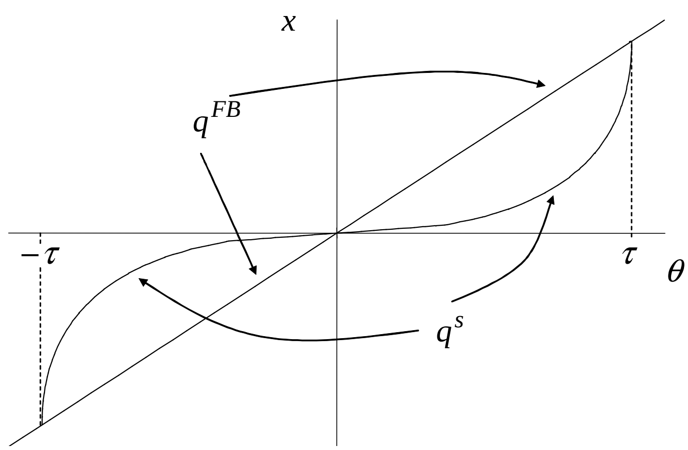

Figure 1: The separating competitive trading schedule q^s for $b \leq 2$. The trading schedule q^s is tangential to the θ-axis at $\theta = 0$.

r in the informed trader's utility function. Note that, as illustrated in the figures, the structure of the separating competitive trading schedules is different for the cases $b \leq 2$ and $b > 2$.

The behavior of the informed trader in a separating competitive trading schedule depends on the characteristics of the distribution of the private information in a limited and particular way. The value of the boundary type completely determines the separating competitive trading schedule, with other characteristics of the distribution function irrelevant. On the other hand, again as usual, increasing the severity of adverse selection by increasing τ does have a significant impact on the separating competitive trading schedule. In particular, it follows from Hellwig (1992) that competitive sequences with separating trading schedules converge to an open market if $b > 2$ and converge to a closed market if $b \leq 2$ (again, see appendix D for the proof):[21]

Lemma 4.3 (Hellwig (1992)) *Suppose $\{(\tau_n, q_n, p_n)\}$ is a competitive sequence with q_n separating for each n. If $b \leq 2$, the competitive sequence converges to a closed market. If $b > 2$, the competitive sequence converges to an open market; in particular $q_n(\theta) \to (b-2)\theta/r$ for all θ.*

Lemma 4.3 implies that for all sequences $\{\tau_n\}$ satisfying $\tau_n \to \infty$ there is an associated competitive sequence converging to an open market when $b > 2$. To

[21] Hellwig (1992) does not provide a complete proof for the case of a continuous type distribution under consideration here. In addition his analysis does not cover the case $b = 2$.

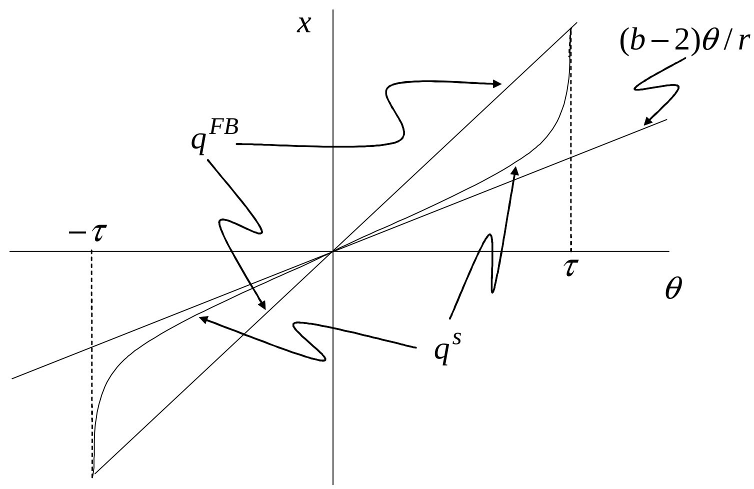

Figure 2: The separating competitive trading schedule q^s for $b > 2$. The trading schedule q^s is tangential to the line $q = (b-2)\theta/r$ at $\theta = 0$.

show that (10) is sufficient for the existence of competitive sequences converging to an open market, it thus suffices to consider the case $b \in (1,2]$ for the remainder of this section.

Remark 4.1 (Discontinuity at infinity) Our focus on large bounded type spaces, rather than unbounded type spaces, is motivated by a lack of continuity as $\tau \to \infty$. For separating competitive schedules, this lack of continuity arises for both large and small b.

Definitions 2.1 and 2.2 apply to the limit market environment where $\tau = \infty$ without change, though the sequentiality condition is trivially satisfied (since the price schedule maps into $\mathbb{R}$).

For $b > 2$, from lemma 4.3, the separating quantities q_n and prices p_n converge pointwise to those of the linear equilibrium in Glosten (1989). The additional separating competitive trading schedules identified by Glosten (1989) for the limit market environment are eliminated as potential limit outcomes by the sequentiality condition, $p_n(x) \in [-\tau_n, \tau_n]$.[22]

For $b \leq 2$, the limit of the separating competitive trading schedules is the trading schedule $q^0 : \mathbb{R} \to \mathbb{R}$ specifying $q^0(\theta) = 0, \forall \theta \in \mathbb{R}$. This limit trading schedule

[22]In contrast, dispensing with the sequentiality condition in our definition of a competitive trading schedule would not affect the conclusion of the lemma for the case $b \leq 2$ (see Hellwig (1992)).

cannot be implemented by any price schedule $p : \mathbb{R} \to \mathbb{R}$ in the limit market environment (u, v, F^*): for any price schedule there exists a (sufficiently large) type such that $q(\theta) = 0$ does not satisfy the implementation condition (7). In this sense, the limit of a sequence of competitive separating trading schedule not only fails to be separating (as noted by Hellwig (1992)) but also fails to correspond to an implementable trading schedule in the limit market environment.[23] The latter issue also arises for the sequence of trading schedules that we use to prove theorem 3.1 for the case $b \in (1, 2]$; see remark 4.3 below.

Analyzing market environments with unbounded supports (instead of considering sequences) would thus lead to misleading conclusion about market environments with large but bounded support. Moreover, unbounded type spaces lead to the interpretation of market breakdown as the non-existence of an equilibrium trading schedule, rather than the more natural interpretation that the only equilibrium trading schedule specifies zero trades for all types.

♦

4.3 Tail-pooling Schedules

For $b \leq 2$, separating competitive sequences converge to a closed market because the distortions required to separate all types become arbitrarily large as the boundary types τ and $-\tau$ become arbitrarily large in absolute value. A natural conjecture (see, for example, Hellwig (1992, footnote 3)) is that pooling extreme types eliminates the negative impact of requiring all types to separate. As we will demonstrate in this subsection, this conjecture is correct in the sense that pooling extreme types does allow us to obtain a *zero-profit* sequence that converges to an open market *if* the market breakdown condition fails. However, pooling extreme types does not generate *competitive* sequences that converge to an open market so the result obtained here falls short of proving theorem 3.1. This defect is rectified in the next subsection, where we build on the insights obtained here to construct a competitive sequence converging to an open market (when the market breakdown condition fails) that also converges to a trading schedule in which extreme types are pooled.

Definition 4.1 *A symmetric τ-trading schedule q is* tail-pooling *if there exists a* cutoff type $\hat{\theta} \in (0, \tau)$ *and a* pooling quantity $\hat{x} > 0$ *such that*

$$q(\theta) = \hat{x}, \quad \forall \theta \in (\hat{\theta}, \tau]$$

[23] If we allow prices to take on values in $\mathbb{R} \cup \{-\infty, +\infty\}$, then the trading schedule q^0 specifying no trade for all types is implementable in the limit market environment (set $p(x) = +\infty$ for $x > 0$ and $p(x) = -\infty$ for $x < 0$). Note, however, that this construction also works when $b > 2$ in which case the trading schedule q^0 does not correspond to the limit of a separating competitive sequence.

and the restriction of q to $[-\hat{\theta}, \hat{\theta}]$ is separating.

Suppose the market breakdown condition (11) fails, so that for some $\hat{\theta} > 0$, $(b-1)\hat{\theta} > e^*(\hat{\theta})$ and let

$$\hat{x} = \underset{x}{\operatorname{argmax}}(b\hat{\theta} - \hat{\theta} - e^*(\hat{\theta}))x - \frac{1}{2}rx^2 = \frac{((b-1)\hat{\theta} - e^*(\hat{\theta}))}{r} > 0. \tag{16}$$

Consider $\tau_n \to \infty$. We now construct, for sufficiently large n, a symmetric tail-pooling zero-profit τ_n-trading schedule q_n with cut-off type $\hat{\theta}$ and pooling quantity $\hat{x}$.

The zero-profit condition requires a price at the quantity $\hat{x}$ of

$$p_n(\hat{x}) = E_n\left[\tilde{\theta} \mid \tilde{\theta} \in (\hat{\theta}, \tau_n]\right] = \hat{\theta} + e_n(\hat{\theta}). \tag{17}$$

The rent for type $\hat{\theta}$ from choosing $\hat{x}$ is then given by

$$R_n(\hat{\theta}) = [(b-1)\hat{\theta} - e_n(\hat{\theta})]\hat{x} - \frac{1}{2}r\hat{x}^2.$$

Because $e_n(\theta) \to e^*(\theta)$ (see lemma B.2 in appendix B) and $\hat{x}$ satisfies (16) we have that $R_n(\hat{\theta})$ converges to

$$R^*(\hat{\theta}) \equiv \frac{((b-1)\hat{\theta} - e^*(\hat{\theta}))^2}{r} > 0.$$

Thus, for sufficiently large n, the rent $R_n(\hat{\theta})$ is strictly positive, and for such n we construct a symmetric zero-profit τ_n-trading schedule.

Set $q_n(\theta) = \hat{x}$ for all $\theta \in (\hat{\theta}, \tau]$. To complete the specification of the trading schedule q_n, let $\bar{x}_n \in (0, \hat{x})$ be the quantity making type $\hat{\theta}$ indifferent between revealing his type at $\bar{x}_n$ and joining the pool, i.e., $\bar{x}_n$ is the (unique) quantity $\bar{x}_n \in (0, \hat{x})$ satisfying $s(\bar{x}_n, \hat{\theta}) = R_n(\hat{\theta})$. From lemma 4.2, there exists a symmetric separating zero-profit $\hat{\theta}$-trading schedule $q_n : [-\hat{\theta}, \hat{\theta}] \to \mathbb{R}$ satisfying the initial condition $q_n(\hat{\theta}) = \bar{x}_n$.

The price schedule is determined for quantities in the range of q_n by using the zero profit condition, i.e., (17) and $p_n(q_n(\theta)) = \theta$ for all $\theta \in [-\hat{\theta}, \hat{\theta}]$. Standard arguments using the single-crossing property of u show that no type has an incentive to choose the quantity of another type. By specifying sufficiently unattractive prices for quantities outside the range of q_n, no type has an incentive to choose quantities outside the range, and so the symmetric tail-pooling schedule constructed in this way is implementable.

Consider now the impact of taking $\tau_n \to \infty$ on the sequence of tail-pooling trading schedules we have just constructed. Since $R_n(\hat{\theta}) \to R^*(\hat{\theta}) > 0$, the limit separating quantity $\lim_n \bar{x}_n$ and the limit rent to $\hat{\theta}$ from the separating quantity $\lim_n s(\bar{x}_n, \hat{\theta})$ must both be strictly positive. Hence, $q_n(\theta) \to q^t(\theta)$ for all θ, where $q^t : [-\hat{\theta}, \hat{\theta}] \to \mathbb{R}$ is (applying lemma 4.2) the symmetric separating zero-profit $\hat{\theta}$-trading schedule satisfying the initial condition $q^t(\hat{\theta}) = \lim_n \bar{x}_n$, $q^t(\theta) = \hat{x}$ for $\theta > \hat{\theta}$, and $q^t(\theta) = -\hat{x}$ for $\theta < -\hat{\theta}$. Since $q^t(\theta) = 0$ only if $\theta = 0$, we have proved the following lemma.

Lemma 4.4 *Suppose there exists $\hat{\theta} > 0$ satisfying $(b-1)\hat{\theta} > e^*(\hat{\theta})$. For every $\tau_n \to \infty$, there exists an associated zero-profit sequence $\{(\tau_n, q_n, p_n)\}$ converging to an open market in which, for n sufficiently large, q_n is a tail-pooling trading schedule with cutoff-type $\hat{\theta}$.*

As we have noted above, lemma 4.4 does not establish theorem 3.1, since it ignores the sequentiality condition. Indeed, the sequence constructed in the proof of lemma 4.4 is *not* competitive. To see this, note that for any price schedule p_n implementing the trading schedule q_n constructed above, the rent obtained by type τ_n is given by

$$R_n(\tau_n) = b[\tau_n - \hat{\theta}]\hat{x} + R_n(\hat{\theta}).$$

As $R_n(\hat{\theta})$ converges to a finite limit, it follows that $R_n(\tau_n)$ is of order $O(\tau_n)$, while from (5), $s^{FB}(\tau_n)$ is of order $O(\tau_n^2)$, so that eventually $R_n(\tau_n) < s^{FB}(\tau_n)$. If p_n is a competitive price schedule, type τ_n can obtain a payoff at least equal to $s^{FB}(\tau_n)$ by choosing $q^{FB}(\tau_n)$, and so for large n, p_n cannot implement q_n, a contradiction. Consequently, any implementing price schedule must violate the sequentiality condition for sufficiently large n. This failure of sequentiality is similar to the failure (discussed in remark 4.1) of the trading schedule q^0, in which all types trade zero, to be implementable.

Remark 4.2 (Competitive tail-pooling and closed markets) It can be shown that all tail-pooling *competitive* sequences converge to a closed market when $b \leq 2$, implying that tail-pooling competitive sequences must converge to a closed market under precisely the same conditions as separating competitive trading schedules. This is despite the fact that for every τ there exist competitive tail-pooling trading schedules interim Pareto-dominating the competitive separating τ-trading schedule.[24] Every tail-pooling competitive sequence converges to a closed market when $b \leq 2$ because sustaining a tail-pool for large τ_n requires a large pooling quantity $\hat{x}_n$ (to ensure that type τ_n is willing to participate in the pool), which in turn requires the cutoff-type, $\hat{\theta}_n$, to also be large. That is, $\lim_n \hat{x}_n = \infty$ and $\lim_n \hat{\theta}_n = \infty$.

[24] That is, every non-zero type of the informed trader achieves a higher rent under the former than under the later. Market makers obtain zero profits under any competitive trading schedule so this is the appropriate notion of interim Pareto-dominance.

But for $b \leq 2$ this implies convergence to a closed market for the same reason that separating competitive sequences converge to a closed market. The tail-pool zero-profit trading schedule of lemma 4.4 converges to an open market because both the pooling quantity and cutoff type are bounded away from infinity as n gets large.

♦

4.4 Semi-pooling Trading Schedules

The difficulty noted after the statement of lemma 4.4 is avoided by adjusting the construction of a tail-pooling trading schedule to allow sufficiently extreme types to separate.

Definition 4.2 *A symmetric τ-trading schedule q is* semi-pooling *if there exists a* pooling interval $(\hat{\theta}, \bar{\theta}]$ *where* $0 < \hat{\theta} < \bar{\theta} < \tau$ *and a pooling quantity* $\hat{x} > 0$ *such that*

$$q(\theta) = \hat{x}, \qquad \forall \theta \in (\hat{\theta}, \bar{\theta}]$$

and the restriction of q to $\theta \in [-\tau, -\bar{\theta}) \cup [-\hat{\theta}, \hat{\theta}] \cup (\bar{\theta}, \tau]$ is separating.

A semi-pooling trading schedule differs from a tail-pooling trading schedule only in that the types $\theta \in (\bar{\theta}, \tau]$ do not choose the pooling quantity $\hat{x}$ but are instead separated.

In conjunction with lemma 4.3 the following result establishes theorem 3.1.

Lemma 4.5 *Suppose $b \in (1, 2]$ and there exists $\hat{\theta} > 0$ satisfying $(b-1)\hat{\theta} > e^*(\hat{\theta})$. For every sequence $\{\tau_n\}$ with $\tau_n \to \infty$, there exists an associated competitive sequence $\{(\tau_n, q_n, p_n)\}$ converging to an open market in which, for n sufficiently large, q_n is a semi-pooling trading schedule.*

We now describe how to construct the competitive semi-pooling trading schedule q_n for sufficiently large n. Lemma D.1 in appendix D ensures that for large n, there exists a triple $(\bar{\theta}_n, \hat{x}_n, \hat{p}_n) \in \mathbb{R}^3$ satisfying the properties required in this construction. The formal argument showing the convergence of such a sequence to an open market is provided in lemma D.2.

Our construction of the competitive semi-pooling schedule q_n is illustrated in in figure 3 for positive types. The solid line depicts the trading schedule. The pooling interval is given by $(\hat{\theta}, \bar{\theta}_n]$ with $\bar{\theta}_n < \tau_n$ and $\hat{\theta}$ satisfying the condition $(b-1)\hat{\theta} > e^*(\hat{\theta})$. The pooling quantity is $\hat{x}_n$. For $\theta > \bar{\theta}_n$ the trading schedule is identical to the unique separating competitive τ_n-trading schedule illustrated in figure 1. Type $\bar{\theta}_n$ is indifferent between his trade in the separating competitive τ_n-trading schedule, $q^s_n(\bar{\theta}_n)$, and trading the pooling quantity at the price

$$\hat{p}_n = E_n\left[\theta|\theta \in (\hat{\theta}, \bar{\theta}_n]\right] < b\hat{\theta}, \tag{18}$$

where the inequality will hold for n sufficiently large. The pooling quantity $\hat{x}_n$ satisfies the condition that it be the optimal quantity for type $\hat{\theta}$ taking the price $\hat{p}_n$ as given:

$$\hat{x}_n = \underset{x}{\operatorname{argmax}}(b\hat{\theta} - \hat{p}_n)x - \frac{1}{2}rx^2 = \frac{(b\hat{\theta} - \hat{p}_n)}{r} > 0,$$

where the inequality is from the inequality in (18). Finally, for $0 \le \theta \le \hat{\theta}$, the trading schedule is given by the zero-profit separating $\hat{\theta}$-trading schedule satisfying the initial condition $q(\hat{\theta}) = \underline{x}_n$ where $\underline{x}_n \in (0, \hat{x}_n)$ satisfies the indifference condition

$$s(\underline{x}_n, \hat{\theta}) = \frac{(b\hat{\theta} - \hat{p}_n)^2}{r},$$

with the inequality in (18) ensuring that the quantity $\underline{x}_n$ is well-defined and strictly positive.

It is immediate from the construction of the semi-pooling trading schedule q_n that the trading schedule is implemented by a price schedule p_n specifying $p_n(\hat{x}_n) = \hat{p}_n$, $p_n(q_n(\theta)) = \theta$ for all θ that are separated,[25] and sufficiently unattractive prices for all quantities q outside the range of the trading schedule. Hence, q_n is a zero-profit trading schedule.

To verify that q_n is in fact competitive, and not just zero-profit, requires extending the zero-profit specification of the price schedule p_n to quantities not in the range of q_n. This extension is illustrated in figure 3 where the heavy dashed lines indicate the specification of the price schedule p_n outside the range of q_n. For quantities x in the interval $(\underline{x}_n, \hat{x}_n)$, the price $p_n(x)$ is set to make $\hat{\theta}$ indifferent between trading x at the price $p_n(x)$ and trading $\hat{x}_n$ at the price $\hat{p}_n$ (equivalently, trading $\underline{x}_n$ at the price $\hat{\theta}$); for quantities $x \in (\hat{x}_n, q^s_n(\bar{\theta}_n))$, the price $p_n(x)$ is set to make $\bar{\theta}_n$ indifferent between trading x at the price $p_n(x)$ and trading $\hat{x}_n$ at the price $\hat{p}_n$ (equivalently, trading $q^s_n(\bar{\theta}_n)$ at the price $\bar{\theta}_n$); and finally, for $x > q^{FB}(\tau_n)$, set $p_n(x) = \tau_n$. In addition to implementing q_n (this is immediate from the single-crossing property of u and footnote 25) the defined price function is increasing and continuous (and so satisfies the sequentiality condition).[26]

[25] Note that we have ensured that type $\hat{\theta}$ (resp., $\bar{\theta}_n$) is indifferent between trading the pooling quantity $\hat{x}$ at price $\hat{p}$ to trading the quantity $\underline{x}_n$ at price $\hat{\theta}$ (resp., to trading the quantity $q^s_n(\bar{\theta}_n)$ at price $\bar{\theta}_n$).

[26] The definition of $\hat{x}_n$ plays a critical role in determining the properties of p_n. In particular, for a pooling interval $(\hat{\theta}, \bar{\theta}_n)$, if we had fixed the pooling quantity larger than $\hat{x}_n$, any implementing price function could not be increasing.

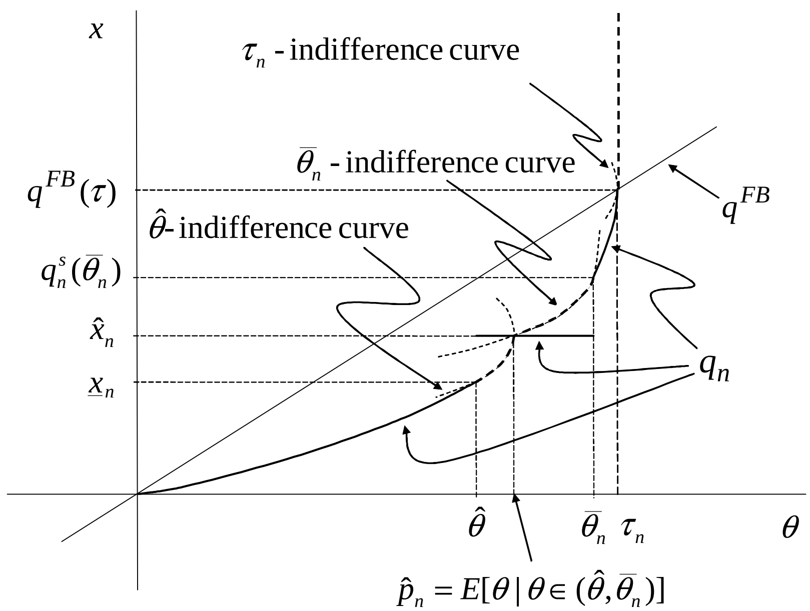

Figure 3: Semi-pooling competitive trading schedule with pooling quantity $\hat{x}_n$ and pooling interval $(\hat{\theta}, \bar{\theta}_n]$. The solid line depicts the trading schedule q_n.

In the proof of lemma D.2, we demonstrate that the sequence $\bar{\theta}_n$ associated with the semi-pooling schedule q_n converges to infinity, implying that the sequence of pooling quantities $\{\hat{x}_n\}$ converges to the strictly positive limit $\hat{x}$ given in (16), and that the sequence of pooling prices $\{\hat{p}_n\}$ converges to $\hat{\theta} + e^*(\hat{\theta})$. The sequence of semi-pooling trading schedules thus converges pointwise to the same limit as the sequence of tail-pooling schedules constructed in section 4.3. In particular, the sequence converges to an open market, thus proving lemma 4.5.

Remark 4.3 We noted that the sequence of tail-pool zero-profit trading schedule constructed in section 4.3 is not competitive (because it violates sequentiality). For a similar reason, the limit of the semi-pooling trading schedules we have constructed here cannot be implemented by any price schedule $p : \mathbb{R} \to \mathbb{R}$ in the limit market environment (u, v, F^*): Sufficiently large types have an incentive to deviate from the pooling quantity. This observation illustrates again (see remark 4.1) that it is essential for our analysis to consider sequences of bounded types spaces instead of working with unbounded types spaces.

♦

5 Convergence to Closed Markets

In this section, we prove theorem 3.2. The results in the previous section established that the existence of θ such that $(b-1)\theta > e^*(\theta)$ is *sufficient* for the existence of a competitive sequence converging to an open market. Here we establish not only that this conditions is *necessary* for the existence of such a competitive sequence, but prove the stronger result that *every* feasible sequence, and hence every competitive sequence, $\{(\tau_n, q_n, p_n)\}$ converges to a *closed* market if the above conditions fail. We recall the statement of theorem 3.2.

Theorem 3.2 *If the* market breakdown condition (11) *holds, then every feasible sequence converges to a closed market.*

We highlight the key steps in the proof here, relegating the more technical arguments to appendix E. Suppose $\{(\tau_n, q_n, p_n)\}$ is a feasible sequence with $q_n(\theta) \to q^*(\theta)$ for all θ.[27] For market environment F_n, denote the market makers' expected profit by

$$\Pi_n \equiv \int_{-\tau_n}^{\tau_n} p_n(q_n(\theta))q_n(\theta) - v(q_n(\theta), \theta)\, dF_n(\theta). \tag{19}$$

[27] The possibility that $\{q_n(\theta)\}$ is not a convergent sequence for some θ is an example of the technical issues dealt with in appendix E.

Since the market environment includes the possibility of both negative and positive types, as well as as the market makers acting as both buyers and sellers, there is a potential for profits from trades on one side of the market to subsidize losses on the other. Suppose, however, that under our feasible sequence $q_n(0) = 0$ and that there is no such cross-subsidization (while this is without loss of generality, see lemma E.2, it is more subtle than the simple argument for zero-profit schedules in section 4.1). We can then restrict attention to $\theta \geq 0$. Since the sequence is feasible, for all n,

$$\begin{aligned} 0 \leq \Pi_n^+ &\equiv \int_0^{\tau_n} p_n(q_n(\theta))q_n(\theta) - v(q_n(\theta),\theta)\,dF_n(\theta) \\ &= \int_0^{\tau_n} VS_n(q_n(\theta),\theta)\,dF_n(\theta), \end{aligned} \tag{20}$$

where

$$\begin{aligned} VS_n(\theta) &\equiv s(q_n(\theta),\theta) - b\frac{(1-F_n(\theta))}{f_n(\theta)}q_n(\theta) \\ &\equiv S_n(\theta) - b\frac{(1-F_n(\theta))}{f_n(\theta)}q_n(\theta) \end{aligned}$$

is the *virtual surplus* and we use lemma 2.2 and integration by parts to obtain (20). When we decorate a trading schedule, such as $\tilde{q}_n$, the corresponding functions defined above are similarly decorated.

We now argue that when the market-breakdown condition is satisfied, Π_n^+ converges to a strictly negative number (and so must eventually be negative) if $q^*(\theta) > 0$ for any $\theta > 0$, contradicting feasibility.

As $\{q_n(\theta)\}$ may be unbounded as a function of θ, we introduce an upper bound on quantities, $x_n^\dagger = q_n(\theta^\dagger)$, with $\theta^\dagger$ to be determined, and define $\tilde{q}_n(\theta) \equiv \min\{q_n(\theta), x_n^\dagger\}$. Recalling the comment about decorations, rewrite the expression for Π_n^+ in (20) as

$$\Pi_n^+ = \int_0^{\tau_n} \widetilde{VS}_n(\theta)\,dF_n(\theta) + \int_0^{\tau_n} \left\{VS_n(\theta) - \widetilde{VS}_n(\theta)\right\} dF_n(\theta). \tag{21}$$

The quadratic nature of the surplus allows us to write the first integral as

$$\int_0^{\tau_n} \{(b-1)\theta f_n(\theta) - b(1-F_n(\theta))\}\,\tilde{q}_n(\theta)\,d\theta - \int_0^{\tau_n} \frac{r}{2}(\tilde{q}_n(\theta))^2\,dF_n(\theta). \tag{22}$$

We now argue to a contradiction from the hypothesis that $q^*(\hat{\theta}) > 0$ for some $\hat{\theta} > 0$, i.e., that the feasible sequence does not converge to a closed market. The last term in (22) satisfies for $\theta^\dagger > \hat{\theta}$ (since q_n is increasing)

$$\begin{aligned}
-\int_0^{\tau_n} \frac{r}{2}(\tilde{q}_n(\theta))^2 f_n(\theta)\,d\theta &\le -\int_{\hat{\theta}}^{\tau_n} \frac{r}{2}(q_n(\hat{\theta}))^2 f_n(\theta)\,d\theta \\
&= -\frac{r(1-F_n(\hat{\theta}))}{2}(q_n(\hat{\theta}))^2 \\
&\to -\frac{r(1-F^*(\hat{\theta}))}{2}(q^*(\hat{\theta}))^2. \qquad (23)
\end{aligned}$$

A calculation (see lemma E.3) shows that the second integral in (21) is, for large n, bounded above by

$$2\int_{\theta^\dagger}^{\infty} s^{FB}(\theta)\,dF^*(\theta). \qquad (24)$$

Because the expected first-best surplus is finite, for sufficiently large $\theta^\dagger$, (24) is sufficiently small that the sum of the bounds (23) and (24) is strictly negative.

It remains only argue that the first integral in (22) cannot dominate the other terms. It is here that the market-breakdown condition is used. By the next lemma, that integral does converge to zero, and so Π_n^+ is eventually negative, contradicting the feasibility of q_n.

Lemma 5.1 *Suppose $\tau_n \to \infty$ and $\{x_n^\dagger\}$ is a sequence of numbers converging to $x^\dagger \ge 0$. Let H_n be the value of the program*

$$\max_{q_n} \int_0^{\tau_n} \{(b-1)\theta f_n(\theta) - b(1-F_n(\theta))\}\, q_n(\theta)\,d\theta \qquad (25)$$

subject to

$$q_n : [0,\tau_n] \to [0, x_n^\dagger] \text{ increasing}. \qquad (26)$$

If $(b-1)\theta \le e^*(\theta)$ for all θ, then $H_n \to 0$.

Proof. The constrained maximization problem described by (25) subject to (26) is a special case of the optimal auction design problem in Myerson (1981). Hence, there exists θ_n such that

$$H_n = x_n^\dagger \int_{\theta_n}^{\tau_n} \{(b-1)\theta f_n(\theta) - b(1-F_n(\theta))\}\,d\theta. \qquad (27)$$

Suppose the sequence $\{H_n\}$ does not converge to zero. As $H_n \ge 0$ for all n, the first moment of F^* is finite, and $x_n^\dagger \to x^\dagger$, the sequence $\{H_n\}$ is bounded. We may thus assume (by taking an appropriate subsequence if necessary) $H_n \to H > 0$.

Suppose, first, the associated sequence $\{\theta_n\}$ satisfying (27) is unbounded. Then, there exists a subsequence $\{\theta_m\}$ such that $\theta_m \to \infty$, implying

$$H_m \le x_m^\dagger \int_{\theta_m}^{\tau_m} (b-1)\theta \, dF_m(\theta) \le \frac{x_m^\dagger}{F^*(\tau_m) - F^*(-\tau_m)} \int_{\theta_m}^{\infty} (b-1)\theta \, dF^*(\theta) \to 0,$$

contradicting the hypothesis $H_n \to H > 0$.

Suppose, then, that the sequence $\{\theta_n\}$ satisfying (27) is bounded. Then there exists a subsequence $\{\theta_m\}$ such that $\theta_m \to \bar{\theta} \ge 0$. Performing an integration by parts on the first term in (27) and using lemma B.1, we obtain

$$H_m = x_m^\dagger (1 - F_m(\theta_m))[(b-1)\theta_m - e_m(\theta_m)]$$

and thus

$$H_m \to x^\dagger (1 - F^*(\bar{\theta}))[(b-1)\bar{\theta} - e^*(\bar{\theta})].$$

By assumption $(b-1)\bar{\theta} - e^*(\bar{\theta}) \le 0$ holds for all $\bar{\theta} \ge 0$, contradicting the hypothesis $H_n \to H > 0$, finishing the proof. ■

6 Concluding Remarks

We, like most of the literature studying adverse selection in financial markets, considered market environments where the informed trader can act as either buyer or seller. It is immediate that our theorems continue to hold as stated if we had restricted the analysis to the buy side of the market while considering type distributions with support in the positive reals.[28] In this sense considering a "two-sided" model is not essential for our analysis. It is also clear that the buy-side and the sell-side of the market can be studied independently for competitive trading schedules (see section 4.1). The possibility of an interaction between the buy-side and the sell-side of the market implies, however, that this separation is not trivially true for feasible trading schedules. Consequently, it is important that our analysis address this possibility. While our proof relies on the assumption that the informed trader's type is symmetrically distributed (for expository convenience), our analysis could be extended to asymmetric distributions, with the market breakdown condition then given by (12). In the absence of symmetry, the interesting possibility arises that only one of the two inequalities in (12) holds. Competitive pricing then implies the informed trader can only be active on one side of the market, whereas, as suggested

[28]Mutatis mutandis the same is true for the sell-side.

by Glosten and Milgrom (1985), cross-subsidies between asset sales and purchases would make it possible to provide liquidity on both sides of the market.

Finally, note that our market breakdown condition hinges on the specific functional form of the informed trader's preferences. We believe that the reformulation of the condition in terms of the trader's marginal willingness-to-pay (see (13)) holds quite generally and provides useful insights into the conditions under which adverse selection will cause market breakdown. For instance, the example we use to prove theorem 3.3 can be used to show that a change in the type distribution leading to a first-order-stochastic dominant increase in the distribution of first-best gains from trade may cause an open market to close. This is surprising and only emphasizes the need to gain a better understanding of the relationship between the distribution of the first-best gains from trade and the occurrence of market-breakdown in more general market environments.

Appendices

A Constructing the Distributions in Remark 2.1

In this section, we describe how to construct zero-mean random variables (t,ω) satisfying (6) given a symmetric distribution for θ with density decreasing in $|\theta|$, and given values for $b > 1$ and $r > 0$.

Let $\hat{\theta} = b\theta$. Because the distribution of $\hat{\theta}$ is symmetric with density decreasing in $|\theta|$, it follows from Eaton (1981, proposition 1) that we may assume the existence of a random variable μ so that the distribution of $(\hat{\theta},\mu)$ is rotation invariant and independent of ε. Let $\alpha \in (0,2\pi)$ satisfy $\tan\alpha = \sqrt{b-1}$ and define random variables x and y as the solution to the equations

$$\begin{aligned} & \mu = x\cos\alpha + y\sin\alpha \\ \text{and} \qquad & \hat{\theta} = -x\sin\alpha + y\cos\alpha. \end{aligned}$$

Because (x,y) is a rotation of $(\hat{\theta},\mu)$, the distribution of (x,y) is identical to the distribution of $(\hat{\theta},\mu)$ and thus, in particular, rotation invariant. Let $t = y\cos\alpha$ and $\omega = x\sin\alpha/r$. As a linear transformation of (x,y), the random variables (t,ω) are elliptically distributed (Fang, Kotz, and Ng, 1990). Because elliptically distributed random variables possess the linear conditional expectation property (Hardin, 1982), (t,ω) have zero mean, and are uncorrelated, we have

$$E[t|t-r\omega] = \frac{\sigma_t^2}{\sigma_t^2 + r^2\sigma_\omega^2}(t-r\omega).$$

As $E[v|t-r\omega] = E[t|t-r\omega]$ and $(t-r\omega) = \hat{\theta} = b\theta$ holds by construction, this implies (6), provided the equality

$$b = \frac{\sigma_t^2 + r^2\sigma_\omega^2}{\sigma_t}^2 \Leftrightarrow \sqrt{b-1} = \frac{r\sigma_\omega}{\sigma_t}$$

is satisfied. This in turn follows from

$$\frac{r\sigma_\omega}{\sigma_t} = \frac{\sigma_x \sin\alpha}{\sigma_y \cos\alpha} = \tan\alpha = \sqrt{b-1},$$

where the second equality uses the fact that the distribution of (x,y) is rotation invariant and the corresponding standard deviations thus satisfy $\sigma_x = \sigma_y$.

B Properties of the Mean Excess Function

For a distribution function F with support $[-\tau,\tau]$, let $e : [0,\tau] \to \mathbb{R}$ be the mean excess function defined by $e(\theta') = E[\tilde{\theta} - \theta' | \tilde{\theta} \geq \theta']$. The mean excess function for F^* (which has support $\mathbb{R}$) is $e^* : [0,\infty) \to \mathbb{R}$ defined by $e^*(\theta') = E^*[\tilde{\theta} - \theta' | \tilde{\theta} \geq \theta']$.

Lemma B.1 *The mean excess function e satisfies*

$$e\left(\theta'\right) = \frac{1}{1 - F\left(\theta'\right)} \int_{\theta'}^{\tau} 1 - F\left(\theta\right) d\theta,$$

and the mean excess function e^* satisfies

$$e^*\left(\theta'\right) = \frac{1}{1 - F^*\left(\theta'\right)} \int_{\theta'}^{\infty} 1 - F^*\left(\theta\right) d\theta.$$

Proof. For $0 \leq \theta \leq \tau$,

$$\begin{aligned}
e\left(\theta'\right) &= E\left[\theta - \theta' | \theta \geq \theta'\right] \\
&= \frac{1}{1 - F\left(\theta'\right)} \int_{\theta'}^{\tau} (\theta - \theta') f\left(\theta\right) d\theta \\
&= \frac{1}{1 - F\left(\theta'\right)} \left\{ -\left(\theta - \theta'\right)\left(1 - F\left(\theta\right)\right)\Big|_{\theta'}^{\tau} + \int_{\theta'}^{\tau} 1 - F\left(\theta\right) d\theta \right\} \\
&= \frac{1}{1 - F\left(\theta'\right)} \left\{ \int_{\theta'}^{\tau} 1 - F\left(\theta\right) d\theta \right\}.
\end{aligned}$$

The argument for e^* is identical, where the last equality is an implication of $\lim_{\theta\to\infty} \theta(1 - F^*(\theta)) = 0$, which follows from the existence of the first moment of F^*. ■

Lemma B.2 *Suppose $\tau_n \to \infty$. Then, the associated sequence of mean excess functions $\{e_n\}$ converges to e^* pointwise.*

Proof. Since

$$e_n(\theta') = \frac{1}{1-F_n(\theta')} \int_{\theta'}^{\infty} \theta \, dF_n(\theta) - \theta',$$

the convergence of $e_n(\theta')$ to $e^*(\theta')$ follows from the convergence of F_n to F^* and of $\int |\theta| \, dF_n(\theta)$ to $\int |\theta| \, dF^*(\theta)$. ■

C The Market Breakdown Condition

Proof of theorem 3.3.

The desired distribution, F^*, is obtained via symmetrization of the density of a translated Pareto distribution with shape parameter β so that

$$F^*(\theta) = 1 - \frac{1}{2}(\theta+1)^{-\beta}, \qquad \theta \geq 0.$$

Let

$$2 < \beta \leq \frac{b}{b-1}. \tag{28}$$

The assumption $b < 2$ ensures that such a β exists; the restriction $\beta > 2$ guarantees the variance of F^* exists.

The mean excess function of F^* is given by

$$e^*(\theta) = \frac{\int_{\theta}^{\infty} \tilde{\theta}\beta(\tilde{\theta}+1)^{-\beta-1} d\tilde{\theta}}{(\theta+1)^{-\beta}} - \theta$$

Since,

$$\begin{aligned}
\int_{\theta}^{\infty} \tilde{\theta}\beta(\tilde{\theta}+1)^{-\beta-1} d\tilde{\theta} &= \int_{\theta}^{\infty} (\tilde{\theta}+1)\beta(\tilde{\theta}+1)^{-\beta-1} d\tilde{\theta} - (\theta+1)^{-\beta} \\
&= \int_{\theta}^{\infty} \beta(\tilde{\theta}+1)^{-\beta} d\tilde{\theta} - (\theta+1)^{-\beta} \\
&= \frac{\beta}{(1-\beta)} \, (\tilde{\theta}+1)^{-\beta+1}\Big|_{\theta}^{\infty} - (\theta+1)^{-\beta} \\
&= \frac{\beta}{(\beta-1)}(\theta+1)^{-\beta+1} - (\theta+1)^{-\beta},
\end{aligned}$$

it follows that

$$e^*(\theta) = \frac{\beta}{(\beta-1)}(\theta+1) - 1 - \theta = \frac{(\theta+1)}{(\beta-1)}.$$

Thus,

$$e^*(\theta) \geq \frac{\theta}{\beta - 1} \geq (b-1)\theta,$$

where the second inequality is from (28). ■

Proof of theorem 3.4. Let $\lim_{\theta\to\infty} g^*(\theta) = g$, where g is possibly infinite. The finiteness of the k^{th}-moment of F^* for $k \geq 2$ implies $g > k \geq 2$ (Lariviere, 2006, theorem 2).[29] Using lemma B.1 for the first equality and applying l'Hôpital's rule to get the second equality, we have

$$\begin{aligned}
\lim_{\theta\to\infty} \frac{e^*(\theta)}{\theta} &= \lim_{\theta\to\infty} \frac{1}{\theta(1-F^*(\theta))} \int_\theta^\infty (1 - F^*(\tilde{\theta}))\, d\tilde{\theta}) \\
&= \lim_{\theta\to\infty} \frac{-(1-F^*(\theta))}{1 - F^*(\theta) - \theta f^*(\theta)} \\
&= \lim_{\theta\to\infty} \frac{-1}{1 - g^*(\theta)} = \frac{1}{g-1}
\end{aligned}$$

and thus

$$\lim_{\theta\to\infty} \frac{e^*(\theta)}{\theta} < \frac{1}{k-1}. \tag{29}$$

For θ sufficiently large, (29) implies

$$e^*(\theta) < \frac{1}{k-1}\theta \leq (b-1)\theta,$$

where the second inequality uses the assumption $b \geq k/(k-1)$. ■

D Competitive Trading Schedules

Proof of lemma 4.2. Since a symmetric separating trading schedule $q^s : [-\tau, \tau] \to \mathbb{R}$ satisfies $q^s(-\theta) = -q^s(\theta)$, it is enough to show the existence of a unique separating trading schedule $q^s : [0, \tau] \to \mathbb{R}_+$ satisfying $q^s(\tau) = \bar{x}$ (the restriction of the range to nonnegative quantities is without loss of generality from lemma 4.1). Such a trading schedule is a one-to-one function solving

$$\theta \in \operatorname*{argmax}_{\theta' \in [0,\tau]} u(q^s(\theta'), \theta) - \theta' q^s(\theta') \tag{30}$$

$$\text{and} \qquad q^s(\tau) = \bar{x}. \tag{31}$$

[29] Lariviere (2006, theorem 2) assumes g^* is increasing, but the proof only uses g^* increasing to conclude that $\lim_{\theta\to\infty} g^*(\theta)$ exists.

The differentiability of any one-to-one function q^s satisfying (30) would follow from Mailath (1987, Theorem 2), except that condition (2), belief monotonicity, is not satisfied. Belief monotonicity requires that the marginal payoff to a change in the beliefs of the uninformed agents (here given by $-x$) never equals 0. However, since single crossing implies a strictly increasing solution to (30), at most $\theta = 0$ can choose $x = 0$, and so belief monotonicity holds for interior types. An examination of the arguments in Mailath (1987) reveals this is enough to obtain differentiability.

We verify existence and uniqueness directly. The maximization problem in (30) implies the first order condition

$$\frac{dq^s(\theta)}{d\theta} = \frac{q^s(\theta)}{(b-1)\,\theta - rq^s(\theta)}. \tag{32}$$

Letting $y(x) = (q^s)^{-1}(x)$ and rearranging, we have

$$xy' - (b-1)y = -rx. \tag{33}$$

Suppose $b \neq 2$. The linear function $rx/(b-2)$ is a particular solution to (33), and βx^{b-1} is a general solution to the homogeneous differential equation $xy' - (b-1)y = 0$. Adding these two yields the general solution

$$y(x) = \frac{r}{(b-2)}x + \beta x^{b-1}$$

(this is well-defined since $x \geq 0$), where β is chosen to satisfy the initial value implied by (31),

$$y(\bar{x}) = \tau. \tag{34}$$

Thus,

$$\beta = \bar{x}^{1-b}\left(\tau - \frac{r}{b-2}\bar{x}\right).$$

Suppose now $b = 2$. Rewrite (33) as $xy' = y - rx$, and differentiate, yielding $y' + xy'' = y' - r$. That is, $y'' = -r/x$. Integrating twice gives $y(x) = -r\int \log x + \alpha x + \kappa$, where α and κ are constants. Equation (33) is only satisfied if $\kappa = 0$.

Hence, the general solution is

$$y(x) = -rx\log x + rx + \alpha x$$

for $x > 0$ with $y(0) = 0$. The parameter α is chosen so that (34) holds.

It remains to verify the uniqueness claim (monotonicity can be verified by calculation). For all $\varepsilon \in (0, \bar{x})$, the equation $(b-1)y/x - r$ is Lipschitz in x for all $x \in [\varepsilon, \bar{x}]$, the initial value problem (33) and (34) has a unique solution on $[\varepsilon, \bar{x}]$.

Letting $\varepsilon \to 0$ gives uniqueness on $[0,\bar{x}]$, and so the initial value problem (32) and (31) has the inverse of y as a unique solution. ■

Proof of lemma 4.3. Fix n. From lemma 4.3, there is a unique symmetric separating competitive trading schedule q_n^s. For $b \neq 2$, from the proof of lemma 4.3, the schedule is implicitly given by, for $\theta \geq 0$,

$$\theta = \frac{-r}{2-b} q_n^s(\theta) + \beta_n(\theta)\left(q_n^s(\theta)\right)^{b-1},$$

where

$$\beta_n(\theta) = \left(\frac{b-1}{r}\right)^{1-b} \frac{1}{2-b} \tau_n^{2-b}.$$

For $b = 2$, the schedule is given by

$$\theta = r q_n^s(\theta)\left[\log \tau_n - \log q_n^s(\theta)\right] + r q_n^s(\theta).$$

For $b < 2$, as $\tau_n \to \infty$, for $\theta > 0$, the coefficient $\beta_n(\theta) \to +\infty$, while for $\theta < 0$, $\beta_n(\theta) \to -\infty$. In other words, for a fixed trade level x, the type choosing that trade diverges. Equivalently, (since the trading schedules are ordered, with $q_n^s(\theta) > q_{n'}^s(\theta)$ for $0 \leq \theta \leq \tau_n$ if $\tau_n < \tau_n'$) the trade of any fixed type converges to 0. Similarly, for $b = 2$, as $\tau_n \to \infty$, for a fixed trade level x, the type choosing that trade diverges. Hence, if $b \leq 2$ every competitive sequence of separating trading schedules converges to a closed market.

Finally, for $b > 2$, $\beta_n(\theta) \to 0$ as $\tau_n \to \infty$, and so $q_n(\theta) \to (b-2)\theta/r$ for all θ in every competitive sequence of separating trading schedules. ■

Lemma D.1 *Let $\hat{\theta} > 0$ satisfy $(b-1)\hat{\theta} > e^*(\hat{\theta})$. For any $\{\tau_n\}$ satisfying $\tau_n \to \infty$ there exists an associated sequence $\{(\bar{\theta}_n, \hat{x}_n, \hat{p}_n)\}$ with $(\bar{\theta}_n, \hat{x}_n, \hat{p}_n) \in \mathbb{R}^3$, satisfying for all n sufficiently large, $\bar{\theta}_n \in (\hat{\theta}, \tau_n)$,*

$$\hat{p}_n = E_n\left[\theta \mid \theta \in (\hat{\theta}, \bar{\theta}_n]\right] < b\hat{\theta}, \tag{35}$$

$$\hat{x}_n = \frac{(b\hat{\theta} - \hat{p}_n)}{r} > 0, \tag{36}$$

and

$$R_n^s(\bar{\theta}_n) = u(\hat{x}_n, \bar{\theta}_n) - \hat{p}_n \hat{x}_n, \tag{37}$$

where $R_n^s : [-\tau_n, \tau_n] \to \mathbb{R}$ is the rent function associated with the unique separating competitive τ_n trading schedule.

Proof of lemma D.1. Observe first that there exists N such that $\hat{\theta} < \tau_n$ for all $n \geq N$.

Consider any sequence $\{\bar{\theta}_n\}$ satisfying $\bar{\theta}_n \in (\hat{\theta}, \tau_n)$ for all $n \geq N$. For all $n \geq N$, determine $(\hat{p}_n, \hat{x}_n)$ by the equalities in (35) and (36). From lemma B.2, we have $e_n(\hat{\theta}) \to e^*(\hat{\theta})$ and thus,

$$(b-1)\hat{\theta} > e_n(\hat{\theta}). \tag{38}$$

for n large. Because

$$E_n\left[\theta \mid \theta \in (\hat{\theta}, \bar{\theta}_n]\right] < \hat{\theta} + e_n(\hat{\theta})$$

it is immediate from (38) that the inequality in (35) and, thus the inequality in (36), holds for all sufficiently large n.

It remains to argue that the sequence $\{\bar{\theta}_n\}$ can be chosen such that (37) holds for n large. Towards this end, note first that since $\hat{x}_n$ is the utility maximizing quantity for trader $\hat{\theta}$ facing a fixed price of $\hat{p}_n \geq \hat{\theta}$, and the trader captures the first best surplus at the price $\hat{\theta}$ when trading the quantity $q^{FB}(\hat{\theta})$, we have $u(\hat{x}_n, \hat{\theta}) - \hat{p}_n \hat{x}_n \leq s^{FB}(\hat{\theta})$. Moreover, for n fixed, $\hat{p}_n$ and $\hat{x}_n$ are continuous functions of $\bar{\theta}_n \in [\hat{\theta}, \tau_n]$.

At the point $\bar{\theta}_n = \hat{\theta}$ we have $\hat{p}_n = \hat{\theta}$ and thus $\hat{x}_n = q^{FB}(\hat{\theta})$, implying that the right side of (37) is strictly larger than the left side (as $R_n^s(\theta) < s^{FB}(\theta)$ for all $\theta \in (0, \tau_n)$). As

$$u(\hat{x}_n, \bar{\theta}_n) - \hat{p}_n \hat{x}_n = (\bar{\theta}_n - \hat{\theta})\hat{x}_n + u(\hat{x}_n, \hat{\theta}) - \hat{p}_n \hat{x}_n \leq (\bar{\theta}_n - \hat{\theta}) q^{FB}(\hat{\theta}) + s^{FB}(\hat{\theta}),$$

the right side of (37) increases linearly with $\bar{\theta}_n$. Consequently, because $R_n^s(\tau_n) = s^{FB}(\tau_n)$ is a quadratic function of τ_n, for n large the left side of (37) is strictly larger than the right side at $\bar{\theta}_n = \tau_n$. As both sides of (37) are continuous in $\bar{\theta}_n$ it then follows from the intermediate value theorem that there exists $\bar{\theta}_n \in (\hat{\theta}, \tau_n)$ such that (37) holds. ∎

Lemma D.2 *The semi-pooling trading schedule constructed in section 4.4 converges to an open market.*

Proof of lemma D.2. Under q_n, the quantity traded by $\hat{\theta}$ is

$$\hat{x}_n > \frac{[(b-1)\hat{\theta} - e_n(\hat{\theta})]}{r}.$$

Let $\eta \equiv [(b-1)\hat{\theta} - e^*(\hat{\theta})]/2 > 0$. Since for large n, $\left|e_n(\hat{\theta}) - e^*(\hat{\theta})\right| < \eta$, the quantity traded by $\hat{\theta}$ is bounded below by

$$\frac{[(b-1)\hat{\theta} - e^*(\hat{\theta}) - \eta]}{r} = \frac{\eta}{r}.$$

It remains to argue that, for $\theta \neq 0$, $q_n(\theta)$ converges to a nonzero quantity.

We claim that $\bar{\theta}_n \to \infty$ as $n \to \infty$: If not, there exists a subsequence with $\bar{\theta}_n \to \bar{\theta} < \infty$. But then $q_n^s(\bar{\theta}_n) \to 0$, and so $R_n^s(\bar{\theta}_n) \to 0$. However, $R_n^s(\bar{\theta}_n) = [2b\bar{\theta}_n - b\hat{\theta} - \hat{p}_n)]\hat{x}_n/2$, the utility from pooling. Since this latter term is no smaller than $(b-1)\bar{\theta}_n\hat{x}_n/2$, which is bounded away from zero, we have a contradiction.

Consequently, q_n converges pointwise to q^t, the tail-pooling trading schedule where all types $\theta \geq \hat{\theta}$ pool on the quantity

$$\hat{x} = \frac{[(b-1)\hat{\theta} - e^*(\hat{\theta})]}{r},$$

types $\theta \leq -\hat{\theta}$ pool on the quantity $-\hat{x}$, and types $|\theta| < \hat{\theta}$ separate.

Since $q^t(\theta) \neq 0$ for all $\theta \neq 0$, $\{q_n\}$ converges to an open market. ■

E Convergence to Closed Markets

E.1 Preliminaries

Before stating and proving the various lemmas needed to prove theorem 3.2, we begin with some preliminary definitions and maintained assumptions.

For any feasible trading and price schedule pair (q_n, p_n), aggregate trading profits are

$$\pi_n(\theta) = p_n(q_n(\theta))q_n(\theta) - v(q_n(\theta), \theta),$$

the surplus function is given by

$$S_n(\theta) = s(q_n(\theta), \theta) = R_n(\theta) + \pi_n(\theta),$$

and the virtual surplus function by

$$VS_n(\theta) \equiv \begin{cases} S_n(\theta) + b\frac{F_n(\theta)}{f_n(\theta)} q_n(\theta), & \text{if } \theta < 0, \\ S_n(\theta) - b\frac{1-F_n(\theta)}{f_n(\theta)} q_n(\theta), & \text{if } \theta > 0. \end{cases}$$

As in the text, when we decorate a trading schedule, such as $\tilde{q}_n$, the corresponding functions defined above are similarly decorated. Note that (8) implies

$$R_n(\theta) = b\int_0^\theta q_n(\tilde{\theta})d\tilde{\theta} + R_n(0)$$

and thus, upon substituting for $S_n(\theta) = u(q_n(\theta), \theta) - v(q_n(\theta), \theta)$ into (19) and integrating by parts aggregate profits,

$$\Pi_n = \Pi_n^- + \Pi_n^+ - R_n(0)$$

where

$$\Pi_n^- \equiv \int_{-\tau_n}^{0} VS_n(\theta)\,dF_n(\theta),$$
$$\Pi_n^+ \equiv \int_{0}^{\tau_n} VS_n(\theta)\,dF_n(\theta).$$

Given F^* and a sequence $\tau_n \to \infty$, we assume that n is sufficiently large that

$$F^*(\tau_n) - F^*(-\tau_n) > \frac{1}{2}. \tag{39}$$

E.2 Two Technical Lemmas

Lemma E.1 *Let $\{(\tau_n, q_n, p_n)\}$ be a feasible sequence. Then for all $\theta \in \mathbb{R}$ the sequence $\{q_n(\theta)\}$ is bounded.*

Proof. Suppose there exists $\hat{\theta} \in \mathbb{R}$ such that $\{q_n(\hat{\theta})\}$ is unbounded above (the case in which $\{q_n(\hat{\theta})\}$ is unbounded below is analogous). There then exists a subsequence $\{q_m\}$ such that $q_m(\hat{\theta}) \to \infty$. For fixed $\theta^\dagger > \hat{\theta}$, since the trading schedules q_m are increasing (lemma 2.1), we have

$$q_m(\theta) \to \infty, \quad \forall \theta \in [\hat{\theta}, \theta^\dagger]. \tag{40}$$

Since $R_m(\theta) \geq 0$ and so $S_m(\theta) \geq \pi_m(\theta)$ for all θ, we have

$$\Pi_m \leq \int_{-\tau_m}^{\tau_m} S_m(\theta) f_m(\theta)\,d\theta. \tag{41}$$

For sufficiently large m, $-\tau_m < \hat{\theta} < \theta^\dagger < \tau_m$ and recalling (39), so

$$\begin{aligned}
\int_{-\tau_m}^{\tau_m} S_m(\theta) f_m(\theta)\,d\theta &= \frac{1}{F^*(\tau_m) - F^*(-\tau_m)} \int_{-\tau_m}^{\tau_m} S_m(\theta) f^*(\theta)\,d\theta \\
&\leq 2 \int_{-\tau_m}^{\tau_m} S_m(\theta) f^*(\theta)\,d\theta \\
&\leq 2 \left[\int_{-\tau_m}^{\tau_m} s^{FB}(\theta) f^*(\theta)\,d\theta + \int_{\hat{\theta}}^{\theta^\dagger} S_m(\theta) f^*(\theta)\,d\theta \right].
\end{aligned}$$

Using (5) we have

$$\int_{-\tau_m}^{\tau_m} s^{FB}(\theta) f^*(\theta)\,d\theta \to \frac{(b-1)^2}{2r}\sigma^2,$$

where σ^2 is the variance of F^*. From (3) and (40) we have $S_m(\theta) \to -\infty$ for all $\theta \in [\hat{\theta}, \theta^\dagger]$ and thus

$$\int_{\hat{\theta}}^{\theta^\dagger} S_m(\theta) f^*(\theta) d\theta \to -\infty.$$

Hence, (41) implies $\Pi_m \to -\infty$, contradicting the hypothesis that $\{q_n\}$ is a feasible sequence. ■

Lemma E.2 *If there exists a feasible sequence not converging to a closed market, then there exists a feasible sequence $\{(\tilde{\tau}_m, \tilde{q}_m, \tilde{p}_m)\}$ satisfying:*

1. *there exist $\hat{\theta} > 0$ and $\hat{x} > 0$ such that $\tilde{q}_m(\hat{\theta}) \to \hat{x}$ as $m \to \infty$,*
2. *$\tilde{q}_m(0) = 0$ for all m, and*
3. *there exists $\tilde{\Pi}^+ \geq 0$ such that $\tilde{\Pi}_m^+ \to \tilde{\Pi}^+$ as $m \to \infty$.*

Proof. The lemma is established in three steps, in which we sequentially construct the sequence, verifying at each step that the desired property holds. Denote by $\{(\tau_n, q_n, p_n)\}$ the feasible sequence not converging to a closed market.

STEP 1 As the sequence $\{(\tau_n, q_n, p_n)\}$ does not converge to a closed market, there is a type θ^* such that $q_n(\theta^*)$ does not converge to zero. From lemma E.1, the sequence $\{q_n(\theta^*)\}$ is bounded, so there exists a subsequence $\{(\tau_m, q_m, p_m)\}$ of $\{(\tau_n, q_n, p_n)\}$ such that $q_m(\theta^*) \to x^* \neq 0$. If $x^* > 0$ and $\theta^* > 0$, then property 1 in the statement of the Lemma holds for the feasible sequence $\{(\tau_m, q_m, p_m)\}$.

If $x^* > 0$ and $\theta^* \leq 0$, consider any $\hat{\theta} > 0 \geq \theta^*$. As q_m is increasing in θ for all m and $\{q_m(\hat{\theta})\}$ is bounded, there exists a subsequence $\{(\tau_k, q_k, p_k)\}$ of $\{(\tau_m, q_m, p_m)\}$ and an $\hat{x} \geq x^* > 0$ such that $q_k(\hat{\theta}) \to \hat{x}$, verifying property 1 for the feasible sequence $\{(\tau_k, q_k, p_k)\}$.

If $x^* < 0$, define a new sequence $\{(\tau_m, q_m^\dagger, p_m^\dagger)\}$ by "flipping" $\{q_m\}$ and $\{p_m\}$, i.e., $q_m^\dagger(\theta) = -q_m(-\theta)$ for all θ and m, $p_m^\dagger(x) = -p_m(-x)$ for all x and m. This sequence then satisfies $q_m^\dagger(-\theta^*) \to -x^* > 0$ and is feasible for $\{F_m\}$, because (u, v, F_m) is a symmetric market environment for all m. Replacing θ^* by $-\theta^*$, x^* by $-x^*$, and $\{q_m\}$ by $\{q_m^\dagger\}$ in the arguments for the case $x^* > 0$ establishes property 1.

STEP 2 By step 1, we can now assume property 1 holds for the original sequence, i.e., there exists $\hat{\theta} > 0$ satisfying $q_n(\hat{\theta}) = \hat{x} > 0$.

Let

$$\tilde{q}_n(\theta) = \begin{cases} \min[q_n(\theta), 0], & \text{if } \theta < 0 \\ 0, & \text{if } \theta = 0 \\ \max[q_n(\theta), 0], & \text{if } \theta > 0 \end{cases}$$

The trading sequence $\{\tilde{q}_n\}$ then satisfies $\tilde{q}_n(0) = 0$ for all n and $\tilde{q}_n(\hat{\theta}) \to \hat{x} > 0$. We show next that $\tilde{q}_n$ is feasible for F_n for each n, establishing the existence of a sequence $\{\tau_n, \tilde{q}_n, \tilde{p}_n)\}$ satisfying properties 1 and 2 in the statement of the lemma. Towards this end note, first, that as q_n is increasing so is $\tilde{q}_n$. Lemma 2.1 implies that, for all n, the trading schedule $\tilde{q}_n$ is implementable. To show that $\{\tilde{q}_n\}$ is feasible, it suffices to show that

$$\tilde{R}_n(\theta) \leq R_n(\theta) \quad \text{and} \quad s(\tilde{q}_n(\theta), \theta) \geq s(q_n(\theta), \theta) \tag{42}$$

and thus $\tilde{\pi}_n(\theta) \geq \pi_n(\theta)$ holds for all θ.

Let $\underline{\theta}_n = \inf\{\theta \mid \tilde{q}_n(\theta) = 0\}$ (we do not exclude the possibility $\underline{\theta}_n = -\tau_n$) and $\bar{\theta}_n = \sup\{\theta \mid \tilde{q}_n(\theta) = 0\}$ (we do not exclude the possibility $\bar{\theta}_n = \tau_n$). For all $\theta \in (\underline{\theta}_n, \bar{\theta}_n)$ (42) holds because for those types $\tilde{R}_n(\theta) = 0 \leq R_n(\theta)$ and $s(\tilde{q}_n(\theta) = 0 \geq s(q_n(\theta), \theta)$, where the latter inequality follows from (3) and observing that $q_n(\theta)\theta \leq 0$ for all types in $(\underline{\theta}_n, \bar{\theta}_n)$. Consider then $\theta > \bar{\theta}_n$. By construction, we have $\tilde{q}_n(\theta) = q_n(\theta)$ and thus $s(\tilde{q}_n(\theta), \theta) = s(q_n(\theta), \theta)$. From (8) we have

$$\tilde{R}_n(\theta^\dagger) - \tilde{R}_n(\bar{\theta}_n) = R_n(\theta^\dagger) - R_n(\bar{\theta}_n), \quad \forall \theta^\dagger > \bar{\theta}_n,$$

implying (42) (because $\tilde{R}_n(\bar{\theta}_n) = 0 \leq R_n(\bar{\theta}_n)$). For $\theta < \underline{\theta}_n$, (42) follows from an analogous argument, establishing the feasibility of $\{\tilde{q}_n\}$.

STEP 3 Let $\{(\tau_n, q_n, p_n)\}$ be feasible for a sequence $\{F_n\}$ approximating F^* and suppose properties 1 and 2 in the statement of the lemma are satisfied. By hypothesis, $q_n(0) = 0$ and thus $R_n(0) = 0$ holds for all n, implying the identity

$$\Pi_n = \Pi_n^- + \Pi_n^+.$$

We next show that the sequences $\{\Pi_n^-\}$ and $\{\Pi_n^+\}$ are bounded so that there exists a subsequence $\{(\tau_m, q_m, p_m)\}$, $\Pi^- \in \mathbb{R}$, and $\Pi^+ \in \mathbb{R}$ such that $\Pi_m^- \to \Pi^-$ and $\Pi_m^+ \to \Pi^+$. Using $q_n(\theta) \geq 0$ for all $\theta \geq 0$ in the first inequality, we have for all n sufficiently large:

$$\begin{aligned}
\Pi_n^+ &\leq \int_0^{\tau_n} [S_n(\theta) f_n(\theta)]\, d\theta \\
&\leq \int_0^{\tau_n} \left[s^{FB}(\theta) f_n(\theta)\right] d\theta \\
&= \frac{1}{F^*(\tau_n) - F^*(-\tau_n)} \int_0^{\tau_n} \left[s^{FB}(\theta) f^*(\theta)\right] d\theta \\
&< \frac{1}{F^*(\tau_n) - F^*(-\tau_n)} \int_0^{\infty} \left[s^{FB}(\theta) f^*(\theta)\right] d\theta \\
&< 2 \int_0^{\infty} \left[s^{FB}(\theta) f(\theta)\right] d\theta
\end{aligned}$$

$$= \frac{(b-1)^2}{2r}\sigma^2,$$

establishing that $\{\Pi_n^+\}$ is bounded above. An analogous argument shows that $\{\Pi_n^-\}$ is bounded above. Because $\{(\tau_n, q_n, p_n)\}$ is feasible we have $\Pi_n = \Pi_n^- + \Pi_n^+ \geq 0$ for all n. It then follows from the fact that $\{\Pi_n^-\}$ (resp. $\{\Pi_n^+\}$) is bounded above that $\{\Pi_n^+\}$ (resp. $\{\Pi_n^-\}$) is bounded below.

Let $\{(\tau_m, q_m, p_m)\}$ be a subsequence of $\{(\tau_n, q_n, p_n)\}$ satisfying $\Pi_m^- \to \Pi^-$ and $\Pi_m^+ \to \Pi^+$. If $\Pi^+ \geq 0$, the sequence $\{(\tau_m, q_m, p_m)\}$ satisfies properties 1–3 in the statement of the lemma and so is the desired sequence $\{(\tilde{\tau}_m, \tilde{q}_m, \tilde{p}_m)\}$.

Finally, suppose $\Pi^+ < 0$. Then, because $\Pi_m \to \Pi^- + \Pi^+$, feasibility of the sequence $\{(\tau_m, q_m, p_m)\}$ implies $\Pi^- > 0$. Consider the "flipped" sequence $\{(\tau_m, q_m^\dagger, p_m^\dagger)\}$ defined by $q_m^\dagger(\theta) = -q_m(-\theta)$ for all θ and m, and $p_m^\dagger(x) = -p_m(-x)$ for all x and m. By construction, this sequence satisfies properties 2 and 3 in the statement of the lemma and, because of symmetry of (u, v, F_m), is feasible for the sequence $\{F_m\}$. We complete our argument by demonstrating that there is a subsequence $\{(\tau_k, \tilde{q}_k, \tilde{p}_k)\}$ of the flipped sequence $\{(\tau_m, q_m^\dagger, p_m^\dagger)\}$ also satisfying property 1. Suppose not. Then we must have, for the unflipped sequence, $q_m(\theta) \to 0$ for all $\theta < 0$. [If not, we can find a type $\tilde{\theta} < 0$ and a subsequence $\{q_k\}$ such that $q_k(\tilde{\theta}) \to \tilde{x} \neq 0$. Because $q_k(\tilde{\theta}) \leq 0$ holds for all k we must have $\tilde{x} < 0$, and so the flipped sequence satisfies property 1.] Let $0 < \varepsilon < \Pi^-$. As the second moment of F^* exists, there exists $\hat{\theta} < 0$ such that

$$2\int_{-\infty}^{\hat{\theta}} s^{FB}(\theta) f^*(\theta)\, d\theta < \varepsilon.$$

Noting that for all m sufficiently large,

$$\Pi_m^- \leq 2\int_{-\infty}^{\hat{\theta}} s^{FB}(\theta) f^*(\theta)\, d\theta + \int_{\hat{\theta}}^{0} S_m(\theta) f_m(\theta)\, d\theta,$$

and that the second integral on the right hand side converges to zero because $q_m(\theta) \to 0$ for all $\theta \in [\hat{\theta}, 0]$, we obtain a contradiction to the hypothesis $\Pi_m^- \to \Pi^- > \varepsilon$. ∎

E.3 Proof of Theorem 3.2

Assume there is a feasible sequence $\{(\tau_n, q_n, p_n)\}$ that does not converge to a closed market. From lemma E.2, we may assume without loss of generality that there exists $\hat{\theta} > 0$ such that $q_n(\hat{\theta}) \to \hat{x} > 0$, $q_n(0) = 0$ for all n, and $\Pi_n^+ \to \Pi^+ \geq 0$. Because $\hat{x} > 0$ and the second moment of F^* exists, there exists $\theta^\dagger > \hat{\theta}$ such that

$$-[1 - F^*(\hat{\theta})]\frac{1}{2} r\hat{x}^2 + 2\int_{\theta^\dagger}^{\infty} s^{FB}(\theta) f^*(\theta) d\theta < 0. \tag{43}$$

From lemma E.1, $q_n(\theta^\dagger)$ is bounded and there thus exists a subsequence $\{(\tau_m, q_m, p_m)\}$ of $\{(\tau_n, q_n, p_n)\}$ such that $q_m(\theta^\dagger) \to x^\dagger$. As every implementable trading schedule is increasing, we have $x^\dagger \geq \hat{x}$.

The following calculations will show that if the market breakdown condition (11) holds, $\{\Pi_n^+\}$ is bounded above by a sequence converging to the left hand side of (43), contradicting the hypothesis $\Pi_n^+ \to \Pi^+ \geq 0$ and thus establishing the desired result.

Define the sequence $\{\tilde{q}_n\}$ by setting $\tilde{q}_n(\theta) = \min[q_n(\theta), q_n(\theta^\dagger)]$ for all θ and n. We recall equation (21),

$$\Pi_n^+ = \int_0^{\tau_n} \widetilde{VS}_n(\theta)\, dF_n(\theta) + \int_0^{\tau_n} \left[VS_n(\theta) - \widetilde{VS}_n(\theta)\right] dF_n(\theta). \tag{44}$$

Lemma E.3 *For n sufficiently large,*

$$\int_0^{\tau_n} \left[VS_n(\theta) - \widetilde{VS}_n(\theta)\right] dF_n(\theta) \leq 2\int_{\theta^\dagger}^{\infty} s^{FB}(\theta)\, dF^*(\theta).$$

Proof. The integrand on the left is equal to zero for all $\theta \in (0, \theta^\dagger]$. For $\theta \geq \theta^\dagger$ we have $q_n(\theta) \geq q_n(\theta^\dagger) = \tilde{q}_n(\theta) \geq 0$ and thus

$$\begin{aligned} VS_n(\theta) - \widetilde{VS}_n(\theta) &= S_n(\theta) - \tilde{S}_n(\theta) - b\frac{1-F_n(\theta)}{f_n(\theta)}[q_n(\theta) - \tilde{q}_n(\theta)] \\ &\leq S_n(\theta) - \tilde{S}_n(\theta) \\ &\leq s^{FB}(\theta) \end{aligned}$$

where the last inequality follows from the calculation

$$\begin{aligned} s(z,\theta) - s(x,\theta) &= (b-1)\theta(z-x) - \frac{1}{2}r(z^2 - x^2) \\ &= (b-1)\theta(z-x) - \frac{1}{2}r(z-x)^2 - \frac{1}{2}r\left(2zx - 2x^2\right) \\ &\leq s^{FB}(\theta) - rx(z-x). \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^{\tau_n} \left[VS_n(\theta) - \widetilde{VS}_n(\theta)\right] dF_n(\theta) &\leq \int_{\theta^\dagger}^{\tau_n} s^{FB}(\theta)\, dF_n(\theta) \\ &\leq 2\int_{\theta^\dagger}^{\infty} s^{FB}(\theta)\, dF^*(\theta). \end{aligned}$$

■

Consider now the first integral in (44). This can be written as

$$\int_0^{\tau_n} \widetilde{VS}_n(\theta)\, dF_n(\theta) = \int_0^{\tau_n} [s(\tilde{q}_n(\theta),\theta) f_n(\theta) - (1-F_n(\theta)) b\tilde{q}_n(\theta)]\, d\theta$$
$$= \int_0^{\tau_n} [(b-1)\theta f_n(\theta) - b(1-F_n(\theta))]\, \tilde{q}_n(\theta) d\theta - \int_0^{\tau^n} \frac{1}{2} r\tilde{q}_n(\theta)^2\, dF_n(\theta).$$

Because $\tilde{q}_n(\theta) \geq q_n(\hat{\theta})$ for all $\theta \geq \hat{\theta}$, we have

$$\int_0^{\tau_n} \frac{1}{2} r\tilde{q}_n(\theta)^2\, dF_n(\theta) \geq \int_{\hat{\theta}}^{\tau_n} \frac{1}{2} r q_n(\hat{\theta})^2\, dF_n(\theta)$$
$$= \{1 - F_n(\hat{\theta})\} \frac{1}{2} r (q_n(\hat{\theta}))^2.$$

Combining our calculations so far, we have

$$\Pi_n^+ \leq G_n + \left\{ -\{1-F_n(\hat{\theta})\} \frac{1}{2} r q_n(\hat{\theta})^2 + 2 \int_{\theta^\dagger}^{\infty} s^{FB}(\theta)\, dF^*(\theta) \right\}, \tag{45}$$

where

$$G_n = \int_0^{\tau_m} [(b-1)\theta f_n(\theta) - b(1-F_n(\theta))]\, \tilde{q}_n(\theta) d\theta.$$

By lemma 5.1, (11) implies that the sequence $\{G_n\}$ is bounded above by a sequence converging to zero. Hence (45) implies that $\{\Pi_n^+\}$ is bounded above by a sequence converging to the left side of (43), completing the proof.

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