

Purification in the Infinitely-Repeated Prisoners' Dilemma

V. Bhaskar*
University of Essex

George J. Mailath†
University of Pennsylvania

Stephen Morris‡
Yale University

January 19, 2004

Abstract

This paper investigates the Harsanyi (1973)-purifiability of mixed strategies in the repeated prisoners' dilemma with perfect monitoring. We perturb the game so that in each period, a player receives a private payoff shock which is independently and identically distributed across players and periods. We focus on the purifiability of a class of one-period memory mixed strategy equilibria used by Ely and Välimäki (2002) in their study of the repeated prisoners' dilemma with private monitoring. We find that the strategy profile is purifiable by perturbed-game finite-memory strategies if and only if it is strongly symmetric, in the sense that after every history, both players play the same mixed action. Thus "most" strategy profiles are not purifiable by finite memory strategies. However, if we allow infinite memory strategies in the perturbed game, then any completely-mixed equilibrium is purifiable.

Key words: Purification, repeated games, belief-free equilibria, imperfect monitoring.

JEL Classification Nos: C72, C73

*Department of Economics, University of Essex, Wivenhoe Park, Colchester C04 3SQ, UK. email: vbhas@essex.ac.uk

†Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104-6297. email: gmailto@econ.upenn.edu

‡Cowles Foundation, Yale University, P.O.Box 208281, New Haven, CT 06520-8281. email: stephen.morris@yale.edu

1 Introduction

Harsanyi's (1973) purification theorem is one of the most compelling justifications for the study of mixed equilibria in finite normal form games. Under this justification, the complete-information normal form game is viewed as the limit of a sequence of incomplete-information games, where each player's payoffs are subject to private shocks. Harsanyi proved that every equilibrium (pure or mixed) of the original game is the limit of equilibria of close-by games with incomplete information. Moreover, in the incomplete-information games, players have essentially strict best replies, and so will not randomize. Consequently, a mixed strategy equilibrium can be viewed as a pure strategy equilibrium of any close-by game of incomplete information. Harsanyi's (1973) argument exploits the regularity (a property stronger than local uniqueness) of equilibria of "almost all" normal form games. As long as payoff shocks generate small changes in the system of equations characterizing equilibrium, the regularity of equilibria ensures that the perturbed game has an equilibrium close to any equilibrium of the unperturbed game.¹

Very little work has examined purification in dynamic games. Even in finite extensive games, generic local uniqueness of equilibria may be lost when we build in natural economic features into the game, such as imperfect observability of moves and time separability of payoffs. Bhaskar (2000) has shown how these features may lead to a failure of local uniqueness and purification: i.e., for a generic choice of payoffs, there is a continuum of mixed strategy equilibria, none of which are the limit of the pure strategy equilibria of a game with payoff perturbations.

For infinitely repeated games, the bootstrapping nature of the system of equations describing many of the infinite horizon equilibria is conducive to a failure of local uniqueness of equilibria. We study a class of symmetric one-period memory mixed strategy equilibria used by Ely and Välimäki (2002) in their study of the repeated prisoners' dilemma with private monitoring. This class fails local uniqueness quite dramatically: there is a two dimensional manifold of equilibria.

Our motivation for studying the purifiability of this class of strategies comes from the recent literature on repeated games with private monitoring. Equilibrium incentive constraints in games with private monitoring are difficult to verify because calculating best replies typically requires understanding the nature of players' beliefs about the private histories of other players. Piccione (2002) showed that by introducing just the right amount of mixing *in every period*, a player's best replies can be made independent of his beliefs, and thus beliefs become irrelevant.² This means in particular that these equilibria of the perfect monitoring game trivially extend to the game with private monitoring. Piccione's (2002)

¹See Govindan, Reny, and Robson (2003) for a modern exposition and generalization of Harsanyi (1973).

²This was not the first use of randomization in repeated games with private monitoring. A number of papers construct nontrivial equilibria using initial randomizations to instead generate uncertainty over which the players can then update (Bhaskar and Obara (2002), Bhaskar and van Damme (2002), and Sekiguchi (1997)).

strategies depend on the infinite history of play. Ely and Välimäki (2002) showed that it suffices to consider simple strategies which condition only upon one period memory of both players' actions. These strategies again make a player indifferent between his actions regardless of the action taken by the other player, and thus a player's incentives do not change with his beliefs. Kandori and Obara (2003) also use such strategies to obtain stronger efficiency results via private strategies in repeated games with imperfect public monitoring.

At first glance, the equilibria of Piccione (2002) and Ely and Välimäki (2002) involve unreasonable randomizations: in some cases, a player is required to randomize differently after two histories, even though the player has identical beliefs over the continuation play of the opponent.³ Moreover, the randomizations involve a delicate intertemporal trade-off. While there are many ways of modelling payoff shocks in a dynamic game, these shocks should not violate the structure of dynamic game. In repeated games, a reasonable constraint is that the payoffs shocks should be independently and identically distributed over time, and moreover, the period t shock should only be realized at the beginning of period t . Our question is: Do the delicate intertemporal trade-offs survive these independently and identically distributed shocks?

Our results show that, in the repeated game with perfect monitoring, most (but not all) of the Ely-Välimäki equilibria can only be purified by infinite horizon strategies, i.e., strategies that are no simpler than those of Piccione (2002). However, while equilibria of the unperturbed perfect monitoring game are automatically equilibria of the unperturbed private monitoring game, our purification arguments do *not* automatically extend to the private monitoring case. We conjecture—but have not been able to prove—that in the repeated game with *private* monitoring all the Ely-Valimäki equilibria will be not be purifiable with finite history strategies but will be purifiable with infinite history strategies.

The paper is organized as follows. In Section 2, we review the completely mixed equilibria of the repeated prisoners' dilemma introduced by Ely and Välimäki (2002). The positive and negative purification results for finite history strategies are in Section 3. In Section 4, we present the positive purification result for infinite history strategies. Finally, in Section 5, we briefly discuss the private monitoring case.

³Anticipating the notation from the next section, this occurs, for example, when $g = \ell$ (the incentive to play D is independent of the action of the opponent), so that $p_{CC} = p_{DC}$ and $p_{CD} = p_{DD}$.

2 Belief-free Equilibria with Perfect Monitoring

Let $\Gamma(0)$ denote the infinitely-repeated perfect-monitoring prisoners' dilemma with stage game:

	C	D
C	$1, 1$	$-\ell, 1 + g$
D	$1 + g, -\ell$	$0, 0$

Each player has a discount rate δ . The class of symmetric mixed strategy equilibria Ely and Välimäki (2002) construct can be described as follows: The profiles have one-period memory, with players randomizing in each period with probability $p_{aa'}$ on C after the action profile aa' . The profile is constructed so that after *each* action profile, the player is indifferent between C and D . Consequently, a player's best replies are independent of his beliefs about the opponent's history, and in this sense the equilibria are, to use the language introduced by Ely, Hörner, and Olszewski (2003) "belief-free." The requirement that after aa' , player 1 is indifferent between playing C and D , when player 2 is playing $p_{a'a}$ yields the following system (where $W_{aa'}$ is the value to a player after aa' , and the second equality in each displayed equation comes from the indifference requirement):

$$\begin{aligned} W_{CC} &= (1 - \delta)(p_{CC} + (1 - p_{CC})(-\ell)) + \delta\{p_{CC}W_{CC} + (1 - p_{CC})W_{CD}\} \\ &= (1 - \delta)p_{CC}(1 + g) + \delta\{p_{CC}W_{DC} + (1 - p_{CC})W_{DD}\}, \end{aligned} \quad (1)$$

$$\begin{aligned} W_{CD} &= (1 - \delta)(p_{DC} + (1 - p_{DC})(-\ell)) + \delta\{p_{DC}W_{CC} + (1 - p_{DC})W_{CD}\} \\ &= (1 - \delta)p_{DC}(1 + g) + \delta\{p_{DC}W_{DC} + (1 - p_{DC})W_{DD}\}, \end{aligned} \quad (2)$$

$$\begin{aligned} W_{DC} &= (1 - \delta)(p_{CD} + (1 - p_{CD})(-\ell)) + \delta\{p_{CD}W_{CC} + (1 - p_{CD})W_{CD}\} \\ &= (1 - \delta)p_{CD}(1 + g) + \delta\{p_{CD}W_{DC} + (1 - p_{CD})W_{DD}\}, \end{aligned} \quad (3)$$

and

$$\begin{aligned} W_{DD} &= (1 - \delta)(p_{DD} + (1 - p_{DD})(-\ell)) + \delta\{p_{DD}W_{CC} + (1 - p_{DD})W_{CD}\} \\ &= (1 - \delta)p_{DD}(1 + g) + \delta\{p_{DD}W_{DC} + (1 - p_{DD})W_{DD}\}. \end{aligned} \quad (4)$$

Subtracting (2) from (1) gives

$$0 = p_{CC}\{(1 - \delta)(-g + \ell) + \delta[(W_{CC} - W_{CD}) - (W_{DC} - W_{DD})]\} - (1 - \delta)\ell + \delta W_{CD} - \delta W_{DD}.$$

Similarly,

$$0 = p_{DC}\{(1 - \delta)(-g + \ell) + \delta[(W_{CC} - W_{CD}) - (W_{DC} - W_{DD})]\} - (1 - \delta)\ell + \delta W_{CD} - \delta W_{DD},$$

$$0 = p_{CD}\{(1 - \delta)(-g + \ell) + \delta[(W_{CC} - W_{CD}) - (W_{DC} - W_{DD})]\} - (1 - \delta)\ell + \delta W_{CD} - \delta W_{DD},$$

and

$$0 = p_{DD}\{(1 - \delta)(-g + \ell) + \delta[(W_{CC} - W_{CD}) - (W_{DC} - W_{DD})]\} - (1 - \delta)\ell + \delta W_{CD} - \delta W_{DD}.$$

Since at least two of the probabilities differ (if not, $p_{aa'} = 0$ for all aa'), the coefficient of $p_{aa'}$ and the constant term are both zero:

$$W_{CD} - W_{DD} = \frac{(1 - \delta)\ell}{\delta} \quad (9)$$

and

$$\begin{aligned} W_{CC} - W_{DC} &= \frac{(1 - \delta)(g - \ell)}{\delta} + W_{CD} - W_{DD} \\ &= \frac{(1 - \delta)g}{\delta}. \end{aligned} \quad (10)$$

Ely and Välimäki (2002) instead work with the values to a player of having his opponent play C and D *this* period, \hat{V}_C and \hat{V}_D . A player is indifferent between C and D when the opponent plays C if

$$\begin{aligned} \hat{V}_C &\equiv (1 - \delta) + \delta W_{CC} \\ &= (1 - \delta)(1 + g) + \delta W_{DC}, \end{aligned}$$

while he is indifferent between C and D when the opponent plays D if

$$\begin{aligned} \hat{V}_D &\equiv (1 - \delta)(-\ell) + \delta W_{CD} \\ &= \delta W_{DD}. \end{aligned}$$

These two equalities are equivalent to (9) and (10), and so (1-8) imply the player is indifferent between C and D , when the opponent is playing C this period, and when he is playing D this period.

Under (9) and (10), the eight equations (1-8) reduce to four (substituting for W_{DC} and W_{DD}):

$$W_{CC} = (1 - \delta)(p_{CC} + (1 - p_{CC})(-\ell)) + \delta \{p_{CC}W_{CC} + (1 - p_{CC})W_{CD}\}, \quad (11)$$

$$W_{CD} = (1 - \delta)(p_{DC} + (1 - p_{DC})(-\ell)) + \delta \{p_{DC}W_{CC} + (1 - p_{DC})W_{CD}\}, \quad (12)$$

$$W_{CC} = (1 - \delta)(p_{CD} + (1 - p_{CD})(-\ell) + g/\delta) + \delta \{p_{CD}W_{CC} + (1 - p_{CD})W_{CD}\}, \quad (13)$$

and

$$W_{CD} = (1 - \delta)(p_{DD} + (1 - p_{DD})(-\ell) + \ell/\delta) + \delta \{p_{DD}W_{CC} + (1 - p_{DD})W_{CD}\}. \quad (14)$$

Treating W_{CC} and W_{CD} parametrically, each equation determines a probability, and so we have a two dimensional manifold of equilibria (the proof is in the Appendix):

Theorem 1 *There is a two-dimensional manifold of mixed equilibria of the infinitely-repeated perfect monitoring prisoners' dilemma: Suppose $W_{CC}, W_{CD} \in (0, 1)$ satisfy the inequalities*

$$W_{CD} - \delta W_{CC} < 1 - \delta, \quad (15)$$

$$\delta W_{CD} + (1 - \delta)g/\delta < (1 - \delta)\ell + W_{CC}, \text{ and} \quad (16)$$

$$(1 - \delta)\ell < \delta W_{CD}. \quad (17)$$

Then, the profile in which player 1 plays C with probability $p_{aa'}$ and player 2 plays C with probability $p_{a'a}$ after aa' in the previous period (and both players play p_{CC} in the first period), where

$$p_{CC} = \frac{(1 - \delta)\ell + W_{CC} - \delta W_{CD}}{(1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD})}, \quad (18)$$

$$p_{DC} = \frac{(1 - \delta)\ell + W_{CD} - \delta W_{CD}}{(1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD})}, \quad (19)$$

$$p_{CD} = \frac{(1 - \delta)(\ell - g/\delta) + W_{CC} - \delta W_{CD}}{(1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD})}, \quad (20)$$

and

$$p_{DD} = \frac{(1 - \delta)\ell(1 - 1/\delta) + W_{CD} - \delta W_{CD}}{(1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD})}, \quad (21)$$

is an equilibrium. Moreover, (15), (16), and (17) are satisfied for any $0 < W_{CD} < W_{CC} < 1$, for δ sufficiently close to 1.

Indeed, for each specification of behavior in the first period, there is a two-dimensional manifold of equilibria. Our analysis applies to all of these manifolds, and for simplicity, we focus on the profiles where both players play p_{CC} in the first period.

For later reference, it is useful to note that, using (9) and (10), the expressions for the probabilities can be written as, for all aa' ,

$$p_{aa'} = \frac{W_{a'a} - \delta W_{DD}}{(1 - \delta)(1 + g) + \delta(W_{DC} - W_{DD})}. \quad (22)$$

3 Finite memory purification

We now argue that if we require that the equilibrium of the perturbed game have finite history dependence, then it is only possible to purify equilibria of the type described in Section 2 when they are strongly symmetric (ie., when $p_{CD} = p_{DC}$).

Let $\Gamma(\varepsilon)$ denote the infinitely-repeated perfect-monitoring prisoners' dilemma with stage game:

	C	D
C	$1 + \varepsilon z_t^1, 1 + \varepsilon z_t^2$	$-\ell + \varepsilon z_t^1, 1 + g$
D	$1 + g, -\ell + \varepsilon z_t^2$	$0, 0$

The payoff shock z_t^i is private to player i , realized in period t , uniformly distributed on $[0, 1]$, independently and identically distributed across players, and histories.

We begin by considering one period memory strategy profiles, where the probability of a player playing C after observing the action profile aa' last period is denoted by $\pi_{aa'}^\varepsilon$. For simplicity we restrict attention to symmetric equilibria, where both players adopt the same strategy. Finally, we focus on completely mixed equilibria, where $\pi_{aa'}^\varepsilon \in (0, 1)$ for every action profile aa' .

Denote the marginal type by \hat{z}_t^i . If $z_t^i \geq \hat{z}_t^i$, then i plays C , and plays D otherwise. Then the probability of C is $\Pr\{z_t^i \geq \hat{z}_t^i\} = 1 - \hat{z}_t^i$.

Let $W_{aa'}^\varepsilon$ denote the *ex ante* value function of a player at the action profile aa' , before the realization of his payoff shock. The *ex post* payoff from C after CC , and given the realization of z_t^1 , is

$$V_{CC}^\varepsilon(z_t^1; C) = (1 - \delta) \{\pi_{CC}^\varepsilon - (1 - \pi_{CC}^\varepsilon) \ell + \varepsilon z_t^1\} + \delta \{\pi_{CC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{CD}^\varepsilon\},$$

while the payoff from D after CC is

$$V_{CC}^\varepsilon(z_t^1; D) = (1 - \delta) \pi_{CC}^\varepsilon (1 + g) + \delta \{\pi_{CC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{DD}^\varepsilon\}. \quad (23)$$

Since \hat{z}_t^1 is indifferent,

$$\begin{aligned} & (1 - \delta) \{\pi_{CC}^\varepsilon - (1 - \pi_{CC}^\varepsilon) \ell + \varepsilon \hat{z}_t^1\} + \delta \{\pi_{CC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{CD}^\varepsilon\} \\ &= (1 - \delta) \pi_{CC}^\varepsilon (1 + g) + \delta \{\pi_{CC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{DD}^\varepsilon\}, \end{aligned}$$

and since $\pi_{CC}^\varepsilon = 1 - \hat{z}_t^1$,

$$\begin{aligned} & (1 - \delta) \{\pi_{CC}^\varepsilon - (1 - \pi_{CC}^\varepsilon) (\ell - \varepsilon)\} + \delta \{\pi_{CC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{CD}^\varepsilon\} \\ &= (1 - \delta) \pi_{CC}^\varepsilon (1 + g) + \delta \{\pi_{CC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{DD}^\varepsilon\}, \end{aligned}$$

or

$$(1 - \delta) \{\pi_{CC}^\varepsilon g + (1 - \pi_{CC}^\varepsilon) (\ell - \varepsilon)\} = \delta \{\pi_{CC}^\varepsilon (W_{CC}^\varepsilon - W_{DC}^\varepsilon) + (1 - \pi_{CC}^\varepsilon) (W_{CD}^\varepsilon - W_{DD}^\varepsilon)\}.$$

Collecting terms gives

$$\begin{aligned} 0 &= \{(1 - \delta) (g - \ell + \varepsilon) - \delta (W_{CC}^\varepsilon - W_{DC}^\varepsilon - (W_{CD}^\varepsilon - W_{DD}^\varepsilon))\} \pi_{CC}^\varepsilon \\ &\quad + (1 - \delta) (\ell - \varepsilon) - \delta (W_{CD}^\varepsilon - W_{DD}^\varepsilon). \end{aligned} \quad (24)$$

Similarly, the payoff from C after DD is

$$(1 - \delta) \{\pi_{DD}^\varepsilon - (1 - \pi_{DD}^\varepsilon) \ell + \varepsilon z_t^1\} + \delta \{\pi_{DD}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{CD}^\varepsilon\},$$

while the payoff from D after DD is

$$(1 - \delta) \pi_{DD}^\varepsilon (1 + g) + \delta \{\pi_{DD}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{DD}^\varepsilon\}.$$

Since \hat{z}_t^1 is indifferent,

$$\begin{aligned} & (1 - \delta) \{ \pi_{DD}^\varepsilon - (1 - \pi_{DD}^\varepsilon) \ell + \varepsilon \hat{z}_t^1 \} + \delta \{ \pi_{DD}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{CD}^\varepsilon \} \\ = & (1 - \delta) \pi_{DD}^\varepsilon (1 + g) + \delta \{ \pi_{DD}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{DD}^\varepsilon \}, \end{aligned}$$

and since $\pi_{DD}^\varepsilon = 1 - \hat{z}_t^1$,

$$\begin{aligned} & (1 - \delta) \{ \pi_{DD}^\varepsilon - (1 - \pi_{DD}^\varepsilon) (\ell - \varepsilon) \} + \delta \{ \pi_{DD}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{CD}^\varepsilon \} \\ = & (1 - \delta) \pi_{DD}^\varepsilon (1 + g) + \delta \{ \pi_{DD}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DD}^\varepsilon) W_{DD}^\varepsilon \}, \end{aligned}$$

or

$$\begin{aligned} & (1 - \delta) \{ \pi_{DD}^\varepsilon g + (1 - \pi_{DD}^\varepsilon) (\ell - \varepsilon) \} \\ = & \delta \{ \pi_{DD}^\varepsilon (W_{CC}^\varepsilon - W_{DC}^\varepsilon) + (1 - \pi_{DD}^\varepsilon) (W_{CD}^\varepsilon - W_{DD}^\varepsilon) \}. \end{aligned}$$

Collecting terms gives

$$\begin{aligned} 0 = & \{ (1 - \delta) (g - \ell + \varepsilon) - \delta (W_{CC}^\varepsilon - W_{DC}^\varepsilon - (W_{CD}^\varepsilon - W_{DD}^\varepsilon)) \} \pi_{DD}^\varepsilon \quad (25) \\ & + (1 - \delta) (\ell - \varepsilon) - \delta (W_{CD}^\varepsilon - W_{DD}^\varepsilon). \end{aligned}$$

Since the equations (24) and (25) have the same structure, if $\pi_{CC}^\varepsilon \neq \pi_{DD}^\varepsilon$, it must be that the coefficient of π^ε and the constant are both zero:

$$(1 - \delta) (g - \ell + \varepsilon) - \delta (W_{CC}^\varepsilon - W_{DC}^\varepsilon - (W_{CD}^\varepsilon - W_{DD}^\varepsilon)) = 0$$

and

$$(1 - \delta) (\ell - \varepsilon) - \delta (W_{CD}^\varepsilon - W_{DD}^\varepsilon) = 0. \quad (26)$$

In other words,

$$W_{CD}^\varepsilon - W_{DD}^\varepsilon = \frac{(1 - \delta) (\ell - \varepsilon)}{\delta} \quad (27)$$

and

$$W_{CC}^\varepsilon - W_{DC}^\varepsilon = \frac{(1 - \delta) g}{\delta}. \quad (28)$$

Turning to the other histories, the payoff from C after CD (so that the opponent sees the history DC) is

$$(1 - \delta) \{ \pi_{DC}^\varepsilon - (1 - \pi_{DC}^\varepsilon) \ell + \varepsilon z_t^1 \} + \delta \{ \pi_{DC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{CD}^\varepsilon \},$$

while the payoff from D after CD is

$$(1 - \delta) \pi_{DC}^\varepsilon (1 + g) + \delta \{ \pi_{DC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{DD}^\varepsilon \}.$$

Since \hat{z}_t^1 is indifferent,

$$\begin{aligned} & (1 - \delta) \{ \pi_{DC}^\varepsilon - (1 - \pi_{DC}^\varepsilon) \ell + \varepsilon \hat{z}_t^1 \} + \delta \{ \pi_{DC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{CD}^\varepsilon \} \\ = & (1 - \delta) \pi_{DC}^\varepsilon (1 + g) + \delta \{ \pi_{DC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{DD}^\varepsilon \}, \end{aligned}$$

and since $\pi_{CD}^\varepsilon = 1 - \hat{z}_t^1$,

$$\begin{aligned} & (1 - \delta) \{ \pi_{DC}^\varepsilon - (1 - \pi_{DC}^\varepsilon) \ell + \varepsilon (1 - \pi_{CD}^\varepsilon) \} + \delta \{ \pi_{DC}^\varepsilon W_{CC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{CD}^\varepsilon \} \\ = & (1 - \delta) \pi_{DC}^\varepsilon (1 + g) + \delta \{ \pi_{DC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{DC}^\varepsilon) W_{DD}^\varepsilon \}, \end{aligned}$$

or

$$\begin{aligned} 0 = & \{ (1 - \delta) (g - \ell + \varepsilon) - \delta (W_{CC}^\varepsilon - W_{DC}^\varepsilon - (W_{CD}^\varepsilon - W_{DD}^\varepsilon)) \} \pi_{DC}^\varepsilon \quad (29) \\ & + \{ (1 - \delta) (\ell - \varepsilon) - \delta (W_{CD}^\varepsilon - W_{DD}^\varepsilon) \} + \varepsilon (1 - \delta) (\pi_{CD}^\varepsilon - \pi_{DC}^\varepsilon). \end{aligned}$$

The first two terms in the above equation have the same structure as in (24), and since the constant term and the coefficient on π^ε in (24) are both zero, these two terms vanish. Thus (29) cannot be true for $\varepsilon > 0$ unless $\pi_{CD}^\varepsilon = \pi_{DC}^\varepsilon$.

Theorem 2 *Let p be a mixed strategy equilibrium of the game with complete information that has one period memory (such as an Ely-Valimaki strategy profile). If $p_a \in (0, 1)$ and $p_{CC} \neq p_{DD}$ and $p_{CD} \neq p_{DC}$, then there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, there is no equilibrium of $\Gamma(\varepsilon)$ with finite memory within $\bar{\varepsilon}$ of p .*

Proof. Fix $\delta = \frac{1}{2} \min \{ p_a, 1 - p_a, |p_{CD} - p_{DC}|, |p_{CC} - p_{DD}| \}$. Consider a profile with memory K , and suppose it is within δ of p . Let π_h^ε be the probability of C after the history h . Then, $\pi_h^\varepsilon \in (0, 1)$ for all $h \in \{C, D\}^{2K}$, $\pi_{hCC}^\varepsilon \neq \pi_{hDD}^\varepsilon$ and $\pi_{hDC}^\varepsilon \neq \pi_{hCD}^\varepsilon$. Then the contradiction obtained above for 1-period profiles also arises after the four histories hCC , hCD , hDC , and hDD , where $h \in \{C, D\}^{2(K-1)}$. ■

We have established that if p_{CC} and p_{DD} are distinct, a necessary condition for purifiability by finite memory strategies is that p must be strongly symmetric, i.e. both players must play the same continuation strategy after every history, even if this history is asymmetric.

3.1 Purification when $p_{DC} = p_{CD}$

It is possible to purify when $p_{CD} = p_{DC}$, since in this case we may choose $\pi_{CD}^\varepsilon = \pi_{DC}^\varepsilon$ in the perturbed game, so that (29) is equivalent to (24), with π_{DC}^ε replacing π_{CC}^ε . There is thus no inconsistency between the conditions for optimality at the four information sets, CC , CD , DC , and DD . Let us assume that the ex ante value functions satisfy (27) and (28). This, in conjunction with $\pi_{CD}^\varepsilon = \pi_{DC}^\varepsilon$ immediately implies that optimality is satisfied at all four information sets. It remains to show that we can choose the strategy profile π^ε in order to generate these ex ante values.

In order to calculate the ex ante values, we need to take into account the dependence of choice on the realized value of the payoff shock. Note first that

$V_{aa'}(z_t^1; D)$, the *ex post* payoff from D after aa' , given the realization of z_t^1 , is independent of z_t^1 , and

$$\begin{aligned} V_{aa'}(z_t^1; C) &= V_{aa'}(\hat{z}_t^1; C) + (1 - \delta) \varepsilon (z_t^1 - \hat{z}_t^1) \\ &= V_{aa'}(D) + (1 - \delta) \varepsilon (z_t^1 - \hat{z}_t^1), \end{aligned}$$

so

$$\begin{aligned} W_{CC}^\varepsilon &= V_{CC}(D) + (1 - \delta) \varepsilon \int_0^1 \max\{z - \hat{z}_t^1, 0\} dz \\ &= V_{CC}(D) + (1 - \delta) \varepsilon \int_{\hat{z}_t^1}^1 z - \hat{z}_t^1 dz \\ &= V_{CC}(D) + (1 - \delta) \varepsilon \left(\frac{z^2}{2} - z\hat{z}_t^1 \right) \Big|_{\hat{z}_t^1}^1 \\ &= V_{CC}(D) + (1 - \delta) \varepsilon \left[\left(\frac{1}{2} - \hat{z}_t^1 \right) + \frac{\hat{z}_t^1 \hat{z}_t^1}{2} \right] \\ &= V_{CC}(D) + \frac{(1 - \delta) \varepsilon}{2} (1 - \hat{z}_t^1)^2 = V_{CC}(D) + \frac{(1 - \delta) \varepsilon}{2} (\pi_{CC}^\varepsilon)^2 \end{aligned}$$

and so (using (23))

$$W_{CC}^\varepsilon = (1 - \delta) \left\{ \pi_{CC}^\varepsilon (1 + g) + \frac{1}{2} \varepsilon (\pi_{CC}^\varepsilon)^2 \right\} + \delta \{ \pi_{CC}^\varepsilon W_{DC}^\varepsilon + (1 - \pi_{CC}^\varepsilon) W_{DD}^\varepsilon \}. \quad (30)$$

Rearranging,

$$(1 - \delta) \left\{ \pi_{CC}^\varepsilon (1 + g) + \frac{1}{2} \varepsilon (\pi_{CC}^\varepsilon)^2 \right\} + \delta \pi_{CC}^\varepsilon (W_{DC}^\varepsilon - W_{DD}^\varepsilon) + \delta W_{DD}^\varepsilon - W_{CC}^\varepsilon = 0,$$

and using (26) and $W_{CD}^\varepsilon = W_{DC}^\varepsilon$,

$$(1 - \delta) \left\{ \pi_{CC}^\varepsilon (1 + g) + \frac{1}{2} \varepsilon (\pi_{CC}^\varepsilon)^2 \right\} + \pi_{CC}^\varepsilon (1 - \delta) (\ell - \varepsilon) + \delta W_{DD}^\varepsilon - W_{CC}^\varepsilon = 0$$

or

$$(1 - \delta) \left\{ \pi_{CC}^\varepsilon (1 + g + \ell - \varepsilon) + \frac{1}{2} \varepsilon (\pi_{CC}^\varepsilon)^2 \right\} + \delta W_{DD}^\varepsilon - W_{CC}^\varepsilon = 0. \quad (31)$$

Proceeding similarly from the value equation for W_{CD}^ε , and using (27) and (28), we have

$$(1 - \delta) \left\{ \pi_{DC}^\varepsilon (1 + g + \ell - \varepsilon) + \frac{1}{2} \varepsilon (\pi_{DC}^\varepsilon)^2 \right\} + \delta W_{DD}^\varepsilon - W_{CC}^\varepsilon + \frac{(1 - \delta)g}{\delta} = 0. \quad (32)$$

From the value equation for W_{DD}^ε ,

$$(1 - \delta) \left\{ \pi_{DD}^\varepsilon (1 + g + \ell - \varepsilon) + \frac{1}{2} \varepsilon (\pi_{DD}^\varepsilon)^2 \right\} - (1 - \delta) W_{DD}^\varepsilon = 0. \quad (33)$$

Thus it suffices to find π_{CC}^ε , π_{CD}^ε , and π_{DD}^ε which solve the quadratics (31), (32) and (33), and which converge to p_{CC} , p_{CD} , and p_{DD} as $\varepsilon \rightarrow 0$. We set $W_{DD}^\varepsilon = W_{DD}$, and $W_{CC}^\varepsilon = W_{CC}$, the values in the unperturbed game for the equilibrium we want to purify. The result then follows from the following lemma (and (22)):

Lemma 1 *Let x_ε solve the quadratic,*

$$a_\varepsilon x^2 + b_\varepsilon x + c_\varepsilon = 0,$$

where a_ε , b_ε , and c_ε all converge as $\varepsilon \rightarrow 0$, and $\lim_{\varepsilon \rightarrow 0} a_\varepsilon = 0$ and $\lim_{\varepsilon \rightarrow 0} b_\varepsilon \neq 0$. Suppose moreover that a_ε , b_ε , and c_ε are all differentiable functions of ε in a neighborhood of 0, with well-defined limits as $\varepsilon \rightarrow 0$, and $\lim_{\varepsilon \rightarrow 0} a'_\varepsilon \neq 0$. Then,

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon = -\frac{\lim_{\varepsilon \rightarrow 0} c_\varepsilon}{\lim_{\varepsilon \rightarrow 0} b_\varepsilon}.$$

Proof. Solving the quadratic gives two candidate solutions for x_ε :

$$x_\varepsilon = \frac{-b_\varepsilon \pm \sqrt{b_\varepsilon^2 - 4a_\varepsilon c_\varepsilon}}{2a_\varepsilon}.$$

Since the denominator goes to zero as $\varepsilon \rightarrow 0$, only the positive root yields a well-defined solution for x_ε in the limit. In this case, both numerator and denominator go to zero, and an application of l'Hopital's rule completes the proof. ■

Thus, we have a purification, for any values of W_{DD}^ε and W_{CC}^ε in the unperturbed game. Since any completely mixed symmetric equilibrium can be parametrized by these two values, we have shown that any such equilibrium can be purified by one period memory strategies in the perturbed game. We state this result as the following theorem:

Theorem 3 *Let $p = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ be a symmetric completely mixed one period memory equilibrium of the unperturbed game $\Gamma(0)$, with $p_{CD} = p_{DC}$. There exists $\bar{\varepsilon} > 0$ so that for $\varepsilon < \bar{\varepsilon}$, there exists $\pi^\varepsilon = (\pi_{CC}^\varepsilon, \pi_{CD}^\varepsilon, \pi_{DC}^\varepsilon, \pi_{DD}^\varepsilon)$, a symmetric one period memory equilibrium of $\Gamma(\varepsilon)$, and $\pi^\varepsilon \rightarrow p$ as $\varepsilon \rightarrow 0$.*

Theorems 2 and 3 show that a strategy is purifiable by finite memory strategies if and only if it is strongly symmetric, i.e., at every information set, the two players must play the same mixed action. This also has implications for the payoffs that may be sustained. In the unperturbed game, any values in the unit interval that satisfy (9) and (10) are equilibrium values. In consequence, if $\delta > \max\{\frac{g}{1+g}, \frac{\ell}{1+\ell}\}$, any value in $(0, 1)$ is an equilibrium value. In the perturbed game, the restriction $W_{CD} = W_{DC}$ implies that we require $\delta > \frac{g+\ell}{1+g+\ell}$. Thus, supporting a non-degenerate set of values requires a higher discount factor.

4 Purification with infinite memory

We now argue that, when we allow the equilibrium of the perturbed game to have infinite history dependence, then it is possible to purify equilibria of the type described in Section 2. Fix an equilibrium with interior probabilities, p_{CC} , p_{CD} , p_{DC} , and $p_{DD} \in (0, 1)$.

We first partition the set of histories, H , into equivalence classes where behavior is identical on elements of the partition. All histories with the same last action profile aa' different from CC are equivalent; denote the associated element of the partition by $(aa', 0)$. We write this as $haa' \in (aa', 0)$ for all h and $aa' \neq CC$. Two histories ending in CC are equivalent if the most recent action profile different from CC in the two histories is the same, aa' say, and if the same number of occurrences of CC occur in the two histories after the last non- CC action profile, aa' . Denote the associated element of the partition by (aa', k) , where k is the number of occurrences of CC after the last non- CC action profile, aa' . Finally, if h is the k -period history in which CC has been played in every period, we write (CC, k) for the singleton element of the partition containing h . Note that the null history is $(CC, 0)$. Note that, any history in an element of the partition (aa', k) with $k \geq 1$ ends in CC .

The strategy in the perturbed game will be measurable with respect to the partition on H just described. Fix $\varepsilon > 0$ and let $\pi_{aa'}^\varepsilon(k)$ denote the probability with which C is played when $h \in (aa', k)$, and let $W_{aa'}^\varepsilon(k)$ denote the ex ante value function of the player at this history. If $\{\pi_{aa'}^\varepsilon(k)\}$ is a sequence (as $\varepsilon \rightarrow 0$) of equilibria purifying $p = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$, then $\pi_{aa'}^\varepsilon(k) \rightarrow p_{CC}$ for all $k \geq 1$ and all aa' , and $p_{aa'}^\varepsilon(0) \rightarrow p_{aa'}$, as $\varepsilon \rightarrow 0$. We will indeed show a *uniform* form of purifiability: the bound on ε required to make $\pi_{aa'}^\varepsilon(k)$ close to p_{CC} is *independent* of k .

The idea is that in the perturbed game, the payoff after a history ending in CC can always be adjusted to ensure that the appropriate realization of z in the previous period is the marginal type to obtain the desired randomization between C and D . We proceed recursively, fixing probabilities after any history in an element of the partition $(aa', 0)$ at their unperturbed levels, i.e., we set $\pi_{aa'}^\varepsilon(0) = p_{aa'}$. In particular, players randomize in the first period with probability p_{CC} on C , and in the second period after a realized action profile $aa' \neq CC$ with probability $p_{aa'}$ on C .⁴ This turns out to determine the value function at histories in $(aa', 0)$ for all aa' ; we write $W_{aa'}^\varepsilon$ for $W_{aa'}^\varepsilon(0)$. In the second period after CC , $W_{CC}^\varepsilon(1)$ is determined by the requirement that the ex ante probability that a player play C in the first period is given by $\pi_{CC}^\varepsilon(0) = p_{CC}$. Given the value $W_{CC}^\varepsilon(1)$, the probability $\pi_{CC}^\varepsilon(1)$ is then determined by the requirement that $W_{CC}^\varepsilon(1)$ be the ex ante value at the history CC . More generally, given a history $h \in (aa', k)$ and a further realization of CC , $W_{aa'}^\varepsilon(k+1)$ is determined by the requirement that

⁴More precisely, player 1 randomizes with probability $p_{aa'}$ and player 2 randomizes with probability $p_{a'a}$.

the ex ante probability that a player play C in the previous period is given by $\pi_{aa'}^\varepsilon(k) = p_{aa'}$, and then $\pi_{aa'}^\varepsilon(k+1)$ is then determined by $W_{aa'}^\varepsilon(k+1)$.

We begin with histories in $(aa', 0)$. Recalling the calculations that led to (30),

$$W_{CD}^\varepsilon = (1 - \delta) \{p_{DC}(1 + g) + \varepsilon p_{CD}^2/2\} + \delta \{p_{DC}W_{DC}^\varepsilon + (1 - p_{DC})W_{DD}^\varepsilon\}, \quad (34)$$

$$W_{DC}^\varepsilon = (1 - \delta) \{p_{CD}(1 + g) + \varepsilon p_{DC}^2/2\} + \delta \{p_{CD}W_{DC}^\varepsilon + (1 - p_{CD})W_{DD}^\varepsilon\}, \quad (35)$$

$$W_{DD}^\varepsilon = (1 - \delta) \{p_{DD}(1 + g) + \varepsilon p_{DD}^2/2\} + \delta \{p_{DD}W_{DC}^\varepsilon + (1 - p_{DD})W_{DD}^\varepsilon\}, \quad (36)$$

and

$$\begin{aligned} W_{aa'}^\varepsilon(k) &= (1 - \delta) \{ \pi_{a'a}^\varepsilon(k)(1 + g) + \varepsilon \pi_{aa'}^\varepsilon(k)^2/2 \} \\ &\quad + \delta \{ \pi_{a'a}^\varepsilon(k)W_{DC}^\varepsilon + (1 - \pi_{a'a}^\varepsilon(k))W_{DD}^\varepsilon \}. \end{aligned} \quad (37)$$

As we indicated above, (34), (35), and (36) can be solved for W_{CD}^ε , W_{DC}^ε , and W_{DD}^ε . Moreover, these solutions converge to W_{CD} , W_{DC} , and W_{DD} (since these are the only solutions to (3), (5) and (7) for fixed p_{DC} , p_{CD} , and p_{DD}). It remains to determine $W_{aa'}^\varepsilon(k)$ and $\pi_{aa'}^\varepsilon(k)$ for $k \geq 1$ ($W_{CC}^\varepsilon(0)$ is also determined, since $\pi_{CC}^\varepsilon(0) = p_{CC}$).

At the history $h = (a'a, k-1)$, the player with payoff realization $z = 1 - \pi_{a'a}^\varepsilon(k-1)$ must be indifferent between C and D :

$$\begin{aligned} &(1 - \delta) \{ \pi_{aa'}^\varepsilon(k-1) + (1 - \pi_{aa'}^\varepsilon(k-1))(-\ell) + \varepsilon(1 - \pi_{a'a}^\varepsilon(k-1)) \} \\ &\quad + \delta \{ \pi_{aa'}^\varepsilon(k-1)W_{a'a}^\varepsilon(k) + (1 - \pi_{aa'}^\varepsilon(k-1))W_{CD}^\varepsilon \} \\ &= (1 - \delta) \pi_{aa'}^\varepsilon(k-1)(1 + g) + \delta \{ \pi_{aa'}^\varepsilon(k-1)W_{DC}^\varepsilon + (1 - \pi_{aa'}^\varepsilon(k-1))W_{DD}^\varepsilon \}. \end{aligned}$$

Solving for $W_{a'a}^\varepsilon(k)$ as a function of $\pi_{aa'}^\varepsilon(k-1)$ and $\pi_{a'a}^\varepsilon(k-1)$ gives

$$\begin{aligned} W_{a'a}^\varepsilon(k) &= \frac{(1 - \delta)(g - \ell)}{\delta} + W_{DC}^\varepsilon + W_{CD}^\varepsilon - W_{DD}^\varepsilon \\ &\quad + \frac{(1 - \delta)\ell - \varepsilon(1 - \pi_{a'a}^\varepsilon(k-1)) - \delta[W_{CD}^\varepsilon - W_{DD}^\varepsilon]}{\delta\pi_{aa'}^\varepsilon(k-1)}. \end{aligned} \quad (38)$$

Given $W_{aa'}^\varepsilon(k)$, (37) is a quadratic in $\pi_{aa'}^\varepsilon(k)$,

$$a_\varepsilon \pi_{aa'}^\varepsilon(k)^2 + b_\varepsilon \pi_{aa'}^\varepsilon(k) + c_\varepsilon(k) = 0,$$

where

$$\begin{aligned} a_\varepsilon &= \varepsilon/2, \\ b_\varepsilon &= (1 - \delta)(1 + g) + \delta(W_{DC}^\varepsilon - W_{DD}^\varepsilon), \end{aligned}$$

and

$$c_\varepsilon(k) = \delta W_{DD}^\varepsilon - W_{aa'}^\varepsilon(k).$$

Applying Lemma 1,

$$\lim_{\varepsilon \rightarrow 0} \pi_{aa'}^\varepsilon(k) = \frac{\lim_{\varepsilon \rightarrow 0} W_{aa'}^\varepsilon(k) - \delta W_{DD}^\varepsilon}{\lim_{\varepsilon \rightarrow 0} (1 - \delta)(1 + g) + \delta(W_{DC}^\varepsilon - W_{DD}^\varepsilon)}.$$

Hence, if $W_{aa'}^\varepsilon(k) \rightarrow W_{CC}$ as $\varepsilon \rightarrow 0$, then $\pi_{aa'}^\varepsilon(k) \rightarrow p_{CC}$.

Theorem 4 *Let $pp = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ be a symmetric completely mixed one period memory equilibrium of the unperturbed game $\Gamma(0)$. For all $\eta > 0$, there is exists $\varepsilon(\eta) > 0$ such that for all $\varepsilon < \varepsilon(\eta)$, the equilibrium of the perturbed game $\Gamma(\varepsilon)$ given by the probabilities $\pi_{aa'}^\varepsilon(k)$ described above satisfies*

$$|\pi_{aa'}^\varepsilon(k) - p_{CC}| < \eta \quad \forall k \geq 1.$$

Proof. The proof of l'Hospital's rule (see Rudin (1976, p. 109), for example) shows the following: Suppose f and g are differentiable on (a, b) , $g'(x) \neq 0$ for all $x \in (a, b)$, and there exists $\delta : [0, \bar{\eta}] \rightarrow \mathfrak{R}_+$ for some $\bar{\eta} > 0$ such that $|x - a| < \delta(\eta)$ implies $|f'(x)/g'(x) - A| < \eta$ for some A for all $\eta \in (0, \bar{\eta})$. If f and g are continuous on $[a, b]$ with $f(a) = g(a) = 0$, then $|f(x)/g(x) - A| < \eta$ for all $|x - a| < \delta(\eta)$ and $\eta \in (0, \bar{\eta})$.

Consequently, it is enough to show that δ can be chosen independently of k in the application of Lemma 1. To apply Lemma 1, we also need to show that b'_ε and c'_ε have well-defined limits as $\varepsilon \rightarrow 0$.

From (34), (35), and (36), there exists $\kappa_{aa'}$ such that $W_{aa'}^\varepsilon = W_{aa'} + \kappa_{aa'}\varepsilon$ for all $aa' \neq CC$. Fix $\eta < \min\{p_{CC}/3, p_{CD}, p_{DC}, p_{DD}\}$. We proceed by induction. We first prove the inductive step, and then the initial step.

Suppose $k \geq 2$, $|\pi_{aa'}^\varepsilon(k-1) - p_{CC}| < \eta$ and $|\varepsilon \frac{d}{d\varepsilon} \pi_{aa'}^\varepsilon(k-1)| < 2\eta$ for all aa' . Since

$$\pi_{aa'}^\varepsilon(k) = \frac{-b_\varepsilon + \sqrt{b_\varepsilon^2 - 2\varepsilon c_\varepsilon(k)}}{\varepsilon},$$

it is enough to consider the behavior of the derivative of the numerator (the derivative of the denominator being 1). The numerator's derivative is

$$-b'_\varepsilon + \frac{1}{2} (b_\varepsilon^2 - 2\varepsilon c_\varepsilon(k))^2 [2b_\varepsilon b'_\varepsilon - 2c_\varepsilon(k) - 2\varepsilon c'_\varepsilon(k)]. \quad (39)$$

If $\varepsilon c_\varepsilon(k)$ and $\varepsilon c'_\varepsilon(k)$ can be made small (we will show that $\varepsilon c_\varepsilon(k)$ and $\varepsilon c'_\varepsilon(k)$ can be made small by choosing ε small, independently of k), the limiting value of this derivative is determined by the limiting value of $c_\varepsilon(k)/b_\varepsilon$ (which is p_{CC}).

First we argue that the rate at which $c_\varepsilon(k)/b_\varepsilon$ converges to its limiting value of p_{CC} is independent of k : Since $\pi_{aa'}^\varepsilon(k-1) > p_{CC}/3 > 0$, as $\varepsilon \rightarrow 0$, the last term in (38) converges to zero (from (9)) uniformly in k . Equations (10) and (9) then imply $W_{aa'}^\varepsilon(k) \rightarrow W_{CC}$ as $\varepsilon \rightarrow 0$.

It is immediate that $c_\varepsilon(k)$ is bounded independently of k ; it remains to bound $c'_\varepsilon(k) = \delta\kappa_{DD} - \frac{d}{d\varepsilon}W_{aa'}^\varepsilon(k)$. Now,

$$\begin{aligned}
\frac{d}{d\varepsilon}W_{aa'}^\varepsilon(k) &= \kappa_{DC} + \kappa_{CD} - \kappa_{DD} \\
&+ \frac{(1-\delta)\ell - \varepsilon(1 - \pi_{a'a}^\varepsilon(k-1)) - \delta[W_{CD}^\varepsilon - W_{DD}^\varepsilon]}{\delta(\pi_{aa'}^\varepsilon(k-1))^2} \frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(k-1) \\
&+ \frac{1}{\delta\pi_{aa'}^\varepsilon(k-1)} \left\{ -1 + \pi_{a'a}^\varepsilon(k-1) + \varepsilon \frac{d}{d\varepsilon}\pi_{a'a}^\varepsilon(k-1) - \delta[\kappa_{CD} - \kappa_{DD}] \right\} \\
&= \kappa_{DC} + \kappa_{CD} - \kappa_{DD} - \frac{(1 - \pi_{a'a}^\varepsilon(k-1) + \delta[\kappa_{CD} - \kappa_{DD}])}{\delta\pi_{aa'}^\varepsilon(k-1)} \\
&+ \frac{\delta(\kappa_{CD} - \kappa_{DD})\varepsilon - \varepsilon(1 - \pi_{a'a}^\varepsilon(k-1))}{\delta(\pi_{aa'}^\varepsilon(k-1))^2} \frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(k-1) \\
&+ \frac{\varepsilon}{\delta\pi_{aa'}^\varepsilon(k-1)} \frac{d}{d\varepsilon}\pi_{a'a}^\varepsilon(k-1)
\end{aligned}$$

(where we have used $W_{CD}^\varepsilon - W_{DD}^\varepsilon = W_{CD} - W_{DD} + \kappa_{CD}\varepsilon - \kappa_{DD}\varepsilon = (1-\delta)\ell/\delta + (\kappa_{CD} - \kappa_{DD})\varepsilon$). Hence,

$$\begin{aligned}
\left| \frac{d}{d\varepsilon}W_{aa'}^\varepsilon(k) \right| &\leq |\kappa_{DC} + \kappa_{CD} - \kappa_{DD}| + \frac{(1 + \delta|\kappa_{CD} - \kappa_{DD}|)}{\delta p_{CC}/3} \\
&+ \frac{(\delta|\kappa_{CD} - \kappa_{DD}| + 1)\varepsilon}{\delta(p_{CC}/3)^2} \left| \frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(k-1) \right| \\
&+ \frac{\varepsilon}{\delta p_{CC}/3} \left| \frac{d}{d\varepsilon}\pi_{a'a}^\varepsilon(k-1) \right| \\
&\leq A,
\end{aligned}$$

(using $|\pi_{aa'}^\varepsilon(k-1) - p_{CC}| < \eta < p_{CC}/3$ in the first inequality, and the bound on $|\varepsilon \frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(k-1)|$ in the second) for some A (independent of k). Hence,

$$|c'_\varepsilon(k)| \leq \delta\kappa_{DD} + A.$$

Thus, there is a bound, $\bar{\varepsilon}(\eta)$, on ε (independent of k) such that $\varepsilon c_\varepsilon(k)$ and $\varepsilon c'_\varepsilon(k)$ are sufficiently small for $\varepsilon < \bar{\varepsilon}(\eta)$ that the expression in (39) is within η of p_{CC} . Hence, from the observation on l'Hopital's rule at the beginning of the proof, $|\pi_{aa'}^\varepsilon(k) - p_{CC}| \leq \eta$.

We also have the bound on the derivative of the probability, since

$$\begin{aligned}
\varepsilon \left| \frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(k) \right| &= \left| \frac{d}{d\varepsilon}\varepsilon\pi_{aa'}^\varepsilon(k) - \pi_{aa'}^\varepsilon(k) \right| \\
&\leq \left| \frac{d}{d\varepsilon}\varepsilon\pi_{aa'}^\varepsilon(k) - p_{CC} \right| + |p_{CC} - \pi_{aa'}^\varepsilon(k)| \leq 2\eta,
\end{aligned}$$

where the last inequality follows from $\frac{d}{d\varepsilon}\varepsilon\pi_{aa'}^\varepsilon(k)$ equalling the expression in (39).

Finally, it remains to verify that $|\pi_{aa'}^\varepsilon(1) - p_{CC}| < \eta$ and $|\varepsilon\frac{d}{d\varepsilon}\pi_{aa'}^\varepsilon(1)| < 2\eta$ for all aa' . It is immediate that for each aa' , there exists $\varepsilon_{aa'}(\eta)$ such that the two inequalities hold for $\varepsilon < \varepsilon_{aa'}(\eta)$. Taking $\varepsilon(\eta) = \min\{\bar{\varepsilon}(\eta), \varepsilon_{CC}(\eta), \varepsilon_{CD}(\eta), \varepsilon_{DC}(\eta), \varepsilon_{DD}(\eta)\}$ completes the proof. ■

5 Private Monitoring

As noted in the introduction, much of the interest in the purifiability of mixed strategy equilibria in repeated games comes from the literature on repeated game with private monitoring. The systems of equations for the perfect monitoring case can be straightforwardly extended to allow for private monitoring. Unfortunately, the particular arguments that we report exploit the perfect monitoring structure to reduce the infinite system of equations to simple difference equations, and somewhat different arguments are required to deal with private monitoring.

We conjecture that the infinite horizon purification results would extend using general methods for analyzing infinite systems of equations. Intuitively, private monitoring will make purification by finite history strategies much harder, as there will be many different histories that will presumably give rise to different equilibrium beliefs that must lead to identical mixed strategies being played, and this should not typically occur. This argument can be formalized for one period histories, but we have not established the argument for arbitrary finite history strategies. However, we believe that the finite history restriction may place very substantial bounds on the set of mixed strategies that can be purified in general repeated games, and we hope to pursue this issue in later work.

A Proof of Theorem 1

Solving (11-14) for the probabilities gives (18-21). By construction, all relevant incentive constraints are satisfied, so it only remains to verify that (15), (16), and (17) imply that the quantities described by (18-21) are indeed well-defined probabilities. Observe first that $p_{CC} > 0$, since

$$\begin{aligned} 0 &< (1 - \delta)\ell + W_{CC} - \delta W_{CD} \\ &\iff \delta W_{CD} < (1 - \delta)\ell + W_{CC}, \end{aligned}$$

which is implied by (16). This then implies that every denominator is positive (since $W_{CC} \leq 1$). Moreover, under this assumption, $p_{CC} < 1$, since

$$\begin{aligned} (1 - \delta)\ell + W_{CC} - \delta W_{CD} &< (1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD}) \\ &\iff (1 - \delta)(W_{CC} - W_{CD}) < (1 - \delta) \\ &\iff W_{CC} - W_{CD} < 1, \end{aligned}$$

which is always satisfied (since $W_{CD} \geq 0$).

Turning to the next quantity, $p_{DC} > 0$, since

$$\begin{aligned} 0 &< (1 - \delta)\ell + W_{CD} - \delta W_{CD} \\ &\iff 0 < \ell + W_{CD}, \end{aligned}$$

which always holds. Moreover, $p_{DC} < 1$, since

$$\begin{aligned} (1 - \delta)\ell + W_{CD} - \delta W_{CD} &< (1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD}) \\ &\iff (1 - \delta)W_{CD} - \delta(W_{CC} - W_{CD}) < (1 - \delta) \\ &\iff W_{CD} - \delta W_{CC} < 1 - \delta. \end{aligned}$$

which is (15).

We also have $p_{CD} > 0$, since

$$\begin{aligned} 0 &< (1 - \delta)(\ell - g/\delta) + W_{CC} - \delta W_{CD} \\ &\iff \delta W_{CD} + (1 - \delta)g/\delta < (1 - \delta)\ell + W_{CC}, \end{aligned}$$

which is (16). Moreover, $p_{CD} < 1$, since

$$\begin{aligned} (1 - \delta)(\ell - g/\delta) + W_{CC} - \delta W_{CD} &< (1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD}) \\ &\iff W_{CC} - W_{CD} < 1 + g/\delta. \end{aligned}$$

Finally, $p_{DD} > 0$ is equivalent to (17), and $p_{DD} < 1$ is implied by (15), since

$$\begin{aligned} (1 - \delta)\ell(1 - 1/\delta) + W_{CD} - \delta W_{CD} &< (1 - \delta)(1 + \ell) + \delta(W_{CC} - W_{CD}) \\ &\iff W_{CD} - \delta W_{CC} < (1 - \delta)(1 + \delta)/\delta. \end{aligned}$$

References

- BHASKAR, V. (2000): “The Robustness of Repeated Game Equilibria to Incomplete Payoff Information,” University of Essex.
- BHASKAR, V., AND I. OBARA (2002): “Belief-Based Equilibria in the Repeated Prisoners’ Dilemma with Private Monitoring,” *Journal of Economic Theory*, 102(1), 40–69.
- BHASKAR, V., AND E. VAN DAMME (2002): “Moral Hazard and Private Monitoring,” *Journal of Economic Theory*, 102(1), 16–39.
- ELY, J. C., J. HÖRNER, AND W. OLSZEWSKI (2003): “Belief-Free Equilibria,” Boston University and Northwestern University.
- ELY, J. C., AND J. VÄLIMÄKI (2002): “A Robust Folk Theorem for the Prisoner’s Dilemma,” *Journal of Economic Theory*, 102(1), 84–105.

- GOVINDAN, S., P. J. RENY, AND A. J. ROBSON (2003): "A Short Proof of Harsanyi's Purification Theorem," *Games and Economic Behavior*, 45(2), 369–374.
- HARSANYI, J. C. (1973): "Games with randomly disturbed payoffs: A new rationale for mixed- strategy equilibrium points," *International Journal of Game Theory*, 2, 1–23.
- KANDORI, M., AND I. OBARA (2003): "Efficiency in Repeated Games Revisited: The Role of Private Strategies," unpublished.
- PICCIONE, M. (2002): "The Repeated Prisoner's Dilemma with Imperfect Private Monitoring," *Journal of Economic Theory*, 102(1), 70–83.
- RUDIN, W. (1976): *Principles of Mathematical Analysis*. McGraw-Hill, New York, 3rd edn.
- SEKIGUCHI, T. (1997): "Efficiency in Repeated Prisoner's Dilemma with Private Monitoring," *Journal of Economic Theory*, 76, 345–361.