Strategic Redistricting †

Faruk Gul

and

Wolfgang Pesendorfer

Princeton University

July 2007

Abstract

We develop and analyze a model of strategic redistricting. Two parties choose optimal redistricting plans for their respective territory. Parties redistrict before aggregate uncertainty is resolved. We show that in the unique equilibrium, parties maximally segregate their opponent's supporters but pool their own supporters into uniform districts. We show that the stronger a party gets, the more it segregates. Hence, of the two competing parties, ceteris paribus, the stronger party segregates more than the weaker one. Finally, we show that if the district level uncertainty is sufficiently small, the stronger party chooses polarizing policies while the weaker party accommodates the stronger party's supporters.

† We thank Thomas Eisenbach, Thomas Palfrey and Andrea Prat for helpful comments on earlier drafts of this paper. This research was supported by grants from the National Science Foundation.

1. Introduction

Periodically, congressional districts are changed to accommodate regional differences in population growth. This process of *redistricting* is a source of intense conflict between political parties. In this paper, we analyze the redistricting process for the House of Representatives.

States face few constraints when setting the boundaries of their congressional districts. Congressional districts must contain the same population and must be contiguous which, in practice, is a fairly permissive constraint. A well-known example, the 4-th congressional district in Illinois, combines two disjoint areas through a very narrow strip. If a political party controls the political institutions of a state, then it has wide latitude in designing a favorable electoral map.

In our model, one of the parties is in full control of each district. Hence, we ignore instances where control of a state's political institutions is divided and redistricting plans are bipartisan. In some cases, independent commissions rather than individual parties have control of the redistricting process in this case. Such bipartisan redistricting can be incorporated into our model by giving parties control of less than 100% of the districts and interpreting the remainder as an exogenous non-partisan redistricting plan.

In general, parties and different agents within parties may evaluate election outcomes in a variety of ways; incumbents may care about protecting their own seats and others may want to maximize the number of seats the party gets in the House of Representatives. However, the most important consequence of an election is that it determines which majority will control the House of Representatives. Hence, maximizing the probability of getting a majority in the House is the most important objective. While other concerns may play a role, focusing on winning a majority facilitates the modelling of political parties as unitary actors and reveals the main issues in strategic redistricting.

Parties base their redistricting plans on a one-dimensional voter type; a higher type indicates a higher probability of voting for party 1. A redistricting plan is an allocation of voter types to districts. A party's probability of winning a particular district is a function of the (post-redistricting) *average type* in that district and two uncertain state variables; one that determines how favorable the aggregate conditions are for each party and one that determines district-level conditions. Since each party is assumed to control a continuum of districts, the law of large numbers ensures that the election outcome is a deterministic function of the redistricting strategy and the aggregate state.

We show that there is a unique equilibrium of the redistricting game. In that equilibrium, parties maximize the number of seats they would get if the realization of aggregate uncertainty is such that both parties get half the seats. We refer to the realization of aggregate uncertainty where both parties get half the seats as the *critical state*. The optimal redistricting plan (for party 1) picks a cutoff type and combines voters with types above the cutoff into uniform districts. Voters with types below the cutoff are maximally segregated into different districts. Hence, parties segregate voters with unfavorable types and combine voters with favorable types. This description of the optimal strategy generalizes Owen and Grofman's (1988) well-known *bipartisan gerrymander*.[1]

A redistricting plan is *biased* if one party wins a majority of seats with a vote share less than $1/2$. We define the partisan bias as the difference between the smallest vote share with which party 1 wins the election and $1/2$. Hence, the election is biased in party 1's favor if the bias is negative and in party 2's favor if the bias is positive.[2]

To understand our comparative statics results, assume that the two parties face the same, symmetric distribution of types. Then, the election will be biased in favor of the party that controls the larger share of districts; the weaker party will choose a more uniform redistricting plan that yields less segregation. Recall that parties, in equilibrium, maximize their seat share at the critical state. The weak party needs a large vote share to win and hence, at the critical state, will have many supporters (favorable types). Therefore, the weak party can create many relatively balanced"winnable" districts and few "unwinnable" ones packed with unfavorable types. Therefore, the weak party's redistricting plan is relatively uniform. By contrast, the stronger party has a relatively small vote share in

[1] When there are two voter types, the optimal strategy is two types of districts, as in the work of Owen and Grofman.

[2] The empirical literature (see, for example, Gelman and King (1990), Cox and Katz (1999)) typically estimates a vote-seat curve that relates a party's vote share to its share of seats. This literature defines bias as the difference between .5 and the seat share of a party when its vote share is .5. This definition of bias is closely related to ours and given an estimated vote-seat curve it is straightforward to determine partisan bias as defined here. Our definition is more appropriate given our focus on the probability of winning rather than the margin of victory.

the critical state and therefore must create more unwinnable districts to maximize its seat share.

The *local bias* of a redistricting plan is the difference between the smallest vote share with which party 1 wins a majority of districts in a territory and $1/2$. The equilibrium redistricting plan is locally biased in favor of the party that controls redistricting. The weaker the party (the smaller the territory it controls) the smaller will be the local bias. Overall bias is related to local bias: if the election is biased in party 1's favor, then party 1's territory will exhibit more bias than party 2's.[3]

Cox and Katz (2002) provide evidence on the evolution of bias after Republican and Democratic redistricting plans between 1946 and 1970. This period encompasses the redistricting revolution (triggered by Supreme Court decisions starting with Baker vs Carr (1962)) which the authors argue greatly strengthened the Democratic party. Their results indicate that the pre-revolutionary Republican redistricting plans' biases were much larger than the post-revolutionary Republican redistricting plans' biases while the opposite holds for Democratic redistricting plans. This finding is consistent with our model's predictions.

Consider, again, the case in which parties control equal size territories and face the same symmetric distribution of voter types. The symmetry ensures that the election will be unbiased. However, in each territory, the local bias favors the party in charge of redistricting. We show that even this local bias disappears as the local uncertainty goes away. That is, if the party that has the higher *expected* vote share in a particular district is virtually certain to win that district, then the local bias is virtually zero. Hence, asymmetry in voter support or party strength is necessary for generating bias and without this asymmetry, local uncertainty is necessary for local bias.

We also examine how the type distribution in a party's territory affects its electoral prospects. For example, suppose party 1's supporters are easier to identify or easier to segregate than party 2's supporters. This could be due to geographic concentration of party 1 supporters or because party 1's support is correlated with some observable variables such as ethnicity. We show that if parties are otherwise in a symmetric situation, then the election will be biased in party 2's favor. The ability to identify a party's own supporters is

[3] This result assumes homogenous populations and symmetric distributions of types.

less valuable than the ability to identify the opponent's supporters: recall that the optimal redistricting plan requires segregating opponent's supporters and pooling the party's own supporters and hence better identifying the party's own supporters has little value since these supporters will be pooled into uniform districts.

Examining Democratic and Republican parties' safe districts provides indirect evidence of asymmetries in their ability to segregate voters. In the 2000 presidential election, the smallest Democratic vote share in any congressional district was 24% while there were 24 districts with a Democratic vote share of over 80% and 5 Districts with a Democratic vote share of over 90%.

Finally, we examine how redistricting plans affect policy choice. We introduce a policy choice to our redistricting game. The policy is one-dimensional; party 1 supporters prefer higher policies while party 2 supporters prefer lower policies. Therefore, party 1 polarizes the electorate if it chooses a high policy while a low policy choice by party 1 has a moderating effect. Conversely, party 2 polarizes if it chooses a low policy and has a moderating effect with a high policy. Parties first choose a redistricting plan and then, after observing the redistricting plan of the opponent, choose a policy. We show that, when local uncertainty is small or when the stronger party is sufficiently strong, the strong party will choose the most polarizing policy while the weak party will choose the most moderating policy.

To understand this result, suppose a single party is in control of all redistricting and policy. Then, there is an obvious benefit to polarization: it further differentiates voter types and makes redistricting more effective. This effect gives the stronger party the incentive to polarize. The effect of polarization is more subtle when polarization increases both the party's and its opponent's ability to segregate. We show that for a given redistricting plan, when local uncertainty is small or one party has a sufficiently large advantage, polarization helps the strong party and hurts the weak party.

Notice that parties in our model have no policy preference. The policy choice simply maximizes the probability of winning. We show that party positions will tend towards the position favored by the stronger party's supporters despite the fact that parties do not care about policy and policy choice has zero net effect on vote shares.

1.1 Related Literature

Our work builds on Owen and Grofman (1988). Their model can be interpreted as a special case of ours with two voter types and one party controlling all districts. For that case, Owen and Grofman show that the optimal plan creates two types of districts, ones that overwhelmingly favor the opponent and others that the party is expected to win.[4]

Gilligan and Matsusaka (1999) and Friedman and Holden (2006) characterize the redistricting plans that maximize a party's expected number of seats. Friedman and Holden (2006) consider a setup in which not only the average type but the entire distribution of types affects a party's probability of winning that district.

Coate and Knight (2006) and Gilligan and Matsusaka (2005) study socially optimal redistricting plans. Epstein and O'Hallaran (2004) analyze how redistricting can be used to enhance the welfare of minorities. Shotts (2002), Katz and Grigg (2005), examine the effect of majority-minority districts. An important constraint in redistricting plans is the mandate to create and maintain districts with a substantial majority of minority voters. Such a constraint amounts to a *lower bound* on segregation. Incorporating this additional constraint into our model is not difficult. For example, if voters with certain types must be fully segregated into their own district, then our characterization of equilibrium redistricting plans would apply to the remaining voter types.

Cox and Katz (2002) provide a comprehensive study of redistricting since the reapportionment revolution of the 1960s. Their model (and much of the literature on redistricting) focuses on the trade-off between bias and responsiveness. There is also a large empirical literature that focuses on the so-called seat-vote curve that is generated by various redistricting plans. (See, for example, Gelman and King (1990 and 1994), King and Browning (1987).

Shotts (2002) and Besley and Preston (2005) model the interaction of redistricting and policy choice. In Shott's model, parties are policy motivated and redistrict to move the median representative closer to their ideal point. Besley and Preston examine the effect of partisan bias on a party's responsiveness to swing voters. In their model, parties have policy preferences but swing voters constrain their extremism. The partisan bias of the

[4] For a different generalization of Owen and Grofman see Sherstyuk (1998).

electoral map affects this constraint and hence affects policy. The mechanism connecting policy and redistricting in our model is different. In our model, polarizing policies make it easier to segregate voters and therefore strong parties polarize.

2. Model and Equilibrium

There are two parties, $i = 1, 2$, each in control of a continuum of districts. The mass of districts under party 1's control is $\lambda \in (0, 1)$ and the mass of districts under party 2's control is $1 - \lambda$. We refer to the districts under party i's control as i's *territory*.

For any voter v, p_v is the probability that v votes for party 1 and $1 - p_v$ is the probability that v votes for party 2. Three factors determine p_v: the voter's type ω_v, local uncertainty in the voter's district s_d, and aggregate uncertainty s. For simplicity, we assume

$$p_v = {}^1\!/_2 + \omega_v + s_d + s \tag{1}$$

Let p_d denote party 1's expected vote share in district d and let ω_d denote the average type in district d. Then, equation (1) implies

$$p_d = {}^1\!/_2 + \omega_d + s_d + s$$

There are many voters in each district and hence the "law of large number" ensures that party 1 wins a district whenever $p_d > {}^1\!/_2$. Therefore, party 1 wins district d whenever

$$s_d > -z_d - s$$

Local and aggregate uncertainty are independent and parties face the same local uncertainty in each district. Hence, the variables s, s_d are independent and every s_d has the same strictly increasing and continuous cumulative L. Since the parties face symmetric local uncertainty, L is symmetric around zero:

$$L(-s_d) = 1 - L(s_d)$$

for all $s_d \in I\!R$. We assume that $Z = [-1/4, 1/4]$ is the support of L. Define, $z_d = \omega_d + s$; we call z_d the district proclivity and ω_d the (average) district type. Then, as a function of the district proclivity, the probability that party 1 wins district d is

$$P_d = 1 - L(-\omega_d - s) = 1 - L(-z_d) = L(z_d)$$

Hence, L translates district proclivities into probabilities of winning that district. Clearly, as p_d increases above $1/2$, its effect on P_d diminishes. Thus, we assume that L is strictly concave on $[0, 1/4]$.

When parties make their redistricting decisions, they face both aggregate uncertainty $s \in S = [-1/8, 1/8]$ and local uncertainty s_d. That is, they do not know the realized values of these variables. But, they observe voters' types $\omega \in \Omega \subset [-1/8, 1/8]$. A party's task is to allocate these types (i.e., voters) among a continuum of equal-sized districts; that is, to choose a district type distribution over their territory.

Let $\mathcal{F}$ denote the collection of cumulative distribution functions (cdfs) with mean zero and support contained in $[-1/8, 1/8]$. A *redistricting plan* is an element, H, of $\mathcal{F}$ and $H(x)$ represents the share of districts within the party's territory that have average type ω_d no greater than x. The segregation constraint represents the most dispersed (or segregated) feasible distribution of district type averages. The cdf $F \in \mathcal{F}$ is the segregation constraint for party 1 and the cdf $G \in \mathcal{F}$ is the segregation constraint for party 2. Any redistricting plan H is feasible for party 1 (party 2) only if F (G) is a mean preserving spread of H. Conversely, any $H \in \mathcal{F}$ such that F (G) is a mean preserving spread of H is a feasible redistricting plan for party 1 (party 2). We write $H' \succeq_2 H$ if H is a mean-preserving spread of H'.

To understand these strategy sets, consider the following example: there are two voter types, $-\omega$ and ω. The segregation constraint F is a two point distribution with mass $1/2$ each at $-\omega$ and at ω. Then, party 1 can create districts by combining voters of type $-\omega$ and voters of type ω. This will yield district means ω_d with $-\omega \leq \omega_d \leq \omega$ and therefore the support of any feasible redistricting plan must be contained in the interval $[-\omega, \omega]$. The redistricting plan must satisfy one further restriction: the average voter type (across all districts) must be the same as the average voter type of F and therefore any feasible redistricting plan must have mean zero.

We say that H is nondegenerate if it has at least two elements in its support; that is, if $0 < H(x) < 1$ for some x. If the segregation constraint F has a single element in its support then F itself is the only feasible redistricting plan. We rule out this trivial case and assume that both parties face a nondegenerate segregation constraint.

In practice, redistricting is done rather infrequently and parties rarely choose their plans simultaneously. Our analysis is robust to the timing of moves: any sequencing of redistricting decisions would lead to the same equilibrium outcome as our simultaneous move game. We have chosen the simultaneous move formulation because it is the simplest.

Parties maximize the probability of winning a majority in the House of Representatives. The redistricting plan H yields the voting proclivity distribution H_s at state s, where

$$H_s(z) = H(z - s) \tag{2}$$

Hence, $H_0 = H$. Since there are a large number of districts and local uncertainty is independently distributed, the "law of large numbers" ensures that party 1 wins $D(H_s)$ districts in its territory in state s with strategy H, where

$$D(H_s) = \int L(z) dH_s(z)$$

Party 1's total seat share (in both territories) in state s given the strategy profile (H, H') is

$$\Delta(H_s, H'_s) = \lambda D(H_s) + (1 - \lambda) D(H'_s)$$

and therefore party 1 wins the election at state s if $\Delta(H_s, H'_s) \geq 1/2$. Hence, party 1 chooses H to maximize

$$\Pr\{s \,|\, \Delta(H_s, H'_s) \geq 1/2\}$$

Party 2 chooses H' to minimize this probability. Parties do not know s and have full support beliefs on S. Beyond the full support assumption, the details of the party's beliefs (about s) play no role in our analysis and therefore we do not specify those beliefs.

A *redistricting game* is a quadruple $\Lambda = (F, G, \lambda, L)$, where F, G are the redistricting constraints of party 1 and 2 respectively, λ is the size of party 1's territory, and L is the cumulative distribution of local uncertainty. When the choice of L is clear, we also omit L.

2.1 An Example with No Local Uncertainty

To illustrate the model, consider the redistricting game $\Lambda = (F, G, \lambda, L^\infty)$ where

$$L^\infty(z) = \begin{cases} 1 & \text{if } z > 0 \\ 1/2 & \text{if } z = 0 \\ 0 & \text{if } z < 0 \end{cases} \tag{3}$$

Hence, there is no local uncertainty; the party that has the higher expected vote share wins the district for sure. Note that L^∞ does not satisfy our assumptions. However, it can be approximated arbitrarily closely by functions that do satisfy them.[5] Using L^∞ simplifies the equilibrium calculations for this example. There are two types of voters, $\Omega = \{-1/8, 1/8\}$ and both segregation constraints (F, G) assign probability $1/2$ to each of the two possible types. Party 1 controls two-thirds of the districts, i.e., $\lambda = 2/3$.

Suppose that party 1 were to construct two kinds of districts in its territory; unfavorable districts "packed" solely with party 2 supporters (i.e., types $\omega = -1/8$) and mixed districts that it expects to win. Since party 1 controls $2/3$ of the electoral map, if the share of the favorable districts in its own territory is above $3/4$, then it wins more than $2/3 \cdot 3/4 = 1/2$ of all districts and therefore wins the election. Conversely, to win the election without winning any districts in party 2's territory, party 1 must win at least $3/4$ of its own districts. To create a $3/4$ proportion of favorable districts, party 1 must combine all of party 1's supporters with half of the party 2 supporters: $1/2 \cdot 1/2 + 1/2 = 3/4$. The average type in these mixed districts will be

$$\omega_d = -1/3 \cdot 1/8 + 2/3 \cdot 1/8 = 1/24$$

[5] For example, the following sequence of functions converges to L^∞. For $z \in Z$,

$$L^n(z) = \frac{(1/4 + z)^n}{(1/4 + z)^n + (1/4 - z)^n}$$

Hence, party 1 will win the election as long as $s > -1/24$ and therefore, party 1's equilibrium payoff is at least $\Pr\{s > -1/24\}$.

Note that there is no strategy for party 1 that enables it to win $3/4$ of the districts in its territory when $s < -1/24$. On the other hand, by creating uniform districts, party 2 can insure that it wins its entire territory whenever $s < 0$. Therefore, party 2 can guarantee winning the election whenever $s < -1/24$. Thus, party 2's equilibrium payoff is no less than $\Pr\{s \leq -1/24\}$. It follows that the equilibrium payoff of party 1 must be $\Pr\{s > -1/24\}$ and the equilibrium payoff of party 2 must be $\Pr\{s \leq -1/24\}$. Moreover, the party 1 strategy described above and the uniform redistricting plan for party 2 constitute an equilibrium. In equilibrium, both parties choose a redistricting plan that maximizes their seat share at $s = -1/24$, i.e., the state at which the election is tied. It is easy to verify that the equilibrium strategy of party 1 is unique. However, at $s = -1/24$ party 2 can choose other redistricting plans and still win all districts in its territory and hence there are multiple equilibrium strategies for party 2 in this example. The strict increasingness of L (i.e., local uncertainty) rules out this multiplicity in Theorem 1.

We show in Theorem 1 below that party 1's equilibrium strategy fully segregates all districts below some critical type ω and create a mass of uniform districts with the same average $x > \omega$. Hence, below ω the redistricting plan coincides with the segregation constraint while above ω all remaining types are combined into a uniform district. To formally define such strategies, we will need the following notation. For any cdf H and $p \in (0, 1)$, let H_+^p be the distribution of the upper $1 - p$-percentile, i.e.,

$$H_+^p(x) := \max\left\{\frac{H(x) - p}{1 - p}, 0\right\}$$

We write $m(H)$ for the mean of H.

Definition: *The $p-$segregation plan for party 1 with constraint F is the distribution F^p, where $F^p = F$ for $p = 1$,*

$$F^p(x) = \begin{cases} F(x) & \text{if } F^{-1}(p) > x \\ p & \text{if } F^{-1}(p) \leq x < m(F_+^p) \\ 1 & \text{if } x \geq m(F_+^p) \end{cases}$$

for $p \in (0,1)$ and F^0 yields $m(F)$ for sure.

To illustrate $p-$segregation strategies, first consider a discrete example with three types, $\Omega = \{-1/8, 0, 1/8\}$. Let F be such that $-1/8$ and $1/8$ have probability .25 each and 0 has probability .5. Then $F^{0.4}$ has support $\{-1/8, 0, 5/96\}$ and yields $-1/8$ with probability .25, 0 with probability .15 and $5/96$ with probability .6.

In Figure 1 below, we illustrate a $p-$segregation strategy for a continuous segregation constraint F.

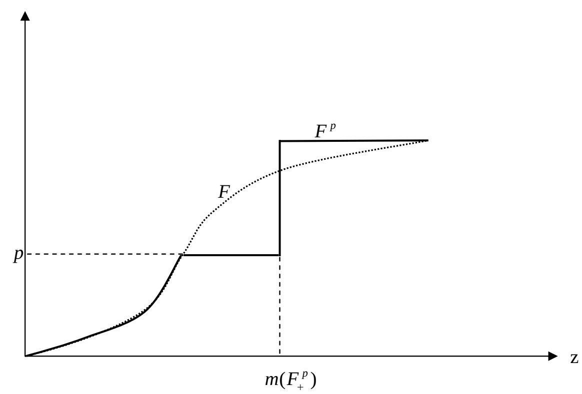

Figure 1

Types that favor party 1 are unfavorable for party 2 and hence a $p-$segregation plan for party 2 fully segregates all districts above some critical ω and creates a mass of uniform districts with the same average below ω. For any distribution H, let $\rho(H)$ denote the corresponding distribution of $-x$. That is, $\rho(H)$ is the unique distribution such that $\rho(H)(x) = 1 - H(-x)$ at every continuity point of $\rho(H)$.[6] If G is the segregation constraint for party 2 then $\rho(G)$ is the translation of G that makes G comparable to F, the segregation constraint of party 1. If $\rho(G) = F$ then both parties face the same segregation constraint.

Definition: *The $p-$segregation plan for party 2 with constraint G is the distribution $\bar{G}^p := \rho[\rho(G)^p]$.*

[6] Note that $\rho(\rho(H)) = H$ and hence $\rho^{-1} = \rho$.

For the segregation constraint in example above with three types $\Omega = \{-1/8, 0, 1/8\}$ the distribution $\bar{G}^{0.4}$ has support $\{-5/96, 0, 1/8\}$. The probability of $-5/96$ is .6, the probability of 1/8 is .25, and the probability of 0 is .15.

Theorem 1 establishes that the equilibrium is unique and that equilibrium strategies are $p-$segregation strategies.

Theorem 1: *(i) There exist p, q such that $(F^p, \bar{G}^q)$ is the unique equilibrium of $\Lambda = (F, G, \lambda, L)$. (ii) In equilibrium, parties maximize their vote shares at the unique s^* that solves $\Delta(F^p_s, \bar{G}^q_s) = 1/2$.*

Theorem 1 shows that a single parameter characterizes a party's optimal strategy. Henceforth, we identify equilibrium strategies with the pair (p, q). We refer to the state of aggregate uncertainty at which the election is tied in the unique equilibrium as the *critical state* and write $s(\Lambda)$ for the critical state in game Λ.[7]

Parties' redistricting plans maximize their seat shares at the critical state. To see why, let s^* be the critical state and assume $\Delta(F^p_{s^*}, \bar{G}^q_{s^*}) = 1/2 < \Delta(H_{s^*}, \bar{G}^q_{s^*})$ for some feasible H. By continuity, $\Delta(H_{\hat{s}}, \bar{G}^q_{\hat{s}}) = 1/2$ for some $\hat{s} < s^*$. Then, with strategy H, party 1 wins the election in all $s > \hat{s}$ and hence H yields a higher payoff for party 1 than F^p.

We can provide a simpler description of party 1's optimal redistricting plan when F has a density $f > 0$ and L is differentiable. Let s^* be the critical state and note that F_{s^*} is the most segregated distribution of voting proclivities at s^*. Let f_* denote the density of F_{s^*} and define

$$z_y = \frac{\int_{z \geq y} t f_*(z) dz}{\int_{z<y} f_*(z) dz}$$

to be the expected proclivity conditional on the proclivity being above y. Then, party 1's optimal strategy is the $p-$segregation strategy such that $p = \int_{t \leq y} f_*(t) dt$ and y satisfies:

$$L'(z_y) = \frac{L(z_y) - L(y)}{z_y - y} \tag{4}$$

[7] Since $L(z) \leq 1/2$ whenever $z \leq 0$ it follows that $D(H_s) \leq 1/2$ if $s = -1/8$ and $D(H_s) \geq 1/2$ if $s = 1/8$ for all $H \in \mathcal{F}$. Since L is continuous and strictly increasing throughout its support, $\Delta(F^p_s, \bar{G}^q_s)$ is continuous and strictly increasing in s. Hence, the critical state is well-defined.

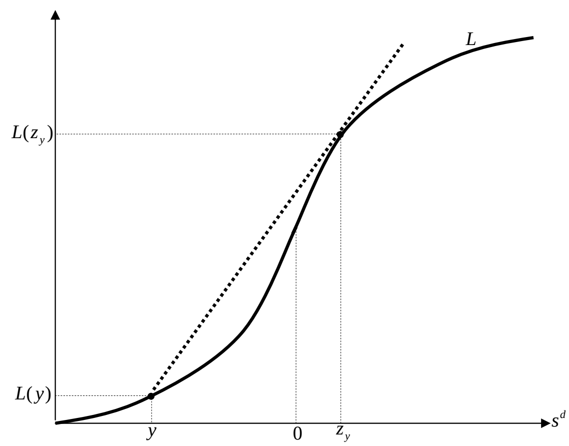

Figure 2

The optimal strategy is the $p-$segregation strategy such that the corresponding y, z_y satisfy the tangency condition (4) illustrated in Figure 2.

3. Bias and Segregation

In this section, we analyze how changing the redistricting game's parameters affects equilibrium outcomes. The main result is that changes that favor party i lead to party i segregating more and party $j \neq i$ segregating less. A parameter change in the redistricting game Λ makes party i *stronger* if it allows i to win over a larger set of states. That is, party 1 becomes stronger if the critical state, $s(\Lambda)$, falls and party 2 becomes stronger if $s(\Lambda)$ rises.

Definition: *Let* $\Lambda = (F, G, \lambda, L)$ *and* $\hat{\Lambda} = (\hat{F}, \hat{G}, \hat{\lambda}, L)$. *We say party* 1 *[party 2] is stronger in* Λ *than in* $\hat{\Lambda}$ *if* $s(\Lambda) < s(\hat{\Lambda})$ $[s(\Lambda) > s(\hat{\Lambda})]$.

Although the probability distribution over the aggregate uncertainty ("states") plays no role in our analysis, it does affect a party's probability of winning. If the probability distribution over states remains constant, then increasing a parties strength increases its probability of winning. However, our analysis remains valid even if the probability distribution over states changes as other parameters change. In that case, a party's strength refers to its ability to win in unfavorable circumstances and not to its probability of winning.

Note that $q > p$ implies $H^p \succeq_2 H^q$ and $\bar{H}^p \succeq_2 \bar{H}^q$; increasing p yields a mean preserving spread of the type distribution. Theorem 2 shows that as a party becomes stronger, the optimal p increases. Hence, the stronger a party gets the more it segregates.

Theorem 2: *Let $\Lambda = (F, G, \lambda)$, $\hat{\Lambda} = (F, \hat{G}, \hat{\lambda})$ and let $p, \hat{p}$ be the corresponding equilibrium strategies of party 1. If party 1 is stronger in Λ than in $\hat{\Lambda}$, then $p \geq \hat{p}$.*

Proof: See Appendix. □

Consider the simple extreme case with almost no local uncertainty; that is, assume that the function L is close to L^∞ as defined in the example of Section 2.1. Since party 1 wins for sure any district with proclivity above $1/2$, it maximizes the number of districts with proclivity just above $1/2$ at the critical state. A symmetric statement holds for party 2. Now, assume that party 1 is stronger than party 2 and hence the critical state is less than $1/2$. If party 2 chooses a uniform redistricting plan ($q = 0$), then since the average proclivity is less than $1/2$, it wins *all seats* in its territory. Obviously, this implies that a uniform redistricting plan is optimal for the weaker party. By contrast, party 1 must segregate voters to win any seats in the critical state. Moreover, a lower critical state implies that party 1 must "give up" more seats and hence choose a larger p.

Theorem 2 shows that this insight holds even with local uncertainty. To gain further intuition for Theorem 2, assume the segregation constraint F has support $\{-\omega, \omega\}$ and $F(-\omega) = 1/2$. Then, the optimal redistricting plan creates two types of districts: a $p-$fraction of districts will contain only types $-\omega$ and the remaining $1 - p-$fraction will contain a mixture of types with average $z^* = \frac{p}{1-p}\omega$. (Note that $p \leq 1/2$). At the critical state s^*, we have

$$L'(z^* + s^*) = \frac{L(z^* + s^*) - L(-\omega + s^*)}{z^* + \omega} \tag{5}$$

A stronger party 1 means a lower critical state. Fixing z^* and decreasing s^* makes the left hand side of (5) larger than the right hand side. Conversely, fixing s^* and increasing z^* makes the right hand larger than the left hand side. Hence, for (5) to hold after a decrease in s^*, z^* must increase and hence p must increase. That is, p increases as party 1 gets stronger.

A party's share of all votes in state s is $s + 1/2$. This follows since the variables ω_d (average type in each district) and s_d (local uncertainty) both have zero mean. The redistricting game is biased in party 1's favor if it needs less than half of the votes to tie the election. That is, the *bias* of a redistricting game is $s(\Lambda)$. The game Λ is biased in party 1's favor if $s(\Lambda) < 0$ and in party 2's favor if $s(\Lambda) > 0$. We say that Λ is more biased than Λ' if $|s(\Lambda)| > |s(\Lambda')|$. Hence, if $s(\Lambda) < 0$ (> 0), then party 1 (2) can win the election even though a majority of voters prefer party 2 (1) and therefore the election is biased in party 1's (2's) favor.

The bias in territory i is defined analogously. Let

$$s_1(\Lambda) := \{s \mid D(F_s^p) = 1/2\}$$
$$s_2(\Lambda) := \{s \mid D(\bar{G}_s^q) = 1/2\}$$

where $(F^p, \bar{G}^q)$ is the unique equilibrium of Λ. Hence, $s_i(\Lambda)$ is the vote share that would yield a tie in territory i. Arguments analogous to the ones made for $s(\Lambda)$ ensure that $s_i(\Lambda)$ is also well defined.

Theorem 3 below establishes that the local bias always favors the redistricting party. Also, it shows that the local bias increases when the redistricting party becomes stronger. Finally, Theorem 3 shows that bias grows as the strong party gets stronger.

Theorem 3: *(i) For any Λ, $s_1(\Lambda) \le 0 \le s_2(\Lambda)$ and $s_1(\Lambda) \le s(\Lambda) \le s_2(\Lambda)$. (ii) Let $\Lambda = (F, G, \lambda)$, $\hat{\Lambda} = (F, \hat{G}, \hat{\lambda})$. If $s(\Lambda) \le s(\hat{\Lambda})$, then $s_1(\Lambda) \le s_1(\hat{\Lambda})$. (iii) The critical state $s(F, G, \lambda)$ is decreasing in λ.*

Proof: See Appendix. □

Theorem 3 relies on two key observations: let $\alpha(s)$ be the p that maximizes the seat share of party 1 in its territory in state s, $D(F_s^p)$. In Theorem 2, we showed that the stronger party 1 is the more it segregates; that is, α is decreasing in s. The second observation is that fixing s, as p increases towards its optimal level, the seat share increases; that is, $D(H_s^p)$ is increasing at $p \le p^*$.

Let $s = s(\Lambda)$ and $s_1 = s_1(\Lambda)$. First, assume that $s < 0$. Since $s < 0$, party 2 must win more than half of the seats in its territory. (For example, a uniform redistricting plan would yield more than half of the seats for party 2.) This, in turn implies that, at s, party 1 must win more than half of the seats in its territory to yield a tied election. Therefore, we have

$$D(F_s^p) \geq \Delta(F_s^p, \bar{G}_s^q) = 1/2 = D(F_{s_1}^p)$$

Then, the monotonicity of D and Δ imply that $s_1 \leq s < 0$. Next, assume $s \geq 0$ and therefore $\alpha(s) \leq \alpha(0)$ and

$$D(F_0^{\alpha(0)}) \geq D(F_0^{\alpha(s)}) \geq D(F_0^0) = 1/2$$

The lasts equality follows since at $s = 0$ a uniform redistricting yields exactly half the seats for each party. It follows that $D(F_0^{\alpha(s)}) \geq 1/2$ and $s_1 \leq s$. Parts (ii) and (iii) follow from similar arguments.

Theorems 2 and 3 offer testable implications of our model. Increasing party 1's strength or bias increases the local bias in territory 1. Cox and Katz (1999) provide evidence on the evolution of bias after Republican and Democratic redistricting plans between 1946 and 1970. This period encompasses the redistricting revolution (triggered by Supreme Court decisions starting with Baker vs Carr (1962)) which the authors argue greatly strengthened the Democratic party in the sense defined above. Their results (Table 3, pg 830) indicate that the pre-revolutionary Republican redistricting plans yielded larger biases than post-revolutionary Republican redistricting plans while the evolution of the biases is exactly reversed for Democratic redistricting plans. Cox and Katz define bias as the seat share of a party when its vote share is one half. They estimate that the bias of Republican plans drops from 8.26% to .092% while the bias of Democratic plans increases from 4.76% to 8.70%. We can use their estimates to compute the estimated bias according to the definition used here.[8] In that case, the estimated bias for Republican plans drops from 2.3% to essentially zero while the estimated bias for Democratic plans increases from 1.1% to 2.1%.

Next, we apply the analysis above to redistricting games in which both parties face the same, symmetric constraint and differ only in the size of their territories. The redistricting

[8] Using their estimated seat-vote curve it is straightforward to compute the corresponding estimated biases as defined in this paper.

game (F, G, λ) is *homogenous* if $F = G$, that is, if both parties face the same redistricting constraint. We let $\Lambda = (F, \lambda, L)$ denote a homogenous redistricting game. Recall that for any distribution $F \in \mathcal{F}, \rho(F)$ denotes the distribution of $-\omega$. The distribution F is *symmetric* if $\rho(F) = F$.

In a homogenous redistricting game (F, λ) with a symmetric constraint F, both parties' situation is identical except for the sizes of their territories. For this special case, the following corollary summarizes the comparative statics results of this section. The election will be biased in favor of the party with the larger territory; the stronger party will choose a more segregating redistricting plan and generate a more biased electoral map in its territory.

Corollary 1: *Let (p, q) be the equilibrium of the homogenous redistricting game $\Lambda = (F, \lambda)$ and assume that F is symmetric. If $\lambda > 1/2$, then*

(i) the election is biased in party 1's favor;

(ii) $p > q$, i.e., party 1 segregates more than party 2;

(iii) bias in territory 1 is greater than bias in territory 2.

The corollary shows how, in a symmetric and homogeneous electorate, the parties' redistricting plans differ in equilibrium. The weaker party favors more uniform redistricting plans while the stronger party favors creates more lopsided districts.

The comparative statics results of this section can also provide some insight into how equilibrium redistricting plans differ from *ex post* seat maximizing redistricting plans. Suppose a particular state s above the critical state is realized, and party 1 wins the election. Party 1's redistricting plan maximizes its seat share at the critical state but not at s. The optimal redistricting plan at s has less segregation (smaller p) than the equilibrium plan and, therefore, the districts that party 1 wins will have a larger margin of victory than would be optimal in the seat maximizing plan. Hence, it may appear as if party 1 is creating overly safe districts. By contrast, party 2's redistricting plan will appear as if it has segregated too little; it's seat share would increase, had it created more safe districts.

We conclude this section by examining the limiting case when local uncertainty disappears. Recall that L^∞ in the example describes the situation without local uncertainty.

Hence, we say that local uncertainty disappears along the sequence $\Lambda^n = (F, G, \lambda, L^n)$ if L^n converges pointwise to L^∞. Corollary 2 generalizes the example and shows that (1) at the critical state, the stronger party wins half of all districts despite not winning any districts in the opponent's territory and (2) the weaker party chooses a uniform redistricting plan as local uncertainty disappears.

Corollary 2: *Suppose uncertainty disappears along $\Lambda^n = (F, G, \lambda, L^n)$ and $\lambda > 1/2$. Let (p^n, q^n) be the equilibrium of Λ^n. Then,*

(i) $\lim p^n = \frac{2\lambda-1}{2\lambda}$ *and* $q^n = 0$ *for all n sufficiently large.*

(ii) $\lim s_1(\Lambda^n) = \lim s(\Lambda^n) \geq \frac{1-2\lambda}{2\lambda}$ *and* $\lim s_2(\Lambda^n) = 0$.

Part (ii) of Corollary 2 shows that, as λ goes to $1/2$, even the strong party will choose uniform redistricting and all biases will be eliminated. Hence, if parties are evenly balanced and local uncertainty is small then competitive redistricting implies small local biases and a small overall bias.

4. Changes in the Segregation Constraint

In this section, we study how changes in the segregation constraint affect the equilibrium outcome. A mean preserving spread of the segregation constraint relaxes the party's constraint and, therefore, must (weakly) increase its seat share in its own territory. Such a change may come about through better information; that is, greater ability to identify voters. However, a greater ability to segregate also helps the opponent. Theorem 4 considers a homogenous redistricting game and shows that when one party's territory is sufficiently larger than its opponents, it benefits from a mean preserving spread of the segregation constraint.

Theorem 4: *Let $\Lambda = (F, \lambda, L), \hat{\Lambda} = (\hat{F}, \lambda, L)$ be homogenous redistricting games such that $F \succeq_2 \hat{F}$. Suppose $(F, 1, L)$ and $(\hat{F}, 1, L)$ do not yield the same equilibrium payoffs. Then, there exists $\lambda^* \in (0, 1)$ such that for all $\lambda > \lambda^*$ party 1 receives a strictly higher equilibrium payoff in $\hat{\Lambda}$ than in Λ.*

Even though the redistricting game is homogenous (hence the two territories have the same distribution of types) the segregation constraint facing the two parties may be asymmetric. For example, assume some supporters of one party are easily identified by their ethnicity or their address while there are no comparably reliable indicators of support for the other party. Asymmetries in the segregation constraint F capture such differences between the two parties' supporters.

Party 2's supporters are easier to segregate if the distribution party 2 supporters (low types) is a mean preserving spread of the distribution of party 1 supporters (low types). Recall that F is symmetric if $\rho(F) = F$. Hence, if F is symmetric, both parties' supporters are equally difficult to segregate. Suppose F is not symmetric. Define $H \in \mathcal{F}$ such that H coincides with F for $\omega < 0$ and is symmetric. Hence, H is the distribution of types that results if both parties' supporters have the distribution of party 2 supporters implied by F. If H is a mean preserving spread of F then we can conclude that party 2 supporters are more "spread out" than party 1 supporters and therefore easier to segregate.

Definition: *Party 2 supporters are easier to segregate at F if there is $H \in \mathcal{F}$ such that $\rho(H) = H$, $H(\omega) = F(\omega)$ for $\omega < 0$ and $F \succeq_2 H$.*

Example: There are 3 types, $\Omega = \{-1/8, 0, 1/16\}$ and F puts probability .25 on $-1/8$; probability .25 on 0 and probability .5 on 1/16. In this case, party 2 supporters are easier to segregate because the symmetric distribution H on $\{-1/8, 0, 1/8\}$ that puts probability .25 on $-1/8$, .5 on 0, and .25 on 1/8 is a mean preserving spread of F.

For US elections, examining the two parties' safe districts reveals evidence of asymmetries in their segregation constraints. In the 2000 presidential election, the smallest Democratic vote share in any congressional district was 24% while there were 24 districts with a Democratic vote share of over 80% and 5 Districts with a Democratic vote share of over 90%. This suggests that there are stronger indicators of Democratic voting proclivities than of Republican voting proclivities.

Theorem 5 examines a situation where both parties control homogenous areas of equal size. If the redistricting game is symmetric, both parties face the same constraint. In that case,

$$s(\Lambda) = 0$$

and hence, the symmetric redistricting game is *unbiased*, i.e., the party with majority support wins the election. When party 2's supporters are easier to segregate, the critical state is less than $1/2$ and the election is biased in party 1's favor.

Theorem 5: *If party 2's supporters are easier to segregate in $\Lambda = (F, 1/2)$, then $s(\Lambda) \leq 0$.*

Theorem 5 establishes that the equilibrium outcome is *biased against* the party whose supporters can be segregated more readily. To understand this result, consider a change that increases both parties' ability to segregate party 2's supporters: this change does not help party 2 in territory 2 because its equilibrium strategy (the $p-$segregation strategy) creates uniform districts of supporters. However, since maximally segregating the opponent's supporters is optimal, party 1 benefits from its increased ability segregate party 2's supporters.

Theorem 5 can be strengthened to establish a strict inequality ($s(\Lambda) < 0$) if the extreme supporters of party 2 are more extreme than the extreme supporters of party one. More formally, let $\underline{\omega}(F)$ be the minimum element in the support of F (the strongest supporter of party 2) and let $\bar{\omega}(F)$ be the maximum element in the support of F (the strongest supporter of party 1). If $\bar{\omega}(F) < -\underline{\omega}(F)$ then party 1 strictly gains from its greater ability to segregate party 2's supporters.

5. Redistricting and Polarization

In this section, we analyze the interaction of redistricting plans and policy choices. Parties first choose a redistricting plan and then make a policy choice. We are interested in why and when a party polarizes; that is, selects a policy that increases the support of voters who favor the party and decreases support of voters who favor the opponent.

In the redistricting-policy game, voting proclivity z depends on the voter's type $\omega \in [-x, x]$, the state $s \in [-1/8, 1/8]$ and the policy choice $\pi_1, \pi_2 \in [0, \bar{\pi}]$ where $x(1 + \bar{\pi}) \leq 1/8$. Let

$$\xi(\omega, \pi_1, \pi_2, s) = s + \omega(1 + \pi_1 - \pi_2)$$

denote type ω voter's proclivity given s, π_1, π_2.

We can interpret this model as one where voter types $\omega > 0$ prefer higher policy values and voter types $\omega < 0$ prefer lower policy values. Types with higher $|\omega|$ respond more to policy changes than those with lower $|\omega|$. By choosing a higher policy, party 1 is catering to its supporters and alienating party 2's supporters. Hence, a higher policy choice is *polarizing* for party 1 and accommodating for party 2. Note that policy choice has a limited impact on voting proclivities: irrespective of π_1, π_2, positive types favor party 1 if $s \geq 0$ and negative types favor party 2 if $s \leq 0$.

As above, F (G) is party 1's (2's) redistricting constraint and is a mean preserving spread of any feasible redistricting plan. In the first stage of the game, the parties choose a redistricting plan. In the second stage, the parties observe each others redistricting plans and choose policies $\pi_i \in [0, \bar{\pi}]$. Then, the state is revealed and in territory i, party 1 wins a seat share of

$$V_i(H, \pi_1, \pi_2, s) = \int L(s + \omega(1 + \pi_1 - \pi_2))dH$$

Party 1 wins if $\lambda V_1 + (1 - \lambda)V_2 > 1/2$ and party 2 wins if this inequality is reversed. We analyze subgame perfect equilibria of this game.

Suppose $\pi_1 = \pi_2$ and hence the voting proclivity is $z = \omega + s$ as in the previous sections. Let $\Lambda = (F, G, \lambda)$ denote the corresponding redistricting game and let (p^*, q^*) denote its equilibrium. Theorem 6 below considers the case where λ is close to one and hence party 1 controls most districts. In that case, the unique equilibrium is the redistricting plan (p^*, q^*) and policies $\pi_1 = \pi_2 = \bar{\pi}$. Hence, party 1 chooses the most polarizing policy while party 2 chooses the most accommodating policy. The equilibrium redistricting plans are segregation plans as in the previous sections. Hence, an equilibrium of the redistricting-policy game is a strategy profile of the form $((\pi_1, p), (\pi_2, q))$.

Theorem 6: *There exists $\lambda^* < 1$ such that for all $\lambda \in (\lambda^*, 1]$ the unique subgame perfect Nash equilibrium of the redistricting-policy game is $((1, p^*), (1, q^*))$.*

Since increased polarization generates a mean-preserving spread in voting proclivities, Theorem 4 suggests that polarization would benefit party 1. However, since policies are chosen *after* redistricting, to prove Theorem 6, we must show that party 2 would not want to choose a more polarizing policy despite party 1's inability to adjust its redistricting plan. We show that for a fixed redistricting plan, more polarization benefits the party that was in charge of the redistricting. The reason is that at the critical state, districts that the redistricting party expects to win are less lopsided than unfavorable districts. Then, the curvature of L implies that more polarization benefits the redistricting party.

Theorem 7 shows that the conclusions of Theorem 6 hold for all $\lambda \neq 1/2$ when local uncertainty is small. Recall that L^∞ describes the no local uncertainty limit and hence uncertainty disappears whenever L^n converges to L^∞ pointwise.

Theorem 7: *Suppose uncertainty disappears along* $\Pi^n = (F, G, \lambda, L^n)$ *and* $\lambda > 1/2$. *Let* $((\pi_1^n, p^n), (\pi_2^n, q^n))$ *be the equilibrium of* Π^n. *Then,*

(i) $\lim p^n = \frac{2\lambda - 1}{2\lambda}$ *and* $q^n = 0$, $\pi_1^n = \pi_2^n = 1$ *for all* n *sufficiently large.*

(ii) $\lim s_1(\Pi^n) = \lim s(\Pi^n) \geq \frac{1-2\lambda}{2\lambda}$ *and* $\lim s_2(\Pi^n) = 0$.

When local uncertainty is small, the election is biased in favor of the party with the larger territory (party 1). Moreover, the weaker party (party 2) chooses a uniform redistricting plan. Therefore, polarization cannot benefit the weak party. On the other hand, the strong party segregates and therefore benefits from polarization.

Theorems 6 and 7 show how redistricting affects policy choice even if parties care only about their probability of winning the election and are indifferent among all policy choices. Equilibrium policies shift towards those favored by the supporters of the stronger party.

In our model, policy choices do not affect the average voting proclivity over all districts. However, extending our model to permit a trade-off between average proclivity (or vote share) and polarization is straightforward. Consider the following example: suppose that there are two voter types ($\Omega = \{-x, x\}$) and let the voting proclivity z be given by

$$z = s + \omega(1 + \pi_1 - \pi_2) - c(\pi_1 - \pi_2) \tag{8}$$

for $c > 0$. In this case, $\pi_1 = \pi_2 = 0$ maximizes party i's vote share. However, for $c < (2\lambda - 1)x$, the equilibrium policies described in Theorem 7 remain an equilibrium. Hence, party 1 sacrifices votes for more polarization while party 2 gives up votes for a more accommodating policy.

6. Conclusion

We have described how aggregate uncertainty creates a strategic interaction between parties' redistricting decisions. This uncertainty ensures that one party's optimal action depends on the redistricting plan of the other even though the fraction of districts a party wins at any particular state s is a separable function of its own and its opponents redistricting plans. Despite the vital role aggregate uncertainty plays in our analysis, equilibrium strategies are independent of the distribution of this uncertainty. It follows that asymmetric information regarding this distribution will have no effect on equilibrium outcomes.

Our model provides a framework for analyzing the interaction between redistricting and other decisions. We have considered one such interaction by adding a policy choice stage to our model. Other decisions such as the allocation of campaign resources across districts or the policy choices of individual candidates who care only about outcome in their own district can also be studied within our framework.

7. Appendix

The following is an obvious consequence of the fact that L is strictly increasing.

Lemma 1A: *For $F, G \in \mathcal{F}$, both $D(F_s, G_s)$ and $\Delta(F_s, G_s)$ are continuous and strictly increasing functions of s.*

Define, for $y \in [-1/4, 0]$ and $z \in [0, 1/4]$,

$$f(y,z) = \begin{cases} \frac{L(z)-L(y)}{z-y} & \text{if } y \neq 0 \text{ or } z \neq 0 \\ L'(0) & \text{if } y = z = 0 \end{cases}$$

Recall that L is strictly concave on $[0, 1/4]$ and symmetric around 0. These two properties ensure that L is differentiable at 0, and therefore f is well-defined. Since L is continuous and differentiable at 0, f is also continuous. Furthermore, the strict concavity of L on $[0, 1/4]$ ensures that for every $y \in [-1/4, 0)$, there is a unique $z \in [0, 1/4]$ that maximizes $f(y, \cdot)$. For $y \in [-1/4, 0)$, let $\phi(y)$ be this maximizer and let $\phi(y) = 0$ for $y \in [0, 1/4]$. By the theorem of the maximum, ϕ is continuous on $[-1/4, 0)$. Below, we show that ϕ is nonincreasing and continuous on the entire interval $Z = [-1/4, 1/4]$.

For any $F \in \mathcal{F}$ let F_-^p be the distribution of the lower $p-$percentile

$$F_-^p(z) := \min\left\{\frac{F(z)}{p}, 1\right\}$$

Recall that F_+^p is the distribution of the upper $1-p$-percentile and therefore

$$F = (1-p)F_+^p + pF_-^p$$

Let $\underline{z}(F)$ denote the minimum of the support of F, $\bar{z}(F)$ denote the maximum of the support of F, and $F^-(y) = \lim_{t \to y^-} F(t)$. For $p \in [0,1]$, let

$$y(p,F) = \{y \in [\underline{z}(F), \bar{z}(F)] \,|\, F^-(y) \le p \le F(y)\}$$

Let $z(p,F) = m(F_+^p)$ and

$$W(p,F) = \{z(p,F) - \phi(y) \,|\, y \in y(p,F)\}$$

When the choice of F is clear, we drop it and write $y(p), z(p)$, and $W(p)$ instead.

A correspondence g from the reals to nonempty subsets of reals is increasing if $x \ge x'$, $w \in g(x)$, $w' \in g(x')$ implies $w \ge w'$ and is strictly increasing if the second inequality above is strict whenever the first one is strict.

Lemma 2A: *ϕ is (ia) decreasing and $\phi(y) < -y$ for $y < 0$ and (ib) continuous. (ii) W is increasing, y is increasing and both are upper-hemicontinuous (uhc).*

Proof: Part (ia) follows from elementary arguments. By (ia), $\lim_{y \to 0^-} \phi(y) = 0$. Hence, ϕ is continuous. Next, we prove (ii). That y is uhc and increasing is obvious. Note that $z(p)$ is the expectation of F conditional on a realization in the top $1-p-$percentile. Hence, it is a continuous and increasing function. Then, by (ib), the correspondence $-\phi(y(\cdot))$ is increasing, W is increasing and both are uhc. □

The next three lemmas characterize the seat maximizing redistricting plan as a function of the state. We do this from party 1's perspective. Below, we omit the reference to party 1, hence we drop the subscript for the party and a $p-$segregation plan is always a $p-$segregation plan for party 1. Define the following maximization problem:

$$\max D(\hat{F}) \quad \text{subject to } \hat{F} \succeq_2 F \tag{$*$}$$

Lemma 1: *For every $F \in \mathcal{F}$, the maximization problem $(*)$ has a unique solution. This solution $F^* = F^p$ for some $p \in [0,1]$ and $F^*(0) \leq p$.*

Proof: First, we note that the set $\{\hat{F} \in \mathcal{F} \,|\, \hat{F} \succeq_2 F\}$ is closed in the topology of weak convergence. Since $\mathcal{F}$ is compact, it follows that the constraint set of $(*)$ is compact. Since D is continuous, a solution exists. Next, we will show that this solution is unique and is a segregation plan.

Step 1: *Let g be an uhc correspondence from $[x, x']$ to nonempty, convex subsets of the reals. If there is $w \in g(x), w' \in g(x')$ such that $w \leq 0 \leq w'$ then there exists $x^* \in [x, x']$ such that $0 \in g(x^*)$.*

Proof: Follows from elementary arguments. □

Note that if $\bar{z}(F) \leq 0$, then the strict convexity of L on $[-1/4, 0]$ ensures that the unique solution to $(*)$ is F^1 and we are done. If $\underline{z}(F) \geq 0$, then the strict concavity of L on $[0, 1/4]$ ensures that the unique solution to $(*)$ is F^0. So, henceforth it is sufficient to consider F such that $\bar{z}(F) > 0 > \underline{z}(F)$.

Step 2: *(i) $z(p) > 0$ for all p such that $0 \in y(p)$. (ii) Either $0 \in W(p^*)$ for some $p^* \in [0,1]$ or $z(0) > \phi(\underline{z}(F))$ and not both. (iii) If the p^* in (ii) exists, it is unique and $z(p^*) = \phi(y)$ implies $y < 0$.*

Proof: Part (i) is immediate since $\bar{z}(F) > 0$. If $z(0) > \phi(\underline{z}(F))$, then $\min W(0) > 0$ and since W is increasing, $w > 0$ for all w, p such that $w \in W(p)$. If $z(0) \leq \phi(\underline{z}(F))$, we have $w \leq 0$ for some $w \in W(0)$. Then, choose $\hat{p}$ such that $0 \in y(\hat{p})$. Since $\phi(0) = 0$ it follows that $z(\hat{p}) - \phi(0) > 0$ by (i) and therefore $\max W(\hat{p}) > 0$. Then, by Lemma 1A, there exists p^* such that $0 \in W(p^*)$. This proves (ii). That p^* is unique is immediate. Since $z(p)$ is increasing and $\phi(y)$ is decreasing, $z(\hat{p}) > 0 = \phi(0)$, part (iii) follows. □

For any F, let $p_F = 0$ if $\min W(0) > 0$ and $p_F = p^*$ (as defined in Step 2) otherwise. Similarly, let $y_F = \underline{z}(F)$ if $\min W(0) > 0$ and $y_F = \min\{y \in y(p^*) | z(p^*) - \phi(y) = 0\}$ otherwise.

Step 3: *F^{p_F} is the unique optimal redistricting plan.*

Proof: Verifying that $F^{p_F} \succeq_2 F$ is straightforward. Define,

$$L^*(z) = \frac{L(z(p_F)) - L(y_F)}{z(p_F) - y_F}(z - y_F) + L(y_F)$$

Hence, L^* is the line that runs through both $(y_F, L(y_F))$ and $(z(p_F), L(z(p_F))$. Note that

$$L(z) \begin{cases} > L^*(z) & \text{whenever } z < y_F \\ = L^*(z) & \text{if } z \in \{y_F, z(p_F)\} \\ < L^*(z) & \text{otherwise} \end{cases} \tag{1}$$

For any cdf $\hat{F}$, let $B(\hat{F})$ denote the correspondence

$$B(\hat{F})(z) = [\hat{F}^-(z), \hat{F}(z)]$$

Clearly, $B(\hat{F})$ is uhc. Consider any optimal F^*. First, we show that for any $z < y_F \leq 0$, $F^*(z) \leq F^{p_F}(z) = F(z)$. To see this suppose $F^*(z_2) > F^{p_F}(z_2)$ for some $z_2 < y_F$. Since $F^* \succeq_2 F$ it follows that $z_2 \neq -1/4$ and there exists $z_1 < z_2$ such that $F^*(z_1) < F^{p_F}(z_1)$. Hence, by Lemma 1A, there exist w such that $\emptyset \neq B(F^*)(w) \cap B(F^{p_F})(w)$; that is, there exist w such that $F(w) \geq [F^{p_F}]^-(w)$ and $F^{p_F}(w) \geq [F^*]^-(w)$. Choose any p such that $p \in B(F^*)(w) \cap B(F^{p_F})(w)$. Clearly, $z_1 < w < z_2$, $0 \neq p \neq 1$, $F_-^{*p} \neq F_-^p$ and $F_-^{*p} \succeq_2 F_-^p$. Then, since L is strictly convex on $[-1/4, y_F]$, we have $D(pF_-^p + (1-p)F_+^{*p}) = pD(F_-^p) + (1-p)D(F_+^{*p}) > pD(F_-^{*p}) + (1-p)D(F_+^{*p}) = D(F^*)$, contradicting the optimality of F^*.

Then, since $F^* \succeq_2 F$, we have $F^*(z) = F(z) = F^{p_F}(z)$ for all $z < y_F$. Next, note that

$$\begin{aligned} D(F^{p_F}) &= \int_{z<y_F} L(z)dF^{p_F}(z) + \int_{z\geq y_F} L(z)dF^{p_F}(z) \\ &= \int_{z<y_F} L(z)dF^{p_F}(z) + \int_{z\geq y_F} L^*(z)dF^{p_F}(z) \\ &= \int_{z<y_F} L(z)dF^{p_F}(z) + \int_{z\geq y_F} L^*(z)dF^{p_F}(z) + \int_Z L^*(z)d[F^* - F^{p_F}](z) \end{aligned}$$

The last equality follows from the fact that L^* is linear and $m(F^*) = m(F^{p_F})$. Hence, we have

$$D(F^{p_F}) = \int_{z<y_F} L(z)dF^{p_F}(z) + \int_{z<y_F} L^*(z)d[F^* - F^{p_F}](z) + \int_{z\geq y_F} L^*(z)dF^*(z)$$
$$\geq \int_{z<y_F} L(z)dF^*(z) + \int_{z\geq y_F} L(z)dF^*(z) = D(F^*)$$

Moreover, unless F^* assigns 0 probability 1 to $(y_F, z(p_F)) \cup (z(p_F), 1/4)$, the inequality above is strict, contradicting the optimality of F^*. Hence, $F^* = F^{p_F}$.

To conclude the proof of the lemma, note that since $y_F \in y(p_F)$, $F^{p_F}(y_F) = p_F$ and since $z(p_F) > 0$, $F^{p_F}(0) = F^{p_F}(y_F) \leq p_F$ as desired. $\square$

By Lemma 1, there exist a function $\alpha : Z \to [0,1]$ such that $F_s^{\alpha(s)}$ is the unique solution to

$$\max D(\hat{F}) \quad \text{subject to } \hat{F} \succeq_2 F_s$$

and $F_s^{\alpha(s)}(0) \leq \alpha(s)$. The following lemma shows that α is decreasing; that is, the stronger a party is, the more it segregates.

Lemma 2: *The function α is strictly decreasing.*

Proof: In Lemma 1, we showed that $\alpha(s) = p_{F_s}$, $y_{F_s} \in y(p_{F_s}, F_s)$ and

$$z(p_{F_s}, F_s) - \phi(y_{F_s}) = 0$$

If $\hat{s} > s$, then

$$z(p_{F_s}, F_{\hat{s}}) = z(p_{F_s}, F_s) + \hat{s} - s > z(p_{F_s}, F_s)$$

Similarly, $y_{F_s} + \hat{s} - s \in y(p_{F_s}, F_{\hat{s}})$ and $\phi(y_{F_s} + \hat{s} - s) \leq \phi(y_{F_s})$ since ϕ is non-increasing and $\hat{s} > s$. Hence,

$$z(p_{F_s}, F_{\hat{s}}) - \phi(y_{F_s} + \hat{s} - s) > 0$$

Then, since W is increasing $y_{F_{\hat{s}}} \in y(p_{F_{\hat{s}}}, F_{\hat{s}})$, and

$$z(p_{F_{\hat{s}}}, F_{\hat{s}}) - \phi(y, F_{\hat{s}}) = 0$$

implies $\alpha(s) = p_F > \hat{p}_F = \alpha(\hat{s})$. $\square$

Lemma 3: *If $\hat{q} < q < \alpha(s)$, then $D(F_s^{\hat{q}}) \leq D(F_s^q)$.*

Proof: Let $F = F_s$. Let $\hat{q} < q \leq p$. Let $y \in y(q, F)$ and let $\hat{y} \in y(\hat{q}, F)$. Furthermore, define

$$L^*(z) = \frac{L(z(q,F)) - L(y)}{z(q,F) - y}(z - y) + L(y)$$

Hence, L^* is the line that runs through both $(y, L(y))$ and $(z(y,F), L(z(y,F)))$. Note that

$$L(z) \begin{cases} > L^*(z) & \text{whenever } z < y \\ = L^*(z) & \text{if } z \in \{y, z(q,F)\} \\ < L^*(z) & \text{if } z \in (y, z(q,F)) \end{cases} \tag{1}$$

since $q \leq p$.

Hence, we have

$$\begin{aligned} D(F^q) &= \int_{z<y} L(z) dF^q(z) + \int_{z \geq y} L^*(z) dF^q(z) + \int_Z L^*(z) d[F^{\hat{q}} - F^q](z) \\ &= \int_{z<y} L(z) dF^q(z) + \int_{z<y} L^*(z) d[F^{\hat{q}} - F^q](z) + \int_{z \geq y} L^*(z) dF^{\hat{q}}(z) \\ &\geq \int_{z<y} L(z) dF^q(z) + \int_{z<y} L(z) d[F^{\hat{q}} - F^q](z) + \int_{z \geq y} L(z) dF^{\hat{q}}(z) \\ &= \int_{z<y} L(z) dF^{\hat{q}}(z) + \int_{z \geq y} L(z) dF^{\hat{q}}(z) = D(F^{\hat{q}}) \end{aligned}$$

Note that the inequality above uses the fact that $F^{\hat{q}} - F^q \leq 0$ for all $z < y$ and $F^{\hat{q}}(z(\hat{q}, F)) = 1$ and $y \leq z(\hat{q}, F) \leq z(q, F)$. $\square$

Symmetric arguments establish that there exist a function β such that $\bar{G}_s^{\beta(s)} := \rho[(\rho(G))^{\beta(s)}]$ minimizes $D(H)$ among all $H \succeq_2 G$, $[\rho(G)]_s^{\beta(s)}(0) \leq \beta(s)$, β is strictly increasing and $D(\bar{G}_s^{\beta(s)}) \leq D(\bar{G}_s^q) \leq D(\bar{G}_s^{\hat{q}})$ whenever $\hat{q} < q \leq \beta(s)$. Henceforth, we will also refer these symmetric statements as Lemmas 1, 2, and 3.

7.1 Proofs of Theorems 1, 2, and 3

Proof of Theorem 1:

By the theorem of the maximum α, β and $D(F_s^{\alpha(s)}), D(\bar{G}^{\beta(s)})$ are continuous functions of s and hence so is $\Delta(F_s^{\alpha(s)}, G_s^{\beta(s)})$. Note that $\Delta(F_s^{\alpha(s)}, G_s^{\beta(s)}) \leq 1/2$ at $s = -1/8$ and

$\Delta(F_s^{\alpha(s)}, G_s^{\beta(s)})) \geq 1/2$ at $s = 1/8$. Hence, there exists s^* such that $\Delta(F_{s^*}^{\alpha(s^*)}, G_{s^*}^{\beta(s^*)}) = 1/2$. To complete the proof we will show that $(F_{s^*}^{\alpha(s^*)}, G_{s^*}^{\beta(s^*)})$ is the unique equilibrium.

Since $\Delta(H_s, \hat{H}_s)$ is strictly increasing in s for all $H, \hat{H}$ (Lemma 1A), party 1's payoff is greater than $\Pr\{s > s^*\}$ if and only if $\Delta(H_s, \hat{H}_s) \geq 1/2$. The strategy $F_{s^*}^{\alpha(s^*)}$ is the unique strategy that ensures $\Delta(F_{s^*}^{\alpha(s^*)}, \hat{H}) \geq 1/2$ for all $\hat{H}$. But since this is a zero sum game, it follows that $F_{s^*}^{\alpha(s^*)}$ is the unique equilibrium strategy for party 1. Symmetric arguments establish that $G_{s^*}^{\beta(s^*)}$ is the unique equilibrium strategy for party 2. □

Proof of Theorem 2: The proof follows immediately from Theorem 1 and Lemma 2. □

Proof of Theorem 3: The proof of part (i) was presented in the discussion following the statement of Theorem 3.

Part (ii): Let $s = s(\Lambda)$, $s_1 = s_1(\Lambda)$, $\hat{s} = s(\hat{\Lambda})$, and $\hat{s}_1 = s_1(\hat{\Lambda})$. Part (i) implies $\hat{s} \geq s \geq s_1$ and hence, Lemma 2 implies $\alpha(\hat{s}) \leq \alpha(s) \leq \alpha(s_1)$. Hence, Lemma 3 implies

$$D(F_{s_1}^{\alpha(\hat{s})}) \leq D(F_{s_1}^{\alpha(s)}) \leq D(F_{s_1}^{\alpha(s_1)})$$

By definition, $D(F_{s_1}^{\alpha(s)}) = 1/2$ and therefore $D(F_{s_1}^{\alpha(\hat{s})}) \leq 1/2 = D(F_{\hat{s}_1}^{\alpha(\hat{s})})$. Then, $\hat{s}_1 \geq s_1$ as required.

Part (iii): Since $s_1 \leq s \leq s_2$, party i's seat share at s in territory i is weakly greater than $1/2$. This implies that the critical state $s(F, G, \lambda)$ is weakly decreasing in λ. □

7.2 Proof of Corollary 2

First, we establish that for large n the critical state s^n satisfies $s^n \leq -\epsilon$ for some $\epsilon > 0$. To see this, let $p = 1 - 1/(2\lambda) - \eta$ be the redistricting plan for party 1. Note that $p > 0$ for small $\eta > 0$ and therefore $z(H, p) > 0$. Let $s = -z(H, p)/2$ and note that $L^n(s + z(H, p)) \to 1$ and therefore at s party 1's seat share converges to a number no less than $(1-p)\lambda = [1/(2\lambda) + \eta]\lambda > 1/2$ and hence for n large, the critical state must be smaller than s.

Elementary arguments show that $\phi(y) \to 0$ as $n \to \infty$ for any $y \in [-1/4, 0]$. Therefore, $s < 0$ and n large imply that $q^n = 0$ for large n.

Since $s < 0$ party 2's seat share in territory 2 converges to 1 (in the critical state). Hence, party 1 must win a seat share of $1/(2\lambda)$ in the critical state in territory 1. Part (i) of the corollary now follows from the fact that local uncertainty disappears as $n \to \infty$.

For part (ii) a straightforward calculation shows that $s(\Lambda) \geq \frac{2\lambda-1}{2\lambda}\frac{1}{8}$. (This bound is achieved for the 2 point distribution of the example of Section 2.1). $\square$

7.3 Proof of Theorems 4-7

Proof of Theorem 4: Any strategy feasible at $\hat{\Lambda} = (\hat{H}, 1, L)$ is feasible at $\Lambda = (H, 1, L)$. Since the games $\Lambda = (H, 1, L)$ and $\hat{\Lambda} = (\hat{H}, 1, L)$ do not yield the same equilibrium payoff, we have $s(H, 1, L) < s(\hat{H}, 1, L)$. The Theorem then follows since $s(H, \cdot, L)$ is continuous. The proof that $s(H, \cdot, L)$ is continuous is straightforward and therefore omitted. $\square$

Lemma 3A: *Let (p, q) be an equilibrium of the redistricting game $\Lambda = (F, G, \lambda, L)$. Then, $p < 1/2, q < 1/2$.*

Proof: Let $s = s(\Lambda)$ be the critical state. Note that

$$D(F_s^p) \leq pL(y(p, F_s)) + (1-p)L(z(p, F_s)) < pL(y(p, F_s)) + (1-p)L(-y(p, F_s))$$

where the last inequality follows from Lemma 2A part (ia). For $p \geq 1/2$ the symmetry of L therefore implies $D(F_s^p) < 1/2$. But this contradicts Theorem 3 which shows that $s_1(\Lambda) \leq s(\Lambda)$ and hence $D(F_s^p) \geq 1/2$. $\square$

Proof of Theorem 5: By Lemma 3A, the optimal strategies satisfy $p, q \leq 1/2$ and therefore $F^{1/2}, \bar{F}^{1/2}$ are mean preserving spreads the parties' optimal strategies.

Consider the strategy $[\rho(F)]^{1/2}$. Since party 2's supporters are easier to segregate at F it follows that $[\rho(F)]^{1/2}$ is a feasible strategy for party 1 and, since $[\rho(F)]^{1/2}$ is a mean preserving spread of $[\rho(F)]^q$ it follows that $[\rho(F)]^q$ is a feasible strategy for party 1. Clearly, if party 1 chooses $[\rho(F)]^q$ and party 2 chooses $\bar{F}^q = \rho([\rho(F)]^q)$ then $s(\Lambda) = 0$ since the strategies are identical. We conclude that $s(F, \lambda) \leq 0$. $\square$

Proof of Theorem 6: Assume $\lambda = 1$. Let $p < 1/2$ denote the optimal redistricting plan for H (assuming $\pi_1 = \pi_2$). First, we show that the unique continuation equilibrium is $\pi_1 = \pi_2 = 1$.

Note $p = 0$ implies $s(H, 1) = 0$. It is easy to see that at $s = 0$ party 1 can guarantee a majority (since H is non-degenerate, this follows from the symmetry of L and the strict convexity of L on $[-1/4, 0]$.) Therefore, $p = 0$ cannot be optimal and hence $p \in (0, 1/2)$ and $s < 0$.

Define

$$\delta := \pi_1 - \pi_2 \geq 0$$

Let $y = \min\{y' \in y(p, H) | z(p, H) - \phi(y) = 0\}$ and let $s = s(H, 1)$ be the critical state; we define

$$\begin{aligned} I(\delta) :=& \int [L(s + \omega(1 + \delta)) - L(s + \omega)]\, dH^p \\ =& \int_{\omega \leq y} [L(s + \omega(1 + \delta)) - L(s + \omega)]\, dH^p \\ &+ (1 - p)\, [L(s + z(p, H)(1 + \delta)) - L(s + z(p, H))] \end{aligned}$$

Since $s < 0$ and $p \leq 1/2$ it follows that $y(p, H) < 0$ and therefore the convexity of L implies for $\omega \leq y$

$$L(s + \omega(1 + \delta)) - L(s + \omega) \geq L(s + y + \delta\omega) - L(s + y)$$

Using Jensen's inequality we conclude that

$$\begin{aligned} I(\delta) \geq& p[L(s + y + \delta m(H^p_-)) - L(s + y()] \\ &+ (1 - p)[L(s + z(p, H)(1 + \delta)) - L(s + z(p, H))] \end{aligned}$$

Note that $pm(H^p_-) + (1 - p)m(H^p_+) = pm(H^p_-) + (1 - p)z(p, F) = 0$, and $0 < z(p, H) < -m(H)$ (Lemma 2A, part (ia) and the fact that $m(H^p_-) \leq y(p, H)$). Therefore, since L is symmetric, concave on $[0, 1/4]$ and since $p < 1/2$ we have

$$\begin{aligned} (1 - p)[L(s + z(p, H)(1 + \delta)) -& L(s + z(p, H))] \\ &> p[L(s + y) - L(s + y + \delta m(H^p_-))] \end{aligned}$$

and therefore $I(\delta) > 0$ for $\delta > 0$ and (by a symmetric argument) $I(\delta) < 0$ for $\delta < 0$. Hence, party 1 seeks to maximize δ and party 2 seeks to minimize δ. We conclude that $\pi_1 = \pi_2 = 1$ is the unique continuation equilibrium.

Next, we show that the $p-$segregation plan is the unique equilibrium redistricting plan in the first stage. Fix $\pi_2 = 1$ and note that (since the game is zero sum) this yields an upper bound to the gains from a possible deviation by party 1. Clearly, if $\pi_1 = 1$ then the deviation cannot be profitable since p is the unique optimal redistricting plan given $\pi_1 = \pi_2 = 1$. Assume $\pi_1 < 1$ and note that the distribution of ω is a mean preserving spread of the distribution of $\omega(1 + d)$ for $d < 0$. Therefore, we may apply Theorem 4 to show that party 1's payoff must decrease when choosing $\pi_1 < 1$.

Hence, we have established that the only Nash equilibrium outcome for $\lambda = 1$ is the one described in the theorem. A straightforward continuity argument yields the same conclusion for λ close to one.

It remains to establish the existence of a subgame perfect Nash equilibrium. In particular, we must show that for every redistricting plan (F, G) there exists an equilibrium in the policy game (in mixed strategies). Note that a party's seat share is a continuous function of the policy and hence we can use a standard argument to show existence of equilibrium in mixed strategies. $\square$

Proof of Theorem 7: Assume party 1 chooses $\pi_1 = 1$ and p^n as in the equilibrium of Corollary 2. Then, party 1 wins for all $s > s^n$ where $s^n < -\epsilon$ for some $\epsilon > 0$. This follows by an argument identical to the one given in the proof of Corollary 2. We conclude that $s(\Pi^n) < -\epsilon$.

Since $s(\Pi^n) < -\epsilon$ the optimal redistricting plan for party 2 is a uniform redistricting plan $q^n = 0$ when n is large. The argument is identical to the one given for the same result in Corollary 2. This implies that the policy choice has no effect on seat share in territory 2. Therefore, we can apply the argument of the proof of Theorem 6 to show that $\pi_1 = \pi_2 = 1$ is the only equilibrium policy choice. The remainder of the Theorem follows from Corollary 2. $\square$

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