Geodesic acoustic mode oscillation in the low frequency rangea)
T. Watari, Y. Hamada, T. Notake, N. Takeuchi, and K. Itoh

Citation: Physics of Plasmas (1994-present) 13, 062504 (2006); doi: 10.1063/1.2206170
View online: http://dx.doi.org/10.1063/1.2206170
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/13/6?ver=pdfcov
Published by the AIP Publishing
I. INTRODUCTION

Since plasma drifts with the velocity $\vec{E} \times \vec{B}$, the rotation speed of the plasma is related to the radial electric field $E_r$. The rotating plasma is provided with a restoring force by the geodesic curvature of the magnetic lines of forces, which exist intrinsically in tori. Therefore, the plasmas are subject to oscillations and many questions have arisen: 1 The asymmetry of the potential field of GAMs, which is enhanced by the coupling with ion acoustic modes. It may affect GAMs in plasmas with electron temperatures higher than those of the ions. (2) The possible existence of GAMs in the lower frequency range: This is discussed in connection with the uniqueness of the kinetic response of the plasma to an external field associated with the geodesic curvature of the magnetic lines of force. (3) The extension of the theory to cover both tokamaks and helical systems: Differences between the helical-type and the tokamak-type GAMs are discussed in terms of their differences in connection length. In a device of mixed helicity, helical natured GAMs are predicted to appear depending on the intensity of the corresponding geodesic curvature and electron temperature. © 2006 American Institute of Physics. [DOI: 10.1063/1.2206170]
(\omega^2 < \omega_{GAM}^2) in search of a new branch of GAM. If there is such a mode with a high quality factor, it will effectively regulate the drift wave turbulence in the low frequency range.\textsuperscript{15} Due to the complexity of the response of the plasma in kinetic analyses, there is a possibility that a GAM-like mode is found around the frequency

\[ \omega \sim k_B T_e = v_T \beta R_q = \sqrt{2 T_e/m_e R_q}, \]

(2)

lower than \omega_{GAM} by a factor of \(1/\beta\). It will fall in the low frequency edge of the turbulence, where drift waves condensate due to inverse cascade processes and are finally dissipated. Therefore, the properties of this mode are important in determining the spectrum of the turbulence and thus in determining transport, even if the mode has a low quality factor.

This paper is organized as follows: In Sec. II, the dispersion relation of the GAM oscillation is formulated, clarifying in Sec. II A the approximation used, giving special attention in Sec. II B to the roles of the nonuniform field, and giving in Sec. II C the dispersion relation of the GAM on which the revised theory is applied to tokamaks. Deriving in Sec. II D, the applicability range of results obtained in this paper is discussed, and the problems to be solved are suggested. Conclusions are presented in Sec. VII.

II. BASIC EQUATIONS

A. Formulation and clarification of assumptions

The extension of GAM theory to helical systems has been addressed in our previous paper,\textsuperscript{32} where it was found that the helical geodesic curvature of the magnetic field is the cause of the geodesic acoustic mode. Therefore, the flux coordinate has been introduced with poloidal and toroidal coordinates \((\theta, \zeta)\) on which a Fourier expansion of the magnetic field intensity is made:

\[ B = B_0 \left( 1 + \sum_{m,n} \delta_{m,n}(\psi) \cos(m\theta - n\zeta) \right), \]

where \(\psi\) is the poloidal flux used as the flux surface label and \(m, n\) are the poloidal and toroidal mode numbers of the confining magnetic field. The drift kinetic equation is solved with a drift velocity of the particles \(\vec{v}_D\) expressed in terms of \(\delta_{m,n}(\psi)\); the latter is associated with the curvature of the magnetic lines of force:

\[ \frac{\partial f_1}{\partial t} + (\vec{u} + \vec{v}_D) \cdot \nabla f_1 = \left\{ e(\vec{u} + \vec{v}_D) \cdot \nabla f_1(\zeta) \right\} \frac{\partial f_0}{\partial \phi} = 0. \quad (4) \]

Electrons and ions are not distinguished in Sec. II A, because their motions are subject to the same equation. Based on the straight magnetic field line coordinate \( (\theta, \zeta, \phi) \), \(\vec{v}_D\) in Eq. (4) is written as follows:

\[ \vec{v}_D \cdot \nabla \psi = 2T_e \left( \frac{m \vec{u}^2 + \frac{1}{2} m \vec{v}_D^2}{2T} \right) \alpha_{m,n} \sin(m\theta - n\zeta) \]

\[ = v_T \left( \frac{\vec{v}_D}{\omega_{e,0}} \right) \gamma \alpha_{m,n} \sin(m\theta - n\zeta), \quad (5) \]

with

\[ \alpha_{m,n} = \alpha_{m,n}(\psi) = \frac{1}{2} \sqrt{g B^2} \]

\[ = 2 \pi \frac{(m \vec{u}^2 + n \vec{v}_D^2)}{q B \vec{u} + B \vec{v}_D} B_0 \delta_{m,n}(\psi) \]

\[ = 2 \pi \frac{(m \vec{u}^2 + n \vec{v}_D^2)}{2T} \gamma \]

\[ = \left( \frac{m \vec{u}^2 + \frac{1}{2} m \vec{v}_D^2}{2T} \right). \quad (6) \]

Here, \( B_{\zeta} \) and \( B_\theta \) are the covariant components of the toroidal and poloidal magnetic field, which are flux functions in Boozer coordinates. \( \sqrt{g} \) is the Jacobean. We specify the coordinate to be Boozer coordinates \( (\sqrt{g} \propto B^2) \), so that

\[ \alpha_{m,n} = \alpha_{m,n}(\psi) \]

\[ \text{is a flux function. We separate the distribution functions into two parts and seek their solutions: } f_1(U \rightarrow), \]

\[ \text{the perturbation caused by the uniform potential field } \langle \phi \rangle \]

\[ \text{and } f_1(NU \rightarrow), \]

\[ \text{the part caused by the nonuniform potential field } (\phi \sim \langle \phi \rangle) \]

\[ \langle A \rangle \]

\[ \text{is the flux surface average of the quantity } A: \]

\[ \frac{\partial f_1(U \rightarrow)}{\partial t} + (\vec{u} + \vec{v}_D) \cdot \nabla f_1(U \rightarrow) = \left\{ e(\vec{u} + \vec{v}_D) \cdot \nabla f_1(\zeta) \right\} \frac{\partial f_0}{\partial \phi} \]

\[ \text{and} \]

\[ \frac{\partial f_1(NU \rightarrow)}{\partial t} + (\vec{u} + \vec{v}_D) \cdot \nabla f_1(NU \rightarrow) \]

\[ = \left\{ e(\vec{u} + \vec{v}_D) \cdot \nabla f_1(\zeta) \right\} \frac{\partial f_0}{\partial \phi}. \quad (8) \]

Here we introduce \( \phi, \phi_0, \text{ and } \phi_0 \) by

\[ (\phi \sim \langle \phi \rangle) = (\phi_0 \sin \theta + \phi_0 \cos \theta) \exp(-i\omega t + ik_q \delta_0), \]

\[ \langle \phi \rangle = \langle \phi \rangle_0 \exp(-i\omega t + ik_q \delta_0). \quad (10) \]

Similarly, we decompose the caused nonuniform distribution function \( f(NU) \) defining \( \vec{f}_d(NU) \) and \( \vec{f}_d(NU \rightarrow) \) in the following equation:


\[ f(\nu U) = [(\tilde{f}_s(\nu U) \sin \theta) + (\tilde{f}_c(\nu U) \cos \theta)] \times \exp(-i\omega t + ik_\theta \phi). \]  

(11)

Equations (8) and (9) have the operators \((\tilde{\nu}_h + \tilde{\nu}_p) \cdot \nabla f_1\) on the left-hand side. These terms are known to create harmonics due to the spatial variation of both \(\tilde{\nu}_h\) and \(\tilde{\nu}_p\). Since they have single-parameter dependences on time \(t\) through the spatial dependence on \((\theta, \zeta)\), their variations have to be considered simultaneously. A more detailed consideration of this problem is introduced in Ref. 33. For this paper, however, the constancy of \(\tilde{\nu}_h\) is assumed and \(\tilde{\nu}_p\) is ignored with respect to the former. The equation is then simplified with the use of the following transformations:

\[ \tilde{\nu}_h \cdot \nabla \cos(\theta - n\zeta) = -k_{l,m,n} \nu_l \sin(\theta - n\zeta), \]  

(12)

\[ \tilde{\nu}_l \cdot \nabla \sin(\theta - n\zeta) = k_{l,m,n} \nu_l \cos(\theta - n\zeta), \]  

(13)

where

\[ k_{l,m,n} = (mB^\theta - nB^\phi)/B \]

with \(B^\theta\) and \(B^\phi\) the contravariant components of the magnetic field. We also assume that \(k_{l,m,n}\) is independent of \((\theta, \zeta)\), though it is correct only to the first order in \(\delta_{m,n}\) on the Boozer coordinates.

Using these assumptions, we obtain

\[ \left( \begin{array}{c} -i\omega & -k_{l,m,n} \nu_l \\ -\omega & \omega \end{array} \right) \tilde{f}_s(U_{\nu U})^{\nu} = \beta_{m,n} \tilde{f}_s(U_{\nu U})^{0 \nu}, \]  

where

\[ \beta_{m,n} = \nu_l \frac{\nu_p}{\omega \nu_c \nu_R} \alpha_{m,n} \gamma, \]  

(14)

with \(\nu_R = \sqrt{2T_m/m}\). The arrow in the bracket \((U_{\nu U})\) indicates that the induced nonuniform perturbation is due to a uniform potential field. Solutions are obtained to the first order in \((\nu_p/\nu_c, 0, k_{l,m,n})\) taking the following forms:

\[ \tilde{f}_s(U_{\nu U}) = \frac{1}{2T_m} \nu_l \frac{\nu_p}{\omega \nu_c \nu_R} \alpha_{m,n} \gamma \left( -i\omega + k_{l,m,n} \nu_l \right) \tilde{f}_s(U_{\nu U})^{0 \nu}, \]  

(15)

\[ \tilde{f}_c(U_{\nu U}) = i(k_{l,m,n} \nu_l/\omega) \tilde{f}_c(U_{\nu U})^{\nu}. \]  

Similarly, the nonuniform distribution functions \(\tilde{f}_c(U_{\nu U})\) and the distribution functions due to the nonuniform potential field \(\tilde{f}_c(U_{\nu U})\) are obtained from the following equations, which are derived from Eq. (9):  

\[ \left( \begin{array}{c} -i\omega & -k_{l,m,n} \nu_l \\ -\omega & \omega \end{array} \right) \tilde{f}_c(U_{\nu U})^{\nu} = -\frac{e}{T} \tilde{f}_c(U_{\nu U})^{0 \nu}. \]  

(16)

The solution is obtained in the following way:

\[ \tilde{f}_s(U_{\nu U}) = -e \frac{\tilde{f}_s(U_{\nu U})^{\nu}}{2T_m} \left( \omega \right) \left( \omega \right) \left( \omega \right) \left( \omega \right), \]  

(17)

and

\[ \tilde{f}_c(U_{\nu U}) = -\frac{e}{T} \tilde{f}_c(U_{\nu U})^{0 \nu} \left( \omega \right) \left( \omega \right) \left( \omega \right) \left( \omega \right). \]  

(18)

B. Determination of the nonuniform field

In determining the nonuniform field, we use quasi-neutrality. This scheme of calculation has been explored by Lebedev et al.34 and was extended to helical systems in a recent work by Sugama et al.15 The sine and cosine components of the density perturbation due to the nonuniform field are obtained by integrating Eqs. (17) and (18):

\[ 4\pi \nu_m \nu_p \left( z_{m,n} \right) = -k_{l,m,n} \nu_l \frac{\nu_p}{\omega \nu_c \nu_R} \alpha_{m,n} \gamma \left( -i\omega + k_{l,m,n} \nu_l \right) \tilde{f}_c(U_{\nu U})^{0 \nu} \times \cos(\theta - n\zeta), \]  

(19)

\[ 4\pi \nu_m \nu_p \left( z_{m,n} \right) = -k_{l,m,n} \nu_l \frac{\nu_p}{\omega \nu_c \nu_R} \alpha_{m,n} \gamma \left( -i\omega + k_{l,m,n} \nu_l \right) \tilde{f}_c(U_{\nu U})^{0 \nu} \times \sin(\theta - n\zeta), \]  

(20)

where \(z_{p}(\xi_{m,n})\) is the ordinary plasma dispersion function, which is defined by

\[ Z_{p}(\xi_{m,n}) = \pi^{-1/2} \int_{-\infty}^{\infty} dx \frac{1}{x - \xi} \exp(-x^2). \]  

(21)

The nonuniform density perturbation caused by the uniform field is obtained by integrating Eq. (15) over velocity space:

\[ 4\pi \nu_m \nu_p \left( z_{m,n} \right) = -k_{l,m,n} \nu_l \frac{\nu_p}{\omega \nu_c \nu_R} \alpha_{m,n} \gamma \left( -i\omega + k_{l,m,n} \nu_l \right) \tilde{f}_c(U_{\nu U})^{0 \nu} \times \cos(\theta - n\zeta), \]  

(22)

Here,
\[ Z_2(\xi) = \pi^{-1/2} \int dx \frac{\left( x^2 + \frac{1}{2} \right)}{x - \xi} \exp(-x^2). \]  

(23)

Summing up the electric charges from electrons and ions induced by the uniform and nonuniform fields and imposing charge neutrality, we obtain

\[ \sum_{i,e} 4 \pi n_i (NU \rightarrow NU) + \sum_{i,e} 4 \pi n_i (U \rightarrow NU) \]

\[ = \left( -k_{D,e}^2 + k_{D,i}^2 \right) \left[ 1 + \xi_{m,n,Z} Z_p(\xi_{m,n}) \right] \bar{\phi}_s + \sum_{i,m,n} \frac{1}{k_{i,m,n} v_T,i} \alpha_{m,n}(k_\phi) Z_2(\xi_{m,n}) \phi_0 \times \sin(m \theta - n \xi) = 0. \]

(24)

Here, the subscript \((i,e)\) has been introduced to designate the contributing species; so far the expressions have been applicable both to electrons and ions. The adiabatic approximation, \(k_{D,e}^2 \left[ 1 + \xi_{m,n,Z} Z_p(\xi_{m,n}) \right] \to k_{D,e}^2 \), has been made in the calculation of the electron density perturbation due to the nonuniform field. Similarly, only the ion contribution is taken into consideration in calculating the density perturbation due to the uniform field; \(Z_2(\xi_{m,n})\) is small due to the small value of \(\xi_{m,n}\). From Eq. (24), we obtain

\[ \bar{\phi}_s = v_T,i \left( k_\phi \frac{v_T}{\omega_{e,i}} \right) \frac{1}{k_{i,m,n} v_T,i} \alpha_{m,n} \chi_{x,n}^2 Z_2(\xi_{m,n}) \bar{\phi}_0. \]

(25)

Here, we have defined

\[ \chi(\xi_{m,n}) = \left( k_{D,e}^2 k_{D,i}^2 + [1 + \xi_{m,n,Z} Z_p(\xi_{m,n})]^{-1} \right) Z_2(\xi_{m,n}) \]

(26)

for uses in the discussions in the following sections. In similar calculations using \(\Sigma_{i,e} 4 \pi n_i (\bar{n}_i (U \rightarrow NU) + \bar{n}_i (U \rightarrow NU)) = 0\), we obtain simultaneously \(\bar{\phi}_s = 0\). It is noted that the only electron-related parameter in Eq. (26) is \(T_e / T_i = (k_{D,e} / k_{D,i})\) and all other quantities are those of ions.

C. Dispersion relation of the GAM

The radial current across the flux surface due to the uniform field, \(J(U)\), is obtained by integrating \(e \bar{v}_D \cdot \nabla \bar{\psi}_s(U \rightarrow NU)\) over the flux surface:

\[ J(U) = 4 \pi \int \bar{j}(NU \rightarrow \cdot) \cdot d\dot{S} = 4 \pi \int \bar{v}_D \cdot \nabla \bar{\psi}_s(U \rightarrow NU). \]

(27)

Substituting Eqs. (5)–(7), we obtain

\[ J(U) = -\frac{k_{D,i}^2}{2} \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \alpha_{m,n}^2 \phi_0 \]

\[ \times \int d^3 \nu_T y^2 \left( \frac{1}{\omega - (k_{i,m,n} u_i)} + \frac{1}{\omega + (k_{i,m,n} u_i)} \right) \]

\[ \times (f \delta n_0) \int \sin^2(m \theta - n \xi) \bar{g} d \theta d \xi \]

\[ = k_{D,i}^2 \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \alpha_{m,n}^2 k_{i,m,n} v_T \]

\[ \frac{1}{2} \int d^3 \nu_T y^2 \left( \frac{1}{\omega - (k_{i,m,n} u_i)} + \frac{1}{\omega + (k_{i,m,n} u_i)} \right) \]

\[ \times (f \delta n_0) \int \sin^2(m \theta - n \xi) \bar{g} d \theta d \xi \]

\[ \frac{1}{2} V'. \]

(28)

Here, we have defined \(Z_1(\xi)\) by

\[ Z_1(\xi) = \pi^{-1/2} \int dx \frac{\left( x^4 + (x^2 + \frac{1}{2}) \right)}{x - \xi} \exp(-x^2) \]

(29)

and made an approximation \(\int \sin^2(m \theta - n \xi) \bar{g} d \theta d \xi \sim \frac{1}{2} V'\) in the last transformation, \(\bar{f}(U \rightarrow NU)\) makes no contributions after integration over velocity space.

Similarly, the current across the flux surface caused by the nonuniform field is obtained as follows by use of Eq. (17):

\[ J(NU) = 4 \pi \int \bar{j}(NU \rightarrow \cdot) \cdot d\dot{S} = 4 \pi \int \bar{v}_D \cdot \nabla \bar{\psi}_s(NU \rightarrow NU) \]

\[ = k_{D,i}^2 \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \alpha_{m,n}^2 \phi_0 \]

\[ \times \int d^3 \nu_T \left( \frac{1}{\omega - (k_{i,m,n} u_i)} + \frac{1}{\omega + (k_{i,m,n} u_i)} \right) \]

\[ \times (f \delta n_0) \int \sin^2(m \theta - n \xi) \bar{g} d \theta d \xi \]

\[ = -k_{D,i}^2 \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \alpha_{m,n}^2 \phi_0 \]

\[ \times \frac{1}{2} V'. \]

(30)

Substituting \(\bar{\phi}_s\) given by Eq. (25) into Eq. (30), the sum of the radial currents \(J_{geo} = J(U) + J(NU)\) is given by

\[ J_{geo} = k_{D,i}^2 \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \alpha_{m,n}^2 k_{i,m,n} v_T \]

\[ \frac{1}{2} \phi_0 \frac{1}{2} V'. \]

(31)

The subscript geo indicates that the quantity is due to the geodesic curvature. Similarly, “geodesic current” shall mean current that is due to the geodesic curvature, distinguished from the classical polarization current. We use the following formula of classical polarization current:

\[ \bar{j}_p = ik_\phi \int \frac{\omega_{i,m,n}^2}{\omega^2} \frac{d \phi_0}{d \psi} d \psi \]

\[ = \frac{i k_{D,i}^2}{2} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \frac{1}{4 \pi} \frac{\omega d \phi_0}{d \psi} d \psi. \]

(32)

and integrate it over the flux surface to calculate the total poloidal current \(J_{pol}\):

\[ J_{pol} = 4 \pi \int \bar{j}_p \cdot \bar{d}S = \frac{\omega}{4 \pi} k_\phi \frac{v_T}{\omega_{e,i}} \int \bar{v}_D \cdot \nabla \bar{\psi}_s^2 \]

\[ = \frac{\omega}{4 \pi} k_{D,i}^2 \frac{1}{k_\phi} \left( k_\phi \frac{v_T}{\omega_{e,i}} \right)^2 \frac{1}{4 \pi} \frac{d \phi_0}{d \psi} d \psi. \]

(33)

One of the geometric factors \(f_\phi\) is defined here by
It is related to the scale factor of the $\psi$ coordinate. However, it may be approximated (for easy calculations) by the peripheral length of the poloidal cross section of the plasma. Through the discussion above, it was shown that all the mechanisms causing radial currents are proportional to $(k_{1,\psi} v_T / \omega_{1,\psi})^2$, indicating that they are a part of the lowest order finite Larmor radius effect. Therefore, only ion contributions are retained, so that the obvious subscript $i$ is saved; $T_e$ appears as a parameter to represent electrons only in $\chi_{m,n}$. The dispersion relation is obtained by balancing the classical polarization current with the geodesic currents:

$$D = 4 \pi J_{pol}(U) + 4 \pi J(U) + 4 \pi J(NU)$$

$$= \frac{1}{2} \sum_{m,n} \left( k_{1,\psi} v_T \frac{\omega_{1,\psi}}{\omega_{1,\psi}^2} \right)^2 \frac{1}{q^2} \int \frac{g d\theta d\xi}{q^2} \nabla \Phi^2 \sim 1 \frac{\omega_{1,\psi}}{q^2} \int \frac{g d\theta d\xi}{q^2} \nabla \Phi^2.$$

(34)

Equation (39) is solved in the dispersive sections for $\xi$, and $\omega$ is calculated by using the relation $\xi = k_{1,\psi} v_T$, i.e., $\xi$ and $\omega$ are identified.

We have introduced three kinds of dispersion functions $Z_p(\xi)$, $Z_{1}(\xi)$, and $Z_{1}(\xi)$ defined by Eqs. (21), (23), and (29), respectively. It is known that the latter two functions can be rewritten by using $Z_p(\xi)$ (see the Appendix for their relations and expressions in series and asymptotic expansion). These three functions are different in the multipliers, $\gamma^2$, $\gamma^1$, and $\gamma^2$, in the integrand. They reflect the multiplicity of the intervention of the geodesic curvature in one process. In Fig. 1, they are plotted versus the argument $\xi$ showing different behaviors in the low frequency range.

B. Characteristics of the GAM for two typical parameters: $\tau_d/T_i=1$ and $T_d/T_i>1$

The formulation in the previous section states that the geodesic current $J_{geo}$ consists of two parts: the part caused by the uniform field, $J(U) = q^2 Z_0(\xi)$, and the other part caused by the nonuniform field, $J(NU) = -q^2 \chi Z_p(\xi)$. Here, the common factor $i/\omega_{1,\psi} v_T$ in Eq. (39) has been dropped; in this framework the polarization current is expressed as $J_{pol}=\xi$. These two geodesic terms are calculated and shown in Figs. 2(a) and 2(b) assuming $T_d/T_i=1$, manifesting that they are different in the dependences on $\xi$ and are similar in amplitudes. Thus, the uniform and nonuniform fields make similar-sized contributions to the radial current in the standard case of $T_d/T_i=1$.

Figures 3(a) and 3(b) show how the GAM frequencies are determined for the two typical cases of $T_d/T_i=1$ and $T_d/T_i=5$, $q=2$ is assumed common to the two cases. The solid curves give the sum of the geodesic currents $J_{geo} = J(U) + J(NU)$ and the dotted curves give the imaginary parts. The straight lines give the classical polarization current $J_{pol}=\xi \chi$ displayed with its sign reversed: As formulated in the previous section, the GAM frequencies are obtained using the $\xi$ values where the curves and straight lines cross. In Figs. 3(a) and 3(b), GAM frequency $\xi = \xi_{GAM}$ is determined to be 3.5 and 5.5, respectively.

These GAM frequencies are interpreted as the ordinary GAM frequencies obtained readily from the MHD equations and classified in this paper, as the first group of solutions.

The second group of solutions is found in the low frequency range: $\xi=1.3$ in Fig. 3(a) and $\xi=2.1$ in Fig. 3(b). They suggest that there are two solutions in the GAM frequency range if there is one solution in the first group. These
solutions of the second group may join the candidates for one of the two frequencies of GAM-like oscillations reported in JIPP-T-IIU$^{19}$ and ASDEX U$^{25}$ experimental results. It is noted that there is always a solution around $\zeta \sim 0$, which may be identified as a stationary zonal flow and classified in this paper as the third group of solutions.

To conclude this section, there are two GAM-like frequencies in addition to the stationary zonal flow for both cases of $T_e/T_i = 1$ and $T_e/T_i = 5$.

C. Interpretation of the calculation results

1. Ordinary GAM

Equation (39) is transformed as

$$\left[ \zeta + q^2 Z_1(\zeta) \right] \left[ \tau_e^{-1} + \left[ 1 + \zeta \cdot Z_p(\zeta) \right] \right] = q^2 \chi \zeta Z_2(\zeta),$$

where we have introduced for simplification a parameter $\tau_e = k_{D_e}^2 / k_{D_e}^2 = Z_2(T_e/T_i)$. If the term on the right-hand side $q^2 \chi \zeta Z_2(\zeta)$ is small, Eq. (41) gives the two decoupled dispersion relations

$$\left[ \zeta + q^2 Z_1(\zeta) \right] = 0$$

and

$$\left\{ \tau_e^{-1} + \left[ 1 + \zeta \cdot Z_p(\zeta) \right] \right\} = 0.$$  

Therefore, Eq. (41) is viewed as an equation for predicting the coupling of the two modes with $q^2 \chi \zeta Z_2(\zeta)$ regarded as a coupling constant. Equation (42) gives the standard kinetic GAM frequency treated in Ref. 29, and Eq. (43), which is the dispersion equation of electrostatic waves, has two solutions in the case of $\tau_e > 1$. The interpretation of results obtained in Sec. III B is simple as to the first group of solutions; we expand asymptotically the three terms composing Eq. (41) to obtain a dispersion relation

$$\left[ \zeta^2 - q^2 \left( \frac{7}{4} + \tau_e \right) \right] \left[ \zeta^2 - \frac{1}{2} \tau_e \right] = \left( \frac{23}{8} q^2 + \frac{3}{4} \tau_e + 2(q^2 \tau_e) + \frac{\tau_e^2}{2} q^2 \right).$$

The following solution results from this equation:

$$\tau^2_{GAM} = \left( \frac{7}{4} q^2 + q^2 \tau_e \right) + \frac{1}{(k_{GAM,0} - \tau_e/2)} \left[ \frac{23}{8} q^2 + \frac{3}{4} \tau_e \right.}

$$

\left. + 2(q^2 \tau_e) - \frac{7}{8} q^2 \tau_e \right].$$

where

$\cdots$
The real and imaginary parts of geodesic current $J_{\text{geo}}=J(U)+J(NU)$ are shown by solid and broken curves. Polarization currents are shown in (a) and (b) by straight lines with their sign reversed. The GAM frequencies are determined as the $\xi$ values where $J_{\text{pol}}$ equals $J_{\text{geo}}=J(U)+J(NU)$. Comparing (a) and (b), it is found that the geodesic current is enhanced in the case of $T_e>T_i$.

\[ a_{\text{GAM}}(\xi) = \left[ q \frac{7}{4} + \tau_{e} \right] . \]  

(46)

This is a kinetic elaboration GAM frequency over the MHD description and fits in the form of Eq. (1) with $\omega_{\text{GAM},0} = \sqrt{2}[T_e+(7/4)T_i]/m_i/R$ in place of $\omega_{\text{MHD},0} = \sqrt{2}[T_e+(5/3)T_i]/m_i/R$ and several of the correction terms having forms of $\delta \omega^2 \propto k_p^2(T_i/m_i)$ and $\propto k_p^2(T_e/m_e)$. $a_{\text{GAM},0}$ accounts for the numerical result in Sec. III B that the GAM frequency is higher if $T_e/T_i$ is larger. It is noted that the correction terms contain various parallel dynamics of the particles, though they play a minor role in the first group of solutions (ordinary GAM) if $q$ is sufficiently large.

2. GAM-like solutions in a lower frequency range

The second group of solutions emerges due to the swing of the geodesic currents, $J(U)$ and $J(NU)$, in the lower frequency range: In order to study the mechanism of these solutions (the GAM-like mode in the lower frequency range), we need to study the properties of $J(U)$ and $J(NU)$ in more detail and consider their relative importance. We have introduced $\chi=N_u/D_e$ in Sec. II B with denominator $D_e = \{k_{D,e}^2/k_{D,i}^2+[1+\xi Z_p(\xi)]\}$ and numerator $N_u=Z_2(\xi)$; $\chi$ is interpreted as the response function of the nonuniform field $\bar{\phi}_0$ to the uniform field $\bar{\phi}_0$. $N_u$ is proportional to the charge directly induced by the external uniform field $\bar{\phi}_0$ and $D_e$ is the shielding factor normalized by $k_{D,e}^2$. It is noted that $\chi$ takes a large value for the $\xi$, satisfying

\[ D_e = \{k_{D,e}^2/k_{D,i}^2+[1+\xi Z_p(\xi)]\} = 0. \]  

(47)

Then, $\chi$ is regarded as an enhancement factor rather than just a shielding factor. To study the total effects of a nonuniform field on geodesic current, we define $\chi^* = N_u/D_e$ with $N_u = \xi Z_2 \cdot Z_3$ so that

\[ J(NU) = -q^2 \chi^*. \]  

(48)

In Fig. 4(a), the real and imaginary parts of $D_e$, Re$[D_e]$ and Im$[D_e]$, are plotted versus $\xi$ in the case of $\tau_{e}=1$. The real part (reactive part) varies from $2(=1+1/\tau_e)$ to $1(=1/\tau_e)$ as $\xi$ varies from 0 to 7 ($\sim$ infinity) corresponding to the variation of $\xi Z_p(\xi)$ from 0 to $\sim 1$. This variation indicates that Debye screening of ions vanishes at a large value of $\xi$ while that of electron persists. In general, $D_e^{-1}$ does not show pe-
cular variations but shows monotonous variations for the case of $\tau_e=1$. The swing of $J(NU)$ in Fig. 3(a) is attributed to the sign change of the numerator $N_\omega^a=\xi_2-Z_2$ as shown in Fig. 4(c); a phase variation in $Z_2(\xi)$ by $\pi/2$ causes that of $N_\omega^a$ by $\pi$.

In the case of $\tau_e=5$, the situation is quite different: In Fig. 4(b), $\text{Re}[D_x]$ and $\text{Im}[D_x]$ are plotted versus $\xi$. $\text{Re}[D_x]$ descends from $1.2=(1+1/\tau_e)$ for the variation of $\xi$ from 0 to 7 and crosses the abscissa twice as indicated by arrows in Fig. 4(b): once at around $\xi\sim 2.0$ (labeled A) and once at $\xi\sim 1.2$ (labeled B). The former coincides with the value of $\xi\sim 2.1$ in Fig. 3(b), where the geodesic current makes a big swing and crosses the abscissa. The meanings of these $\xi$ values are understood by expanding $\text{Re}[D_x]$ asymptotically and in series using the expression given in the Appendix.

We obtain the following two forms suitable respectively for the larger and smaller solutions of $\text{Re}[D_x]=0$:

$$\text{Re}[D_x] = \frac{1}{\tau_e} \left( 1 - \frac{1}{2} \tau_e \xi^2 - \tau_e \frac{3}{4} \xi^4 \right) = 0 \quad (49)$$

and

$$\text{Re}[D_x] = \frac{1}{\tau_e} \left( 1 + \tau_e - 2 \tau_e \xi^2 \right) = 0. \quad (50)$$

The solutions to Eqs. (49) and (50) for $\tau_e=5$ are $\xi=1.89$ and $\xi=0.77$, well approximating the corresponding values $\xi\sim 2.0$ and $\xi\sim 1.0$ obtained in Fig. 4(b). [The slight disagreements between them are attributed to inaccuracies of the approximations, Eqs. (49) and (50).] Since the larger and smaller solutions have asymptotic forms $\omega=k_\phi/(Z_2 T_e/m_i)$ and $\omega=k_\phi(2T_e/m_i)$ for large $\tau_e$, we refer to these solutions in this paper as the electron and the ion branch of sound waves, respectively. The electron branch is known as the ion acoustic wave, having weak damping if $\tau_e>1$. The ion branch is less popular and categorized as a class of quasimode having heavier damping. They are inclusively referred to as sound waves. The inversion of $J(NU)$ occurs in the case of $\tau_e>1$ at the ion acoustic wave frequency. Though this inversion of the sign occurs also at the sound wave frequency of ion branch, $J(NU)$ does not change sign there in the case of $\tau_e>1$, because it coincides with that where numerator $\xi_2-Z_2$ changes sign.

The radial current due to the uniform field $J(U)$ also changes sign in the low frequency range. This is attributed to the uniqueness of the $Z_2(\xi)$, where faster and slower groups of particles carry radial currents in opposite directions. The inversion of the direction occurs due to the population change of ions belonging to the two groups in the lower frequency range.

To summarize this subsection, the geodesic radial currents, $J(U)$ and $J(NU)$, eventually have similar frequency dependences, having currents in the sense of polarization current in the lower frequency range; this is a quite interesting feature that MHD equations cannot describe. The GAM-like mode in the low frequency range is thus attributed to this uniqueness of kinetic responses.

It is also pointed out that the coupling to the branches of sound waves occurs under the condition of $\tau_e>3$. It is supposed that the experimentally observed GAM oscillations under the condition of $T_e/T_i>1$ may contain modes influenced by the ion acoustic mode.

### 3. Nonuniformity of the potential field

The presence of this effect may be checked in the experiments or in the results of computer simulations by diagnosing the up-down asymmetry of the potential. From the discussion in the previous section we obtain

$$\frac{\phi_x}{\phi_0} = \nu_T \left( k_{\phi,0} \frac{1}{\omega_{\phi,0}} \omega_{\phi,0} \alpha_{m,n} \chi_{m,n} \right)$$

$$\sim 2 \pi \left( k_{\phi,0} \frac{\nu_T}{\omega_{\phi,0}} \right) m q B_0^n B_0^m - n B_0^{q-1} \delta_{m,n} \chi_{m,n}. \quad (51)$$

For tokamaks of a circular plasma cross section, we assume $(m,n)=(1,0)$ and make a substitution $k_{\phi} \sim k_\phi (dr/d\phi) \sim k_\phi (1/2 \pi RB_0)$. For a simple tokamak case, we obtain by dropping out subscripts $(m,n)=(1,0)$

$$\frac{\phi_x}{\phi_0} \sim \left( k_\phi \frac{\nu_T}{\omega_{\phi,0}} B_0^0 B_0^1 \right) \chi \sim - \left( k_\phi \frac{\nu_T}{\omega_{\phi,0}} B_0^1 B_0^0 \right) \frac{1}{\xi}. \quad (52)$$

where an approximation $\chi=(Z_2 T_e/T_i)Z_2 \sim -(Z_2 T_e/T_i) (1/\xi)$ has been made, which is derived from Eq. (A7) for a large value of $\xi$.

According to recent experimental observations that $(k_{\phi,0} \nu_T/\omega_{\phi,0}) \sim 1/q$, $\phi_x/\phi_0$ may be of the order of $1/q$. Though the latter is regarded as a parameter indicating smallness, $\phi_x/\phi_0$ may be detectable in experiments. If the observed GAM is affected by the ion acoustic mode, the nonuniformity is even enhanced due to the increased $\chi$, making it easier to detect in experiments.

We obtain, for the case of $T_e/T_i=1$, the window in which the GAM can exist as follows:

$$0.83 < q. \quad (53)$$

It is expected that GAM may exist in a more limited parameter range than Eq. (53), if the damping is taken into account, and may be practically prevented from existing in the core region of the plasma. On consideration of the possible enhancement of polarization current due to neoclassical effects, the window of the GAM existence may be even narrower. Even under such conditions, the presence of the coupling of GAMs to ion acoustic waves ($T_e/T_i>1$) may enhance the geodesic current $J(U)$ and allow the presence of the GAM.

### IV. APPLICATION TO HELICAL SYSTEMS

#### A. Applications to straight helical systems

We first consider a straight helical device with dominant mode numbers $(m,n)=(M,N)$, which is determined by the helical pitch numbers of the device. This gives a good reference for the understanding of the GAM in toroidal helical systems.

By using these integral numbers, the dispersion relation may be written as
\[ D = i\hat{\sigma} \left( \omega + \frac{az^2}{(k_{j,M,N} v_f)^2} \left[ Z_1(\xi_{M,N}) - \chi_{M,N} \xi_{M,N} Z_2(\xi_{M,N}) \right] \right). \]  

(54)

Since single helicity is assumed, the structure of the equation is the same as that of tokamaks. The variable \( \xi = \omega/k_{\perp,1,0} \) used in the previous section is simply replaced by \( \xi_{M,N} = \omega/k_{j,M,N} \).

Though the formulation has the same structure, the wave numbers are quite different. While \( k_{\perp,1,0} = B^0/B \) in tokamaks, \( k_{j,M,N} = (MB^0 - NB^\perp)/B \sim -NB^\perp/B \) in helical systems.

For standard linear helical systems, which have large toroidal pitch numbers, the latter wave number takes on much larger values than in tokamaks. This gives the GAM in helical systems different features from those of tokamaks. Since the pressure asymmetry is more easily canceled along the magnetic lines of force, the geodesic radial current has more difficulty balancing the polarization current.

The condition of the GAM existence is obtained analytically for the case of \( T_e/T_i = 1 \) as

\[ 0.7 < \left( \frac{\delta_{M,N}}{(l/2\pi R_0)^2} \right) \frac{\xi_{M,N}}{(k_{j,M,N} v_f)^2}. \]  

(55)

Substituting \( \delta_{M,N}^2 = (l/2\pi R_0)^2 \sim 1 \), \( (MB^0 - NB^\perp)/B \sim MB^\perp \), and \( k_{j,M,N}^2 \sim -N/R_0 \), Eq. (53) is approximated by

\[ 0.83 < \frac{\delta_{M,N}}{(l/2\pi R_0)} \frac{M}{N}. \]  

(56)

Since \( M/N \) is usually small in standard helical systems, the parameter range of GAM existences will be narrower than that of tokamaks. The previous work, Ref. 32, is based on an asymptotic expansion and therefore caution must be taken in applications to the standard parameter \( (T_e/T_i = 1) \).

An exception is the radial domain where \( MB^0 - NB^\perp = 0 \), in which the connection length becomes larger and this boundary of GAM existence is relaxed.

Figures 5(a) and 5(b) show the determination of GAM frequencies in a straight helical system: \( T_e/T_i = 1 \) and \( T_e/T_i = 5 \). The geodesic current \( J_{\text{geo}} \) and the polarization currents \( J_{\text{pol}} \) are plotted versus \( \xi \) (in place of \( \xi_{M,N} \)) to facilitate a smooth connection to the next subsection. Shortage of \( J_{\text{geo}} \) relative to \( J_{\text{pol}} \) is evident in Fig. 5(a), where \( T_e/T_i = 1 \) is assumed. Figure 5(b) shows that a helical-type GAM can exist in the cases of \( T_e/T_i > 1 \). This is partly due to the general enhancement of the geodesic current, but more importantly, due to the coupling with ion acoustic mode.

Returning to Eqs. (47) and (49), we may write

\[ D_e = \{ \tau_e^{-1} + [1 + \xi Z_p(\xi)] \} = \left[ \tau_e^{-1} - \left( \frac{1}{2} \frac{1}{\xi} + \frac{3}{4} \frac{1}{\xi^2} \right) \right] \]

\[ = \frac{2}{\xi} \tau_e^{-1}[(\xi - \xi_0) + i\xi_0^2 \tau_e \delta/2]. \]  

(57)

Here, \( \xi_0 = \frac{3}{2} \tau_e + 3 \) is the frequency of the ion acoustic wave and \( \delta = \text{Im}[Z_p(\xi)] \) is the imaginary part of \( Z_p(\xi) \) calculated at \( \xi = \xi_0 \).

![Fig. 5(a)](image)

![Fig. 5(b)](image)

![Fig. 5(c)](image)

**FIG. 5.** Determination of the GAM frequency in helical systems. Real parts of \( J_{\text{ geo}}[\xi] \) and \( J_{\text{pol}}[\xi] \) are plotted vs \( \xi \) for three cases: (a) and (b) show the calculation in the cases of a straight helical system; \( T_e/T_i = 1 \) is assumed in (a) and \( T_e/T_i = 5 \) in (b). In the latter case, with high \( T_e/T_i \), geodesic current is enhanced and two solutions emerge in the helical GAM frequency range. (c) is the calculation for a mixed helicity case; the two modes in the different frequency ranges are nearly independent and they affect each other perturbatively.

Equation (41) is then approximated as follows in the vicinity of \( \xi = \xi_0 \):

\[ \text{Re} \left( \frac{1}{(\xi - \xi_0) + i(\tau_e \xi_0 \delta/2)} \right) = b = \frac{2a}{\xi_0^2} \tau_e^{-1}. \]  

(58)

Here, \( a = \text{Re}[(\xi_0 + q^2 Z_p(\xi_0))/q^2 Z_p(\xi_0)] \) and is regarded to be positive and small in the cases of inferiority of the geodesic currents to the polarization current.

We obtain a quadratic equation

\[ (\xi - \xi_0)^2 - \frac{\xi_0}{2a} \tau_e (\xi - \xi_0) + (\tau_e \xi_0 \delta/2)^2 = 0 \]  

(59)

to which the following two solutions follow:

\[ \xi - \xi_0 = \frac{1}{2} \left( \frac{\xi_0}{2a} \tau_e \pm \sqrt{\left( \frac{\xi_0}{2a} \tau_e \right)^2 - (\tau_e \xi_0 \delta/2)^2} \right). \]  

(60)

Since the right-hand side is positive independent of the sign, two solutions are predicted to be above the ion acoustic wave frequency: one above \( \xi_0 \) by \( \delta \tau_e \sim \tau_e \xi_0 / 2a \) and the other...
by \( \delta \zeta \sim r_a a_s^3 \dot{\phi}^2 / 2 \). The former solution (+) is interpreted as a continuous modification from the ordinary GAM in tokamaks. The latter (−) is conjectured to have heavier damping. With the present assumptions, \( a > 1 \) and \( \delta \ll 1 \), neither of the two solutions is far from \( \zeta_0 \).

Coupling of GAM with the ion acoustic mode in helical systems has more significant effects than in tokamaks: It not only causes the frequency shift of GAM but also enhances the geodesic current to allow its existence. As is shown in Eq. (51), such GAM will have a larger fraction of nonuniform field \( \phi_0 / \tilde{\theta}_0 \propto \sin (M \theta - N \zeta) \) than that in the tokamaks \( \phi_0 / \tilde{\phi}_0 \propto \sin \theta \).

B. Applications to toroidal helical systems

Practical helical systems are toroidal and intrinsically have toroidal curvature. Therefore it is necessary to consider at least two kinds of curvatures: For the CHS device \((m,n) = (M,N) = (2,8) \) and \((m,n) = (1,0) \). The sizes of the tokamak and helical ripples are similar in size \( \delta^2_{MN} = \delta^2_0 \) at around the \( r/a \sim 2/3 \), where \( r \) is the averaged minor radius and \( a \) is that of the outermost closed flux surface. As estimated in the previous examples, the curvature due to the toroidal ripple plays a more important role due to the longer connection length. From this point of view, toroidal helical systems are expected to have at least tokamak-type GAMs even in the standard range of plasma parameters, \( T_e / T_i \sim 1 \). They appear in the similar frequency range of tokamak GAM and are classified as a tokamak-type GAMs in toroidal helical systems.

With the coupling of GAM with the ion acoustic mode, helical ripples may play a substantial role. We start with the following model equation:

\[
D = i \sigma \left( \omega + \omega_G^2 \left( \frac{\tilde{\eta}}{k_{1,0} v_T} \right) [Z_1(\xi_{1,0}) + \chi_{1,0} \xi_{1,0} Z_2(\xi_{1,0})] \\
+ \omega_G^2 \left( \frac{\tilde{\eta}^2}{k_{1,0} v_T} \right) [Z_1(\xi_{MN}) + \chi_{MN} \xi_{MN} Z_2(\xi_{MN})] \right).
\]

(61)

Figure 5(c) shows the calculations to determine the GAM frequency in the case of \( T_e / T_i = 5 \), where \( \zeta = \xi_{m=1,n=0} = 5.5 \) is taken on the abscissa and radial curvatures on the ordinate. The solid curve shows the sum of the geodesic currents due to the tokamak curvature \((m,n) = (1,0) \) and helical curvature \((m,n) = (M=2,N=8) \). The straight line shows the classical polarization current with a reversed sign. \( \tilde{\eta}_{MN} = 3 \) is assumed to specify the strength of the helical ripple relative to the tokamak ripple, which is a value representing one third minor radius.

In Fig. 5(c), a pair of solutions is found around \( \zeta = \xi_{m=1,n=0} = 5.5 \), where the toroidal \((m,n) = (1,0) \) magnetic ripple plays a dominant role. The GAM will then have the features of the tokamak type. Another pair of solutions is found around \( \zeta = \xi_{m=1,n=0} = 3.2 \) in Fig. 5(c), where the radial current due to \((m,n) = (M=2,N=8) \) magnetic ripple plays a dominant role. The GAM will thus have features of helical GAM. It is noted that this helical-type GAM is also dependent on \( \tilde{\eta}_{MN} \), the parameter representing the intensity of the helical geodesic curvatures relative to that of a tokamak. Thus it is predicted that a helical-type GAM emerges as the helical ripple is increased. Such a change of the mode structure may occur in heliotron-type devices spatially, because it is known that the relative intensity of the helical ripples increases with the minor radius.

V. FULL COMPLEX ANALYSES

A. Simple tokamak case

Full complex analyses are required for more exact examinations of the dispersion relation. In order to do this, we formally recover an external driving term \( 4 \pi J_{ext}(\omega) \) on the right-hand side of Eq. (36) and write

\[
\frac{d\tilde{\phi}(\omega)}{d\phi} = \frac{4 \pi J_{ext}(\omega)}{D(\zeta(\omega))} = R(\omega) J_{ext}(\omega).
\]

(62)

Here, \( R(\omega) \propto D(\omega)^{-1} \) is regarded as the response function of the electric field \( d\tilde{\phi}(\omega) / d\phi \) to the external current \( J_{ext}(\omega) \), which is assumed to be caused by turbulence through nonlinear phenomena including Reynolds stress and Stringer mechanisms. \( \omega \) and \( \zeta \) are identified due to the relation \( \zeta = \omega / k_{1,0} v_T \). In Secs. III and IV, we derived the dispersion relation assuming \( \text{Re}(D) = 0 \); if \( \zeta \) is real it is equivalent to \( D^H = 0 \) with \( D^H \) the Hermitian part of \( D \).

Figure 6(a) shows the calculation made in a tokamak case: the response function \( R(\zeta) \) is shown in a three-dimensional (3D) plot over \( \zeta = \text{Re}(\zeta) \) and \( \zeta = \text{Im}(\zeta) \). Here, parameters \( q = 2 \) and \( \tau_e = 5 \) are used.
This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to  IP: 133.75.110.204 On: Tue, 07 Jan 2014 01:29:11

FIG. 7. Full complex calculation of the response function in a helical system (mixed-helicity case): Total response function $|R(\xi)|$ is calculated over complex $\xi$ plane; parameter sets ($T_e/T_i = 1.0, \eta_{MN} = 3.0$, $N = 8$) and ($T_e/T_i = 5.0, \eta_{MN} = 3.0$, $N = 8$) were used in calculating (a) and (b), respectively. Two major peaks appear in each figure corresponding to tokamak and helical ripples. In (b), where $T_e/T_i$ is larger, the geodesic current is magnified and a helical-type GAM solution appears with a smaller $\text{Im}[\xi]$ value.

In this display, the presence of peaks suggest the possible presence of poles of $R(\omega)$, i.e., the frequency of the GAM can be determined. There are two outstanding peaks found in the fourth quadrant of the $\omega$ plane: the one at around $(\xi_r = 5.5, \xi_i = 0.0)$ corresponds to the ordinary GAM, having a similar value of $\xi$ that was obtained from Fig. 3(b). The other one at around $(\xi_r = 0.0, \xi_i = 0.0)$ corresponds to the stationary zonal flow. Within the framework of collisionless damping (plateau regime), the stationary zonal flow has a finite damping rate. Figure 6(a) also has small peaks, suggesting the presence of another group of poles. Figure 6(b) is an expanded view of Fig. 6(a) in the form of a contour plot; each of the peaks is confirmed to be a pole by using Cauchy’s theorem. Thus it is found that the GAM-like mode in the lower frequency range, which we discussed in Sec. III, is transformed into modes with substantial damping rate in the full complex analysis.

B. Mixed helicity case

GAM in mixed helicity configurations has been studied using the same full complex analyses as described in Sec. V A. Figures 7(a) and 7(b) give global views of the absolute value of the response function $|R(\xi)|$, shown in a 3D plot over $\text{Re}[\xi]$ and $\text{Im}[\xi]$. The parameters used in the calculations are commonly $q = 2.0, (N,M) = (2,8)$, and $\eta_{MN} = 3.0$, and the parameter $T_e/T_i$ is different between the two cases: Equation (61) shows that $P$ is a function of $\omega$ via the dependences on $\zeta_{m,0}(\omega)$ and $\zeta_{MN}(\omega)$ distinguishing if the cause of their contributions is due to tokamak ripple or helical ripple.

Since they are mutually related by

$$\zeta_{m,n} = (M - qN)\zeta_{MN},$$

$\zeta_{m,n} = 0$ is chosen as a single variable $\xi$. $|R(\xi)|$ is then 3D displayed over $\text{Re}[\xi]$ and $\text{Im}[\xi]$.

In Fig. 7(a), where we assume $T_e/T_i = 1$, the helical-type GAM is found at $(\xi_r, \xi_i) = (25.0, -8.0)$. It is noted that a solution is given in the full complex analysis under the condition of $T_e/T_i = 1$, where no solution was found in Sec. IV [see Fig. 5(a)]; the solution has instead a significantly larger damping rate. We can also recognize two satellite poles, which, however, may be more difficult to find in experiments because they are deeper in the lower half $\xi$ plane.

In Fig. 7(b), where we assume $T_e/T_i = 5.0$, the helical-type GAM is found at frequency $(\xi_r, \xi_i) = (32.0, -0.0)$. The real frequency is approximately the same as the one we found in Fig. 5(c), and since $\xi_i = 0$, the GAM is predicted to be well defined. We can also recognize two satellite poles like the ones we find in Fig. 6(a), which also may be difficult to find in experiments having a large imaginary part. From the comparison of Figs. 7(a) and 7(b), it is found that the GAM frequency is higher and the damping rate is lower with increasing $T_e$.

In the lower frequency range, there are tokamak-type GAMs in both Figs. 7(a) and 7(b). Since this frequency range is crowded, an expanded view of Fig. 7(b) is shown in Fig. 8, facilitating a direct comparison with Fig. 6(a) with the same $\tau_r$ assumed. It is found that both of the two major poles corresponding to a tokamak-type GAM and stationary zonal flow have moved deeper into the lower half plane. The only difference in Fig. 8 from Fig. 6(a) is the presence of an added helical ripple in the former. In the low frequency range the tokamak ripple plays a dominant role and no significant difference is found in the real frequency. A significant difference between the two cases is however found in $\text{Im}[\omega]$, which is attributed to the additional damping mechanism due to the helical ripple, which can be treated perturbatively as has been shown in Refs. 32, 35, and 36.
VI. DISCUSSIONS

The possible presence of GAM in the lower frequency was suggested in this paper by using the drift kinetic equation. This mode, however, turned out to have a significant damping rate in a full complex analysis, and therefore it is difficult to convincingly predict that it will be observed in experiments. However, it may be noted that these predictions are results of solving the drift kinetic equation in the plateau regime.

Therefore, analyses in a collisionless regime are required in order to clarify the characteristics of GAM-like mode in the lower frequency range. It has been shown in this paper that geodesic currents have a maximum at a certain value of ω and changes sign in the low frequency range. These results will remain true even with the inclusion of a probable neo-classical polarization current. Therefore, there is a certain possibility that a new low frequency of GAM branch will be found in the collisionless (banana) regime with a damping rate smaller than that in the plateau regime.

VII. CONCLUSIONS

The geodesic mode was reconsidered in this paper in order to find some clues to understanding experimental observations of the GAM, which have different features depending on the device and operational parameter range: First, the role of the nonuniform potential field associated with the GAM was analyzed and its relation to coupling of GAMs with the two branches of sound waves were discussed. Specifically, coupling of the GAM with the ion acoustic wave was found to have strong effects in experiments carried out under the condition that the electron temperature is higher than that of ions. We therefore propose to measure the asymmetry of the potential for the clarification of this mechanism. In the present analyses, made avoiding the asymptotic expansions, it was found that there are typically two GAM frequencies: The one with higher frequency is the well-known GAM and the one found in the lower frequency range has been given little attention. The existence of the GAM in the low frequency range is attributed to the unique responses of the ions in this frequency range. The analyses with a full complex calculation showed that this mode has substantial damping. It is due to the rather strong dissipative response of ions in the low frequency range. It may, however, emerge as a well-defined mode in the treatment in a banana regime. The analyses developed in this paper allow inclusion of the multi-helicity problem; in applications to toroidal helical systems, it was shown that helical devices have at least one tokamak-type GAM. It was found that a GAM of a purely helical nature occurs only under limited conditions (T_e > 3T_i); it emerges as the geodesic current is enhanced by the coupling with ion acoustic wave and has a strong helical asymmetry. In the systems of mixed helicity, the properties of the GAM changes from that of the tokamak type to that of the helical type as the helical geodesic curvature is increased.

ACKNOWLEDGMENTS

This paper is based on the material presented at the Second IAEA Technical Meeting on the Theory of Plasma Instabilities: Transport, Stability, and Their Interaction. The authors acknowledge stimulating discussions with Dr. T. S. Hahm, Dr. S. Sharapov, and Dr. B. Scott. T.W. thanks Dr. H. Sugama for useful discussions in revising the paper and Dr. B. J. Peterson for careful reading and grammatical advice.

This work was partly supported by the JSPS-Core University Program in the field of [Plasma and Nuclear Fusion].

APPENDIX: PROPERTIES OF THE DISPERSION FUNCTIONS

Three kinds of the dispersion functions Z_1, Z_2, and Z_p have been used in the main text. They differ in terms of the multiplicity with which the particle drift motion played a role:

\[ Z_1 = \frac{1}{\sqrt{\pi}} \int \frac{1}{x-\xi} \left( (x)^4 + (x)^2 + \frac{1}{2} \right) \exp(-x^2) dx, \]  
(\text{A1})

\[ Z_2 = \frac{1}{\sqrt{\pi}} \int \frac{1}{x-\xi} \left( (x)^2 + \frac{1}{2} \right) \exp(-x^2) dx, \]  
(\text{A2})

\[ Z_p = \frac{1}{\sqrt{\pi}} \int \frac{1}{x-\xi} \exp(-x^2) dx. \]  
(\text{A3})

Here, their relationship and expansion formula are tabulated: Z_p is the well-known plasma dispersion function. Z_1 and Z_2 are expressed by using it as follows:

\[ Z_1 = \frac{3}{2} \xi + \frac{3}{2} \xi + \left( (\xi)^4 + (\xi)^2 + \frac{1}{2} \right) Z_p(\xi), \]  
(\text{A4})

\[ Z_2 = \xi + \left( (\xi)^2 + \frac{1}{2} \right) Z_p(\xi). \]  
(\text{A5})

1. Asymptotic expansion

Asymptotic expansions of Z_1 and Z_2 are obtained from that of the plasma dispersion function Z_p:

\[ Z_1 \approx \left( \frac{7}{4} \xi + \frac{23}{8} \xi \right) + i \sqrt{\pi} \left( (\xi)^4 + (\xi)^2 + \frac{1}{2} \right) \exp(-\xi^2), \]  
(\text{A6})

\[ Z_2 \approx \left( \frac{1}{4} \xi + \frac{1}{2} \xi \right) + i \sqrt{\pi} \left( (\xi)^2 + \frac{1}{2} \right) \exp(-\xi^2), \]  
(\text{A7})

\[ Z_p \approx \left( \frac{1}{4} \xi + \frac{3}{4} \xi + \frac{15}{4} \xi \right) + i \sqrt{\pi} \exp(-\xi^2). \]  
(\text{A8})
2. Series expansions

Series expansions of $Z_1$ and $Z_2$ are also obtained from that of the plasma dispersion function $Z_p$:

$$Z_1 = \frac{1}{2} \xi - \frac{1}{3} \xi^3 + i \sqrt{\pi} \left( \frac{1}{2} + \xi^2 + \xi^4 \right) \exp(-\xi^2), \quad \text{(A9)}$$

$$Z_2 = -\frac{4}{3} \xi^3 + i \sqrt{\pi} \left( \frac{1}{2} + \xi^2 \right) \exp(-\xi^2), \quad \text{(A10)}$$

$$Z_p = -\left(2\xi - \frac{4}{3} \xi^3 \right) + i \sqrt{\pi} \exp(-\xi^2). \quad \text{(A11)}$$

These three dispersion functions are calculated and shown in Fig. 1 versus $\xi$. The most notable feature is that $Z_1$, the ion current caused by the uniform potential field, increases in proportion to $\xi(\neq \omega)$ in the sense of polarization current in the low frequency range. This is quite different from the case of $Z_p$, where the current varies with the opposite coefficient of proportionality. It is also noted that $Z_2$ does not have a term proportional to $\xi$. These unique features of the geodesic response function originate from the multiplication factors in Eqs. (23) and (29).