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THE HEBREW UNIVERSITY OF JERUSALEM

BARGAINING SETS OF MAJORITY VOTING GAMES

by

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Discussion Paper # 410 November 2005

מרכז לחקר הרציונליות

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Bargaining Sets of Majority Voting Games*

Ron Holzman† Bezalel Peleg‡ Peter Sudhölter§

Abstract

Let A be a finite set of m alternatives, let N be a finite set of n players and let R^N be a profile of linear preference orderings on A of the players. Let u^N be a profile of utility functions for R^N. We define the NTU game V_{u^N} that corresponds to simple majority voting, and investigate its Aumann-Davis-Maschler and Mas-Colell bargaining sets.

The first bargaining set is nonempty for $m \leq 3$ and it may be empty for $m \geq 4$. However, in a simple probabilistic model, for fixed m, the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one if n tends to infinity.

The Mas-Colell bargaining set is nonempty for $m \leq 5$ and it may be empty for $m \geq 6$. Furthermore, it may be empty even if we insist that n be odd, provided that m is sufficiently large. Nevertheless, we show that the Mas-Colell bargaining set of any simple majority voting game derived from the k-th replication of R^N is nonempty, provided that $k \geq n + 2$.

Keywords: NTU game, voting game, majority rule, bargaining set

Journal of Economic Literature Classification: C71, D71

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*This is an enhanced and modified version of Discussion Paper # 376, Center for the Study of Rationality, The Hebrew University of Jerusalem

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1 Introduction

The Voting Paradox prevents us from applying the majority voting rule to choice problems with more than two alternatives. The standard way to avoid the paradox is to assume that the preferences of the voters are restricted so that the method of decision by majority yields no cycles (see Gaertner (2001) for a recent comprehensive survey). In this paper we follow a different path. It is well-known that the Voting Paradox is equivalent to the emptiness of the core of the corresponding cooperative majority voting game. We investigate two bargaining sets which contain the core.

We shall now review our results. At the end of the review we shall present our main conclusions.

In Section 2 we derive the exact form of the cooperative NTU games which correspond to simple majority voting.[1] We also recall the definitions of the Aumann-Davis-Maschler and Mas-Colell bargaining sets of cooperative NTU games.

The Voting Paradox with three voters and three alternatives is analyzed in Section 3 with respect to these two bargaining sets.

Section 4 addresses the existence question for the Aumann-Davis-Maschler bargaining set of a simple majority voting game. We show that it is nonempty when there are at most three alternatives, but may be empty when there are four or more alternatives.

The same question for the Mas-Colell bargaining set is addressed in Section 5. It turns out that the boundary between existence and non-existence is somewhat higher in this case: We prove existence for up to five alternatives, and give examples of emptiness for six or more alternatives.

In these examples, there is an even number of voters. This raises the question, addressed in Section 6, of whether the Mas-Colell bargaining set of a simple majority voting game with an odd number of voters may be empty. This indeed turns out to be the case, but showing this requires a much more elaborate construction and huge numbers of alternatives and voters.

We conclude in Section 7 with existence results for two models in which there are many voters, whose preferences are drawn in a specified way. In one of them, a simple probabilistic model, we show that both bargaining sets are nonempty with probability tending to one as the number of voters tends to infinity. In the other, a replication model, we prove that the Mas-Colell bargaining set is nonempty for any k-fold replication with k sufficiently large.

An individually rational payoff vector belongs to the bargaining set if (i) it is (weakly) Pareto optimal and (ii) for every objection (in the sense of the bargaining set under consideration) there

[1] Similar derivations may be carried out for other voting rules. Here we concentrate on the most natural voting rule, simple majority. We refer the reader to an earlier version of this manuscript (available as Discussion Paper # 376, Center for the Study of Rationality, The Hebrew University of Jerusalem) for a treatment of plurality voting and approval voting.

is a counter objection. Our study proves that the tension between (i) and (ii) is so strong that for six or more alternatives both bargaining sets may be empty. This is our first conclusion. Our second conclusion is more vague: If the number of players tends to infinity and the number of alternatives is held fixed, then the bargaining sets of simple majority voting games are likely to be nonempty.

2 Preliminaries

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of voters, also called players, and let $A = \{a_1, \ldots, a_m\}$, $m \geq 2$, be a set of m alternatives. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on S. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ imply that $y \in C$. An *NTU game* with the player set N is a pair (N, V) where V is a function which associates with every *coalition* S (that is, $S \subseteq N$ and $S \neq \emptyset$) a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$, such that

(1) $V(S)$ is closed and comprehensive;

(2) $V(S) \cap (x + \mathbb{R}^S_+)$ is bounded for every $x \in \mathbb{R}^S$.

We shall now assume that each $i \in N$ has a linear preference R^i on A. Thus, for every $i \in N$, R^i is a complete, transitive, and antisymmetric binary relation on A. Moreover, let u^i, $i \in N$, be a utility function that represents R^i. We shall always assume that

$$\min_{\alpha \in A} u^i(\alpha) = 0 \text{ for all } i \in N. \tag{2.1}$$

We consider a situation in which every player votes for some alternative in A. If a strict majority of voters agrees on $\alpha \in A$, then the outcome is α, and every voter i gets utility $u^i(\alpha)$. Otherwise, if no majority forms, a deadlock results and every voter gets utility 0. Given any utility profile $u^N = (u^i)_{i \in N}$ that satisfies (2.1), this naturally leads (via α-effectiveness) to the following definition of the NTU game (N, V_{u^N}) associated with choice by simple majority voting and called *simple majority voting game* (see Aumann (1967)):

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid x \leq 0\} \text{ if } S \subseteq N, 1 \leq |S| \leq \frac{n}{2}; \tag{2.2}$$

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq u^S(\alpha)\} \text{ if } S \subseteq N, |S| > \frac{n}{2}, \tag{2.3}$$

where $u^S(\alpha) = (u^i(\alpha))_{i \in S}$.

Notation 2.1 In the sequel let $L = L(A)$ denote the set of linear preferences on A. For $R \in L$ and for $k \in \{1, \ldots, m\}$, let $t_k(R)$ denote the k-th alternative in the order R. If $R^N \in L^N$ and $\alpha, \beta \in A$, $\alpha \neq \beta$, then α *dominates* β (abbreviated $\alpha \succ_{R^N} \beta$) if $|\{i \in N \mid \alpha \; R^i \; \beta\}| > \frac{n}{2}$. We shall say that an alternative $\alpha \in A$ is a *weak Condorcet winner* (with respect to R^N) if $\beta \not\succ_{R^N} \alpha$ for all $\beta \in A$. Also, if $R^N \in L^N$, then denote

$$\mathcal{U}^{R^N} = \{(u^i)_{i \in N} \mid u^i \text{ is a representation of } R^i \text{ satisfying (2.1) } \forall i \in N\}.$$

Let (N, V) be an NTU game. The pair (N, V) is *zero-normalized* if $V(\{i\}) = -\mathbb{R}_+^{\{i\}}$ for all $i \in N$. Also, (N, V) is *superadditive* if for every pair of disjoint coalitions S, T, $V(S) \times V(T) \subseteq V(S \cup T)$. It should be remarked that the NTU games defined by (2.2) and (2.3) are zero-normalized and superadditive.

Now we shall recall the definitions of two bargaining sets introduced by Davis and Maschler (1967) and by Mas-Colell (1989), following the general approach delineated by Aumann and Maschler (1964). Let (N, V) be a zero-normalized NTU game and $x \in \mathbb{R}^N$. We say that x is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if x is weakly Pareto optimal in $V(N)$;
- an *imputation* if x is an individually rational preimputation.

We also use the natural analogue of the Pareto optimality notion with respect to $V(S)$, where $\emptyset \neq S \subseteq N$.

A pair (P, y) is an *objection* at x if $\emptyset \neq P \subseteq N$, y is Pareto optimal in $V(P)$, and $y > x^P$. An objection (P, y) is *strong* if $y \gg x^P$. The pair (Q, z) is a *weak counter objection* to the objection (P, y) if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z \geq (y^{P \cap Q}, x^{Q \setminus P})$. A weak counter objection (Q, z) is a *counter objection* to the objection (P, y) if $z > (y^{P \cap Q}, x^{Q \setminus P})$. A strong objection (P, y) is *justified in the sense of the bargaining set* if there exist players $k \in P$ and $\ell \in N \setminus P$ such that there does not exist any weak counter objection (Q, z) to (P, y) satisfying $\ell \in Q$ and $k \notin Q$. The *bargaining set* of (N, V), $\mathcal{M}(N, V)$, is the set of all imputations x that do not have strong justified objections at x in the sense of the bargaining set (see Davis and Maschler (1967)). An objection (P, y) is *justified in the sense of the Mas-Colell bargaining set* if there does not exist any counter objection to (P, y). The *Mas-Colell bargaining set* of (N, V), $\mathcal{MB}(N, V)$, is the set of all imputations x that do not have a justified objection at x in the sense of the Mas-Colell bargaining set (see Mas-Colell (1989)).

Remark 2.2 The original definition of Mas-Colell considered preimputations, not just imputations. In restricting our attention to imputations we follow Vohra (1991). In any case, all our results about existence and non-existence are valid for both variants of the definition.

Remark 2.3 For a given $R^N \in L^N$, the particular choice of a representation $u^N \in \mathcal{U}^{R^N}$ is essentially immaterial: different representations lead to NTU games that are derived from each other by ordinal transformations, and so are their bargaining sets.

3 The Voting Paradox

In this section we shall compute the bargaining sets of the Voting Paradox and interpret the results.

Let $A = \{a, b, c\}$, let $n = 3$, and let $R^N \in L^N$ be given by Table 3.1.

Table 3.1: Preference Profile of the Voting Paradox

R^1	R^2	R^3
a	c	b
b	a	c
c	b	a

For $i \in N$ let u^i be a utility representation of R^i satisfying (2.1) and let $V = V_{u^N}$ (see (2.2) and (2.3)).

We claim that $\mathcal{M}(N, V) = \{0\}$. Indeed, it is straightforward to verify that $0 \in \mathcal{M}(N, V)$. In order to show the opposite inclusion let $x \in \mathcal{M}(N, V)$. Then there exists $\alpha \in A$ such that $x \leq u^N(\alpha)$. Without loss of generality we may assume that $\alpha = a$. Assume, on the contrary, that $x > 0$. If $x^1 > 0$, then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection of 3 against 1 at x in the sense of the bargaining set. If $x^1 = 0$ and, hence, $x^2 > 0$, then $(\{1, 3\}, u^{\{1,3\}}(b))$ is a justified objection of 1 against 2.

In order to compute the Mas-Colell bargaining set, we define $x = (u^1(b), u^2(a), 0)$ and claim that $x \in \mathcal{MB}(N, V)$. Indeed, let (P, y) be an objection at x. Then $|P| \geq 2$. As y is Pareto optimal in $V(P)$, $y \in \{u^P(\alpha) \mid \alpha \in A\}$. If $y = u^P(a)$, then (P, y) is countered by $(\{2, 3\}, u^{\{2,3\}}(c))$. If $y = u^P(b)$, then $y > x^P$ implies that $P = \{1, 3\}$. In this case (P, y) is countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $y = u^P(c)$, then $y > x^P$ implies that $P = \{2, 3\}$ and that (P, y) is countered by $(\{1, 3\}, u^{\{1,3\}}(b))$.

In order to show that every $\hat{x} \in \mathbb{R}^N$ satisfying $0 \leq \hat{x} \leq x$ is an element of $\mathcal{MB}(N, V)$, it should be noted that each objection at $\hat{x}$ is also an objection at x if $\hat{x}^1 > 0$ and $\hat{x}^2 > 0$. If $\hat{x}^1 = 0$ and $\hat{x}^2 > 0$, then the additional objections are of the form $(P, u^P(c))$ for some $P \subseteq N$ and these objections can be countered by $(\{1, 3\}, u^{\{1,3\}}(b))$. Similarly, if $\hat{x}^1 > 0$ and $\hat{x}^2 = 0$, then the additional objections can be countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $\hat{x} = 0$, then each additional objection can be countered by one of the foregoing pairs $(\{1, 3\}, u^{\{1,3\}}(b))$ or $(\{1, 2\}, u^{\{1,2\}}(a))$.

Similarly, for $y = (u^1(b), 0, u^3(c))$ and $z = (0, u^2(a), u^3(c))$ we have that every $\hat{y} \in \mathbb{R}^N$ satisfying $0 \leq \hat{y} \leq y$ and every $\hat{z} \in \mathbb{R}^N$ satisfying $0 \leq \hat{z} \leq z$ is in $\mathcal{MB}(N, V)$.

We shall show now that there are no other elements in $\mathcal{MB}(N, V)$. Indeed, any remaining individually rational $\widetilde{x} \in V(N)$ must have a coordinate that is higher than the utility of that voter's second best alternative. Say, without loss of generality, that $\widetilde{x}^1 > u^1(b)$. Then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection in the sense of the Mas-Colell bargaining set at $\widetilde{x}$. We conclude that $\mathcal{MB}(N, V)$ is the intersection of $\mathbb{R}^N_+$ and the comprehensive hull of $\{x, y, z\}$.

Discussion: The singleton $\mathcal{M}(N, V)$ tells us that in order to achieve (coalitional) stability the players have to give up any profit above their individually protected levels of utility. There is no hint how an alternative of A will be chosen. The message of $\mathcal{MB}(N, V)$ is much more detailed. For example, the element $x = (u^1(b), u^2(a), 0)$ tells us that the alternative a may be chosen provided player 1 disposes of $u^1(a) - u^1(b)$ utiles. Thus, we also see here that lower utility levels guarantee stability. Actually, x implies that there is an agreement between 1 and 2, the alternative a is chosen as a result of the agreement, and the utility of 1 is reduced (because of the agreement) from $u^1(a)$ to $u^1(b)$. Note that cooperative game theory does not specify the details of agreements that support stable payoff vectors.

In this example (and indeed in many other examples) the Mas-Colell bargaining set is much larger than the Aumann-Davis-Maschler one. However, it is intersting to note that $\mathcal{MB}(N, V)$ need not contain $\mathcal{M}(N, V)$ in general, as shown by the following example.

Example 3.1 Let $n = 4$ and let R^N be given by Table 3.2.

Then $x = (\min\{u^i(b), u^i(a)\})_{i \in N} \in \mathcal{M}(N, V)$, because there is no strong objection at x. However, $x \notin \mathcal{MB}(N, V)$ because $(N, u^N(a))$ is a justified objection in the sense of the Mas-Colell bargaining set at x.

Nevertheless, it can be shown that when the number of alternatives is three and there is no weak Condorcet winner, then in the associated NTU game (N, V) we have $\mathcal{M}(N, V) \subseteq \mathcal{MB}(N, V)$.[2]

[2] We refer the reader to an earlier version of this manuscript (available as Discussion Paper # 376, Center for the Study of Rationality, The Hebrew University of Jerusalem), where this fact is derived from a detailed (though incomplete) description of the bargaining sets of simple majority voting games in the case of three alternatives.

Table 3.2: Preference Profile of a 4-Person Voting Problem

R^1	R^2	R^3	R^4
a	a	c	c
b	b	b	b
c	c	a	a

4 The Bargaining Set

Throughout this section let $R^N \in L(A)^N$, $\succ \; = \; \succ_{R^N}$, $u^N \in \mathcal{U}^{R^N}$ (see Notation 2.1), $V = V_{u^N}$ (see (2.2) and (2.3)).

Theorem 4.1 *If* $|A| \leq 3$*, then* $\mathcal{M}(N, V) \neq \emptyset$*.*

Proof: If there exists a weak Condorcet winner $\alpha \in A$, then $u^N(\alpha) \in \mathcal{M}(N, V)$. So we may assume that $|A| = 3$ and for every $\alpha \in A$ there exists $\beta \in A$ such that $\beta \succ \alpha$. We claim that for any $\alpha \in A$ there exists $i \in N$ such that $t_3(R^i) = \alpha$. Indeed, if $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$ and if $\beta \succ \alpha$, then $|\{i \in N \mid \beta = t_1(R^i)\}| > \frac{n}{2}$ and β is a Condorcet winner which was excluded. We conclude that $0 \in \mathbb{R}^N$ is weakly Pareto optimal. Hence $0 \in \mathcal{M}(N, V)$. **q.e.d.**

Example 4.2 Let $A = \{a, b, c, d\}$, let $n = 3$, and let R^N be given by Table 4.1.

Table 4.1: Preference Profile of a 4-Alternative Voting Problem

R^1	R^2	R^3
a	c	b
b	a	c
d	d	d
c	b	a

We claim that $\mathcal{M}(N, V) = \emptyset$. Let x be an imputation of (N, V). In order to show that $x \notin \mathcal{M}(N, V)$ we may assume without loss of generality that $x^1 \geq u^1(d)$. We distinguish the following possibilities:

(1) $x \leq u^N(a)$ or $x \leq u^N(d)$. Then $\big(\{2,3\}, u^{\{2,3\}}(c)\big)$ is a justified objection (in the sense of the bargaining set) of 3 against 1.

(2) $x \leq u^N(b)$. If $x^3 < u^3(c)$, then we may use the same justified strong objection as in the first possibility. If $x^3 \geq u^3(c)$, then $\big(\{1,2\}, u^{\{1,2\}}(a)\big)$ is a justified objection of 2 against 3.

Example 4.2 shows the tension between (weak) Pareto optimality and stability may result in an empty bargaining set.

Example 4.2 may be generalized to any number $m \geq 4$ of alternatives. Indeed, let $A = \{a, b, c, d_1, \ldots, d_k\}$, where $k = m - 3$, and define R^N by

$$\begin{aligned} R^1 &= (a, b, d_1, \ldots, d_k, c), \\ R^2 &= (c, a, d_1, \ldots, d_k, b), \\ R^3 &= (b, c, d_1, \ldots, d_k, a), \end{aligned}$$

and note that $\mathcal{M}(N, V) = \emptyset$. More interestingly, Example 4.2 can be generalized to yield an empty bargaining set for simple majority voting games on four alternatives with infinitely many numbers of voters.

Example 4.3 (Example 4.2 generalized) Let

$$R_1 = (a, b, d, c), \quad R_2 = (a, c, d, b), \quad R_3 = (b, a, d, c),$$

$$R_4 = (b, c, d, a), \quad R_5 = (c, a, d, b), \quad R_6 = (c, b, d, a),$$

and let $k \in \mathbb{N}$. Let $N = \{1, \ldots, 6k - 3\}$ and let $R^N \in L^N$ satisfy

$$|\{j \in N \mid R^j = R_i\}| = \begin{cases} k & , \text{ if } i = 1, 4, 5, \\ k - 1 & , \text{ if } i = 2, 3, 6. \end{cases}$$

Then $\mathcal{M}(N, V) = \emptyset$. Indeed, $k = 1$ coincides with Example 4.2. The reader may check e.g. the case $k = 2$ (see Table 4.2) by repeating the arguments of Example 4.2.

5 The Mas-Colell Bargaining Set

We shall show that $\mathcal{MB}$ is nonempty for any simple majority voting game on less than six alternatives. Also, we shall show that there is a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty. We shall always assume that $R^N \in L(A)^N$, $\succ \; = \; \succ_{R^N}$, $u^N \in \mathcal{U}^{R^N}$, and $V = V_{u^N}$. We start with the following simple lemma.

Table 4.2: Preference Profile for $k = 2$

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8	R^9
a	a	b	b	c	c	a	c	b
b	c	a	c	a	b	b	a	c
d	d	d	d	d	d	d	d	d
c	b	c	a	b	a	c	b	a

Lemma 5.1 *Assume that there is no weak Condorcet winner. If $x \in \mathbb{R}^N_+$ satisfies $x^i \leq u^i\left(t_{m-1}(R^i)\right)$ for all $i \in N$ and if x is weakly Pareto optimal in $V(N)$, then $x \in \mathcal{MB}(N, V)$.*

Proof: If (S, y) is an objection at x, then $|S| > n/2$ and there exists $\alpha \in A$ such that $u^S(\alpha) = y$. Choose $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$ such that $u^T(\beta) \gg u^T(\alpha)$. Thus, $(T, u^T(\beta))$ is a counter objection. **q.e.d.**

Theorem 5.2 *If $|A| \leq 5$, then $\mathcal{MB}(N, V) \neq \emptyset$.*

Proof: If $|A| \leq 3$, the proof that we gave for $\mathcal{M}$ (Theorem 4.1) works for $\mathcal{MB}$, too. In order to prove the theorem for $m = 4$ we may assume that there is no weak Condorcet winner. Then, for each $\alpha \in A$,

$$\text{there exists } i \in N \text{ such that } \alpha \in \{t_3(R^i), t_4(R^i)\}. \tag{5.1}$$

Indeed, if for some $\alpha \in A$, $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$, then $\beta \succ \alpha$ implies that β is a Condorcet winner which was excluded. For $\alpha \in A$, define $x_\alpha = \left(\min\{u^i(\alpha), u^i(t_3(R^i))\}\right)_{i \in N}$. By Lemma 5.1, $x_\alpha \in \mathcal{MB}(N, V)$, if x_α is weakly Pareto optimal. Hence, in order to complete the proof for $m = 4$, it suffices to show that there exists $\alpha \in A$ such that x_α is weakly Pareto optimal. Two possibilities may occur: If there exists $\alpha \in A$ such that $\alpha \neq t_4(R^i)$ for all $i \in N$, then, by (5.1), x_α is weakly Pareto optimal. Otherwise, any x_α is weakly Pareto optimal.

Now, let $m = 5$, let $A = \{a_1, \ldots, a_5\}$, and assume that $\mathcal{MB}(N, V) = \emptyset$. Then, for each $\alpha \in A$

(1) there exists $\beta \in A$ such that $\beta \succ \alpha$;

(2) $u^N(\alpha)$ is Pareto optimal (because $\mathcal{MB}$ is nonempty when we restrict our attention to the game corresponding to the restriction of u^N to $A \setminus \{\alpha\}$).

For $\alpha \in A$ denote $\ell(\alpha) = \max\{k \in \{1, \ldots, 5\} \mid \exists i \in N : t_k(R^i) = \alpha\}$. Let $\ell_{min} = \min_{\alpha \in A} \ell(\alpha)$. We distinguish cases:

(i) $\ell_{min} \geq 4$: Then there exists a weakly Pareto optimal $x \in V(N)$ such that $x^i \leq u^i(t_4(R^i))$ for all $i \in N$ which is impossible by Lemma 5.1.

(ii) $\ell_{min} \leq 2$: Let $\alpha, \beta \in A$ such that $\ell(\alpha) = \ell_{min}$ and $\beta \succ \alpha$. Then β is a Condorcet winner, which is impossible by (1).

(iii) $\ell_{min} = 3$: Let $B = \{\beta \in A \mid \ell(\beta) = 3\}$. If $|B| = 3$, then any $\alpha \in A \backslash B$ violates (2). If $|B| = 2$, let us say $B = \{\alpha, \beta\}$, then we may assume without loss of generality that $\alpha \not\succ \beta$. Let $\gamma \in A$ such that $\gamma \succ \beta$. Then none of the remaining $\delta \in A \setminus (\{\gamma\} \cup B)$ dominates any of the elements α, β, γ. By (1) we conclude that $\gamma \succ \beta \succ \alpha \succ \gamma$. Then $(\min\{u^i(\alpha), u^i(\beta)\})_{i \in N} \in \mathcal{MB}(N, V)$.

Now we turn to the case $|B| = 1$, let us say $B = \{a_3\}$. Let $\widehat{S} = \{i \in N \mid t_3(R^i) = a_3\}$. For any $k \in \widehat{S}$ there exists $x_k \in \mathbb{R}^N$ such that x_k is weakly Pareto optimal, $x_k^k = u^k(a_3)$, and $x_k^i \leq u^i(t_4(R^i))$ for all $i \in N \setminus \{k\}$. As $x_k \notin \mathcal{MB}(N, V)$, there exists a justified objection $(S, u^S(\alpha))$ for some $S \subseteq N$, $|S| > n/2$, and some $\alpha \in A$. Let $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$, such that $u^{S \cap T}(\beta) \gg u^{S \cap T}(\alpha)$ and $u^{T \setminus S}(\beta) \geq (u^i(t_4(R^i)))_{i \in T \setminus S}$. As $(T, u^T(\beta))$ is not a counter objection, we conclude that $k \in T$, $t_4(R^k) = \beta$, and $t_5(R^k) = \alpha$. We conclude that for any $k \in \widehat{S}$ the alternative $t_5(R^k)$ is only dominated by $t_4(R^k)$. If n is odd, we may now easily finish the proof by the observation that α dominates all other alternatives except β, and therefore $(\min\{u^i(\alpha), u^i(\beta)\})_{i \in N} \in \mathcal{MB}(N, V)$. Hence we may assume from now on that n is an even number. As $a_3 \not\succ \alpha$, $\{i \in N \mid u^i(\alpha) > u^i(a_3)\} \cap \{i \in N \mid u^i(\beta) > u^i(\alpha)\} \neq \emptyset$. Thus, there exists $j \in \widehat{S}$ such that $t_1(R^j) = \beta$ and $t_2(R^j) = \alpha$. So far we have for any $k \in \widehat{S}$, where $\alpha = t_5(R^k), \beta = t_4(R^k)$:

$$\alpha \text{ is only dominated by } \beta; \tag{5.2}$$

$$\text{There exists } j \in \widehat{S} \text{ such that } t_1(R^j) = \beta, t_2(R^j) = \alpha; \tag{5.3}$$

$$|\{i \in N \mid u^i(\alpha) > u^i(a_3)\}| \geq \tfrac{n}{2}. \tag{5.4}$$

Now, let $k, j \in \widehat{S}$ have the foregoing properties, let us say $k = 1$ and $j = 2$. We also may assume that $t_4(R^1) = a_4$, $t_5(R^1) = a_5$, $t_4(R^2) = a_1$, $t_5(R^2) = a_2$ (hence $R^2 = (a_4, a_5, a_3, a_1, a_2)$). So, for any $k \in \widehat{S}$, we have

$$\{t_4(R^k), t_5(R^k)\} = \{a_4, a_5\} \quad \Rightarrow \quad t_4(R^k) = a_4 \tag{5.5}$$

$$t_5(R^k) = a_5 \quad \Rightarrow \quad t_4(R^k) = a_4 \tag{5.6}$$

$$\{t_4(R^k), t_5(R^k)\} = \{a_1, a_2\} \quad \Rightarrow \quad t_4(R^k) = a_1 \tag{5.7}$$

$$t_5(R^k) = a_2 \quad \Rightarrow \quad t_4(R^k) = a_1 \tag{5.8}$$

We are now going to show that there exists $k \in \widehat{S}$ such that $t_5(R^k) \notin \{a_5, a_2\}$. Assume the contrary. Then $\{i \in N \mid u^i(a_5) > u^i(a_3)\} \cap \{i \in N \mid u^i(a_2) > u^i(a_3)\} = \emptyset$ and, by (5.4), $a_5 \not\succ a_3$ and $a_2 \not\succ a_3$. Hence, by (1), $a_1 \succ a_3$ or $a_4 \succ a_3$. However, note that by our assumption $u^i(a_1) > u^i(a_3)$ implies $u^i(a_1) > u^i(a_5)$ for all $i \in N$. Thus, if $a_1 \succ a_3$, then $a_1 \succ a_5$ which contradicts (5.2). Similarly, $a_4 \succ a_3$ can be excluded.

Hence, we may assume without loss of generality, that there exists $k \in \widehat{S}$ such that $t_5(R^k) = a_1$. We now claim that there exists $j \in \widehat{S}$ such that $t_5(R^j) = a_4$. By (5.2) and the fact that $a_1 \succ a_2$, $t_4(R^k) \in \{a_4, a_5\}$. If $t_4(R^k) = a_4$, then by (5.3) there exists $j \in \widehat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_5\}$. By (5.6), $a_5 \neq t_5(R^j)$, and by (5.8), $a_2 \neq t_5(R^j)$. Hence this possibility is ruled out. We conclude that $t_4(R^k) = a_5$. By (5.3) there exists $j \in \widehat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_4\}$. By (5.8), $t_5(R^j) = a_4$. So our claim has been shown.

So far we have deduced there exist $k_j \in \widehat{S}$, $j = 1, 2, 4, 5$, such that $t_5(R^{k_j}) = a_j$. By (5.4), $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| \geq \frac{n}{2}$ for all $j = 1, 2, 4, 5$. We conclude that $a_3 = t_3(R^i)$ for all $i \in N$ and $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| = \frac{n}{2}$ for all $j = 1, 2, 4, 5$. Therefore a_3 is not dominated by any alternative, which contradicts (1). **q.e.d.**

We shall now present an example of a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty.

Table 5.1: Preference Profile leading to an empty $\mathcal{MB}$

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
c	c	c	b
b	b	b	a_4
a_3	a_2	a_1	c
a_4	a_3	a_2	a_1

Example 5.3 Let $n = 4$, $A = \{a_1, \ldots, a_4, b, c\}$, and let $R^N \in L^N$ be given by Table 5.1. We claim that $\mathcal{MB}(N, V) = \emptyset$. Note that the proof below is similar to the proof of the emptiness of an extension of the Mas-Colell bargaining set of a game derived from a 4-person voting problem on ten alternatives (see Section 3 of Peleg and Sudhölter (2005)).

Assume that there exists $x \in \mathcal{MB}(N, V)$. Let $\alpha \in A$ such that $x \leq u^N(\alpha)$. Let

$$S_1 = \{1, 2, 3\}, S_2 = \{1, 2, 4\}, S_3 = \{1, 3, 4\}, S_4 = \{2, 3, 4\}.$$

We distinguish the following possibilities:

(1) $x \leq u^N(a_1)$. In this case $(S_4, u^{S_4}(a_4))$ is an objection at x. As there must be a counter objection to this, we conclude that $(S_3, u^{S_3}(a_3))$ is a counter objection, and therefore also

an objection at x. Hence, $x^1 \leq u^1(a_3)$. To this objection, too, there must be a counter objection. We conclude that $(S_2, u^{S_2}(a_2))$ is a counter objection. Hence, $x^2 \leq u^2(a_2)$ and therefore $x \ll u^N(b)$ and the desired contradiction has been obtained in this case.

(2) The possibilities $x \leq u^N(\alpha)$ for $\alpha \in \{a_2, a_3, a_4\}$ may be treated similarly.

(3) $x \leq u^N(b)$. Then $(S_1, u^{S_1}(c))$ is an objection at x. There are several possibilities for a counter objection to this. Each of them involves player 4 and one of the alternatives a_1, a_4, or c. We conclude that, in any case, $x^4 \leq u^4(a_4)$. Hence, $(S_4, u^{S_4}(a_4))$ is an objection at x. Now we conclude that $(S_3, u^{S_3}(a_3))$ must be a counter objection and, hence, an objection at x. We continue by concluding that $(S_2, u^{S_2}(a_2))$ must be an objection and that, hence, $(S_1, u^{S_1}(a_1))$ is a counter objection. Therefore, $x \ll u^N(b)$ and the desired contradiction has been obtained.

(4) $x \leq u^N(c)$. We consecutively deduce that $(S_4, u^{S_4}(a_4)), \ldots, (S_1, u^{S_1}(a_1))$ are objections. The desired contradiction again is obtained by the observation that $x \ll u^N(b)$. **q.e.d.**

Example 5.3 may be generalized to any number $m \geq 6$ of alternatives. Also, it may be generalized to any even number $n \geq 4$ of voters: if $R_i = R^i$ for $i = 1, \ldots, 4$, if

$$R_5 = (a_2, a_1, c, b, a_3, a_4), R_6 = (a_4, a_3, c, b, a_1, a_2),$$

if $n = 4 + 2k$ for some $k \in \mathbb{N}$, if $\widetilde{R}^N \in L^N$ such that

$$|\{j \in N \mid \widetilde{R}^j = R_i\}| = \begin{cases} k & , \text{if } i = 5, 6, \\ 1 & , \text{if } i = 1, 2, 3, 4, \end{cases}$$

and if $\widetilde{V} = V_{u^N}$ for some $u^N \in \mathcal{U}^{\widetilde{R}^N}$, then $\mathcal{MB}(N, \widetilde{V}) = \emptyset$.

6 The Mas-Colell Bargaining Set for an Odd Number of Voters

The examples that we just gave for emptiness of the Mas-Colell bargaining set have an even number of voters. The most natural setting for simple majority rule is when the number of voters is odd. It is therefore desirable to study the existence question for $\mathcal{MB}$ in the class of simple majority voting games with an odd number of voters. Attempts to construct small explicit counterexamples, similar to those above, seem to fail. We take a different approach, that leads to the construction of a profile of preferences with an odd number of voters, whose associated simple majority voting game has an empty Mas-Colell bargaining set. This construction is difficult to visualize, and its presentation requires several preparatory steps. It also requires huge numbers of voters and alternatives.

Throughout this section we shall always assume that A is a finite set of $m \geq 2$ alternatives and that $N = \{1, \ldots, n\}$ for some odd $n \in \mathbb{N}$. Recall that $T = (A, \succ)$ is a *tournament* on A if $\succ$ is an irreflexive, asymmetric and complete relation on A (that is, $\alpha, \beta \in A, \alpha \neq \beta$ implies that exactly one of $\alpha \succ \beta$, $\beta \succ \alpha$ holds).

The following lemmata and remarks are useful.

To put our first lemma into context, we recall that McGarvey (1953) proved that every tournament may be obtained as the domination relation $\succ_{R^N}$ of some profile of preferences R^N. Our lemma strengthens this result by insisting that the contest between any two alternatives be tight, i.e., decided by a one-vote difference.[3]

Lemma 6.1 *For every tournament $T = (A, \succ)$ there exists a finite set N of voters and a preference profile $R^N \in L(A)^N$ such that n is odd and for all $\alpha, \beta \in A$,*

$$\alpha \succ \beta \Rightarrow |\{i \in N \mid \alpha \ R^i \ \beta\}| = \frac{n+1}{2}. \tag{6.1}$$

Proof: We proceed by induction on $m = |A|$. If $m = 2$, then $\succ$ is a linear preference and the statement is true (with $n = 1$ and $R^1 = \ \succ$). If $m > 2$, then select $\alpha_0 \in A$, define $A_0 = A \setminus \{\alpha_0\}$ and let $\succ_0$ be the restriction of $\succ$ to A_0. By the inductive hypothesis there is a set N_0 with an odd number of elements and $R_0^{N_0} \in L(A_0)^{N_0}$ such that

$$\alpha, \beta \in A_0, \alpha \succ_0 \beta \Rightarrow |\{i \in N_0 \mid \alpha \ R_0^i \ \beta\}| = \frac{n_0+1}{2}.$$

Let $n = n_0 + 2$, $B = \{\beta \in A_0 \mid \alpha_0 \succ \beta\}$, and let $R_0 \in L(A_0)$ such that, for all $\alpha \in B$ and all $\beta \in A_0 \setminus B$, $\alpha \ R_0 \ \beta$. Put $k_0 = |A_0 \setminus B|$. Moreover, let R_0^* be the reverse linear preference of R_0. Now, define $R^i \in L(A)$ for all $i \in N$ as follows (see Table 6.1). If $i \leq \frac{n_0+1}{2}$, then let R^i be the linear preference that coincides with R_0^i on A_0 and ranks α_0 first, that is, $t_1(R^i) = \alpha_0$ and $t_{k+1}(R^i) = t_k(R_0^i)$ for $k = 1, \ldots, m-1$. If $\frac{n_0+1}{2} < i \leq n_0$, then let R^i be the linear preference that coincides with R_0^i on A_0 and ranks α_0 last, that is, $t_k(R^i) = t_k(R_0^i)$ for $k = 1, \ldots, m-1$ and $t_m(R^i) = \alpha_0$. Also, let R^{n_0+1} be the ordering that coincides with R_0 on A_0 and ranks α_0 last, that is, $t_k(R^{n_0+1}) = t_k(R_0)$ for $k = 1, \ldots, m-1$ and $t_m(R^{n_0+1}) = \alpha_0$. Finally, let R^n be the ranking that coincides with R_0^* on A_0 and ranks α_0 between the elements of $A_0 \setminus B$ and the members of B, that is, $t_i(R^n) = t_i(R_0^*)$ for $i = 1, \ldots, k_0$, $t_{k_0+1}(R^n) = \alpha_0$, and $t_{j+1}(R^n) = t_j(R_0^*)$ for $j = k_0 + 1, \ldots, m-1$. The pair (N, R^N) satisfies the desired properties. **q.e.d.**

Notation 6.2 Let $(A, \succ)$ be a tournament and $\beta \in A$. Denote

$$A_{\succ}^{+}(\beta) = A^{+}(\beta) = \{\alpha \in A \mid \beta \succ \alpha\}, \ A_{\succ}^{-}(\beta) = A^{-}(\beta) = \{\alpha \in A \mid \alpha \succ \beta\}.$$

[3] Incidentally, the smallest number of voters n that is needed for McGarvey's theorem (as a function of m) has been studied in the literature. McGarvey's original proof (which allowed also to prescribe ties between pairs of alternatives) required $n = m(m-1)$, and subsequent research (see Stearns (1959) and Erdős and Moser (1964)) has shown that $n = O(\frac{m}{\log m})$ suffices and is the right order of magnitude. Our proof requires $n = 2m - 3$.

Table 6.1: Sketch of a Profile R^N

R^1	$\cdots$	$R^{\frac{n_0+1}{2}}$	$R^{\frac{n_0+3}{2}}$	$\cdots$	R^{n_0}	R^{n_0+1}	R^n
α_0	$\cdots$	α_0	$t_1\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_1(R_0^{n_0})$	$t_1(R_0)$	$t_{m-1}(R_0)$
$t_1(R_0^1)$	$\cdots$	$t_1\left(R_0^{\frac{n_0+1}{2}}\right)$	$t_2\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_2(R_0^{n_0})$	$t_2(R_0)$	$t_{m-2}(R_0)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$t_{k_0-1}(R_0^1)$	$\cdots$	$t_{k_0-1}\left(R_0^{\frac{n_0+1}{2}}\right)$	$t_{k_0}\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_{k_0}(R_0^{n_0})$	$t_{k_0}(R_0)$	$t_{m-k_0}(R_0)$
$t_{k_0}(R_0^1)$	$\cdots$	$t_{k_0}\left(R_0^{\frac{n_0+1}{2}}\right)$	$t_{k_0+1}\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_{k_0+1}(R_0^{n_0})$	$t_{k_0+1}(R_0)$	α_0
$t_{k_0+1}(R_0^1)$	$\cdots$	$t_{k_0+1}\left(R_0^{\frac{n_0+1}{2}}\right)$	$t_{k_0+2}\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_{k_0+2}(R_0^{n_0})$	$t_{k_0+2}(R_0)$	$t_{m-k_0-1}(R_0)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$t_{m-2}(R_0^1)$	$\cdots$	$t_{m-2}\left(R_0^{\frac{n_0+1}{2}}\right)$	$t_{m-1}\left(R_0^{\frac{n_0+3}{2}}\right)$	$\cdots$	$t_{m-1}(R_0^{n_0})$	$t_{m-1}(R_0)$	$t_2(R_0)$
$t_{m-1}(R_0^1)$	$\cdots$	$t_{m-1}\left(R_0^{\frac{n_0+1}{2}}\right)$	α_0	$\cdots$	α_0	α_0	$t_1(R_0)$

Our next lemma asserts the existence of tournaments in which every alternative beats exactly half of the other alternatives, and it never happens that all the alternatives that beat a given alternative are in turn beaten by (or equal to) another alternative.

Lemma 6.3 *There exist infinitely many positive integers m such that there exists a tournament $T = (A, \succ)$ with $|A| = m$ that satisfies the following properties:*

$$|A^+(\alpha)| = |A^-(\alpha)| = \tfrac{m-1}{2} \textit{ for all } \alpha \in A. \tag{6.2}$$

$$A^-(\alpha) \neq A^+(\beta) \textit{ for all } \alpha, \beta \in A. \tag{6.3}$$

$$\textit{For all } \alpha \in A \textit{ and } \beta \in A^-(\alpha) \textit{ there exists } \gamma \in A^-(\alpha) \setminus \{\beta\} \textit{ such that } \gamma \succ \beta. \tag{6.4}$$

Proof: The set $Q = \{p \in \mathbb{N} \mid p \text{ is a prime such that } p \equiv 3 \bmod 4\}$ is infinite. Let $p \in Q$, $p > 3$. Let $\mathbb{Z}_p = \{0, \ldots, p-1\}$ denote the field of residue classes modulo p and let $A = \mathbb{Z}_p$. Let $\succ$ on A be defined by $\alpha \succ \beta$ iff $\alpha, \beta \in A$ and $\alpha - \beta$ is a quadratic residue modulo p (for the definition of quadratic residues and their basic properties that we use below, see e.g. Chapter VI of Hardy and Wright (1979)). It suffices to prove that $(A, \succ)$ satisfies the desired properties.

The fact that $(A, \succ)$ is a tournament that satisfies property (6.2) is an immediate consequence of the following claim.

Claim 1: The set of quadratic residues mod p contains exactly one element of every set $\{t, p-t\}$ for every $t \in A \setminus \{0\}$.

Assume the contrary. As there are $\frac{p-1}{2}$ quadratic residues mod p, there exists $t \in A \setminus \{0\}$ such that t and $p-t$ are both quadratic residues. So, there are $a, b \in A$ such that $a^2 \equiv t \bmod p$ and $b^2 \equiv -t \bmod p$. Thus, $a^2 \equiv -b^2 \bmod p$. Let $c \in A$ be the inverse of b, that is, $bc \equiv 1 \bmod p$. Then $(ac)^2 \equiv -1 \bmod p$. We conclude that $(ac)^{p-1} \equiv (-1)^{\frac{p-1}{2}} \bmod p$. As $p \equiv 3 \bmod 4$, $\frac{p-1}{2}$ is odd and, hence, $(ac)^{p-1} \equiv -1 \bmod p$. On the other hand, by Fermat's theorem, $(ac)^{p-1} \equiv 1 \bmod p$ and the desired contradiction has been obtained.

The following claim enables us to show that (6.3) and (6.4) are satisfied.

Claim 2: The prime p divides the sum of all quadratic residues mod p.

If s denotes this sum, then since every quadratic residue is the square of two residues modulo p, $2s \equiv \sum_{a \in \mathbb{Z}_p} a^2 \bmod p$. As $\mathbb{Z}_p$ is a field and $p \neq 2$,

$$4 \sum_{a \in \mathbb{Z}_p} a^2 = \sum_{a \in \mathbb{Z}_p} (2a)^2 \equiv \sum_{a \in \mathbb{Z}_p} a^2 \bmod p.$$

We conclude that $3s \equiv 0 \bmod p$. As $p > 3$, $s \equiv 0 \bmod p$.

In order to show (6.3) we assume, on the contrary, that $A^-(\alpha) = A^+(\beta)$. By Claim 2,

$$\sum_{\gamma \in A^-(\alpha)} (\gamma - \alpha) = \sum_{\gamma \in A^-(\alpha)} \gamma - \frac{p-1}{2}\alpha \equiv 0 \bmod p$$

and

$$\sum_{\gamma \in A^+(\beta)} (\beta - \gamma) = \frac{p-1}{2}\beta - \sum_{\gamma \in A^+(\beta)} \gamma \equiv 0 \bmod p.$$

By the assumption, $\frac{p-1}{2}(\beta - \alpha) \equiv 0 \bmod p$, which is impossible.

In order to show (6.4) we assume, on the contrary, that there exists $\beta \in A^-(\alpha)$ such that $\beta \succ \gamma$ for all $\gamma \in A^-(\alpha) \setminus \{\beta\}$. Hence, $A^-(\alpha) \setminus \{\beta\} = A^+(\beta) \setminus \{\alpha\}$. Claim 2 yields

$$\sum_{\gamma \in A^-(\alpha) \setminus \{\beta\}} (\gamma - \alpha) = \sum_{\gamma \in A^-(\alpha) \setminus \{\beta\}} \gamma - \frac{p-3}{2}\alpha \equiv (\alpha - \beta) \bmod p$$

and

$$\sum_{\gamma \in A^+(\beta) \setminus \{\alpha\}} (\beta - \gamma) = \frac{p-3}{2}\beta - \sum_{\gamma \in A^+(\beta) \setminus \{\alpha\}} \gamma \equiv (\alpha - \beta) \bmod p.$$

By the assumption, $\frac{p+1}{2}(\beta - \alpha) \equiv 0 \bmod p$, which is impossible. **q.e.d.**

For any set A of m alternatives let prob_A be the uniform probability measure on $L(A)$, that is, $\text{prob}_A : 2^{L(A)} \to \mathbb{R}$ is defined by $\text{prob}_A(T) = \frac{|T|}{m!}$ for all $T \subseteq L(A)$.

The next few lemmata and remarks establish some facts about the relative frequency of linear preferences that display some desirable patterns. These facts are conveniently expressed in terms of the uniform probability measure on $L(A)$.

Remark 6.4 Let $\alpha, \gamma \in A$, $\alpha \neq \gamma$, and let $Z \subseteq A \setminus \{\alpha, \gamma\}$. Then

$$\text{prob}_A(\{R \in L(A) \mid \exists\, \zeta \in Z \text{ such that } \alpha\ R\ \zeta \text{ and } \gamma\ R\ \zeta\}) = \frac{|Z|}{|Z|+2}. \tag{6.5}$$

Indeed, we may assume that $A = Z \cup \{\alpha, \gamma\}$. Let $z = |Z|$. There are $(m-1)!$ elements R of $L(A)$ such that $t_m(R) = \alpha$. A similar statement is valid for γ. We conclude that

$$|\{R \in L(A) \mid t_m(R) \in Z\}| = m! - 2(m-1)! = (m-2)(m-1)! = z(m-1)!$$

and, hence, (6.5) is true.

Lemma 6.5 *Let $t \in \mathbb{Z}$ such that $t \geq 0$ and $2t+1 \leq m$. Let $\alpha, \beta_r, \gamma_r \in A$, $r = 1, \ldots, t$, be $2t+1$ distinct elements and define for any $r = 0, \ldots, t$,*

$$c_r = \text{prob}_A(\{R \in L(A) \mid \exists\, k \in \{1, \ldots, r\} \textit{ such that } \alpha\ R\ \gamma_k\ R\ \beta_k\}).$$

Then $c_0 = 0$, and

$$c_r = \frac{1}{2r+1}\left(\frac{2^r - 1}{2^r} + 2rc_{r-1}\right) \textit{ for all } r = 1, \ldots, t. \tag{6.6}$$

Proof: Clearly $c_0 = 0$. Let $r \in \{1, \ldots, t\}$. We may assume that $m = 2r+1$. There are $\frac{2^r-1}{2^r}(m-1)!$ preferences $R \in L(A)$ with the properties that $t_1(R) = \alpha$ and that $\gamma_k\ R\ \beta_k$ for some $k = 1, \ldots, r$. Also, for every $k = 1, \ldots, r$, there are $(m-1)c_{r-1}(m-2)!$ preferences $R \in L(A)$ such that $t_1(R) = \beta_k$ and $\alpha\ R\ \gamma_\ell\ R\ \beta_\ell$ for some $\ell \in \{1, \ldots, r\} \setminus \{k\}$, because the rank of γ_k is any element of $2, \ldots, m$. The same number of preferences occurs, if γ_k is ranked first. We conclude that there are

$$d_r = \frac{2^r - 1}{2^r}(m-1)! + 2rc_{r-1}(m-1)!$$

preferences $R \in L(A)$ such that $\alpha\ R\ \gamma_k\ R\ \beta_k$ for some $k = \{1, \ldots, r\}$. Equation (6.6) follows, because $c_r = \frac{d_r}{m!}$. **q.e.d.**

Remark 6.6 Let $c_0 = 0$. Successive computation of $c_1, \ldots, c_6$ via (6.6) yields that $c_6 > \frac{1}{2}$.

Lemma 6.7 *For any tournament $T = (A, \succ)$ with $m \geq 453$ that satisfies (6.2) – (6.4) the following holds true: For every $\alpha \in A$ and every mapping $h : A^-(\alpha) \to A$ such that $h(\beta) \succ \beta$ for all $\beta \in A^-(\alpha)$,*

$$\text{prob}_A(\{R \in L(A) \mid \exists\, \beta \in A^-(\alpha) \textit{ such that } \alpha\ R\ h(\beta)\ R\ \beta\}) > \frac{1}{2}. \tag{6.7}$$

Proof: Two cases may be distinguished.

Case 1: There exists $\gamma \in A$ such that $|h^{-1}(\gamma)| \geq 23$. By Lemma 6.3 there exists $\beta \in A^-(\alpha)$ such that $\gamma \notin \{\beta, h(\beta)\}$. Let $Z = h^{-1}(\gamma)$. Let

$$L_1 = \{R \in L(A) \mid \exists\, \zeta \in Z \text{ such that } \alpha\ R\ \gamma\ R\ \zeta\} \text{ and } L_2 = \{R \in L(A) \mid \gamma\ R\ \alpha\ R\ h(\beta)\ R\ \beta\}.$$

Then $L_1 \cap L_2 = \emptyset$. As $Z \subseteq A^-(\alpha)$ and as $\beta \in A^-(\alpha)$, it suffices to show that $\text{prob}_A(L_1) + \text{prob}_A(L_2) > \frac{1}{2}$. Now, $\text{prob}_A(L_2) = \frac{1}{4!}$ and, by Remark 6.4,

$$\text{prob}_A(L_1) = \frac{1}{2}\frac{|Z|}{|Z|+2} = \frac{1}{2} - \frac{1}{|Z|+2} \geq \frac{1}{2} - \frac{1}{25} > \frac{1}{2} - \frac{1}{4!},$$

where $|Z| \geq 23$ implies the weak inequality.

Case 2: For all $\gamma \in A$, $|h^{-1}(\gamma)| \leq 22$. In this case, we may choose pairwise distinct alternatives $\beta_1, \gamma_1, \beta_2, \gamma_2, \ldots, \beta_6, \gamma_6$ so that $\beta_k \in A^-(\alpha)$ and $\gamma_k = h(\beta_k)$ for $k = 1, \ldots, 6$. Such a choice may be achieved inductively, by selecting

$$\beta_k \in A^-(\alpha) \setminus \bigcup_{i=1}^{k-1} [\{\gamma_i\} \cup h^{-1}(\{\beta_i, \gamma_i\})] \tag{6.8}$$

and letting $\gamma_k = h(\beta_k)$. By the assumption of this case, the set appearing in square brackets in (6.8) has at most 45 elements, and therefore the union in (6.8) has at most 225 elements. As we are assuming that $m \geq 453$, we have $|A^-(\alpha)| = \frac{m-1}{2} \geq 226$, and therefore the choice indicated in (6.8) is feasible. Now, the probability in question is at least

$$\text{prob}_A(\{R \in L(A) \mid \exists\ k \in \{1, \ldots, 6\} \text{ such that } \alpha\ R\ \gamma_k\ R\ \beta_k\}).$$

The proof is complete by Lemma 6.5 and Remark 6.6. **q.e.d.**

Now we are able to construct simple majority voting games with an odd number of players whose Mas-Colell bargaining sets are empty. Let $T = (A, \succ)$ be a tournament with $m \geq 453$ that satisfies (6.2) – (6.4). Lemma 6.3 guarantees the existence of T. Let N_0, n_0 odd, and $Q_0^{N_0} \in L(A)^{N_0}$ be such that (6.1) is satisfied (for $N = N_0$ and $R^N = Q_0^{N_0}$). Lemma 6.1 guarantees the existence of N_0 and $Q_0^{N_0}$. Let N be obtained from N_0 by adding $k \cdot m!$ new voters and let Q^N be obtained from $Q_0^{N_0}$ by assigning each preference of $L(A)$ to k of the new voters. Note that (6.1) remains valid for $R^N = Q^N$. Moreover, we assume that k is sufficiently large such that the following condition is satisfied. The empirical distribution of preferences in Q^N is close enough to the uniform distribution so that the conclusion of Lemma 6.7 holds true when prob_A is replaced by this empirical distribution, that is, by the probability measure prob on $L(A)$ that is determined by $\text{prob}(\{R\}) = \frac{|\{i \in N | R = Q^i\}|}{n}$ for all $R \in L(A)$. Lemma 6.7 guarantees the existence of k.

In order to continue our construction, the following definitions and simple lemma are useful.

A vector $\vec{\alpha} = (\alpha_i)_{i \in N}$, $\alpha_i \in A$ for all $i \in N$, is a *position*. Let $\vec{\alpha}$ be a position and $\beta \in A$. We say that $\vec{\alpha}$ *enhances* β (at Q^N) if for every $\gamma \in A$ such that $\gamma \succ \beta$ there exists $i \in N$ such that $\alpha_i\ Q^i\ \gamma\ Q^i\ \beta$ and $\alpha_i \neq \gamma$. Note that the set of positions is partially ordered. Indeed, let $\vec{\alpha}$ and $\vec{\beta}$ be positions. Then define $\vec{\alpha} \geq \vec{\beta}$ iff $\alpha_i\ Q^i\ \beta_i$ for all $i \in N$. Note that if $\vec{\alpha} \geq \vec{\beta}$ and $\vec{\beta}$ enhances an alternative γ, then $\vec{\alpha}$ enhances γ as well.

We call a position $\vec{\alpha}$ *non-enhancing* (at Q^N) if it does not enhance any $\beta \in A$. If, in addition, every position $\vec{\beta} \geq \vec{\alpha}$, $\vec{\beta} \neq \vec{\alpha}$, enhances some $\gamma \in A$, then we call $\vec{\alpha}$ *maximal* non-enhancing (MNE). Note that for any non-enhancing position $\vec{\alpha}$ there exists an MNE position $\vec{\beta}$ such that $\vec{\beta} \geq \vec{\alpha}$.

Lemma 6.8 *If $\vec{\alpha}$ is a non-enhancing position and if $\alpha \in A$, then* $|\{i \in N \mid \alpha_i \ Q^i \ \alpha\}| < \frac{n}{2}$.

Proof: Let $S = \{i \in N \mid \alpha_i \ Q^i \ \alpha\}$ and $\beta \in A^-(\alpha)$. As $\vec{\alpha}$ does not enhance β there exists $h(\beta) \in A$ such that $h(\beta) \succ \beta$ and, for all $i \in N$, if $\alpha_i \ Q^i \ h(\beta) \ Q^i \ \beta$, then $\alpha_i = h(\beta)$. As $h(\beta) \neq \alpha$, $\{i \in N \mid \alpha \ Q^i \ h(\beta) \ Q^i \ \beta\} \subseteq N \setminus S$. Therefore $h : A^-(\alpha) \to A$ is a function as in Lemma 6.7 and $\{i \in N \mid \exists \ \beta \in A^-(\alpha)$ such that $\alpha \ Q^i \ h(\beta) \ Q^i \ \beta\} \subseteq N \setminus S$. By Lemma 6.7 and construction, $|N \setminus S| > \frac{n}{2}$, and the proof is complete. **q.e.d.**

Construction (cont.): Let $A^* = \{\vec{\alpha}^* \mid \vec{\alpha}$ is an MNE position of $Q^N\}$ be a set, whose cardinality is the number of MNE positions, of alternatives such that $A \cap A^* = \emptyset$. For every voter $i \in N$ let $R^i \in L(A \cup A^*)$ be a preference that arises from Q^i by inserting every alternative in A^* into Q^i in such a way that

$$\vec{\alpha}^* \ R^i \ \alpha \Leftrightarrow \alpha_i \ Q^i \ \alpha \text{ for all } \alpha \in A \text{ and all } \vec{\alpha}^* \in A^*. \tag{6.9}$$

In other words, the new alternative that corresponds to the position $\vec{\alpha}$ is inserted just above $\vec{\alpha}$. The internal order between new alternatives that are inserted in the same slot is immaterial. Note that, by Lemma 6.8, in the tournament associated with R^N, $\succ_{R^N}$ (see Notation 2.1), every $\alpha \in A$ beats any $\vec{\alpha}^* \in A^*$, i.e.,

$$\alpha \succ_{R^N} \vec{\alpha}^* \text{ for all } \alpha \in A, \vec{\alpha}^* \in A^*. \tag{6.10}$$

We proceed to show that the Mas-Colell bargaining set of the simple majority voting game that corresponds to R^N via some utility representation is empty. We first present the idea of the construction and proof verbally. In Q^N, the non-enhancing positions correspond to payoff vectors that admit no justified objection. In order to prevent those vectors from belonging to $\mathcal{MB}$, we added in R^N new alternatives that render them non-weakly Pareto optimal. Of course there is a danger that by doing this we introduce new candidates for belonging to $\mathcal{MB}$. Condition (6.10) is crucial for avoiding this, and in order to guarantee it we had to do the long preparatory work.

Let $u^N \in \mathcal{U}^{R^N}$ and $V = V_{u^N}$.

Proposition 6.9 $\mathcal{MB}(N, V) = \emptyset$.

Proof: Assume, on the contrary, that there is some $x \in \mathcal{MB}(N, V)$. Let $y \in \mathbb{R}^N$ be defined by $y^i = \min\{u^i(\alpha) \mid \alpha \in A \cup A^*, u^i(\alpha) \geq x^i\}$ for all $i \in N$. Then $y \in \mathcal{MB}(N, V)$ as well. Moreover, there is a position $\vec{\alpha}$ of R^N such that $y^i = u^i(\alpha_i)$ for all $i \in N$. As $y \in \mathcal{MB}(N,V)$, the position $\vec{\alpha}$ has the following properties:

$$\exists\ \alpha \in A \cup A^* \text{ such that } \alpha\ R^i\ \alpha_i\ \forall\ i \in N. \tag{6.11}$$

$$\nexists\ \beta \in A \cup A^* \text{ such that } \beta\ R^i\ \alpha_i \text{ and } \beta \neq \alpha_i\ \forall\ i \in N. \tag{6.12}$$

$$\nexists\ \beta \in A \text{ such that } |\{i \in N \mid \beta\ R^i\ \alpha_i, \beta \neq \alpha_i\}| > \tfrac{n}{2} \text{ and } \vec{\alpha} \text{ enhances } \beta \text{ at } R^N. \tag{6.13}$$

Indeed, (6.11) and (6.12) are true, because $y \in V(N)$ and y is weakly Pareto optimal. In order to show (6.13) let $\beta \in A$ satisfy $|S| > \frac{n}{2}$, where $S = \{i \in N \mid \beta\ R^i\ \alpha_i, \beta \neq \alpha_i\}$. Then $(S, u^S(\beta))$ is an objection against y. Hence, there exist $\gamma \in A \setminus \{\beta\}$ and $T \subseteq N$, $|T| > n/2$ such that $u^i(\gamma) \geq \max\{y^i, u^i(\beta)\}$ for all $i \in T$ (note that γ must be in A, rather than A^*, due to (6.10)). By Lemma 6.1, $T = \{i \in N \mid \gamma\ R^i\ \beta\}$ and $|T| = \frac{n+1}{2}$. Hence, $\vec{\alpha}$ does not enhance β.

Claim 1: The position $\vec{\alpha}$ does not enhance any $\beta \in A$.

In view of (6.13) we may assume that $|\{i \in N \mid \alpha_i\ R^i\ \beta\}| > \frac{n}{2}$. Let $\alpha \in A \cup A^*$ satisfy (6.11). Then $|\{i \in N \mid \alpha\ R^i\ \beta\}| > \frac{n}{2}$. Therefore, either $\alpha = \beta$ or $\alpha \succ_{R^N} \beta$. If $\alpha = \beta$, then $\alpha_i\ R^i\ \gamma\ R^i\ \beta$ for some $i \in N$ implies that $\alpha_i = \gamma$. If $\alpha \succ_{R^N} \beta$, then $\alpha \in A$ by (6.10), and $\alpha_i\ R^i\ \alpha\ R^i\ \beta$ for some $i \in N$ implies $\alpha_i = \alpha$. So, in both cases $\vec{\alpha}$ does not enhance β.

Claim 2: There exists a position $\vec{\beta}$ satisfying $\beta_i \in A$ and $\beta_i\ R^i\ \alpha_i$ for all $i \in N$ such that $\vec{\beta}$ does not enhance any member of A (that is, $\vec{\beta}$ is non-enhancing at Q^N).

Let $i \in N$ and let $\delta_i = t_1(Q^i)$ (that is, i's best alternative in A). We show now that $\delta_i\ R^i\ \alpha_i$. Assume, on the contrary, $\alpha_i\ R^i\ \delta_i$, $\alpha_i \neq \delta_i$. Let γ_i be i's lowest alternative in A, that is, $\gamma_i = t_m(Q^i)$. If $\delta \in A \cup A^*$ satisfies $\delta \succ_{R^N} \gamma_i$, then $\delta \in A$ by (6.10). Moreover, $\alpha_i\ R^i\ \delta\ R^i\ \gamma_i$ and $\alpha_i \neq \delta$. Hence, $\vec{\alpha}$ enhances γ_i and a contradiction to Claim 1 is established. Let β_i be i's lowest alternative in A weakly above α_i, that is, $\beta_i \in A$, $\beta_i\ R^i\ \alpha_i$, and $\beta_i'\ R^i\ \alpha_i$ implies $\beta_i'\ Q^i\ \beta_i$ for all $\beta_i' \in A$. By construction, since $\vec{\alpha}$ does not enhance any $\beta \in A$, neither does $\vec{\beta}$. Claim 2 has been shown.

Select any MNE position $\vec{\beta}$ at Q^N that satisfies the conditions of Claim 2. By (6.9), $\vec{\beta}^*\ R^i\ \beta_i$ and $\vec{\beta}^* \neq \beta_i$ for all $i \in N$. Combined with the fact that $\beta_i\ R^i\ \alpha_i$ holds for all $i \in N$ this contradicts (6.12). **q.e.d.**

7 Two Models with Many Voters

We present here two models, in which special assumptions about the distribution of preferences in the population of voters lead to existence results when there are many voters.

The first model is probabilistic. Let A be a fixed set of m alternatives, and let $L = L(A)$. We assume that each $R \in L$ appears with positive probability $p_R > 0$ in the population of potential voters, where $\sum_{R\in L} p_R = 1$. Now let $(\mathcal{R}^i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables such that $Pr(\mathcal{R}^i = R) = p_R$ for all $i \in \mathbb{N}$, $R \in L$. Let $\mathcal{R}^N = (\mathcal{R}^1, \ldots, \mathcal{R}^n)$ be the corresponding random profile of preferences for n voters, and let $(N, V(\mathcal{R}^N))$ be the random simple majority voting game that is associated via some utility representation for each realization R^N of $\mathcal{R}^N$.

Theorem 7.1 $\lim_{n\to\infty} Pr\left(\mathcal{M}\left(N, V\left(\mathcal{R}^N\right)\right) \neq \emptyset\right) = \lim_{n\to\infty} Pr\left(\mathcal{MB}\left(N, V\left(\mathcal{R}^N\right)\right) \neq \emptyset\right) = 1.$

Proof: Call $R^N \in L^N$ *good* if for all $\alpha \in A$ there exists $i \in N$ such that $\alpha = t_m(R^i)$. If R^N is good, then $0 \in \mathcal{M}(N, V_{u^N})$ for any $u^N \in \mathcal{U}^{R^N}$. Regarding $\mathcal{MB}(N, V_{u^N})$ when R^N is good and $u^N \in \mathcal{U}^{R^N}$, we distinguish two cases. If there is a weak Condorcet winner α, then $u^N(\alpha) \in \mathcal{MB}(N, V_{u^N})$. If no such α exists, then $0 \in \mathcal{MB}(N, V_{u^N})$. Thus we see that in order to prove both parts of the theorem, it suffices to show that $\mathcal{R}^N$ is good with probability tending to 1 as n tends to infinity. This fact is easily checked. **q.e.d.**

The second model involves replication. Let A be a fixed set of m alternatives, and let $L = L(A)$. Let $N = \{1, \ldots, n\}$, let $R^N \in L^N$, and let $u^N \in \mathcal{U}^{R^N}$. In order to replicate the simple majority voting game (N, V_{u^N}), let $k \in \mathbb{N}$ and denote

$$kN = \{(j,i) \mid i \in N,\ j = 1, \ldots, k\}.$$

Furthermore, let $R^{(j,i)} = R^i$ and $u^{(j,i)} = u^i$ for all $i \in N$ and $j = 1, \ldots, k$. Then $(kN, V_{u^{kN}})$ is the *k-fold replication* of (N, V_{u^N}).

Theorem 7.2 *If* $k \geq \left\{\begin{array}{ll} n+2 & \text{, if } n \text{ is odd,} \\ \frac{n}{2}+2 & \text{, if } n \text{ is even,} \end{array}\right\}$ *then* $\mathcal{MB}(kN, V_{u^{kN}}) \neq \emptyset$.

Proof: If α is a weak Condorcet winner with respect to R^N, then $u^{kN}(\alpha) \in \mathcal{MB}(kN, V_{u^{kN}})$. Hence we may assume that for every $\alpha \in A$ there exists $\beta(\alpha) \in A$ such that $\beta(\alpha) \succ_{R^N} \alpha$. Let $\widetilde{x} \in \mathbb{R}^N_+$ be any weakly Pareto optimal element in $V_{u^N}(N)$. We define $x \in \mathbb{R}^{kN}$ by $x^{(1,i)} = \widetilde{x}^i$ and $x^{(j,i)} = 0$ for all $i \in N$ and $j = 2, \ldots, k$ and claim that $x \in \mathcal{MB}(kN, V_{u^{kN}})$. Let (P, y) be an objection at x. Then there exists $\alpha \in A$ such that $y = u^P(\alpha)$. Let $\beta = \beta(\alpha)$ and let $T = \{i \in N \mid \beta\ R^i\ \alpha\}$. Then

$$|T| \geq \begin{cases} \frac{n+1}{2} & \text{, if } n \text{ is odd,} \\ \frac{n}{2}+1 & \text{, if } n \text{ is even.} \end{cases} \tag{7.1}$$

Let $Q = \{(j,i) \mid i \in T, j = 2, \ldots, k\}$ and define $z \in \mathbb{R}^Q$ by $z^{(j,i)} = u^i(\beta)$ for all $i \in T$ and $j = 2, \ldots, k$. Then $|Q| = (k-1)|T|$ and $z > (y^{P\cap Q}, x^{Q\setminus P})$. By (7.1), $|Q| \geq \frac{kn+1}{2}$. So, (Q, z) is a counter objection to (P, y). **q.e.d.**

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