Substitute Valuations, Auctions, and Equilibrium with Discrete Goods*

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Abstract

For economies in which goods are available in several (discrete) units, this paper identifies *two* notions of substitutes. The weaker notion guarantees monotonicity of tâtonnement processes and convergence of clock auctions to a *pseudo-equilibrium*, but only the stronger notion, which treats each unit traded as a distinct good with its own price, guarantees that every pseudo-equilibrium is a Walrasian equilibrium, the Vickrey outcome is in the core, and the "law of aggregate demand" is satisfied. The paper provides several characterizations and properties of weak and strong substitutes.

Keywords: Substitute Valuation, Auction, Discrete Goods, Nonlinear Price, Submodular Dual, Walrasian Equilibrium, Pseudo-Equilibrium, Law of Aggregate Demand, Vickrey Auction.

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1 Introduction

In the neoclassical theory of the firm, the notion of inputs is defined only after classes of goods are specified. Inputs are substitutes if when the price of one type of input rises, the number of units demanded of the other types cannot fall. But what are "types" of inputs? If electricity generated at locations A and B are perfectly substitutable in production, should we regard these as one class of input or two? The answer would seem to depend on whether the inputs can have different prices. To investigate the distinction, we will say that the firm has a *weak-substitute valuation* when the substitutes condition is satisfied for distinct types of goods and a *strong-substitute valuation* when, in addition, it is satisfied even for individual goods of the same type. The biggest surprises in our analysis are that even in very ordinary-looking problems, the two notions of substitution have very different implications, even for the study of *linear* pricing equilibria.

We illustrate the distinction with simple examples. Suppose that the price of output is one and that the amount of output produced by a firm $f(x, y)$ is a function of two types of discrete inputs $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$, as follows:

f	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0	1	$\sqrt{2}$
$x = 1$	1	1	$\sqrt{2}$

f is submodular in its two arguments and has nonincreasing marginal returns.[1] The firm chooses x and y to maximize $f(x, y) - rx - wy$. Since f is submodular, the inputs x and y are substitutes.[2] Substitutes means that when comparing any two price vectors p and p' for which the firm's optimum is unique, if $p \geq p'$ and $p_i = p'_i$, then the demand for good i is weakly higher at prices p.

Next, consider a formulation in which the two units of input y are treated as distinct.

[1]It is easy to see that functionally identical inputs can fail to be substitutes for one another in the usual sense of price theory when there are increasing marginal returns to that type of input. In order to be clear that this is not what underlies our example, we chose f with nonincreasing marginal returns.

[2]Submodularity guarantees the substitutes property when the economy has two goods.

Let $y = y_1 + y_2$ and suppose $y_1, y_2 \in \{0, 1\}$. In this formulation, the prices are also potentially distinct, so the firm maximizes $f(x, y_1 + y_2) - rx - w_1y_1 - w_2y_2$. It is as if we had distinguished blue and red versions of the input, where the color is devoid of any consequences for production. It is easy to check that if the input prices are $(r, w_1, w_2) = (0.2, 0.3, 0.2)$, then the firm's unique profit-maximizing input vector is (0,1,1), but if $(r, w_1, w_2) = (0.2, 0.3, 0.7)$, then the profit-maximizing choice is (1,0,0). This demonstrates that an increase in the price of input y_2 reduces the demand for input y_1: different units of the same type of good may fail to be substitutes.

Examples of this sort are hardly rare. For instance, an airline that is acquiring landing slots at a hub airport may wish to have some number N of slots, for illustration N=2, within a particular period, say from 2:00pm to 2:15pm or from 3:00pm to 3:15pm. The two periods define weak substitutes if when slots at 2-2:15 are expensive, the airline will substitute slots at 3-3:15. Slots within a given time period, however, need not be substitutes. As in our example, the airline may demand both or neither and this can happen even with diminishing returns to additional slots in the same time period. Because clock auctions have been proposed for just this sort of application, it is important to investigate how these auctions perform in settings where slots are weak substitutes but not strong substitutes.

Despite the practical significance of the weak substitutes condition, it has not been analyzed in previous studies of ascending clock auctions, which have instead emphasized the strong substitutes case. Ausubel (2006, p. 16) mentions that his clock auction design, which applies when goods are distinct and substitutes, can also be applied when there are multiple units of each good. As our examples show, however, that extension entails a stronger condition than weak substitutes and even than weak substitutes plus diminishing marginal returns to each good. Gul and Stacchetti (2000) restrict their auction design to nonidentical goods, in effect assuming strong substitutes.

One important difference between the weak and strong substitutes arises when studying the existence of market-clearing prices. Using models in which goods are priced individually, Kelso and Crawford (1982) establish that when distinct goods are substitutes, market-clearing prices always exist. Gul and Stacchetti (2000) and Milgrom (2000) display monotonic auction processes that converge to exact or approximate market-clearing prices.[3] In all of those formulations, substitutes means strong substitutes: the results do

[3] The Gul-Stacchetti analysis assumes that bidder values are integers and their algorithm requires that bidders report their entire demand set at each point. When interpreted as an auction, the requirement

not extend to the case of weak substitutes. For suppose in our example that good y is treated as a single class and that the available supply for the two classes of goods is given by the vector $(1, 2)$. Suppose that firm 1 has valuation f as before, and that there is a second firm with unit valuation $g(x, y) = 1_{y \geq 1}$ (thus, firm 2 is only interested in getting one unit of good y). At the efficient allocation, firm 2 uses one unit of y and firm 1 uses one unit of x. To induce firm 1 to make this choice, the price of input y must be strictly positive, but then firm 1 will strictly prefer not to buy any units of input y and firm 2 will strictly prefer to buy exactly one unit. Hence, there will be a strict excess supply of y: no market clearing prices exist.

In our example, if the supply vector is anything else besides $(1, 2)$, then not only does a market clearing price vector exist, but more is true. First, the set of market clearing price vectors is a sublattice. Second, a continuous *tâtonnement* or *clock auction* process beginning with low prices converges monotonically upward to the minimum market clearing price vector. A similar process beginning with high prices converges monotonically downward to the maximum market clearing price vector. Similar conclusions have been derived in the past using strong substitutes, but not for the weak substitutes of this example.

How does the clock auction perform when there are no market clearing prices? Suppose that firm 1 has valuation f as above, firm 2 has valuation $v(x, y) = .05 \times 1_{y \geq 1}$, supply is $(1, 1)$ and we initially set the input price vector to $(0, 0)$. At that price there is strict excess demand for good y but not for good x. The price of good y is gradually increased. When p_y becomes greater than .05, firm 2's demand drops to 0 units of good y. Eventually, the price reaches a level $\bar{p}_y$ at which firm 1 is indifferent between buying one unit of x or two units of y, as determined by the equation $1 = \sqrt{2} - 2\bar{p}_y$. At that price, firm 1 is indifferent between 1 unit of x or two units of y, while firm 2 is indifferent between 0 and 1 unit of good x (since $p_x = 0$). Aggregate demand thus consists of the bundles $(2, 1)$, $(2, 0)$, $(0, 1)$ and $(0, 2)$, hence contains supply $(1, 1)$ in its convex hull. We define such a situation as a *pseudo-equilibrium*.[4] In this example, there is no Walrasian equilibrium and the clock auction terminates at the minimum pseudo-equilibrium price vector.

Examples of this sort are potentially significant for the design of activity rules in auctions. At prices $(p_x, p_1, p_2) = (.4, .4, .41)$, firm 1 demands $(x, y_1, y_2) = (0, 1, 1)$ while at prices

that demand sets be reported makes their procedure different from any auction process in current use. In contrast, the clock auctions we analyze allow bidders to report a single demand vector for every price vector.

[4]Precisely, this property is equivalent to our definition of pseudo-equilibrium. See Definition 11.

$(p_x, p_1, p_2) = (.4, .5, .41)$, firm 1 demands $(x, y_1, y_2) = (1, 0, 0)$. Suppose these two price vectors represent successive prices in an ascending auction and that the next price vector is $(.5, .5, .41)$. The firm's demand now shifts to $(0, 1, 1)$: its total demand rises from 1 unit to 2 units. Hatfield and Milgrom (2005) had shown that the strong substitutes property implies that a profit-maximizing firm satisfies the *law of aggregate demand*: as prices rise, *the sum of the quantities of goods demanded does not increase.* Activity rules for ascending auctions with or without clocks typically require that the demand expressed during an auction must satisfy that law,[5] and our example shows that such rules can block straightforward bidding when goods are weak substitutes (but not when they are strong substitutes).

These observations herald more general results, which are the subject of this paper. Section 2 defines weak-substitute valuations, based on a multi-unit formulation of the economy, and strong-substitute valuations, based on a binary formulation. Section 3 characterizes weak-substitute and strong-substitute valuations in terms of the firm's dual profit function, which adds transparency to some of our central results. Section 4 further studies the concepts of substitutes and how they are related. Gul and Stacchetti had shown that strong substitutes is equivalent to a certain *single-improvement property* defined using nonlinear prices. We show that it is also equivalent to a similar property defined using only linear prices that strong substitutes is equivalent to weak substitutes plus two additional conditions, and that while the law of aggregate demand may fail with weak substitutes, it always holds when a certain additional assumption is made, which we call the *consecutive-integer property.* This property means that if bundles x and y are both optimal at some given price and $x_k < y_k$ for some good k, then there are also optimal bundles requiring a quantity z_k of good k, for $z_k \in \{x_k, x_k + 1, \ldots y_k\}$. Section 5 treats the implications of weak and strong substitutes for aggregate demand. We show that the strong substitutes condition is sufficient and necessary (in a quantified sense) for the robust existence of market-clearing prices, that the weak substitutes condition implies that the set of pseudo-equilibrium price vectors is a non-empty sublattice, that this set coincides with the set of equilibrium prices whenever an equilibrium exists, and that the strong-substitutes is a sufficient and, in a similar quantified sense, necessary condition for Vickrey payoffs to be in the core. Section 6 presents our analysis of clock auctions when bidders have weak-substitute valuations. We first introduce a continuous model to represent clock auctions. We show that weak substitutes is necessary and sufficient for the monotonicity of a certain tâtonnement-like clock auction and that continuous descending

[5] An exception is the revealed-preference activity rule of Ausubel and Milgrom (2002).

or ascending clock auctions always terminate at a pseudo-equilibrium. In one version of the clock auction model, the auction terminates at the smallest pseudo-equilibrium price. We then show how the analysis can be applied to the case in which prices follow small bid increments bidders only need to announce one optimal bundle, rather than their entire indifference set of optimal bundles. Section 7 concludes and compares our results to the very different results of the divisible goods case obtained by Milgrom and Strulovici (2006).

2 Definitions

Consider an economy with K goods, in which good k is available in N_k units for $k \in \mathcal{K} = \{1, \ldots, K\}$. Let $\mathcal{X} = \Pi_{k\in\mathcal{K}}\{0, 1, \ldots, N_k\}$ and $\tilde{\mathcal{X}} = \Pi_{k\in\mathcal{K}}\{0, 1\}^{N_k}$ represent the space of possible bundles of the exchange economy in its *multi-unit* and *binary* formulations. The obvious correspondence between these formulations is represented by the function $\phi : \tilde{\mathcal{X}}$ into $\mathcal{X}$. Formally, $x_k = \varphi_k(\tilde{x}) = \sum_{j=1}^{N_k} \tilde{x}_{kj}$.

Definition 1 (Multi-Unit Valuation) *A multi-unit valuation v is a mapping from $\mathcal{X}$ into $\mathbb{R}$.*

Definition 2 (Binary Valuation) *A binary valuation $\tilde{v}$ is a mapping from $\tilde{\mathcal{X}}$ into $\mathbb{R}$.*

The binary valuation $\tilde{v}$ *corresponds to* the multi-unit valuation v, if for every $\tilde{x}$, $\tilde{v}(\tilde{x}) = v(\varphi(\tilde{x}))$. We denote by $\mathcal{V}$ the space of multi-unit valuations and $\tilde{\mathcal{V}}$ the space of corresponding binary valuations. Similarly, $\mathcal{P} = \mathbb{R}_+^K$ and $\tilde{\mathcal{P}} = \Pi_{k\in\mathcal{K}}\mathbb{R}_+^{N_k}$ denote the respective price spaces of the multi-unit and binary economies. The first formulation permits only linear prices for each category of goods, while the second effectively allows nonlinear prices for each class of goods, with the marginal price for each good weakly increasing. Throughout the paper, we assume that agents have quasi-linear utilities.

Assumption 1 (Quasi-linearity) The utility of an agent with multi-unit valuation v acquiring a bundle x at price p is $u(x, p) = v(x) - px$. Similarly, the utility of an agent with binary valuation $\tilde{v}$ acquiring a bundle $\tilde{x}$ at price $\tilde{p}$ is $\tilde{u}(\tilde{x}, \tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}$.

Given a binary valuation $\tilde{v}$ and a price vector $\tilde{p} \in \tilde{\mathcal{P}}$, define the demand of the agent at price $\tilde{p}$ by $\tilde{D}(\tilde{p}) = \arg\max_{\tilde{x}\in\tilde{\mathcal{X}}}\{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}$.

Similarly, we define the multi-unit demand D of an agent with valuation v as $D(p) = \arg\max_{x \in \mathcal{X}}\{v(x) - px\}$.

With quasi-linear preferences, there is no distinction to be made between gross and net substitutes, so we drop the modifier and make the following definitions.

Definition 3 (Strong-Substitute Valuation) *A multi-unit valuation v is a* strong-substitute valuation *if its binary form $\tilde{v}$ satisfies the* binary substitutes property*: for any prices $\tilde{p}$ and $\tilde{q}$ in $\tilde{\mathcal{P}}$ such that $\tilde{p} \le \tilde{q}$, and $x \in \tilde{D}(\tilde{p})$, there exists a bundle $\tilde{x}' \in \tilde{D}(\tilde{q})$ such that $\tilde{x}'_{kj} \ge \tilde{x}_{kj}$ for all (k, j) such that $\tilde{p}_{kj} = \tilde{q}_{kj}$.*

Definition 4 (Weak-Substitute Valuation) *A multi-unit valuation v is a* weak-substitute valuation *if it satisfies the* multi-unit substitutes property*: for all prices p and q such that $p \le q$ and $x \in D(p)$, there exists a bundle $x' \in D(q)$ such that $x'_k \ge x_k$ for all k in $\mathcal{K} = \{\kappa \in \mathcal{K} : p_\kappa = q_\kappa\}$.*

The strong substitutes condition is at least weakly more restrictive than the weak substitutes condition, because the latter applies only for linear prices while the former applies also for nonlinear prices. Moreover, the weak substitutes condition only compares units of distinct goods, while the strong substitutes condition requires that units of the *same* good be substitutes. Section 1 illustrates that the two conditions are not equivalent. In particular, weak-substitute valuations can violate the law of aggregate demand, but strong substitute valuations cannot.

3 Duality Results

To any multi-unit valuation v we associate the dual profit function $\pi : \mathcal{P} \to \mathbb{R}$ such that $\pi(p) = \max_{x \in \mathcal{X}}\{u(x, p) = v(x) - px\}$. Similarly, to any binary valuation $\tilde{v}$ we associate the dual profit function $\tilde{\pi}(\tilde{p}) = \max_{\tilde{x} \in \tilde{\mathcal{X}}}\{\tilde{u}(\tilde{x}, \tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}$.

Definition 5 (Multi-Unit Concavity) *A multi-unit valuation is concave if it can be extended to a concave function on $\mathbb{R}^K$.*

Theorem 1 *Let v be a multi-unit valuation and π be its dual profit function. Then, for all $x \in \mathcal{X}$, $v(x) \le \min_{p \in \mathcal{P}}\{\pi(p) + px\}$. Moreover, v is concave if and only if*

$$v(x) = \min_{p \in \mathcal{P}}\{\pi(p) + px\} \text{ for all } x \in \mathcal{X}. \tag{1}$$

Proof. The first claim follows from the definition of π. The second claim is proved by applying the separating-hyperplane theorem. ■

Ausubel's and Milgrom's dual characterization of strong substitute valuations extends straightforwardly to the cases treated here.

Theorem 2 (Ausubel and Milgrom (2002)) *v is a weak-substitute valuation if and only if π is submodular, and this holds if and only if the dual profit function $\tilde{\pi}$ of the corresponding binary form $\tilde{v} = \phi(v)$ is submodular on the restricted domain where goods of the same type have equal prices. In addition, v is a strong-substitute valuation if and only if the dual profit function $\tilde{\pi}$ of its binary form $\tilde{v} = \phi(v)$ is submodular.*

Proof. The proofs of the two statements follow the proof of Theorem 10 in Ausubel and Milgrom (2002). ■

The preceding theorem relies on the idea that one can characterize weak substitutes by focusing on the subset $\mathcal{P}_L$ of the price space in the binary formulation $\tilde{\mathcal{P}}$ in which goods of the same type have the same price. This subset is isomorphic to the set $\mathcal{P}$ of linear prices used in the multi-unit economy. The weak-substitute property then corresponds to the requirement that the dual profit function is submodular on $\mathcal{P}_L$, while the strong-substitute property requires submodularity on the whole price space. An immediate consequence of this alternative formulation is the following:

Theorem 3 *Any strong-substitute valuation is also a weak-substitute valuation.*

The converse is not true. For example, suppose there is only one type of good, so that every valuation v is a weak-substitute valuation. Let $v(0) = 0$, $v(1) = 1$ and $v(2) = 3$ and suppose prices are $(p_1, p_2) = (1.4, 1.4)$, at which both units are demanded. Increasing p_1 to 1.7 would reduce demand to 0, thus violating the strong-substitute property. The same example establishes that a multi-unit valuation can be submodular even when the related binary valuation is not.

We have seen than weak-substitute valuations need not be submodular. The following result shows that adding the requirement that v is concave does yield submodularity.

Theorem 4 *Any concave weak-substitute valuation is submodular.*

Proof. From Theorem 1 $v(x) = \min_{p \in \mathcal{P}}\{\pi(p) + px\} = \max_p\{-\pi(p) - px\}$. From Theorem 2, π is submodular. Therefore, v is the maximum over p of a function that is supermodular in p and $-x$, which implies that v is supermodular in $-x$ or, equivalently, submodular in x. ■

THEOREM 5 *Let $\tilde{v}$ be a strong-substitute valuation. Then $\tilde{v}(\tilde{x}) = \min_{\tilde{p} \in \tilde{\mathcal{P}}}\{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}$.*

Proof. Given $\tilde{x}$, define $\tilde{p}$ as $\tilde{p}_a = 0$ if $\tilde{x}_a = 1$ and $\tilde{p}_a = \infty$ if $\tilde{x}_a = 0$. Clearly, $\tilde{x} \in \tilde{D}(\tilde{p})$. The rest of the proof is identical to the proof of Theorem 1. ■

Underlying Theorem 4 is the fact that concavity allows v to be expressed by formula (1). As Theorem 5 shows, concavity is not required in the binary form to obtain that equation, which offers a way to understand why strong substitutes implies submodularity.

4 Relations between Concepts of Substitutes

Gul and Stacchetti (1999) introduced the *single-improvement property* for binary valuations, which requires that if some vector x is not demanded at price vector p, then there is a vector y that is strictly preferred to x and entails increasing the demand for at most one good and decreasing the demand for at most one other good, as follows.

DEFINITION 6 (BINARY SINGLE-IMPROVEMENT PROPERTY) *A binary valuation $\tilde{v}$ satisfies the single-improvement property if for any price vector $\tilde{p}$ and $\tilde{x} \notin \tilde{D}(\tilde{p})$, there exists $\tilde{y}$ such that $u(\tilde{y}, \tilde{p}) > u(\tilde{x}, \tilde{p})$, $\|(\tilde{y} - \tilde{x})_+\|_1 \leq 1$, and $\|(\tilde{x} - \tilde{y})_+\|_1 \leq 1$.*

Gul and Stacchetti also showed that this single-improvement property is equivalent to the strong substitutes property:

THEOREM 6 (GUL AND STACCHETTI (1999)) *A monotonic valuation is a strong-substitute valuation if and only if it satisfies the binary single-improvement property.*

We now extend these results to multi-unit economies.

DEFINITION 7 (MULTI-UNIT SINGLE-IMPROVEMENT PROPERTY) *A valuation v satisfies the multi-unit single-improvement property if for any p and $x \notin D(p)$, there exists x' such that $u(x', p) > u(x, p)$, $\|(x' - x)_+\|_1 \leq 1$, and*[6] *$\|(x - x')_+\|_1 \leq 1$.*

[6]Here the norm is defined on $\mathbb{R}^K$, whereas it was defined on $\mathbb{R}^{\sum_k N_k}$ in the binary setting.

The only difference in the definitions of binary and multi-unit single-improvement properties resides in the price domain where the property has to hold.

Throughout the paper, we will denote by e_k the vector of $\mathbb{R}^K$ whose k^{th} component equals one and whose other components equal zero.

THEOREM 7 *If v satisfies the multi-unit single-improvement property then it is a weak-substitute valuation.*

Proof. Suppose by contradiction that the weak-substitute property is violated: there exist p, k, a small positive constant ε, and a bundle x such that $x \in D(p)$ and for all $y \in D(p + \varepsilon e_k)$, there exists $j \neq k$ such that $y_j < x_j$. Set $\hat{p} = p + \varepsilon e_k$. We have $x \notin D(\hat{p})$ and $y_k < x_k$ for all $y \in D(\hat{p})$ (since D(p) clearly contains bundles with strictly less than x_k units of good k). Therefore x is only dominated by bundles y that have strictly less units of at least two goods, implying that $\|(x - y)_+\|_1 \geq 2$, which violates the single-improvement property. ■

The converse in not true. In the first example of Section 1, the valuation is submodular in a two-good economy, thus satisfies the weak substitutes property. However, for $r = 0.2$ and $w = 0.3$, the bundle $(1, 0)$ is only dominated by the bundle $(0, 2)$, which violates the single-improvement property.

DEFINITION 8 (MULTI-UNIT SUBMODULARITY) *A multi-unit valuation v is submodular if for any vectors x and x' of $\mathcal{X}$, $v(x) + v(x') \geq v(x \wedge x') + v(x \vee x')$.*

The next theorem contains a key result for the existence of Walrasian equilibria in multi-unit economies. The proof uses Gul and Stacchetti's characterization theorem (Theorem 6) and thus requires monotonicity of v. Throughout the rest of the paper, we assume that v is nondecreasing.

ASSUMPTION 2 *Agent valuations are nondecreasing.*

THEOREM 8 *If v is a strong-substitute valuation, then any bundle x is optimal at some linear price.*

Proof.

Let x be any bundle, and $\tilde{x}$ be a binary representation of this bundle. From Theorem 5, we have

$$v(x) = \tilde{v}(\tilde{x}) = \min_{\tilde{p}}\{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}. \tag{2}$$

Since v is a strong substitutes valuation, $\tilde{\pi}$ is submodular, so the objective in (2) is submodular. By a theorem of Topkis (1998), the set M of minimizers of a submodular function is a sublattice and, since the objective is continuous, the sublattice is closed. Therefore, it has a largest element $\tilde{p}$. We claim that this element is a linear price which supports $\tilde{x}$. Linearity means that for any good k such that $x_k \geq 1$, $\tilde{p}_{ki} = \tilde{p}_{kj}$ whenever $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$.

Suppose by contradiction that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units i, j of some good k such that $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units i and j of good k, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a minimizer of (2). Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also in M, which contradicts maximality of $\tilde{p}$. We have thus shown that $\tilde{p}$ is linear on the support of $\tilde{x}$: for each good k there exists a price p_k such that $\tilde{p}_{ki} = p_k$ for all i such that $\tilde{x}_{ki} = 1$. Obviously, $\tilde{p}_{kl} = +\infty$ whenever $\tilde{x}_{kl} = 0$. For any good k such that $x_k \in \{1, N_k - 1\}$, the firm is indifferent, at $\tilde{p}$, between x and some bundle y^k such that $y^k_k < x_k$, otherwise it would be possible to increase p_k, which would contradict maximality of $\tilde{p}$. We can choose y^k so that it is optimal if we slightly increase the price of some particular unit of good k. Since $\tilde{v}$ is a strong substitute valuation, we can choose y such that $y^k_k = x_k - 1$, and $y^k_j \geq x_j$ for all j. Since $\tilde{p}_{kl} = +\infty$ outside of the support of $\tilde{x}$, we necessarily have $y^k_j = x_j$ for $j \neq k$. This shows that $y^k = x - e_k$. Such indifference bundles exist for all goods k such that $1 \leq x_k \leq N_k - 1$.

We now prove that x is optimal for the linear price vector $p = (p_k)_{k \in \mathcal{K}}$, where $p_k = +\infty$ when $x_k = 0$, $p_k = 0$ when $x_k = N_k$, and p_k is defined as above when $1 \leq x_k \leq N_k - 1$. That is, we can impose $\tilde{p}_{kl} = p_k$ for all units, including those for which $\tilde{x}_{kl} = 0$, and preserve optimality of x. For all goods such that $x_k \in [1, N_k - 1]$, reset all unit prices outside the support of $\tilde{x}$ from $+\infty$ to p_k. This change does not affect optimality of x among bundles z such that $z \leq x$, and it does not affect indifference between x and the bundles y^k. For any good k, consider the bundle $z^k = x + e_k$. Since $\tilde{v}$ is submodular, Theorem 11 implies that v is component-wise concave (see p. 14). Therefore, $v(z^k) - v(x) \leq v(x) - v(y^k) = p_k$, which implies that z^k is weakly dominated by x. Now for two goods $k \neq j$ such that $x_k \geq 1$ and $x_j < N_j$, consider the bundle $z^{kj} = x - e_k + e_j$. We claim that z is also weakly dominated by x. To see this, we use the following Lemma, whose proof is in the

Appendix.[7]

LEMMA 1 *If v is a strong-substitute valuation, k and j are two goods and x is a bundle such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$, then $v(x+e_k+e_j) - v(x+e_k) \geq v(x+2e_j) - v(x+e_j)$.*

Applying Lemma 1 to the bundle $x - e^j - e_k$ yields $v(x) - v(y^j) \geq v(z^{kj}) - v(y^k)$, which implies, along with $v(x) = v(y^j) + p_j = v(y^k) + p_k$, that $v(x) - p_k \geq v(z^{kj}) - p_j$. Thus, x weakly dominates z. This shows that $\tilde{x}$ has no single improvement. From Theorem 6, $\tilde{v}$ satisfies the single-improvement property. Therefore, $\tilde{x}$ must be optimal at the linear price $\tilde{p}$ such that $\tilde{p}_{kl} = p_k$ for all $l \in \{1, \ldots, N_k\}$. Equivalently, the bundle x is optimal at price $p = (p_k)$, which concludes the proof. ■

We can now state the properties of strong-substitute valuations in linear-pricing economies.

THEOREM 9 *Suppose that v is a strong-substitute valuation. Then it satisfies the following properties:*

[Concavity] *v is concave.*

[Weak-Substitute Property.] *For any $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, and $x \in D(p)$, there exists $x' \in D(p + \varepsilon e_k)$ such that $x'_j \geq x_j$ for all $j \neq k$.*

[Law of Aggregate Demand.] *For any $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, and $x \in D(p)$, there exists $x' \in D(p + \varepsilon e_k)$ such that $\|x'\|_1 \leq \|x\|_1$.*

[Consecutive-Integer Property.] *For any $p \in \mathcal{P}$ and $k \in \mathcal{K}$, the set $D_k(p) = \{z_k : z \in D(p)\}$ consists of consecutive integers.*

Proof. Theorem 3 implies that v satisfies the weak-substitute property, and Hatfield and Milgrom (2005) show that v must satisfy the law of aggregate demand. Therefore, it remains to show that v is concave and satisfies the consecutive-integer property.

We first show that v is concave. Theorem 8 implies that for any x there exists p such that $\pi(p) = v(x) - px$, where π is the dual profit function defined in Section 3. From the first part of Theorem 1, $v(x) \leq \min_p \pi(p) + px$. Combining the two equations above yields $v(x) = \min_p \pi(p) + px$ for all x. Applying the second part of Theorem 1 then proves that v is concave.[8]

[7] As can be easily checked, the proof of Lemma 1 is independent of the proof of the present theorem.

[8] As can be easily verified, the proof of Theorem 1 is independent of the present proof.

Last, we show the consecutive-integer property. Suppose by contradiction that there exist p, k, and two bundles x and y in $D(p)$ such that $x_k \geq y_k + 2$ and $z \in D(p) \Rightarrow z_k \notin (y_k, x_k)$. Consider the binary price vector $\tilde{p}$ that is linear and equal to p_j for all good $j \neq k$, and that equals p_k for the first x_k units of good k and $+\infty$ for the remaining units of good k. Clearly, there exist binary forms $\tilde{x}$ and $\tilde{y}$ of x and y that belong to $\tilde{D}(\tilde{p})$, and there is no bundle $\tilde{z}$ in $\tilde{D}(\tilde{p})$ such that $z_k \in (y_k, x_k)$. If the price of one unit of good k is slightly increased, the demand for good k thus falls directly below z_k, implying that the demand of another unit of good k, whose price had not increased, has strictly decreased, which violates the strong-substitute property for $\tilde{v}$. ■

The consecutive-integer property is not implied by concavity of v. For example, in a (multi-unit) two-good economy, concavity is compatible with the demand set $D(p) = \{(1,0),(0,2)\}$. However, this demand set violates the consecutive-integer property: the set $D_2(p) = \{0,2\}$ does not consist of consecutive integers. The consecutive-integer property rules out valuations causing a sudden decrease in the consumption of a good (independently of the consumption of other goods). For example, there are no prices at which the firm is indifferent between bundles containing, say, 5 and 10 units of a good, but strictly prefers these bundles to any bundle containing between 6 and 9 units of that good. In that sense, there are no "holes" in the demand set with respect to any good. In terms of demand, the property implies a progressive reaction to price movements: as the price of a good increases, the optimal demand of that good decreases unit by unit. By contrast, concavity is not required for the law of aggregate demand.

THEOREM 10 *If v is a weak-substitute valuation that satisfies the consecutive-integer property, then it satisfies the law of aggregate demand.*

Proof. See the Appendix.

The weak-substitute property and the law of aggregate demand do not imply the consecutive-integer property. For example, in an economy with one good available in two units, consider the non-concave valuation $v(0) = 0$, $v(1) = 1$, and $v(2) = 4$. v is trivially a substitutes valuation, and satisfies the law of aggregate demand. However, at price $p = 2$, the demand set is $\{0, 2\}$, which violates the consecutive-integer property. This is also an example of a weak-substitute valuation that is not concave.

To obtain sharp results, we consider the concept of component-wise concavity, which is weaker than concavity and entails diminishing marginal returns in each component separately.

Definition 9 (Component-wise Concavity) *A multi-unit valuation v is component-wise concave if for all x and k, $v(x_k+1,x_{-k})-v(x)\geq v(x_k+2,x_{-k})-v(x_k+1,x_{-k})$.*

Theorem 11 *A multi-unit valuation v is submodular and component-wise concave if and only if its binary form $\tilde{v}=\phi(v)$ is submodular.*

Proof. By a theorem of Topkis (1998), it is sufficient to consider binary bundles x and y that differ in just two components. If the two components represent the same good, then submodularity of the binary form is the same as component-wise concavity. If the two components represent different goods, then submodularity of the binary form is implied by submodularity of the multi-unit form (and conversely). ■

The last three properties listed in Theorem 9 describe the demands corresponding to a strong-substitute valuation in linear-pricing economies. Even though strong-substitute valuations are defined by their demands in response to nonlinear prices, the identified properties turn out to be sufficient to characterize strong substitutes. That is the essential content of Theorem 12 below.

Before proving this theorem, we state a new "minimax" result, in which one of the choice set is a lattice and the other choice set consists of nonlinear prices. The proof of this result is in the Appendix.

If x is a multi-unit bundle and $\tilde{p}$ is a nonlinear price vector, let $(\tilde{p},x)$ denote the cost of acquiring bundle x under $\tilde{p}$. That is,

$$(\tilde{p},x)=\sum_{k\in\mathcal{K}}\sum_{i=1}^{x_k}\tilde{p}_{k(i)},$$

where $\tilde{p}_{k(i)}$ is the price of the i^{th} cheapest unit of good k.

Proposition 1 (Minimax) *Suppose that v is a concave weak-substitute valuation satisfying the consecutive-integer property, and let $\tilde{p}$ be a nonlinear price vector. Then,*

$$\max_x \min_p\{\pi(p)+px-(\tilde{p},x)\}=\min_p \max_x\{\pi(p)+px-(\tilde{p},x)\}$$

Theorem 12 *Let v be a multi-unit valuation. The following properties are equivalent.*

(i) v is a strong-substitute valuation.

(ii) v is a concave weak-substitute valuation, and satisfies the consecutive-integer property.

Proof. We know from Theorem 9 that (i) implies (ii). We now show that (ii) implies (i). From Theorem 2, it is enough to show that $\tilde{\pi}$ is submodular. Consider any nonlinear price vector $\tilde{p}$. We have

$$\tilde{\pi}(\tilde{p}) = \max_{\tilde{x}}\{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\} = \max_{x}\{v(x) - (\tilde{p}, x)\}.$$

Since v is concave, Theorem 1 implies that

$$\tilde{\pi}(\tilde{p}) = \max_{x}\{\min_{p}\{\pi(p) + px\} - (\tilde{p}, x)\} = \max_{x}\{\min_{p}\{\pi(p) + px - (\tilde{p}, x)\}\}.$$

From Proposition 1, the max and min operators can be swapped:

$$\tilde{\pi}(\tilde{p}) = \min_{p}\{\max_{x}\{\pi(p) + px - (\tilde{p}, x)\}\} = \min_{p}\{\pi(p) + \max_{x}\{px - (\tilde{p}, x)\}\}.$$

As can be easily verified, the inner maximum equals

$$\sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+.$$

Therefore,

$$\tilde{\pi}(\tilde{p}) = \min_{p}\left\{\pi(p) + \sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+\right\}.$$

Since v is a weak-substitute valuation, π is submodular by Theorem 2. Moreover, the function $(x, y) \to (x - y)_+$ is submodular as a convex function of the difference $x - y$. Therefore, $\tilde{\pi}(\tilde{p})$ is the minimum over p of an objective function that is submodular in p and $\tilde{p}$, which shows that it is submodular in $\tilde{p}$.[9] ■

It turns out that, *given* concavity and the weak-substitute property, the law of aggregate demand is equivalent to the consecutive integer property. Some of the main results above are combined and extended in the following theorem.

Theorem 13 (Equivalence of Substitute Concepts) *Let v be a multi-unit valuation. The following statements are equivalent.*

(i) v satisfies the binary single-improvement property.

[9] See Topkis (1968).

(ii) v is a strong-substitute valuation.

(iii) v is a concave weak-substitute valuation and satisfies the consecutive-integer property.

(iv) v is a concave weak-substitute valuation and satisfies the law of aggregate demand.

(v) v is concave and satisfies the multi-unit single-improvement property.

Proof. $(i) \Leftrightarrow (ii)$ is Gul and Stacchetti's theorem (see Theorem 6). $(ii) \Leftrightarrow (iii)$ is a restatement of Theorem 12. Theorem 10 shows that (iii) implies (iv). For the converse, the weak-substitute property implies[10] for all p that any edge E of $D(p)$ has direction e_i or $e_i - \alpha e_j$ for some goods i, j. In the first case, concavity implies that all integral bundles on the edge belong to the demand. In the second case, $\alpha = 1$. Otherwise, slightly modifying the price would reduce demand to that edge, and increasing p_i if $\alpha > 1$ or p_j if $\alpha < 1$ would violate the law of aggregate demand. This, along with concavity, implies that the consecutive-integer property holds along all edges, and thus for $D(p)$. $(i)-(iv)$ implies (v): (i) clearly implies the multi-unit single-improvement property, and (iii) implies concavity. We conclude by showing that (v) implies (iii). We already know from Theorem 7 that if v satisfies (v), then it is a weak-substitute valuation. Therefore, there only remains to show that v satisfies the consecutive-integer property. Suppose it doesn't. There exists a price vector p, a good k, and a unit number d such that $D_k = \{z_k : z \in D(p)\}$ is split by d: the sets $D_k^- = D_k \cap [0, d-1]$ and $D_k^+ = D_k \cap [d+1, N_k]$ are disjoint and cover D_k. Now slightly increase p_k. The new demand set D' is such that $D'_k \subset D_k^-$. Pick any bundle y that is optimal under the new price within the set $\{x \in \mathcal{X} : x_k \geq d\}$. Then $y_k > d$, because p_k has only been slightly increased and any bundle with d units of good k was strictly dominated by D_k^+. At the new price, y is dominated but cannot be strictly improved upon with reducing the amount of good k by at least two units, which violates the single-improvement property. ■

The multi-unit single-improvement property alone is not equivalent to strong substitutes. For example, in an economy with two goods available in two units, consider the valuation v defined by $v(x) = \|x\|_1 - .1r(x)$, where $r(x)$ equals 1 if x contains exactly one unit of each good, and 0 otherwise. The valuation is not concave, and therefore cannot be a strong-substitute valuation. However, one can easily verify that v satisfies the multi-unit single-improvement property.

[10] See the proof of Proposition 1.

We conclude this section with a property of concave, weak-substitute valuations. For any (multi-unit) bundle x, let $\mathcal{P}(x)$ denote the set of price vectors such that $x \in D(p)$.

Theorem 14 *If v is a weak-substitute valuation, then for all x, $\mathcal{P}(x)$ is either the empty set or the complete sublattice of $\mathcal{P}$ given by $\mathcal{P}(x) = \arg\min\{\pi(p) + px\}$.*

Proof. Fix $x \in \mathcal{X}$. From Theorem 5, $v(x) \leq \min_p\{\pi(p)+px\}$. Suppose that the inequality is strict. Then $v(x) - px < \pi(p)$ for all p, so $\mathcal{P}(x)$ is the empty set. Now suppose that $v(x) = \min_p\{\pi(p) + px\}$. Then, for all $p \in \arg\min\{\pi(p) + px\}$, $v(x) - px = \pi(p)$, so $x \in D(p)$. Conversely, if $x \in D(\bar{p})$ for some price $\bar{p}$, then $\arg\min\{\pi(p) + px\} = v(x) = \pi(\bar{p}) + \bar{p}x$. Therefore, $\mathcal{P}(x) = \arg\min\{\pi(p) + px\}$. From Theorem 2, $\pi(p)$ is submodular. Therefore $\mathcal{P}(x)$ is the set of minimizers of a submodular function over a sublattice $\mathcal{P}$; hence, it is a sublattice of $\mathcal{P}$. Completeness is obtained by a standard limit argument. ∎

In the binary formulation, all bundles can be achieved through nonlinear pricing, by setting some unit prices to zero and others to infinity. Therefore, Theorem 14 takes a simpler form. For any binary bundle $\tilde{x}$, let $\tilde{\mathcal{P}}(\tilde{x})$ denote the set of price vectors such that $\tilde{x} \in \tilde{D}(\tilde{p})$.

Theorem 15 *If $\tilde{v}$ is a binary valuation satisfying the strong substitutes, then $\tilde{\mathcal{P}}(\tilde{x})$ is a complete, non-empty lattice for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

Proof. For any bundle $\tilde{x}$, there exists a price $\tilde{p}$ such that $\tilde{x} \in \tilde{D}(\tilde{p})$. Therefore, $\tilde{\mathcal{P}}(\tilde{x})$ is nonempty. The rest of the proof is similar to the proof of Theorem 14. ∎

5 Aggregate Demand and Equilibrium Analysis

The first theorem of this section extends results by Gul and Stacchetti and by Milgrom asserting *necessary* conditions for the existence of Walrasian equilibrium in the binary formulation. These theorems assume that individual valuations are drawn from a set that includes all unit-demand valuations (Gul and Stacchetti), which are defined next, or all additive valuations (Milgrom).[11] They establish that if the set of valuations includes any that are not strong substitutes, then there is a profile of valuations to be drawn from the set such that no competitive equilibrium exists.

[11] An *additive* valuation is a valuation with the property that the value of any set is equal to the sum of the values of the singletons in the set.

These results are unsatisfactory for our multi-unit context, because they allow preferences to vary among identical items and the constructions used in those papers hinge on that freedom. The next theorem extends the earlier results by including the restriction that firms' binary valuations are consistent with some multi-unit valuation, that is, that firms treat all units of the same good symmetrically.

Definition 10 *A unit-demand valuation is such that for all price p and $x \in D(p)$, $\|x\|_1 \leq 1$.*

Let $N = \sum_k N_k$ denote the total number of units in the economy.

Theorem 16 *Consider a multi-unit endowment $\mathcal{X}$ and a firm having a concave, weak-substitute valuation v_1 on $\mathcal{X}$ that is not a strong-substitute valuation. Then there exist I firms, $I \leq N$, with unit-demand valuations $\{v_i\}_{i \in I}$, such that the economy $E = (\mathcal{X}, v_1, \ldots, v_{I+1})$ has no Walrasian equilibrium.*

Proof. See the Appendix.

Since preferences are assumed to be quasi-linear, one can conveniently analyze equilibrium prices and allocations in terms of the solutions to certain optimization problems. With that objective in mind, consider an economy consisting of n firms with valuations $\{v_i\}_{1 \leq i \leq n}$. The valuations v_i are defined for $\{x \in \mathbb{N}^K : x_k \leq N_k \ \forall k \in \mathcal{K}\}$. It is convenient to extend the domain of v_i by setting $v(x) = v(x \wedge (N_1, \ldots N_K))$ for all x in $\mathbb{N}^K$. We now define the market-valuation v of the economy by

$$v(x) = \max\left\{\sum v_i(x_i) : \sum x_i = x \text{ and } x_i \in \mathbb{N}^K\right\}.$$

and the market dual profit function of the economy by $\pi(p) = \max_{x \in \mathbb{N}^K}\{v(x) - px\}$. The function π is convex, as can be checked easily.

Theorem 17 *For all $p \in \mathcal{P}$, $\pi(p) = \sum_{1 \leq i \leq n} \pi_i(p)$.*

Proof.

$$\begin{aligned}
\pi(p) &= \max_x \left\{\max\{\textstyle\sum_i v_i(x_i) : \sum_i x_i = x\} - px\right\} \\
&= \max_{x_1, \ldots, x_n} \textstyle\sum_i \{v_i(x_i) - px_i\} \\
&= \textstyle\sum_i \pi_i(x_i),
\end{aligned}$$

which concludes the proof. ■

Theorem 17 cannot be extended to nonlinear prices. To see this we observe, for example, that the cheapest unit of a given good can only be allocated to a single firm when computing the market dual profit function, whereas it is included in all individual dual profit functions involving at least one unit of this good. It is thus easy to construct examples where the market dual profit function is strictly lower than the sum of individual dual profit functions, the latter allowing each firm to use the cheapest units.

COROLLARY 1 *If all firms have weak-substitute valuations, then the market valuation v is also a weak-substitute valuation.*

Proof. If individual firms have substitute valuations, Theorem 2 implies that individual profit functions are submodular. By Theorem 17, the market dual profit function is therefore a sum of submodular functions, and so itself submodular. Theorem 2 then allows us to conclude that v is a substitute valuation. ■

DEFINITION 11 *A price vector p is a* pseudo-equilibrium price *of the economy with endowment $\bar{x}$ if $p \in \arg\min\{\pi(p) + p\bar{x}\}$.*

Section 6 uses the following characterization of pseudo-equilibrium prices.

PROPOSITION 2 *p is a pseudo-equilibrium price if and only if $\bar{x}$ is in the convex hull of $D(p)$.*

Proof. By definition p minimizes the convex function $f : p \to \pi(p) + p\bar{x}$. Therefore, 0 is in the subdifferential of f at p.[12] That is, $0 \in \partial\pi(p) + \bar{x}$. The extreme points of $-\partial\pi(p)$ are bundles that are demanded at price p. Moreover, $-D(p) \subset \partial\pi(p)$. Therefore $-Co(D(p)) = \partial\pi(p)$. Combining these results yields $\bar{x} \in Co(D(p))$. ■

Let $\mathcal{P}(\bar{x})$ denote the set of pseudo-equilibrium prices.

PROPOSITION 3 *If all firms have weak-substitute valuations, then $\mathcal{P}(\bar{x})$ is a complete sublattice of $\mathcal{P}$.*

Proof. Individual weak-substitute valuations imply that π_i is submodular for all i by Theorem 2. Therefore, π is submodular. The proof is then identical to the proof of Theorem 14. ■

[12] See for example Rockafellar (1970).

Theorem 18 *The economy with endowment $\bar{x}$ has a Walrasian equilibrium if and only if $v(\bar{x}) = \min_p\{\pi(p) + p\bar{x}\}$. Moreover, if the economy with endowment $\bar{x}$ has a Walrasian equilibrium, then the set of Walrasian equilibrium prices is exactly the set $P(\bar{x})$ of pseudo-equilibrium prices.*

Proof. Theorem 1 implies that $v(\bar{x}) \leq \min_p\{\pi(x)+p\bar{x}\}$. Suppose that $v(\bar{x}) = \pi(p)+px$ for some p. Let $\bar{x}_i$ denote the bundle received by firm i for some fixed allocation maximizing the objective in the definition of v. For all i, $v_i(\bar{x}_i) - p\bar{x}_i \leq \pi_i(p)$. Summing these inequalities yields $v(\bar{x}) \leq \pi(p) - p\bar{x}$. By assumption, the last inequality holds as an equality, which can only occur if $v_i(\bar{x}_i) - p\bar{x}_i = \pi(p)$ for all i, implying that $(p, \bar{x}_1, ..., \bar{x}_n)$ is a Walrasian equilibrium. To prove the second claim, suppose that $(\{\bar{x}_i\}_{1\leq i\leq n}, p)$ is a Walrasian equilibrium. Then, $v_i(\bar{x}_i) = \pi_i(p) + p\bar{x}_i$ for all i. Summing these equalities yields $v(\bar{x}) = \pi(p) + p\bar{x}$, which implies that $v(\bar{x}) = \min_p\{\pi(p) + p\bar{x}\}$ (since the minimum is always above $v(\bar{x})$). It is clear from the first part of the proof that if the economy has a Walrasian equilibrium, the set of Walrasian prices is exactly the set of pseudo-equilibrium prices. ∎

Theorem 18 shows that whenever a Walrasian equilibrium exists, the concepts of pseudo-equilibrium and equilibrium coincide. In binary economies, where nonlinear pricing is available, the question of the existence of a Walrasian equilibrium have been solved by Gul and Stacchetti (1999) and Milgrom (2000), who both show that equilibrium exists in the binary formulation when goods are strong substitutes and establish the two partial converses described above.

For the multi-unit formulation, we have already established the partial converse in Theorem 16. We now consider the other direction: we prove that strong substitutes implies the existence of a Walrasian equilibrium with *linear* pricing. This result is then used to prove the stronger theorem that strong-substitute valuations are closed under aggregation: if all valuations satisfy strong-substitutes, then so does the market valuation.

Theorem 19 (Linear-Pricing Walrasian Equilibrium) *In a multi-unit exchange economy with individual strong-substitute valuations, there exists a Walrasian equilibrium with linear prices.*

Proof. Considering the binary form of the economy, Gul and Stacchetti (1999, Corollary 1) have shown that the set of (nonlinear pricing) Walrasian equilibria is a complete lattice. In particular, it has smallest and largest elements. We now prove that these two elements consist of linear prices, which proves the result. Suppose by contradiction that the largest element $\tilde{p}$ is such that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units i, j of some good k. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units i and j of good k, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a Walrasian equilibrium. Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also a Walrasian equilibrium, which contradicts maximality of $\tilde{p}$. Linearity of the smallest element is proved similarly. ■

Corollary 2 (Concavity of Aggregate Demand) *In a multi-unit exchange economy with individual strong-substitute valuations, the market valuation is concave.*

Proof. Denote by x the total endowment of the economy, and n the number of firms. We show that for all y such that $0 \leq y \leq x$, there exists a linear price vector p such that y is in the demand set of the market valuation. From Theorem 19, we already know that the result is true if $y = x$. Thus suppose that $y < x$. Consider an additional firm with valuation $v_{n+1}(z) = Kz \wedge (x - y)$, where K is a large constant, greater than the total value of other firms for the whole endowment x. One can easily check that v_{n+1} is an assignment valuation, and therefore a strong-substitute valuation (see Hatfield and Milgrom (2004)). Applying Theorem 19 to the economy with $(n+1)$ firms, there exists a Walrasian equilibrium with linear price vector p. At this price, the additional firm obtains the bundle $x - y$ since its marginal utility dominates all other firms' for any unit up to this bundle, and vanishes beyond this bundle. This implies that the remaining firms ask for y at price p, or equivalently, that y belongs to the demand set of n firms' market valuation at price p. Concavity is then obtained as in the proof of Theorem 9. ■

Theorem 20 (Aggregation) *If individual firms have strong-substitutes valuations, then the market valuation v is a strong-substitute valuation.*

Proof. Let $\{v_i\}_{1 \leq i \leq n}$ denote the family of individual valuations and v denote the market valuation, defined by $v(x) = \max\{\sum_i v_i(x_i) : \sum x_i = x, x_i \in \mathbb{N}\}$. From Theorem 12, we will prove the result if we show that v is a concave weak-substitute valuation that satisfies the consecutive-integer property. Corollary 2 states that v is concave. From Corollary 1, v is a weak-substitute valuation. It thus remains to show that v satisfies the consecutive-integer property. For any price p, the demand set of v is the solution of

$$\max_x\{v(x) - px\} = \max_x\left\{\max\left\{\sum_i v_i(x_i) : \sum_i x_i = x\right\} - px\right\} = \sum_i \max_{x_i} v_i(x_i) - px_i.$$

Therefore, $D(p) = \sum_i D_i(p)$. In particular, the projection of D on the k^{th} coordinate satisfies $D_k = \sum_i D_{i,k}$. The sets $D_{i,k}$ consist of consecutive integers by Theorem 9, implying that D_k also consists of consecutive integers. ■

Finally, we examine the connections between strong-substitute valuations and the structure of the core of the associated cooperative game. The setting considered in this section is the same as Ausubel and Milgrom (2002), but with the multi-unit formulation replacing their binary formulation. We first recall the definitions of coalitional value functions, the core, and Vickrey payoffs.

Suppose that, in addition to bidders, there exists a single owner (labeled "0") of all units of all goods, who has zero utility for her endowment.

DEFINITION 12 *The coalitional value function of a set S of bidders is defined by* $w(S) = \max\left\{\sum_{i\in S} v_i(x_i) : \sum x_i \in \mathcal{X}\right\}$ *if* $0 \in S$*, and* $w(S) = 0$ *otherwise.*

Denote L the set consisting of all bidders and the owner of the good.

DEFINITION 13 *The core of the economy is the set*

$$Core(L, w) = \left\{\pi : w(L) = \sum_{l\in L} \pi_l, w(S) \leq \sum_{l\in S} \pi_l \text{ for all } S \subset L\right\}.$$

DEFINITION 14 *The Vickrey payoff vector is given by* $\bar{\pi}_l = w(L) - w(L \setminus l)$ *for* $l \in L \setminus 0$*, and* $\bar{\pi}_0 = w(L) - \sum_{l\in L\setminus 0} \bar{\pi}_l$.

Ausubel and Milgrom (2002) show that this is the payoff at the dominant-strategy solution of the generalized Vickrey auction.

DEFINITION 15 *The coalitional value function w is bidder-submodular if for all* $l \in L \setminus 0$ *and sets S and S' such that* $0 \in S \subset S'$, $w(S) - w(S \setminus l) \geq w(S') - w(S' \setminus l)$.

THEOREM 21 *Suppose that there are at least* $2 + \max_k N_k$ *bidders. If any bidder has a concave, weak-substitute valuation that is not a strong-substitute valuation, then there exist linear or unit-demand valuations for remaining bidders such that the coalitional value function is not bidder-submodular and the Vickrey payoff vector is not in the core.*

Proof. See the Appendix.

THEOREM 22 *If all bidders have strong-substitute valuations, then the coalitional value function is bidder-submodular and the vector of Vickrey payoffs is in the core.*

Proof. From Ausubel and Milgrom (2002, Theorem 7), it is enough to show that the coalitional value function is bidder-submodular. By assumption, the binary form $\tilde{v}_i$ of each bidder satisfies the substitutes property. Therefore, applying Theorem 11 in Ausubel and Milgrom (which is valid for the binary formulation) implies that the coalitional value function is bidder-submodular (a property which is independent of the formulation (binary or multi-unit). ■

6 Walrasian Tâtonnement and Clock Auctions

This section analyzes auctions where goods are available in multiple units and prices are linear. The goods are summarized by a vector $\bar{x} \in \mathcal{X} = \mathbb{N}_{++}^{K}$. We propose a class of algorithms guaranteeing monotonic convergence of the auction to a pseudo-equilibrium whenever bidders have weak-substitute valuations. Combining that with the the results of Section 5 leads to the conclusion that if bidders have strong-substitute valuations, the auctions converge to a Walrasian equilibrium.

For the present analysis, we define a clock auction as a price adjustment process in which the path of prices is monotonic—either increasing or decreasing. In practice this monotonicity and other features, especially activity rules for bidders (see Milgrom (2000)), differentiate clock auctions from a Walrasian tâtonnement. In order to understand the relation between substitute valuations and clock auctions, it is useful to start the analysis with Walrasian tâtonnement and only later to impose monotonicity on the process.

6.1 Continuous time and price

We begin by analyzing an idealized economy with prices changing continuously through time and where bidders submit their entire demand set. Later, we adapt the results to economies with a discrete price and time, and where bidders only demand a single bundle for each announced price vector.

There are n bidders with valuations $\{v_1, ..., v_n\}$ and a corresponding market valuation v. At any time t, a price vector $p(t)$ is posted. We limit attention to linear pricing. Each bidder submits his demand set, resulting in an aggregate demand $x(t)$ in the demand set $D(p(t))$ of v.

The goal of this section is to construct algorithms that are monotonic and converge to a pseudo-equilibrium. We focus on algorithms for which initial price is low, then increases and converges to the smallest pseudo-equilibrium price $\underline{p}$. Reverse algorithms, where price decreases and converges to the largest pseudo-equilibrium price can be constructed in a similar way.

We have seen that pseudo-equilibrium prices are the minimizers of the convex function $f : p \to \pi(p) + \bar{x}p$. Among the general algorithms to find such minimizers are steepest-descent algorithms. At any time, price changes are determined by the gradient of f whenever f is differentiable, and by the vector of smallest norm of its subdifferential otherwise.[13] Such algorithms are a particular Walrasian tâtonnement, as they adjust prices to eliminate excess demand. Moreover, they follow the steepest descent and are therefore particularly efficient. For any price vector p, we denote by $z(p)$ the point of smallest norm in the opposite of the differential of f at p. When f is differentiable, z corresponds to the excess (aggregate) demand $D(p) - \bar{x}$. In general, z is the vector of smallest norm in the convex hull of the set of excess demand. Intuitively, an algorithm is a procedure that determines the evolution of price through time as a function of excess demand $D(p) - \bar{x}$ and of time itself. In continuous time, an algorithm would then be defined by a function F such that $\dot{p}(t) = F(D - \bar{x}, t)$. However, this definition is not formally satisfactory in our setting, because F need not be continuous. The steepest-descent algorithm, in particular, follows discontinuous changes of direction. In general, we will say that an algorithm is *well-defined* if, from any initial price, it generates a unique trajectory in the price space. The previous considerations lead to the following definition.

Definition 16 *A continuous, correspondence-based, steepest descent algorithm is defined by*

$$\dot{p}_r(t) = \alpha(t, p(t))z(p(t)), \tag{3}$$

where the subscript r denotes right derivative, the function $\alpha : (t,p) \to \alpha(t,p)$ is real-valued and continuous, and takes values in $[\underline{\alpha}, \bar{\alpha}]$ for some $0 < \underline{\alpha} < \bar{\alpha}$.

Using right derivatives addresses discontinuities of $z(p)$. The lower bound $\underline{\alpha}$ ensures that the algorithm does not stall at a suboptimal price, and the upper bound ensures that that the equation is integrable. The following theorem states that, starting from any

[13]By definition, the subdifferential $\partial f(p)$ at p of a convex function f is the set of vectors x such that $f(q) - f(p) \geq x(q - p)$ for all q. The subdifferential is always a nonempty convex set, and coincides with f's gradient whenever it is differentiable.

sufficiently low price, the algorithm is well defined, monotonic and converges to the lowest pseudo-equilibrium price, $\underline{p}$. Let $\mathcal{L} = \{p : p \leq \underline{p}$ and $z(p) \geq 0\}$.

THEOREM 23 *Any continuous, correspondence-based, steepest-descent algorithm is well defined. Suppose that bidders have weak-substitutes valuations. For any such algorithm, if* $p(0) \in \mathcal{L}$, *then* $p(t) \in \mathcal{L}$ *for all* t, $p(t)$ *is increasing and converges to* $\underline{p}$ *in finite time.*

The proof is in the Appendix. Theorem 23 implies that, when bidders have weak-substitute valuations, any steepest-descent algorithm starting from low prices is an ascending clock auction and converges to the smallest pseudo-equilibrium price. This result is important in practice, and can be reformulated as follows. We define a continuous, correspondence-based, ascending clock auction as a continuous, correspondence-based steepest-descent algorithm, except that (3) is replaced by $\dot{p}_r(t) = \max\{\alpha(t, p(t))z(p(t)), 0\}$, where the maximum is taken componentwise.

COROLLARY 3 *If bidders have weak-substitute valuations, any continuous, correspondence-based clock auction starting from a price in* $\mathcal{L}$ *converges to the smallest pseudo-equilibrium price.*

In particular, if goods are weak substitutes, ascending clock auctions will find an equilibrium whenever there exists one. By contrast, it is easy to build examples of valuations violating weak-substitutes such that a Walrasian equilibrium exists but ascending clock auctions fail to find it.

Our result extends Ausubel (2006) in three ways. First, it searches on the space of linear prices, while Ausubel's algorithm specifies separate prices for each unit of the good. Second and more importantly, it relies only on the assumption of weak substitutes, where Ausubel's analysis requires on the stronger assumption of strong substitutes. Third, it shows the the process converges monotonically to pseudo-equilibrium prices, which always exist in this setting and which are equilibrium prices whenever an equilibrium exists.

In theory, $\mathcal{L}$ depends on bidder valuations, which may see problematic, given that the auctioneer does know them. In practice, the assumption $p_0 \in \mathcal{L}$ means that the clock auction can start at any price low enough to guarantee that there is excess demand in all goods. This obviously includes zero initial prices, but also "reasonably low" reserve prices.

6.2 Discrete time and price

We now consider the case in which prices evolve on a grid. In such setting, it is natural to consider discrete-time models, as nothing happens in any interval of time during which prices remain constant. We thus consider a discrete time scale, where prices are adjusted at each period.[14] The first goal of this section is to show that the results derived in the previous section are approximately true, in the sense that trajectories obtained with discretized algorithms are very close to those generated by continuous algorithms, provided that the price grid is fine enough. The second goal of the section is to show that the algorithm still works if bidders only announce one desired bundle at each period, rather than their entire demand set, consistent with what is observed in current practice.

A price grid is a lattice $\mathcal{P}_\eta = (\eta\mathbb{N})^K$, where η is a small positive constant. A discrete algorithm generates a sequence of prices $\{p_t : t = 0, 1, \ldots\}$ in $\mathcal{P}_\eta$, whose evolution is determined by excess demand at any period. In a discrete setting, algorithms are always well-defined. A new issue is that price changes, which are restricted to a grid, may not be able to follow exactly the gradient z. In general, vector directions can be approximated up to the fineness of the grid, which is indexed by η. The following lemma goes further by showing that, provided the grid is fine enough, even the *exact* direction is feasible. Following the previous section, we let $z(p)$ denote the vector of smallest norm in the convex hull of the excess (aggregate) demand set $D(p) - \bar{x}$. The proof of the lemma is in the Appendix.

Lemma 2 (Feasible directions of descent) *Suppose that the number of bidders is less than some constant $N > 0$, and that no bidder can demand more than overall supply $\bar{x}$. Then, for any grid $\mathcal{P}_\eta$, there exists $\alpha(\eta) > 0$ such that $\alpha(\eta)z(p) \in \mathcal{P}_\eta$ for all p and all bidder valuations. Moreover, α can be chosen such that $\alpha(\eta) \to 0$ as $\eta \to 0$.*

In the rest of this section, we may therefore assume that the price grid is fine enough for price changes to exactly follow steepest-descent directions and be arbitrarily small. In order to stay exactly on the grid, we assume from now on that step sizes are integer multiples of $\alpha(\eta)$. Another issue is that discrete algorithms sometimes "overshoot", meaning that the discrete price sequence crosses a region boundary while the continuous algorithm follows the boundary, causing the discrete algorithm to enter regions where some goods

[14]The lapse between two periods has no importance, and in fact could in principle vary during the auction, possibly stochastically.

are in excess supply, and where the algorithm gradient z, which is not continuous, takes very different values from the gradient of the continuous algorithm. The purpose of the following lemma is to show that such overshoots are not important, as nearby trajectories of any discrete steepest-descent algorithm stay close to each other. Let $\{p(t)\}_{t\in\mathbb{N}}$ and $\{q(t)\}_{t\in\mathbb{N}}$ denote the trajectories generated by a given steepest-descent algorithm, starting from respective initial prices $p(0)$ and $q(0)$.

LEMMA 3 (NEARNESS LEMMA) *Suppose that the number of bidders is less than some constant $N > 0$, that no bidder can demand more than aggregate supply $\bar{x}$, and that there exists a vector $M \in \mathbb{R}_+^K$ such that bidders demand none of good i whenever $p_i > M_i$. Then, for any $\varepsilon > 0$, there exists $\bar(\eta) > 0$ and $\bar{\alpha} > 0$ such that for all $\eta < \bar(\eta)$ and step sizes less than $\bar{\alpha}$, $\|p(0) - q(0)\| < \varepsilon$ implies $\|p(t) - q(t)\| < \varepsilon$ for all periods and all bidder valuations.*

Proof. See the Appendix.

The nearness lemma states that overshooting is not going to affect the trajectory by more than some arbitrarily small constant. This leads to the following theorem, which states that the discrete algorithm essentially follows the continuous one. For any price p_0, denote by $T(p_0) = \{p(t) : t \in \mathbb{R}_+, p(0) = p_0\}$ the trajectory generated by the continuous, correspondence-based steepest-descent algorithm of the previous section, and let $T(p_0, \varepsilon) = \cup_{p\in T(p_0)} B(p, \varepsilon)$ denote the tube[15] of radius ε around $T(p_0)$.

THEOREM 24 (DISCRETE STEEPEST-DESCENT ALGORITHM) *For any $\varepsilon > 0$, there exists $\eta > 0$ and $\bar{\alpha} > 0$ such that for any grid finer than η, step size less than $\bar{\alpha}$, and initial price p_0, the trajectory generated by the discrete steepest descent algorithm is contained in $T(p_0, \varepsilon)$.*

Proof. Starting in the same region, trajectories of both algorithms are undistinguishable, since they follow the same direction. Let t_0 denote the first time that the trajectory T of the discrete algorithm overshoots, causing the two paths to have distinct vectors. Let $\epsilon > 0$ be a positive constant (to be chosen later), and denote by p_{t_0} the price of the discrete algorithm, and by q_{t_0} a price on $T(p_0)$ such that $\|p_{t_0} - q_{t_0}\| < \epsilon$. Such a price exists if the step size $\bar{\alpha}(\epsilon)$, which gives an upper bound on the overshoot, is small enough. Let T_1 denote the trajectory that the discretized algorithm would generate if it were starting from q_{t_0}. By construction T_1 coincides with $T(p_0)$ until there is a second overshoot. By the

[15] $B(p, \varepsilon)$ is the open ball centered at p and radius ε.

nearness lemma, T and T_1 are within ε from each other. Therefore, when T_1 overshoots, at time t_1, there is a price q_{t_1} of $T(p_0)$ such that $\|p(t_1) - q_{t_1}\| < 2\epsilon$. Iterating the process, we thus prove that, up to the k^{th} overshoot, we have $T \subset T(p_0, k\epsilon)$ when T is truncated at $t = t_k$. The number of overshoots is bounded above by the number R of regions (since any region is visited at most once by the continuous algorithm, see proof of Theorem 23). Therefore, the result obtains by setting $\epsilon = \varepsilon/R$. ■

As a by-product of Theorem 24, we can get rid at little cost of the assumption that bidders submit their entire demand set. Bidder valuations can be seen as vectors of the finite-dimensional space $\mathcal{V} = \mathbb{R}^{\bar{x}}$. A property of an algorithm holds "for almost all economies" if it holds for all bidder valuations, except for a subset of Lebesgue measure zero of $\mathcal{V}^n$, where n is the number of bidders. A singleton-based steepest-descent algorithm, is the same as the discrete steepest-descent algorithm, except that bidders ask only one bundle at each period. Concretely, this means that instead of using the vector of smallest norm of the excess demand set, the algorithm may follow any vector of that set. The following result shows that this information loss does not affect Theorem 24 except possibly on a set of economies with Lebesgue measure zero.

Theorem 25 (Singleton-Based Algorithm) *Under the assumptions of Theorem 24, let p_0 be any initial price of the algorithm. The trajectory of a singleton-based steepest-descent algorithm is contained in $T(p_0, \varepsilon)$ for almost all economies.*

The proof is based on the following proposition.

Proposition 4 *For all $(v_1, \ldots, v_n) \in \mathcal{V}^n$, the demand correspondence $p \to D(p)$ is single-valued almost everywhere in $\mathcal{P}$ with respect to the Lebesgue measure on this set.*

Proof. We suppose first that there is a unique bidder. For any two bundles x and x', the subset $P(x, x')$ of $\mathcal{P}$ defined by $P(x, x') = \{p : p(x - x') = v(x) - v(x')\}$, is the intersection of a hyperplane with the positive orthant $\mathcal{P}$, and has therefore zero Lebesgue measure. Since the number of possible bundles is finite, the set

$$Q = \bigcup_{x \neq x'} P(x, x'),$$

which contains all prices at which the bidder's demand is multi-valued, also has zero Lebesgue measure. For a countable (in particular, finite) number of bidders, the set of prices where aggregate demand is multi-valued is contained in $Q^a = \cup Q_i$, which has zero Lebesgue measure. ■

Proposition 4 implies that the set of economies such that $Q^a \cap \mathcal{P}_\eta \neq \emptyset$ has Lebesgue measure zero. Therefore, singleton-based and correspondence-based algorithms are identical in almost all economies.

In practice, the auctioneer does not know bidder valuations. Theorem 25 implies that for any belief that is absolutely continuous with respect to the Lebesgue measure, the algorithm is arbitrarily close to the continuous, correspondence-based steepest descent algorithm of the ideal economy. In particular, the algorithm completely ignores bidders' indifference sets. This feature contrasts with Gul and Stacchetti (2000), whose algorithm gives much importance to indifference sets.

7 Conclusion

The substitutes concepts play a critical role in equilibrium theory. For discrete economies, strong substitutes is necessary for the robust existence of equilibrium and weak substitutes drive the monotonicity that is exploited by current auction algorithms. Strong substitutes is also the condition that determines whether the Vickrey outcome is in the core. A related concept–the law of aggregate demand–has been the informal justification for the wide adoption of activity rules in practical auctions. Among our findings is that the law of aggregate demand is precisely the additional property that converts a concave weak substitute valuation to a strong-substitute valuation when goods are discrete. Adapting results from Hatfield and Milgrom (2005), it is also possible to show that strong substitutes is necessary for the existence of stable matchings in their contracting model. The analysis also showed that the two concepts of substitutes are closed under aggregation.

Milgrom and Strulovici (2006) extend the analysis to divisible goods. In particular, for divisible goods and concave valuations, a natural extension of strong substitutes coincides with weak substitutes. In that case, the law of aggregate demand and its unit-free extensions generally fail. Thus, for concave valuations, the law of aggregate demand characterizes the difference between the cases of discrete goods and divisible goods.

8 Appendix: Proofs

8.1 Section 4

Proof of Lemma 1. Consider a bundle x such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$. Take any binary representant $\tilde{x}$ of x, and call l and m two units of good j not in $\tilde{x}$, and s a unit of good k not in $\tilde{x}$. Since $\tilde{v}$ satisfies the gross-substitute property, the triple

$$\{\tilde{v}(\tilde{x} + e_l + e_m) - \tilde{v}(x + e_l) - \tilde{v}(\tilde{x} + e_m), \tilde{v}(\tilde{x} + e_l + e_s) - \tilde{v}(\tilde{x} + e_l) - \tilde{v}(\tilde{x} + e_s),$$
$$\tilde{v}(\tilde{x} + e_m + e_s) - \tilde{v}(\tilde{x} + e_m) - \tilde{v}(\tilde{x} + e_s)\} \quad (4)$$

has at least two maximizers. Symmetry of $\tilde{v}$ implies that the last two arguments of that quantity are equal, and therefore greater than or equal to the first one. That is, written in multi-unit form $v(x + e_k + e_j) - v(x + e_k) - v(x + e_j) \geq v(x + 2e_j) - 2v(x + ej)$ which concludes the proof after simplification. ■

Proof of Theorem 10. Suppose by contradiction that the law of aggregate demand is violated: there exist k, p and x such that for all ε small enough, we have (i) $x \in D(p - \varepsilon e_k)$, and (ii) for all $y \in D(p + \varepsilon e_k)$, $\|y\|_1 > \|x\|_1$. Clearly, for any such y, we have $y_k < x_k$. Let $D_k = D_k(p)$, $\underline{d} = \min D_k$ and $\bar{d} = x_k = \max D_k$. By continuity, we have (i) $x \in D(p)$, (ii) there exists some $y \in D(p)$ such that $y_k < x_k$, and (iii) for all $y \in D$ such that $y_k = \underline{d}$, $\|y\|_1 > \|x\|_1$.

For each $d \in D_k$, define $g(d) = \min\{\|y_{-k}\|_1 : y_k = d \text{ and } y \in D(p)\}$. Let $\gamma : [\underline{d}, \bar{d}] \to \mathbb{R}$ denote the largest convex function such that $\gamma(d) \leq g(d)$ for all $d \in D_k$. The function γ is well defined and piecewise affine: there exists a partition $\Delta = \{\delta_l\}_{l \in \Lambda}$ of $[\underline{d}, \bar{d}]$ such that γ is affine on $[\delta_l, \delta_{l+1}]$. Moreover, $\bar{d}$ and $\underline{d}$ are elements of Δ: there exist $\underline{l}$ and $\bar{l}$ such that $\underline{d} = \delta_{\underline{l}}$ and $\bar{d} = \delta_{\bar{l}}$. For $l \in \{\underline{l} + 1, \bar{l}\}$, denote $H(l)$ the hyperplane containing the two $(K-2)$-dimensional affine varieties

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_l) \text{ and } z_k = \delta_l\}$$

and

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_{l-1}) \text{ and } z_k = \delta_{l-1}\}.$$

There exists a unique hyperplane containing these two affine varieties, so $H(l)$ is well defined. Moreover, $H(l)$ lies below $D(p)$ and contains at least two elements z and y of $D(p)$ such that $z_k = \delta_l$ and $y_k = \delta_{l-1}$.

We claim that there exists $l \in \{\underline{l}+1, \bar{l}\}$ such that $\gamma(\delta_{l-1}) - \gamma(\delta_l) > \delta_l - \delta_{l-1}$. Suppose that the contrary holds. Then, $\gamma(\underline{d}) - \gamma(\bar{d}) \leq \bar{d} - \underline{d} = x_k - \underline{d}$. But then, there exists y in $D(p)$ such that $y_k = \underline{d}$ and $\|y_{-k}\|_1 = \gamma(\underline{d})$, implying that $\|x\|_1 = x_k + \gamma(\bar{d}) \geq \underline{d} + \gamma(\underline{d}) = \|y\|_1$, which contradicts the hypothesized violation of the law of aggregate demand.

Consider an index l as in the previous paragraph, and modify p slightly so that the demand set becomes $D(p) \cap H(l)$. The price vector can be further modified so that the remaining bundles in the demand set are aligned on a unique straight line and, for the new price $\bar{p}$, there still exist z and y in $D(\bar{p})$ such that $z_k > y_k$ and $\|z\|_1 < \|y\|_1$. There are two cases: either there are two indices i and j such that $y_i > z_i$ and $y_j > z_j$, or there exists an index i such that $y_i - x_i > x_k - y_k$. Since optimal bundles are aligned, the same properties hold for the extremities bundles of the segment containing $D(\bar{p})$, so we assume without loss of generality that z and y are these extreme bundles. In the first case, increasing p_i slightly violates the weak-substitute property, as the optimal quantity of good j also decreases. In the second case, the convex-demand property is violated: the set $D_i(\bar{p})$ contains a hole between z_i and y_i. ■

Proof of Proposition 1 Trivially,

$$\max_x \min_p \{\pi(p) + px - (\tilde{p}, x)\} \leq \min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}. \tag{5}$$

We need to prove that the reverse inequality also holds. We fix $\tilde{p}$ throughout the proof. Consider a price p solving $\min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}$. Let $N(p) = \arg\max_x \{px - (\tilde{p}, x)\}$. $N(p)$ is a hyper-rectangle: there exist two bundles r and R with $r \leq R$ such that $N(p) = \{z \in \mathbb{Z}^K : r \leq z \leq R\}$.

Suppose that there exists a bundle x in $N(p) \cap D(p)$. Then, the right-hand side of (5) equals $\pi(p) + px - (\tilde{p}, x) = v(x) - (\tilde{p}, x)$, where the last equality comes from the fact that x belongs to $D(p)$. Now consider any linear price vector q. We have $\pi(q) + qx - (\tilde{p}, x) \geq v(x) - (\tilde{p}, x)$, by definition of $\pi(q)$. This last inequality implies that the left-hand side of (5) is actually greater than or equal to its right-hand side. Therefore, we will have concluded the proof if we show that $N(p) \cap D(p)$ is nonempty, which we now turn to.

Let $Co(D(p))$ and $Co(N(p))$ denote the convex hulls of $D(p)$ and $N(p)$. We first show that $Co(D(p)) \cap Co(N(p))$ has a nonempty intersection. Suppose by contradiction that $Co(D(p)) \cap Co(N(p)) = \emptyset$. Then, since these two sets are closed and convex, the separating-hyperplane theorem implies that there exists a direction δ and a number a such that $y\delta < a$ for $y \in N(p)$ and $x\delta > a$ for $x \in D(p)$. Now modify p by an infinitesimal amount along the direction δ, yielding a new level $q = p + \varepsilon\delta$. The objective function $\pi(p) + \max_z\{pz - (\tilde{p}, z)\}$ is affected by this change in two ways. First, through the sensitivity of π with respect to p. Taking any $x \in D(q) \subset D(p)$, we have $\pi(p) = v(x) - px$ and $\pi(q) = v(x) - qx$. Therefore, the change of π is $-\varepsilon x\delta$. Second, through the sensitivity of $\max_z\{pz - (\tilde{p}, z)\}$ with respect to p. There exists $y \in N(p)$ such that $\max_z\{pz - (\tilde{p}, z)\} = py - (\tilde{p}, y)$ throughout the price change. Therefore, the effect on this term equals $\varepsilon y\delta$. The overall change of the objective function is then $\varepsilon(y - x)\delta < 0$, implying that q leads to a strictly lower objective function than p, which contradicts optimality of p.

We have proved that the sets $Co(D(p))$ and $Co(N(p))$ have a non empty intersection. We now prove that this intersection contains a point with integer coordinates. Consider any polytope of $\mathbb{R}^K$. We say that an edge (i.e. a segment joining two vertices of the polytope) is *simply oriented* if either (i) it is parallel to one coordinate axis $\{\lambda e_i : \lambda \in \mathbb{R}\}$ of the space or (ii) there exist two coordinates i and j such that the edge is parallel to $e_i - e_j$. We say that a polytope is simply oriented if all its edges are simply oriented. Last, we recall that a polytope all of whose vertices have integer coordinates is called a *lattice polytope*.

LEMMA 4 *If a lattice polytope P is simply oriented, and H is the half space $\{x : x_k \geq q\}$, where $k \in \{1, \ldots, K\}$ and q is an integer, then $P \cap H$ is either the empty set, or a simply oriented, lattice polytope.*

Proof. Suppose that $Q = P \cap H$ is nonempty. Its vertices are either vertices of P, in which case they are integral, or new vertices belonging to H. We prove that any such vertex also has integer coordinates. Any new vertex x is the intersection of H with an edge E of P that is not parallel to H. In particular, there exists an integral vertex y of P such that $x - y$ is parallel to E. Moreover, $y_k \neq q$, since the edge is not parallel to H. The edge is either parallel to e_k or to $e_k - e_i$ for some $i \neq k$. In the first case, we have $x_j = y_j \in \mathbb{Z}$ for all $j \neq k$ and $x_k = q \in \mathbb{Z}$, so x has integer coordinates. In the second case, $x_j = y_j \in \mathbb{Z}$ for all $j \notin \{i, k\}$, $x_k = q \in \mathbb{Z}$, and $x_i = y_i + (y_k - x_k) \in \mathbb{Z}$, so x also has integer coordinates. We now prove that the edges of Q are simply oriented. Thus consider an edge E of Q, joining vertices x and y. If either x or y are vertices of P, then E is either an edge of P, or the result of such an edge being cut by H. In either case, it is simply oriented because P is simply oriented. If both x and y are new vertices, E is the intersection of a two-dimensional face F of P with H, where F is not parallel to H. F is defined by two linearly independent edges E' and E'' of P which are simply oriented, and at least one of which contains e_k. Suppose first that either E' or E'', say E', is orthogonal to e_k. Then it is easy to show that E is parallel to E'' and therefore simply oriented. Now suppose that both E' and E'' have a nonzero k^{th} component. Because they are linearly independent, there exist i and j such that F is generated by $e_k - e_i$ and $e_k - e_j$ (where the signs come from the fact that P is simply oriented). In that case, as can be easily verified, E is parallel to $e_i - e_j$, and therefore simply oriented. ■

We observe that Lemma 4 still holds if the inequality sign is reversed in the definition of H.

$Co(D(p))$ is a lattice polytope since $D(p)$ consists of integral vectors. We now prove that $Co(D(p))$ is simply oriented. Thus consider any edge E of $Co(D(p))$. There exists a vector δ of $\mathbb{R}^K$ such that E is included in some straight line $\Delta = \{x_0 + \lambda\delta\}_{\lambda\in\mathbb{R}}$. We first show that δ has at most two nonzero components. Suppose on the contrary that δ has at least three components, say i, j, and k. Without loss of generality assume that δ_i and δ_j are positive. Since E is a face of $Co(D(p))$, there exists an infinitesimal modification of the price vector p, such that $D(p) = E$. Moreover, E contains two vectors x and y such that $x - y = \lambda\delta$ for some $\lambda > 0$. If we slightly increase p_i, x becomes suboptimal, so the optimal quantity of good j decreases, which violates the weak-substitute property. Thus, δ has at most two nonzero components. We now prove that E is simply oriented. If δ has only one nonzero component, the claim is trivial. Suppose that δ has two positive components, say i and j. We show that $\delta_i = -\delta_j$. Since E has integer vertices, we can assume that δ_i and δ_j are integers.[16] If $\delta_i\delta_j > 0$, slightly increasing p_i reduces the optimal quantity of good j which violates the weak-substitute property. Thus, δ_i and δ_j have opposite signs. Now suppose that $|\delta_i| < |\delta_j|$. This implies that for all integral vectors x and y in E, we have $|x_j - y_j| \geq 2$, which violates the consecutive-integer property. Thus, $\delta_i = -\delta_j$, which concludes the proof.

We have shown that $Co(D(p))$ is a simply oriented lattice polytope. Since $Co(N(p))$ is a hyperrectangle of the form $\{x \in \mathbb{R}^K : a \leq x \leq b\}$ for some integral vectors a and b, we have, denoting $H(k,q)_+ = \{x : x_k \geq q\}$ and $H(k,q)_- = \{x : x_k \leq q\}$,

$$Co(D(p)) \cap Co(N(p)) = Co(D(p)) \bigcap_{1\leq k\leq K} \left(H_+(k,a_k) \cap H_-(k,b_k)\right).$$

Iterating Lemma 4 $2K$ times implies that $Co(D(p)) \cap Co(N(p))$ is either the empty set

[16] See for example Korte and Vygen (2000).

or a lattice polytope. Since we have already shown that this intersection is nonempty, it must contain an integral point, which concludes the proof of Proposition 1. ■

8.2 Section 5

PROOF OF THEOREM 16. We extend part of the proof of Theorem 2 in Gul and Stacchetti (1999) to a multi-unit context. By assumption, there exist a price vector $\bar{p}$, a good k, and bundles x and x' such that (i) $\{x, x'\} \in D(\bar{p})$, (ii) $x'_k - x_k \geq 2$, and (iii) for all z in $D(\bar{p})$, $z_k \notin (x_k, x'_k)$. This implies that at the price $p = \bar{p} - \eta e_k$, x is only dominated by bundles z such that $z_k \geq x_k + 2$. In particular, the single-improvement property is violated by x at price p. Therefore, any bundle y that solves $\min_z \sum_k |x_k - z_k|$ subject to $u_1(z,p) > u_1(x,p)$ satisfies $y_k \geq x_k + 2$.

Let $\rho = \sum_j (y_j - x_j)_+$. By hypothesis, $\rho \geq 2$. Let $\varepsilon = \frac{u_1(y,p) - u_1(x,p)}{2\rho}$. Let $I_+ = \{j : x_j < y_j\}$, $I_- = \{j : x_j > y_j\}$, and $I_0 = \{j : x_j = y_j\}$. If $j \in I_+$, introduce $N_j - y_j$ firms, call them "C_j", with unit-demand valuation $v_1(\mathcal{X}) + 2$ for a single unit of good j. If $j \in I_+ \setminus \{k\}$, introduce $y_j - x_j$ firms, call them "c_j", with unit-demand valuation $p_j + \varepsilon$ for a single unit of good j. If $j = k$, introduce $y_k - x_k - 1$ firms ("c_k") with unit-demand valuation $p_k + \varepsilon$ for a single unit of good k. If $j \in I_-$, introduce $N_j - x_j$ firms (C_j) with unit-demand valuation $v_1(\mathcal{X}) + 1$ for a single unit of good j, and $x_j - y_j$ firms (c_j) with unit-demand valuation p_j for a single unit of good j. If $j \in I_0$, introduce $N_j - x_j$ firms with unit-demand $v_1(\mathcal{X}) + 1$. Last, introduce a special firm, "firm 2", with unit-demand $p_k + v_1(\mathcal{X}) + 1$ for a single unit of good k.

Now suppose that there exists a Walrasian equilibrium with price vector t, and let X_i denote the bundle of the equilibrium received by firm i. Necessarily, $(X_1)_j \geq \min\{x_j, y_j\}$ for all j, since even if all unit-demand firms get one unit, there remain $\min\{x_j, y_j\}$ units of good j. Define a new price vector as follows: $q_j = t_j$ for $j \notin I_-$ and $q_j = p_j$ for $j \in I_-$. For $j \in I_-$, $N_j - x_j$ units go to firms C_j. The remaining x_j units are shared between firm 1 and firms c_j, with at least y_j units for firm 1. Now, if firm 1 has none of the remaining $x_j - y_j$ units, it means that $t_j \leq p_j$, and this share remains optimal when t_j is increased to p_j. If firm 1 has all of the remaining units, it means that $t_j \geq p_j$, and this share remains optimal when t_j is decreased p_j. If firm 1 has only a part of these remaining units, it means that t_j is already equal to p_j. Thus (X, q) is also a Walrasian equilibrium, such that $X_1 \geq x \wedge y$. Moreover, all C_j get their units, so that $X_1 \leq x \vee y$. Therefore

$$x \wedge y \leq X_1 \leq x \vee y. \tag{6}$$

Firm 2 necessarily gets a unit of good $k \in I_+$. Therefore, $X_{1k} < y_k$. This, together with (6), implies that $\sum_k |x_k - X_{1k}| < \sum_k |x_k - y_k|$, and thus

$$u(X_1, p) \leq u(x, p). \tag{7}$$

Suppose that there exist some goods j in I_+ such that $X_{1j} > x_j$. This implies that $q_j \geq p_j + \varepsilon$, since firms c_j would otherwise want to get all the units. Combining these price inequalities with (7) yields $u_1(X_1, q) < u_1(x, q)$, which contradicts optimality of X_1 for firm 1.

Suppose instead that $X_{1j} \leq x_j$ for all j. Then, all units between x_j and y_j for $j \in I_+$ are consumed by firms c_j and by firm 2. For $j \neq k$, this implies that c_j have a positive value for the good: $q_j \leq p_j + \varepsilon$. For $j = k$, even though firm 2 takes one units of the $y_k - x_k$ available units of k, the fact that $y_k \geq x_k + 2$ implies that there is also a firm c_k taking one unit of good k, which implies that $q_k \leq p_k + \varepsilon$. Since $X_1 = x$ on I_+ and $p_j = q_j$ for $j \notin I_+$, (7) implies $u_1(X_1, q) \leq u_1(x, q)$. Since $q_j \leq p_j + \varepsilon$ for all $j \in I_+$, the value initially chosen for ε implies that $u_1(x, q) < u_1(y, q)$, and thus $u_1(X_1, q) < u_1(y, q)$, which contradicts optimality of the bundle X_1 for firm 1. ■

Proof of Theorem 21 From A&M Theorem 7 (which allows for multiple units of goods), the vector of Vickrey payoff vector is in the core if and only if the coalitional value function is bidder-submodular. We show that under the assumptions of Theorem 21, there always exist bidder valuations such that the coalitional value function is not bidder-submodular. Suppose that bidder 1's valuation violates the consecutive-integer property. There exist $\hat{p}$ and k such that $D_k(\hat{p})$ does not consist of consecutive integers. Let $p = \hat{p} + \varepsilon e_k$ for ε small enough. Then there exists x and z such that $x_k \geq z_k + 2$, and

$$v(z) - pz > v(x) - px > v(y) - py \tag{8}$$

for all y such that $y_k \in (z_k, x_k)$. Introduce a second bidder with linear valuation $v_2(x) = p_{-k}x_{-k}$, and $x_k - z_k$ unit-demand bidders who only value good k. The total number of bidders is $x_k - z_k + 2 \leq N_k + 2 \leq \max_k N_k + 2$. From (8), we have

$$v(x) + p_{-k}(\bar{x} - x)_{-k} \geq v(y) + p_{-k}(\bar{x} - y)_{-k} + p_k(x_k - y_k)$$

whenever $x_k - y_k \leq x_k - z_k - 1$, and

$$v(z) + p_{-k}(\bar{x} - z)_{-k} p_k(x_k - z_k) > v(x) + p_{-k}(\bar{x} - x)_{-k}.$$

Therefore, denoting S the set consisting of bidders 1, 2 and the $x_k - z_k - 2$ unit-demand firms, and s and t the last two unit-demand bidders, we have $w(S \cup \{s\}) = w(S)$ and $w(S \cup \{s,t\}) > w(S \cup \{t\})$, showing that w is not bidder-submodular. ■

8.3 Section 6

Proof of Theorem 23. The proof is based on three lemmas, proving respectively well-definedness, monotonicity, and confinement in $\mathcal{L}$.

Lemma 5 (Well-definedness) *The continuous SDA algorithm is well defined.*

Proof. On any region of the price space where excess demand is constant, the algorithm defines a straight trajectory of direction z, and is thus well-defined.[17] The only possible problem, thus, is to rule out the possibility that there are infinitely many region changes in an arbitrarily small amount of time. With the steepest-descent algorithm, the norm of z is nondecreasing in time. Since z is constant over any region where aggregate demand is constant, and the norm of z strictly decreases each time it changes, any region that is left is never visited again. ■

Lemma 6 (Monotonicity) *When bidders have weak-substitute valuations and $z(0) \geq 0$, $p(\cdot)$ is nondecreasing.*

Proof. Suppose by contradiction that $z(t)$ fails to be nonnegative at some time t, and take the smallest such time. Since $z(0) \geq 0$, $t > 0$. By construction, $z(s) \geq 0$ on a left neighborhood of t. Let $m = z(t)$, $x = z(t_-)$, and P be the opposite of the subdifferential of f at $p(t)$. P is a convex polytope, whose vertices are elements of the excess demand at $p(t)$, and m is the element of P with smallest norm. By assumption, x is nonnegative. By continuity of demand, x must also belong to P. Let $J = \{k : m_k < 0\}$. By assumption, $J \neq \emptyset$. Let H be the affine hyperplane going through (the point) m and orthogonal to (the vector) m. By assumption, P is on one side of H and touches H at m. Let F be the largest face of P contained in H, y be any vertex of F, and

[17] The scalar function α is immaterial, as long as it is bounded away from 0 and $+\infty$.

$C_y = \{z : \sum_J m_k z_k \geq \|m\|^2 - \sum_{J^c} m_s y_s\}$. Since $y - m$ is orthogonal to m, C_y is a cone with vertex y. We will show that C_y contains P but not x, a contradiction.

Since $y-m$ is orthogonal to m, we have $\|m\|^2 - \sum_{J^c} m_s y_s = \sum_J m_k y_k = m_J y_J$, where the components of m_J are equal to those of m on J and vanish on J^c, and a similar definition for y_J. By convexity of F, $m = y + \sum_l \alpha_l E_l$, where $\{E_l\}$ is the family of direction vectors of the edges of F emanating from y. Taking the scalar product of the previous equality with m_J yields $m m_J = y_J m_J + \sum_l \alpha_l E_l m_J$. We now prove that $E_l m_J = 0$ for all l. By construction of F,

$$m.E_l = 0. \tag{9}$$

Moreover the weak substitute property implies that E_l has at most two nonzero components, and any two nonzero components are of opposite sign (see the proof of Proposition 1). If E_l has one nonzero component, it must be in J^c, otherwise it would violate (9). If it has two nonzero components, then either they are both in J or both in J^c, for otherwise (9) would be violated. In any case, this implies that $E_l.m_J = 0$. Thus, $m_J v_J = m_J^2 > 0$. In particular $C_y = \{z : \sum_J m_k z_k \geq m_J^2\}$. Since the components of x are nonnegative by construction, x cannot belong to C_y.

To conclude the proof, we show that C_y contains P. By convexity of P, it is enough to show that all edges of P emanating from y are going in the cone C_y. This will be the case if we show that for any such edge with direction δ (away from y), we have

$$\delta m_J \geq 0. \tag{10}$$

By definition of F, we have $\delta m \geq 0$ (i.e. any edge from y must point outwards from H). Since bidders have weak-substitute valuations, δ has at most two nonzero components. Suppose first that it has exactly two components, δ_i and δ_j. If i, j are in J, then (10) trivially holds. If i, j are in J^c, then (10) is an equality. If $i \in J$ and $j \in J^c$, then $\delta m \geq 0$ and the fact that $\delta_i \delta_j < 0$ (by weak-substitutes) implies that $\delta_i < 0$, and thus that (10) holds. If there is only one nonzero component, (10) holds trivially. ■

Lemma 7 (Confinement) *If bidders have weak-substitute valuations, $p(0) \leq \underline{p}$ and $z(0) \geq 0$, then $p(t) \leq \underline{p}$ for all $t \geq 0$.*

Proof. Suppose not: there exists a time t such that $p(t)$ crosses the hyperrectangle $R = \{z : z \leq \underline{p}\}$ from inside out. In particular, the index subset $I = \{j : p_j(t) = \underline{p}_j\}$ is nonempty, and we have $p_j(t) < \underline{p}_j$ for $j \notin I$. Moreover, $p(s) \not\leq \underline{p}$ for s in a right neighborhood of t: there exists a nonempty subset $J \subset I$ such that $p_{s,j} > \underline{p}_j$ for $j \in J$

and $s \in (t, t+\varepsilon)$. By construction of the algorithm, this means that the vector n of smallest norm in the opposite of the subdifferential of $p(t)$ satisfies $n_j > 0$ for $j \in J$. We will contradict this statement by showing that the vector m defined by $m_j = n_j$ for $j \notin J$ and $m_j = 0$ for $j \in J$ is in the opposite of the subdifferential. m's norm is strictly smaller than n's, contradicting the assumption that n is of smallest norm in the opposite of the subdifferential. By definition of the subdifferential, we need to show that, letting $p = p(t)$,

$$m(q-p) \geq f(p) - f(q) \tag{11}$$

for all q. We first show this inequality in a neighborhood of p. By construction of n, $n(q-p) \geq f(p) - f(q)$ for all q. Therefore, (11) is automatically satisfied for q such that $q_j \leq p_j$ for $j \in J$. Now consider the case where $q_j > p_j$ for a subset $J(q)$ of J. Consider the vector q' such that $q'_j = q_j$ for $j \notin J(q)$ and $q'_j = p_j$ for $j \in J(q)$. Since we are in a neighborhood of p, $q_j \leq \underline{p}_j$ for all $j \notin J(q)$. This implies that $q' \leq \underline{p}$ and, therefore, that $q' = q \wedge \underline{p}$. Submodularity of f implies $f(\underline{p} \wedge q) + f(\underline{p} \vee q) \leq f(\underline{p}) + f(q)$. The inequality, combined with the fact that $\underline{p}$ is a minimum of f, implies that $f(q') \leq f(q)$. By construction of q',

$$m(q-p) = m(q'-p) \geq n(q'-p) \geq f(p) - f(q') \geq f(p) - f(q),$$

which concludes the proof on a neighborhood of p. To prove the result globally, consider any vector q and let $q_\lambda = \lambda q + (1-\lambda)p$ where $\lambda \in (0,1)$. From the previous analysis, we have for λ small enough $m(q_\lambda - p) \geq f(p) - f(q_\lambda)$. By convexity of f, $f(q_\lambda) \leq \lambda f(q) + (1-\lambda) f(p)$. Combining the previous two inequalities and dividing by λ yields the result. ■

We now conclude the proof of the theorem. Since $p(t)$ is nondecreasing and bounded, it must converge to some limit in $\mathcal{L}$. Since α is bounded away from zero, the rate of change of p is bounded away from zero on any closed subset of the price space that does not contain any pseudo-equilibrium price. Since the only pseudo-equilibrium price contained in $\mathcal{L}$ is $\underline{p}$, this has to be the limit.

Proof of Lemma 2. By assumption, the excess demand set is an integer polytope of $\mathbb{R}^K$, bounded by the rectangle $[-\bar{x}, N\bar{x}]$. Therefore, z can only take finitely many values. Since any such z is the vector of minimum norm of an integral polytope, it has rational coordinates. Therefore, its direction can always be achieved on any regular lattice. That is, there exists a positive number $\alpha(z)$ such that $\alpha(z)z$ is the difference vector of two points of the lattice. Moreover, the smallest such $\alpha(z)$ gets arbitrarily small as the grid gets arbitrarily thin. Since there are finitely many values of z, $\max_z\{\alpha(z)\}$ goes to zero as the grid thinness η goes to zero. ■

PROOF OF LEMMA 3. Without loss of generality, we can restrict attention to price vectors less than M. Since the number of bidders is finite, the function $f : p \to \pi(p) + \bar{x}p$ is piecewise affine, with finitely many regions. Moreover, directions of the hyperplanes supporting f are determined by excess demand vectors, which take finitely many values (cf. proof of Lemma 2). Since z is in the opposite of the differential of f, $f(q) - f(p) \geq z(p)(p-q)$ for all q, with strict inequality if p and q are in distinct regions. The fact that p is bounded by M and that there are finitely many possible slopes for f implies the existence of a constant $\rho > 0$ such that

$$f(q) - f(p) \geq \rho + z(p)(p-q) \tag{12}$$

whenever p and q are not in the same region. We now consider paths of the discrete steepest-descent algorithm starting from respective initial price vectors p_0 and q_0, with $\|p_0 - q_0\| < \varepsilon$. Trajectories are parallel until the two prices reach different regions, and thus leave the vector $p_t - q_t$ unchange until that time. Let $s \geq 0$ denote the first time that the two paths hit distinct regions. (12) implies $f(q_s) - f(p_s) \geq \rho + z(p_s)(p_s - q_s)$ and $f(p_s) - f(q_s) \geq \rho + z(q_s)(q_s) - p_s)$. Summing these inequalities yields[18] $(z(p_s) - z(q_s))(p_s - q_s) \leq -2\rho$. Let α be the step size[19] of the steepest-descent algorithm: $p_{s+1} = p_s + \alpha z(p_s)$, and $q_{s+1} = q_s + \alpha z(q_s)$

$$\|p_{s+1} - q_{s+1}\|^2 = \|p_s - q_s\|^2 + \|\alpha(z(p_s) - z(q_s))\|^2 + 2\alpha(z(p_s) - z(q_s)) \cdot (p_s - q_s).$$

Therefore,

$$\|p_{s+1} - q_{s+1}\|^2 - \|p_s - q_s\|^2 \leq -4\rho\alpha + O(\alpha^2),$$

which is negative for α small enough, which we impose by appropriately setting $\bar{\alpha}$. Thus, we have proved that $\|p_t - q_t\|$ remains constant when prices are in the same region, and decreases otherwise. ■

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[18] This proof strategy introduces a strict version of the theory of maximally monotone mapping. See Rockafellar (1970).

[19] The result holds if α depends on t and p, as long as it is continuous in p.

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