

METASTABLE EQUILIBRIA

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ABSTRACT. We define a refinement of Nash equilibria called metastability. This refinement supposes that the given game might be embedded within any global game that leaves its local best-reply correspondence unaffected. A selected set of equilibria is metastable if it is robust against perturbations of every such global game; viz., every sufficiently small perturbation of the best-reply correspondence of each global game has an equilibrium that projects arbitrarily near the selected set. Metastability satisfies the standard decision-theoretic axioms obtained by Mertens' (1989) refinement (the strongest proposed refinement), and it satisfies the projection property in Mertens' small-worlds axiom: a metastable set of a global game projects to a metastable set of a local game. But the converse is slightly weaker than Mertens' decomposition property: a metastable set of a local game contains a metastable set that is the projection of a metastable set of a global game. This is inevitable given our demonstration that metastability is equivalent to a strong form of homotopic essentiality. Mertens' definition invokes homological essentiality whereas we derive homotopic essentiality from primitives (robustness for every embedding). We argue that this weak version of decomposition has a natural game-theoretic interpretation.

1. INTRODUCTION

This article contributes to the refinement of the Nash equilibria of a finite game. For a critical review of equilibrium refinements see Hillas and Kohlberg [9]. The initial sections of Hillas, Jansen, Potters, and Vermuelen [8] review further those refinements based on perturbations of a game's best-reply correspondence, which is the formulation adopted here.

We define a new refinement called *metastability*. Our definition builds on those proposed by Hillas [7] and variants studied by Hillas, Jansen, Potters, and Vermuelen [8]. However, metastability is a substantially stronger refinement because we invoke a natural generalization—the Embedding Principle described below—of the Invariance and Small Worlds axioms proposed by Kohlberg and Mertens [10] and Mertens [15].

Our main results establish that metastability satisfies the standard decision-theoretic axioms considered by Mertens [13, 14]. It also satisfies the projection property and a slightly weaker version of the decomposition property—the two parts of Mertens' [15] Small Worlds axiom. Due to the latter, metastability is slightly weaker than the refinements proposed by

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Mertens, but in a companion paper [5] we prove that metastability coincides with Mertens' stability for generic extensive-form games. Mertens invokes homological essentiality as an integral part of the definitions of his refinements, whereas here we derive a strong form of homotopic essentiality from our definition. As described below, the definition imposes two basic requirements called Embedding and Robustness.

Briefly (a precise definition is provided later):

A connected set of the equilibria of a game G is metastable if every neighborhood contains the projection of an equilibrium for each sufficiently small perturbation of the best-reply correspondence of any global game \tilde{G} in which G is embedded.

By an embedding we mean a trivial embedding in that the optimal strategies of players in G are not affected; that is, their best-reply correspondence is not affected.

This definition invokes two principles.¹

(1) **Embedding.** Any game G can be construed as a local version of a global game \tilde{G} with additional features that do not affect optimal behavior in G . This principle subsumes those axioms requiring that a refinement is not affected by extraneous features:

- *Small Worlds.* The additional features could be actions of players in \tilde{G} who are not players in G , provided their actions have no effect on the optimal strategies of players in G .
- *Invariance.* The additional features could be redundant strategies of players in G , such as a pure strategy whose payoffs are replicated by some mixed strategy.
- *Rationality.* The additional features could be presentation effects, behavioral anomalies, or subjective beliefs that are not relevant for optimal play in G .

Metastability ensures that a selected set of equilibria of a global game \tilde{G} projects to a selected set of equilibria of any embedded game G . As Mertens [15, p. 555] remarks regarding the Small Worlds axiom:

“... such a property is essential if one wants to speak of a ‘solution concept.’

Indeed, otherwise one could never apply the ‘solution concept’ to a given

¹The requirement that a metastable set is connected excludes the set of all equilibria, which trivially satisfies (1) and (2). It also reflects the fact that all equilibria in a single connected component of the equilibria of a generic extensive-form game have the same outcome (Kreps and Wilson [11]), and more fundamentally, the fact that the uniformly hyperstable sets of Nash equilibria are necessarily connected since they are precisely the essential sets of fixed points of any map whose fixed points are the Nash equilibria (Govindan and Wilson [4]). Connectedness excludes the minimal stable sets studied by Kohlberg and Mertens [10]—Mertens [15] argues that minimality violates the ordinal properties of players' preferences.

game; one would first have to embed this game in some ‘universal game’ of everything going on in the world.”

- (2) **Robustness.** Nearby global games should induce nearby equilibria of the local game. Specifically, each neighborhood of a selected set should include the projection of an equilibrium of a sufficiently small perturbation of a global game in which G is embedded. That is, global perturbations induce small perturbations of optimal behavior in the local game.

We do not provide here a decision-theoretic justification for Robustness, but in [6] we establish that two properties (one of which is Invariance) imply robustness with respect to perturbations of strategies in the weaker refinement called stability by Kohlberg and Mertens’ [10].

Although Robustness is a weak requirement, we show in §4 that in combination with Embedding it is equivalent to a succinct mathematical test. This test is a strong form of homotopic essentiality called *stable essentiality*. In technical terms, stable essentiality says that the strategy set of the given game remains in the range of every homotopic deformation of any suspension of the local projection map from the graph of equilibria to the space of strategy perturbations. Although stable essentiality is weaker than the homological essentiality invoked in Mertens’ definition, it suffices to assure that the same decision-theoretic axioms are satisfied.

The fact that Embedding and Robustness imply stable essentiality has a precedent. In [4] Invariance and robustness with respect to payoff perturbations are shown to imply homological essentiality; viz., a uniformly hyperstable set (Kohlberg and Mertens [10]) of equilibria is an essential component of every map whose fixed points are the Nash equilibria. In contrast, Mertens [13] directly imposes essentiality in his definition of a stable set.

As one knows from Mertens’ work, essentiality of the projection map enables verification that decision-theoretic axioms are satisfied because it ensures that the fixed-point problems they pose have solutions. In §5 we establish the following properties:

- *Admissibility, Perfection, and Backward Induction.* A metastable set includes only perfect (hence admissible) equilibria, and includes a proper equilibrium that induces a sequential equilibrium in every extensive-form game with G as its strategic form.
- *Iterative Elimination of Weakly Dominated Strategies and Never Weak Best Replies, and Forward Induction.* A subset of a metastable set survives iterative elimination of weakly dominated strategies and strategies that are inferior replies at every equilibrium in the set.

Additional properties include the axioms of Ordinality and Player-Splitting.

Metastability differs from Mertens' stability chiefly in that it satisfies a version of the Small Worlds axiom that is weaker than the one proposed by Mertens. As stated above, it satisfies the first part of Mertens' axiom.

Projection Property. A metastable set of a global game projects to a metastable set of an embedded local game.

That is, if G is embedded in \tilde{G} and \tilde{S} is a metastable set for \tilde{G} then $S \equiv \text{proj}_G(\tilde{S})$ is a metastable set for G . Mertens' stability satisfies the Projection Property and also its converse in the following strong form:

- (1*) Each stable set of G is the projection of a stable set of each game in which G is embedded.
- (2*) If S is a stable set of G then for any stable set S' of any other game G' the product $S \times S'$ is a stable set of the product game $G \times G'$.

The Projection Property along with (1*) is the *Small Worlds Property*. The Projection Property along with (2*) is the *Decomposition Property*. Metastability satisfies the following weaker versions of (1*) and (2*).

- (1) Each metastable set of G contains the projection of a metastable set of each game in which G is embedded.
- (2) If S is a metastable set of G then for any other game G' there exists a metastable set S' of G' such that $S \times S'$ is a metastable set of the product game $G \times G'$.

Thus (1) states that if S is a metastable set of G and G is embedded in \tilde{G} then there is a metastable set \tilde{S} of \tilde{G} such that $S \supseteq \text{proj}_G(\tilde{S})$, where the Projection Property assures that $\text{proj}_G(\tilde{S})$ is itself a metastable set for G — and analogously in (2).

Our view is that these weaker versions are natural from a game-theoretic viewpoint. A metastable set is intended to be a collection of possible outcomes that can be refined further only with additional information. For instance, in an extensive-form game the play off the equilibrium path is typically indeterminate, but in specific contexts additional considerations might lead to selection of (say) minimal or maximal 'punishments' for deviations from the equilibrium path. Analogously, embedding a game in a particular global game provides such a context that can select a metastable set (the projection of a metastable set of the global game) that is a strict subset of another metastable set. The decomposition property is thus seen as unduly strong when the product game $G \times G'$ allows correlated selections of metastable sets for the two embedded games; i.e. correlated selections in the product game destroy some of the presumed independence between play in the component games.

Even so, these considerations are relevant only for non-generic extensive-form games, since metastability agrees with Mertens' stability for generic extensive-form games.

Our definition of metastability has two key aspects. One is that (as in Govindan and Mertens [3]) we use the best-reply correspondence of a game as the primitive, rather than a formulation in terms of payoffs. The second is that (as in Hillas [7] and Hillas, Jansen, Potters, and Vermuelen [8]) we test for Robustness using perturbations of the best-reply correspondence. The relevant mathematical tool is then homotopy theory rather than the stronger homology theory invoked by Mertens [13] for perturbations of strategies.

The key step of our technical development is the demonstration that metastability (like Mertens' stability) of a connected set of equilibria is a property of its *germ*, i.e. its neighborhood in the graph of the equilibrium correspondence over the space of strategy perturbations. Theorem 4.2 characterizes metastability in terms of the property that the local projection map from the graph of the equilibrium correspondence is stably essential in the sense of homotopy theory. That is, as defined in Appendix A's Definition A.8, the map remains essential when it is embedded in a space with extra dimensions—using the formal definition of a 'suspension' from algebraic topology. (For the applications here, stably essential is the same as essential when the dimensions of the domain and range are the same, but not when the domain has higher dimension than the range.) This characterization is especially relevant for our companion paper [5] on extensive-form games, since nearby points in the germ (i.e., equilibria of nearby games) induce the beliefs that support sequential equilibria.

After the formulation and definition of metastability and related refinements are established in §2, technical aspects of the Robustness condition are established in §3. In particular, §3.4 shows that the Projection Property is equivalent to stable essentiality of the projection map from the graph of the equilibrium correspondence. Then in §4 metastability is also characterized in terms of the stable essentiality of the local projection map. Also in §4, the relationships of metastability to the stronger refinement of Mertens' stability and the weaker refinement of Hillas et al.'s BR-stability are established. Then in §5 we verify that metastability satisfies the decision-theoretic axioms listed by Mertens [13]. §6 provides concluding remarks.

Appendix A provides mathematical background regarding homotopic essentiality and defines stable essentiality of a map. Appendix B is a brief summary of the properties of multsimplicial and polyhedral complexes invoked in the proofs in §3.

2. FORMULATION AND DEFINITIONS OF REFINEMENTS

2.1. Preliminaries. Throughout, by a map we mean a continuous function. By a correspondence we mean an upper-semicontinuous correspondence whose domain is compact, whose range is a compact convex subset of a Euclidean space, and whose values are nonempty compact convex sets. For any two correspondences $\varphi, \varphi' : X \rightarrow Y$ their distance is $d(\varphi, \varphi') = \sup_{x \in X} d_H(\varphi(x), \varphi'(x))$ where d_H is the Hausdorff distance between compact sets. A correspondence φ' is a δ -perturbation of another φ if $d(\varphi, \varphi') \leq \delta$.

We consider a fixed finite game G in strategic form. The player set is $\mathcal{N} = \{1, \dots, N\}$. Player n 's mixed strategy set is Σ_n and its vertices comprise his pure strategies Σ_n° . Let $\Sigma = \prod_n \Sigma_n$. Player n 's payoff function is given by a multilinear function $G_n : \Sigma \rightarrow \mathbb{R}$. We use $R : \Sigma \rightarrow \Sigma$ to denote the best-reply correspondence of the game G .

For each $\delta \in [0, 1]$ let Σ_δ be the set of $\sigma \in \Sigma$ such that $\sum_{s_n \in \Sigma_n^\circ \setminus R_n(\sigma)} \sigma_{n, s_n} \leq \delta$ for each player n , i.e. the total probability of player n 's strategies that are suboptimal against σ is at most δ . Observe that all fixed points of a δ -perturbation of R are contained in Σ_δ . Say that a closed subset V of Σ_δ is *admissible* if $V \setminus \partial\Sigma$ is connected and dense in V .

For each nonnegative integer k , let Λ^k be the k -dimensional unit simplex in \mathbb{R}^{k+1} . (Λ^0 is a single point.) Let $R^k : \Sigma \times \Lambda^k \rightarrow \Sigma$ be the trivial extension of R , namely, $R^k(\sigma, \lambda) = R(\sigma)$. Suppose $\varphi^k \times \pi^k : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$ is a correspondence where φ^k is a δ -perturbation of R^k . Observe that if (σ, λ) is a fixed point of $\varphi^k \times \pi^k$ then $\sigma \in \Sigma_\delta$.

2.2. Definition of Metastability. We interpret R^k as the best-reply correspondence for the players in \mathcal{N} when G is embedded in a global game \tilde{G} with k additional dimensions parameterized by $\lambda \in \Lambda^k$. A perturbation of the best-reply correspondence of the global game is then represented by a correspondence $\varphi^k \times \pi^k$ where φ^k is a δ -perturbation of R^k and some correspondence $\pi^k : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ describes behavior on other dimensions.

Basically, metastability requires that for each neighborhood V of a selected set S and any k , every correspondence π^k and sufficiently small δ -perturbation φ^k of R^k should have an equilibrium (i.e. a fixed point of $\varphi^k \times \pi^k$) whose projection is in V . The formal definition is:

Definition 2.1 (metastability). A closed set $S \subseteq \Sigma$ is a metastable set if there exists a sequence δ_i converging to zero and a corresponding sequence of closed subsets V_i of Σ_{δ_i} converging to S such that for each i :

- (1) Connexity: V_i is admissible, i.e. $V_i \setminus \partial\Sigma$ is connected and dense in V_i .
- (2) Robustness: V_i contains for every k the projection of a fixed point of the product $\varphi^k \times \pi^k$ of every δ_i -perturbation φ^k of R^k and every correspondence $\pi^k : \Sigma \times \Lambda^k \rightarrow \Lambda^k$.

Among the ways one might consider modifying the Robustness condition, one seems stronger and the other weaker. First, we could require that for each correspondence $\psi^k : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$ such that $\text{proj}_\Sigma \circ \psi^k$ is a δ_i -perturbation of R^k there exists a fixed point of ψ^k in $V_i \times \Lambda^k$. As Theorem 3.2 shows, this is equivalent to the Robustness condition as stated. Second, we could require that for each k there exists $i(k)$ such that for each $i \geq i(k)$ and each correspondence $\varphi^k \times \pi^k : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$, where φ^k is a δ_i -perturbation of R^k , there exists a fixed point in $V_i \times \Lambda^k$. As we show at the end of §4 characterizing metastability, this too is equivalent to the condition as stated.

From the definition one might infer that verifying whether a set of equilibria is metastable is a formidable task. In §4 we show that a verification is accomplished by checking that the local projection map from the graph of the equilibrium correspondence to the space of perturbed is stably essential in the sense of homotopy. Basically, this requires that the projection map remains essential if it is embedded in a space with extra dimensions. The following is a version of Definition A.8 in Appendix A (cf. also Lemma A.6).

Definition 2.2 (stably essential map). Given a map $p : (E, \partial E) \rightarrow (P, \partial P)$ where $(P, \partial P)$ is a ball pair with $p(E \setminus \partial E) \subseteq P \setminus \partial P$, and a k -dimensional simplex Λ^k , let $p^k : (E, \partial E) \times (\Lambda^k, \partial \Lambda^k) \rightarrow (P, \partial P) \times (\Lambda^k, \partial \Lambda^k)$ be the trivial extension for which $p^k(e, \lambda) = (p(e), \lambda)$. The map p is stably essential in homotopy if every trivial extension is essential in homotopy.

2.3. Related Refinements. To enable later comparisons we now state the definitions of the weaker refinement proposed by Hillas et al. [8] and the stronger refinement proposed by Mertens.

The following definition is due to Hillas et al. [8].

Definition 2.3 (BR-set, BR-stable set). A closed subset S of Σ is a best-response set (BR-set) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each δ -perturbation of R has a fixed point within ε of S . A BR-set is BR-stable if it is a connected set of perfect equilibria.

Hillas et al. [8] show that a BR-stable set satisfies several of the properties listed by Mertens [13]. BR-stability is a weaker refinement in that metastability invokes the Embedding Principle described in the Introduction. In effect, metastability requires BR-stability for any global game in which the given game might be embedded.

To present Mertens' definition we need some notation. For each player n and each $0 < \delta \leq 1$, let $P_\delta = \{\varepsilon\tau \mid 0 \leq \varepsilon \leq \delta, \tau \in \Sigma\}$ and denote its topological boundary by ∂P_δ . For $\eta \in P_1$, $\bar{\eta}_n \equiv \sum_{s \in \Sigma_n^c} \eta_{n,s}$ is the minimum error probability. $\bar{\eta}_n$ is constant across players so we denote this number by $\bar{\eta}$. Given any $\eta \in P_1$ define the perturbed game $G(\eta)$ to be the

game where the strategy set of each player n is the same as in G , but where the payoff from a strategy profile τ is the payoff in G from the profile $\sigma = (1 - \bar{\eta})\tau + \eta$. Then we say that σ is a perturbed equilibrium of $G(\eta)$ if τ is an equilibrium of $G(\eta)$. Let \mathcal{E} be the graph of the perturbed equilibrium correspondence over P_1 , i.e.,

$$\mathcal{E} = \{(\eta, \sigma) \in P_1 \times \Sigma \mid \sigma \text{ is a perturbed equilibrium of } G(\eta)\}.$$

For $(\eta, \sigma) \in \mathcal{E}$ we use $\tau(\eta, \sigma) \equiv (1 - \bar{\eta})^{-1}(\sigma - \eta)$ to denote the corresponding equilibrium of $G(\eta)$. Observe that a pure strategy s of player n is in the support of $\tau_n(\eta, \sigma)$ only if it is an optimal reply to σ in G . Denote by p the natural projection from \mathcal{E} to P_1 . For $E \subseteq \mathcal{E}$, let $E_0 = \{(0, \sigma) \in E\}$, and for $0 < \delta \leq 1$ let $(E_\delta, \partial E_\delta) = p^{-1}(P_\delta, \partial P_\delta) \cap E$. That is, as described in the Introduction, E_δ is a germ.

Let \check{H} refer to Čech cohomology with integer coefficients. As a mnemonic we refer to Mertens' definition of $*$ -stability as \check{H} -stability.

Definition 2.4 (\check{H} -stable set). $S \subseteq \Sigma$ is an \check{H} -stable set if for some closed subset E of \mathcal{E} with $E_0 = \{0\} \times S$:

- (1) Connexity: For every neighborhood V of E_0 in E , the set $V \setminus \partial E_1$ has a connected component whose closure is a neighborhood of E_0 in E .
- (2) Cohomological Essentiality: $p^* : \check{H}^*(P_\delta, \partial P_\delta) \rightarrow \check{H}^*(E_\delta, \partial E_\delta)$ is nonzero for some $\delta > 0$.

Mertens [13, 14] proposes several definitions of stability in which the essentiality requirement is cast in terms of singular homology with coefficients in an Abelian group M . He then shows that \check{H} -stability is the union over M of all these refinement concepts and thus is the most inclusive solution concept.

Instead of working with the graph of the perturbed equilibrium correspondence we could equivalently work with the graph of the equilibrium correspondence, i.e. the set of $(\eta, \tau) \in P_1 \times \Sigma$ such that τ is an equilibrium of $G(\eta)$ and thus $\sigma = (1 - \bar{\eta})\tau + \eta\tau$ is a perturbed equilibrium of $G(\eta)$. There is an obvious homeomorphism between the two spaces that commutes with the projections to P_1 . Hence we obtain the same \check{H} -stable sets if we use the graph of equilibria. This observation is true for all 'stability' definitions involving subsets of \mathcal{E} .

Mertens [14] does not explicitly define stability using essentiality in homotopy but the following definition is implicit.

Definition 2.5 (homotopy-stable set). $S \subseteq \Sigma$ is a homotopy-stable (h-stable) set if for some closed subset E of \mathcal{E} with $E_0 = \{0\} \times S$:

- (1) Connexity: For every neighborhood V of E_0 in E , the set $V \setminus \partial E_1$ has a connected component whose closure is a neighborhood of E_0 in E .
- (2) Homotopic Essentiality: $p : (E_\delta, \partial E_\delta) \rightarrow (P_\delta, \partial P_\delta)$ is essential in homotopy for some $\delta > 0$.

In Definitions 2.4 and 2.5, if p is essential for some δ then it is essential for all smaller $\delta' < \delta$. Mertens [14, Section 5] proves this for the case of \check{H} -stability. Here we show that it is true for homotopy stability. Suppose the projection is inessential in homotopy for some $0 < \delta' < \delta$. We prove that the projection is then inessential in homotopy for δ . Let $q : E_{\delta'} \rightarrow \partial P_{\delta'}$ be a map that is homotopic to $p : E_{\delta'} \rightarrow P_{\delta'}$ relative to $\partial E_{\delta'}$. The map q extends to a map over E_δ when we let it coincide with p over $E_\delta \setminus E_{\delta'}$. Moreover p and q are now homotopic as maps from E_δ relative to ∂E_δ . Now define $r : E_\delta \rightarrow \partial P_\delta$ as follows: pick a point $\eta^0 \in P_{\delta'} \setminus \partial P_{\delta'}$; then $r(\eta, \sigma)$ is the unique point in ∂P_δ that is nearer to $q(\eta, \sigma)$ than η^0 on the line segment through η and η^0 . r agrees with q over ∂E_δ , and there is a linear homotopy between q and r . Therefore, the projection from E_δ is inessential.

Since essentiality in cohomology implies essentiality in homotopy, \check{H} -stable sets are also h-stable. If E_δ is semialgebraic and has the same dimension as P_δ then the converse is true. (In [5] we show that this is precisely the case for generic extensive-form games.) However, the converse is not true in general since for pathological games E_δ can have greater dimension than P_δ —see [14, Section 4] for details.

As mentioned, Appendix A, Definition A.8, defines the stronger property of stably essential in homotopy. A map is stably essential if it remains essential when its domain and range are extended trivially to higher dimensional spaces. Using this property, the following strengthens the definition of an h-stable set in Definition 2.5.

Definition 2.6 (stably essential set). $S \subseteq \Sigma$ is a stably essential set if for some closed subset E of \mathcal{E} with $E_0 = \{0\} \times S$:

- (1) Connexity: For every neighborhood V of E_0 in E , the set $V \setminus \partial E_1$ has a connected component whose closure is a neighborhood of E_0 in E .
- (2) Stable Essentiality: the projection map $p : (E_\delta, \partial E_\delta) \rightarrow (P_\delta, \partial P_\delta)$ is stably essential in homotopy for some $\delta > 0$.

As before, one can show that if the k -th suspension (as defined in Appendix A) of the projection from $(E_\delta, \partial E_\delta)$ is essential then so is that from $(E_{\delta'}, \partial E_{\delta'})$ for any smaller δ' .

In §4 we prove that metastable sets are the limits of stably essential sets.

3. SUSPENSIONS OF THE BEST-REPLY CORRESPONDENCE

The purpose of this Section is to obtain results about the Robustness condition in the definition of metastability. In §3.4 these are applied to obtain a characterization of the Small Worlds axiom in terms of a stably essential projection from the equilibrium graph. This section is mostly technical and so might be skipped on first reading.

Throughout this section, V is a closed subset of Σ . Consider the following two properties.

- (1 k) There exists $0 < \delta_k \leq 1$ such that for each δ_k -perturbation φ^k of R^k and each correspondence $\pi^k : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, $\varphi^k \times \pi^k$ has a fixed point in $V \times \Lambda^k$.
- (2) There exists $0 < \delta \leq 1$ such that for each k , each δ -perturbation φ^k of R^k , and each correspondence $\pi^k : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, $\varphi^k \times \pi^k$ has a fixed point in $V \times \Lambda^k$.

Asking for Property (2) to hold is obviously stronger than requiring Property (1 k) to hold for each k , since it is uniform in k .

Remark 3.1. Suppose that property (1 k) does not hold. Then for $l > k$, property (1 l) does not hold. Indeed, fix $\delta > 0$ and let suppose φ^k is a δ -perturbation of R^k such that for some π^k , $\varphi^k \times \pi^k$ does not have a fixed point in $V \times \Lambda^k$. For $l > k$, let $\rho^l : \Lambda^l \rightarrow \Lambda^k$ be a retraction (viewing Λ^k as a face of Λ^l). Define the correspondence φ^l (resp. π^l) from $\Sigma \times \Lambda^l$ to Σ (resp. Λ^l) by $\varphi^l(\sigma, \lambda) = \varphi^k(\sigma, \rho^l(\lambda))$ (resp. $\pi^l(\sigma, \lambda) = \pi^k(\sigma, \rho^l(\lambda))$). Then φ^l is a δ -perturbation of R^l and $\varphi^l \times \pi^l$ does not have a fixed point in $V \times \Lambda^l$.

We begin with some preliminary results about Property (1 k). The analogous results for Property (2) should be obvious and we omit them.

Theorem 3.2. *Property (1 k) holds iff there exists $\delta_k > 0$ such that every correspondence $\psi^k : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$ where $\varphi^k \equiv \text{proj}_\Sigma \circ \psi^k$ is a δ_k -perturbation of R^k has a fixed point in $V \times \Lambda^k$.*

Proof. The sufficiency part is obvious. As for necessity, given such a correspondence ψ^k , let W be the set of $(\sigma, \lambda) \in V \times \Lambda^k$ such that $\sigma \in \varphi^k(\sigma, \lambda)$. W is a closed subset of $V \times \Lambda^k$. For $(\sigma, \lambda) \in W$, let $\pi^k(\sigma, \lambda)$ be the set of $\lambda' \in \Lambda^k$ such that $(\sigma, \lambda') \in \psi^k(\sigma, \lambda)$. $\pi^k : W \rightarrow \Lambda^k$ is a well-behaved correspondence when W is nonempty. Extend it to a correspondence over $\Sigma \times \Lambda^k$, denoted still by π^k . By assumption $\varphi^k \times \pi^k$ has a fixed point $(\sigma, \lambda) \in V \times \Lambda^k$. The fact that $\sigma \in \varphi^k(\sigma, \lambda)$ means that $(\sigma, \lambda) \in W$. Since $\lambda \in \pi^k(\sigma, \lambda)$, we therefore have that $(\sigma, \lambda) \in \psi^k(\sigma, \lambda)$ and thus (σ, λ) is a fixed point of ψ^k in $V \times \Lambda^k$. \square

Theorem 3.3. *Property (1 k) holds iff there exists $\delta_k > 0$ such that for every δ_k -perturbation φ^k of R^k and every function $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, $V \times \Lambda^k$ contains a fixed point of $\varphi \times f$.*

Proof. The necessity of the condition is obvious. As for sufficiency, consider a correspondence $\pi^k : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ and a δ_k -perturbation φ^k of R . By McLennan [12, Proposition 2.25], for each positive integer m there exists a function $f_m : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ whose graph is contained in the m^{-1} -neighborhood of the graph of π^k . By assumption, there exists a fixed point $(\sigma_m, \lambda_m) \in V \times \Lambda^k$ of $\varphi^k \times f_m$. If necessary by passing to a subsequence, the limit of the sequence (σ_m, λ_m) is a fixed point of $\varphi^k \times \pi$ that belongs to $V \times \Lambda^k$. Hence, Property (1k) holds. \square

Remark 3.4. In view of the above Theorem, we could, in studying Properties (1k) and (2), replace Λ^k with some $\tilde{\Lambda}^k$ that is homeomorphic to Λ^k as long as we require π^k to be a function. (If we allow π^k to be a correspondence then $\tilde{\Lambda}^k$ must be convex too.) By a slight abuse of notation, given such a set $\tilde{\Lambda}^k$, we still use R^k to denote the correspondence from $\Sigma \times \tilde{\Lambda}^k \rightarrow \Sigma$ that ignores the coordinates in $\tilde{\Lambda}^k$. (The exact domain of R^k should be clear from the context.)

Though we do not use the result, the following Theorem shows that the properties we are studying are related to suspensions of R .

Theorem 3.5. *Let $Id : \Lambda^k \times \Lambda^k$ be the identity function. Property (1k) holds iff there exists $\delta_k > 0$ such that each δ_k -perturbation of $R^k \times Id$ has a fixed point in $V \times \Lambda^k$.*

Proof. The necessity follows from Theorem 3.2. As for sufficiency, suppose Property (1k) does not hold. Then, by Theorem 3.3, for each $\delta_k > 0$, there exists a δ_k -perturbation φ^k of R^k and a function $f : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$ such that $\varphi^k \times f$ does not have a fixed point in $V \times \Lambda^k$. Let $g : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ be the function $g(\sigma, \lambda) = (1 - \delta_k)\lambda + \delta_k f(\sigma, \lambda)$. Then, $\varphi^k \times g$ is a perturbation of $R^k \times Id$ that does not have a fixed point in $V \times \Lambda^k$. \square

3.1. Essentiality of Projections. In this subsection, we show the connection among Properties (1k) and (2) and the essentiality of suspensions of the projection map from \mathcal{E} , the graph of the perturbed equilibrium correspondence. Throughout this section, let $E = \{(\eta, \sigma) \in \mathcal{E} \mid \sigma \in V\}$. p denotes the natural projection from E . In order to make the domain clear, we sometimes write p_δ to denote the projection from $(E_\delta, \partial E_\delta)$ to $(P_\delta, \partial P_\delta)$. $S^k p_\delta$ refers to the k -th suspension of p_δ and $p_\delta^k : (E_\delta, \partial E_\delta) \times (\Lambda^k, \partial \Lambda^k) \rightarrow (P_\delta, \partial P_\delta) \times (\Lambda^k, \partial \Lambda^k)$ is the map $p_\delta((\eta, \sigma), \lambda) = (\eta, \lambda)$. By Lemma A.6, $S^k p_\delta$ is essential in homotopy iff p_δ^k is.

For each $0 < \delta \leq 1$, let $Q_\delta = \{\eta \in P \mid \bar{\eta} = \delta\}$ and denote its boundary by ∂Q_δ . Q_δ is homeomorphic to the strategy space Σ . Let $(F_\delta, \partial F_\delta) = p^{-1}(Q_\delta, \partial Q_\delta) \cap E_\delta$. $q_\delta : (F_\delta, \partial F_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ denotes the natural projection. We define $S^k q_\delta$ and q_δ^k like their counterparts above. As before, $S^k q_\delta$ is essential in homotopy iff q_δ^k is.

Observe that if $(\eta, \sigma) \in E_\delta$ then $(\eta', \sigma) \in F_\delta$, where $\eta' = (1 - \bar{\eta})^{-1}((\delta - \bar{\eta})\sigma + (1 - \delta)\eta)$. Indeed, if a pure strategy s is not a best reply for player n against σ then $\eta_{n,s} = \sigma_{n,s} = \eta'_{n,s}$.

Lemma 3.6. *Suppose p_δ^k is essential for some k and $\delta > 0$. Then q_δ^k is essential for each $0 < \delta' \leq \delta$.*

Proof. As in the remarks following Definition 2.5, one verifies straightforwardly that $p_{\delta'}^k$ is essential for all $\delta' < \delta$. Therefore, it is sufficient to prove that essentiality of p_δ^k implies that of q_δ^k . Suppose q_δ^k is inessential. Then, by Lemma A.3 there exists a correspondence $\psi : F_\delta \times \Lambda^k \rightarrow Q_\delta \times \Lambda^k$ such that ψ has no point of coincidence with q_δ^k . Construct now a correspondence $\tilde{\psi} : E_\delta \times \Lambda^k \rightarrow P_\delta \times \Lambda^k$ as follows: for each $\tilde{\psi}((\eta, \sigma), \lambda) = \psi((1 - \bar{\eta})^{-1}((\delta - \bar{\eta})\sigma + (1 - \delta)\eta), \sigma, \lambda)$. By our previous observation, $\tilde{\psi}$ is a well-defined correspondence whose image is contained $Q_\delta \times \Lambda^k$. Obviously, it too does not have a point of coincidence with p_δ^k and therefore p_δ^k is inessential. \square

Theorem 3.7. *Suppose that $p_{\delta_k}^k$ or $q_{\delta_k}^k$ is essential in homotopy for some $\delta_k > 0$. Then Property (1k) holds. In particular, Property (2) holds if either p_δ or q_δ is stably essential for some $\delta > 0$.*

Proof. The second statement follows trivially from the first. We prove the first statement. Our assumption along with Lemma 3.6 implies that $q_{\delta_k}^k$ is essential. Let φ^k be a δ_k -perturbation of $R^k : \Sigma \times \Lambda^k \rightarrow \Sigma$ and let $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ be a function. Define $\psi : F_{\delta_k} \times \Lambda^k \rightarrow Q_{\delta_k} \times \Lambda^k$ as follows: $\psi((\eta, \sigma), \lambda)$ is the set of all $(\eta', f(\sigma, \lambda)) \in Q_{\delta_k} \times \Lambda^k$ such that there exists $\sigma' \in \Sigma$ such that $\sigma' \in \varphi^k(\sigma, \lambda)$ and $\eta' = \sigma' - (1 - \delta_k)\tau(\eta, \sigma)$. (Recall that $\tau(\eta, \sigma)$ is the equilibrium of $G(\eta)$ that corresponds to the perturbed equilibrium σ , i.e. $\sigma = (1 - \bar{\eta})\tau(\eta, \sigma) + \eta$.) It follows from its definition that ψ is compact and convex valued and upper-semi-continuous. Thus, to show that ψ is a correspondence, there remains to check that ψ is nonempty valued. Fix $((\eta, \sigma), \lambda)$. Let s^1, \dots, s^{k_1} the set of all pure strategy profiles that are best replies against σ . Since φ^k is a δ_k -perturbation of R^k , there exists for each $1 \leq i \leq k_1$ a mixed strategy profile $\sigma^i \in \varphi^k(\sigma, \lambda)$ that is within δ_k of s^i (when viewing s^i as a point in Σ). Therefore, $\eta^i \equiv \sigma^i - (1 - \delta_k)s^i$ belongs to Q_{δ_k} for all i . Let $\tau(s^i)$ be the probability of s^i in $\tau(\eta, \sigma)$. Obviously $\sum_i \tau(s^i) = 1$, because $\tau(\eta, \sigma)$ is the equilibrium of $G(\eta)$ that corresponds to the perturbed equilibrium σ . Since φ^k is convex valued, we now have that $\sum_i \tau(s^i)\sigma^i$ belongs to $\varphi^k(\sigma, \lambda)$. Moreover, $\sum_i \tau(s^i)\sigma^i - (1 - \delta_k)\tau(\eta, \sigma) = \sum_i \tau(s^i)\eta^i \in Q_{\delta_k}$. Hence $(\sum_i \tau(s^i)\eta^i, f(\sigma, \lambda))$ belongs to $\psi((\eta, \sigma), \lambda)$ and ψ is nonempty valued. Our assumption and Lemma A.2 imply that ψ has a point of coincidence with $q_{\delta_k}^k$: there exists (η, σ, λ) such that $(\eta, \lambda) \in \psi((\eta, \sigma), \lambda)$ and $\bar{\eta} = \delta_k$. By the definition of ψ , there exists $\sigma' \in \Sigma$ such

that: (a) $\sigma' \in \varphi^k(\sigma, \lambda)$; and (b) $\eta = \sigma' - (1 - \delta)\tau(\eta, \sigma)$. Therefore, using the definition of $\tau(\eta, \sigma)$ we have $\eta = \sigma' - (1 - \delta_k)(1 - \bar{\eta})^{-1}(\sigma - \eta) = \sigma' - \sigma + \eta$. Thus, $\sigma' = \sigma \in \varphi^k(\sigma, \lambda)$. Also, since (σ, η, λ) is a point of coincidence between ψ and $q_{\delta_k}^k$, we have that $\lambda = f(\sigma, \lambda)$. (σ, λ) is then a fixed point of $\varphi^k \times f$. Since $(\eta, \sigma) \in E$, $\sigma \in V$ by definition, and the proof is complete. \square

The following theorem gives a partial converse to the previous theorem.

Theorem 3.8. *Let d be the dimension of P_1 .*

- (1) *If Property (1k) holds for some $k \geq d$, then p_δ^{k-d} is essential in homotopy for all $0 < \delta \leq \delta_k$.*
- (2) *If Property (1k) holds for some $k \geq d - 1$, then q_δ^{k-d-1} is essential in homotopy for all $0 < \delta \leq \delta_k$.*
- (3) *If Property (2) holds, then $p_{\delta'}$ and $q_{\delta'}$ are stably essential for all $0 < \delta' \leq \delta$. Moreover, if V is semialgebraic then it contains a stably essential set.*

Proof. As before, the first statement in 3 follows from statements 1 and 2. We prove statement 1. (The proof for statement 2 is analogous.) By Lemma A.3 it is sufficient to show that every function $f : E_\delta \times \Lambda^{k-d} \rightarrow P_\delta \times \Lambda^{k-d}$ has a point of coincidence with the function p_δ^k . Accordingly, fix such a function f . Extend f to a function from $P_\delta \times \Sigma \times \Lambda^{k-d} \rightarrow P_\delta \times \Lambda^{k-d}$, denoting it still by f .

Let $\varphi^k : \Sigma \times P_\delta \times \Lambda^{k-d} \rightarrow \Sigma$ be defined as follows. For each (σ, η, λ) , letting η' be the projection of $f(\eta, \sigma, \lambda)$ to P_δ , $\varphi^k(\sigma, \eta, \lambda)$ is the set of $\tau \geq \eta'_{n,s}$ such that $\tau_{n,s} = \eta'_{n,s}$ if strategy s is not a best reply for player n against σ . Then φ^k is a δ -perturbation of $R^k : \Sigma \times P_\delta \times \Lambda^{k-d} \rightarrow \Sigma$. Using Remark 3.4, our assumption implies the existence of a fixed point (σ, η, λ) of the correspondence $\varphi^k \times f$ where $\sigma \in V$. Since $f(\sigma, \eta, \lambda) = (\eta, \lambda)$, we have by the construction of φ^k that (η, σ) belongs to E_δ and hence that (σ, η, λ) is a point of coincidence between p_δ^{k-d} and f , which proves that p_δ^{k-d} is essential in homotopy.

We now prove the second part of statement 3. Suppose now that V is semialgebraic. Then E is semialgebraic. Let $S = \{\sigma \mid (0, \sigma) \in E\}$. Obviously $S \subseteq V$. We now show that S contains a stably essential set. Since Property (2) holds, p_δ is stably essential, as we have just seen. Let X be the closure of $E \setminus \partial E_1$. Clearly, X is a compact semialgebraic set and $\{\sigma \mid (0, \sigma) \in X\} \subseteq S$. Moreover, by Remark A.7 the projection from X is also stably essential.

Let $f : X \rightarrow \mathbb{R}$ be the function $f(\eta, \sigma) = \bar{\eta}$. By definition, we have that for each $\delta \geq 0$, $f^{-1}([0, \delta]) = X_\delta$. By [15, Lemma 2] there exist a positive integer l , a real number $\delta_1 > 0$, and semialgebraic sets X^1, \dots, X^l , such that for each $0 < \delta \leq \delta_1$: (i) for each i , $X_\delta^i \setminus \partial X_\delta^i$ is

connected and dense in X_δ^i , where $\partial X_\delta^i = \partial X_\delta \cap X_\delta^i$; (ii) $i \neq j$ implies $X_\delta^i \cap X_\delta^j \subseteq \partial X_1$; and (iii) $\cup_i X_\delta^i = X_\delta$. By property (i), each of the X^i 's satisfies the connexity requirement of Definition 2.6. Since X satisfies the essentiality property of Definition 2.6, by Properties (ii) and (iii), there exists i such that X^i satisfies the stable essentiality property and $\{\sigma \mid (0, \sigma) \in X^i\}$ is a stably essential set. \square

Remark 3.9. There is a certain asymmetry in the preceding two theorems. While the essentiality of $S^k p$ or $S^k q$ implies the Robustness property for R^k , our proof of Lemma 3.8 requires the Robustness property for R^{k+d} —not just that of R^k —to obtain the essentiality of $S^k p$. It is not clear to us if Robustness of R^k suffices.

If Property (1k) holds for each k then, by the previous theorem, $p_{\delta_k}^k$ is essential for each k . As k grows large, if δ_k goes to zero, then p_δ is not stably essential. In other words Property (2) fails to obtain. We do not have an example exhibiting this phenomenon. Our next theorem gives sufficient conditions when this does not happen.

Theorem 3.10. *Suppose V is a semialgebraic set. And suppose Property (1k) holds for each k . Then Property (2) holds.*

The proof of this theorem uses the following lemma, which is stated in a slightly more general form here because it is used in the next section. Let X be a closed semialgebraic subset of \mathcal{E} . For each $0 < \delta \leq 1$, let $(Y_\delta, \partial Y_\delta)$ be the inverse image of $(Q_\delta, \partial Q_\delta)$ under the projection map from X to P_1 , and let $q_\delta : (Y_\delta, \partial Y_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ be the natural projection. For each k , q_δ^k and p_δ^k are defined exactly as we defined them for the sets F and E respectively.

Lemma 3.11. *There exists $\delta_0 > 0$ such that for each k and $0 < \delta \leq \delta_0$, q_δ^k is essential iff $q_{\delta_0}^k$ is. Moreover, $p_{\delta_0}^k$ is essential iff $q_{\delta_0}^{k+1}$ is.*

Proof of the Lemma. Since X is semialgebraic, for each maximal proper face F_α of P_1 other than Q_1 , $X^\alpha \equiv X \cap p^{-1}(F^\alpha)$ is semialgebraic as well. Let $\varepsilon : X \rightarrow [0, 1]$ be the function $\varepsilon(\eta, \sigma) = \bar{\eta}$. By the Generic Local Triviality Theorem [1, Proposition 9.3.2] there exists $\delta_0 > 0$, a semialgebraic fiber C , with for each maximal proper face $F^\alpha \neq Q_1$ a closed semialgebraic subset C^α of C , and a homeomorphism $h : (0, \delta_0] \times C \rightarrow \varepsilon^{-1}(0, \delta_0]$, such that: (i) for each α , h maps $(0, \delta_0] \times C^\alpha$ into X^α ; (ii) h maps $\{\delta\} \times C$ onto Y_δ for $0 < \delta \leq \delta_0$.

Let $\partial C = \cup_\alpha C^\alpha$. For each $0 < \delta \leq \delta_0$, define $h_\delta : (C, \partial C) \rightarrow (Y_\delta, \partial Y_\delta)$ be the map the $h_\delta(c) = h(\delta, c)$. h_δ is then a homeomorphism. And q_δ is essential iff $q_\delta \circ h_\delta$ is. For each k , we now have a homeomorphism $h_\delta^k : (C, \partial C) \times (\Lambda^k, \partial \Lambda^k) \rightarrow (Y_\delta, \partial Y_\delta) \times (\Lambda^k, \partial \Lambda^k)$ with the identity function on the factor Λ^k . And q_δ^k is essential iff $q_\delta^k \circ h_\delta^k$ is. Define $f_\delta : (C, \partial C) \rightarrow (Q_\delta, \partial Q_\delta)$

by $f_\delta = r_\delta \circ q_{\delta_0} \circ h_{\delta_0}$, where $r_\delta : (Q_{\delta_0}, \partial Q_{\delta_0}) \rightarrow (Q_\delta, \partial Q_\delta)$ is the function $r_\delta(\eta) = \delta\eta/\delta_0$. We claim that f_δ is linearly homotopic to $q_\delta \circ h_\delta$. Indeed, this claim follows from the fact that if $c \in C^\alpha$ for some α , then its images under $q_\delta \circ h_\delta$ and $q_{\delta_0} \circ h_{\delta_0}$ (and hence also f_δ) belong to F^α . Define f_δ^k in the obvious way and we have that q_δ^k is essential iff f_δ^k is essential. Observe that r_δ is a homeomorphism. Therefore, f_δ is essential iff $q_{\delta_0}^k \circ h_{\delta_0}^k$ is essential, i.e. iff $q_{\delta_0}^k$ is.

We now turn to the second statement. As we saw above, $q_{\delta_0}^{k+1}$ is essential iff $q_{\delta_0}^{k+1} \circ h_{\delta_0}^{k+1}$ is. Let $(Z, \partial Z) = ([0, \delta_0], \{0, \delta_0\}) \times (C, \partial C)$. For each k , let $f^k : (Z, \partial Z) \times (\Lambda^k, \partial \Lambda^k) \rightarrow ([0, \delta_0], \{0, \delta_0\}) \times (Q_{\delta_0}, \partial Q_{\delta_0}) \times (\Lambda^k, \partial \Lambda^k)$ be the map given by $f^k(\delta, c, \lambda) = (\delta, h(\delta_0, c), \lambda)$. Using Lemma A.6, therefore, $q_{\delta_0}^{k+1}$ is essential iff f^k is essential. Define \bar{Z} be the quotient space of $Z \equiv [0, \delta_0] \times C$ obtained by collapsing $\{0\} \times C$ to a point, call it c_0 . And let $\partial \bar{Z}$ be the image of $(([0, \delta_0] \times \partial C) \cup (\{0, \delta_0\} \times C))$ under the quotient map. Likewise there is a quotient map from $([0, \delta_0], \{0, \delta_0\}) \times (Q_{\delta_0}, \partial Q_{\delta_0})$ to $(P_{\delta_0}, \partial P_{\delta_0})$ that sends (δ, η) to $\delta\eta$. Define $\bar{f}^k : (\bar{Z}, \partial \bar{Z}) \times (\Lambda^k, \partial \Lambda^k) \rightarrow (P_{\delta_0}, \partial P_{\delta_0}) \times (\Lambda^k, \partial \Lambda^k)$ as follows: $\bar{f}^k(c_0, \lambda) = (0, \lambda)$; for $0 < \delta \leq \delta_0$, $c \in C$, $\lambda \in \Lambda^k$, $\bar{f}^k(\delta, c, \lambda) = f^k(\delta, c, \lambda)$. By Lemma A.5, f^k is essential iff \bar{f}^k is.

Let \bar{X} be the quotient space of X_{δ_0} obtained by collapsing $p_{\delta_0}^{-1}(0)$ to a point, x_0 , and let $\partial \bar{X}$ be the image of ∂X_{δ_0} under the quotient map. Let \bar{p}_{δ_0} be corresponding projection map to P_{δ_0} : it maps x_0 to 0 and each other point (η, σ) to η . For each k , $\bar{p}_{\delta_0}^k$ is also defined in an analogous way. By Lemma A.5, for each k , $p_{\delta_0}^k$ is essential iff $\bar{p}_{\delta_0}^k$ is essential.

The map h extends to a homeomorphism \bar{h} between $(\bar{Z}, \partial \bar{Z})$ and $(\bar{X}, \partial \bar{X})$ by letting $\bar{h}(c_0) = x_0$. For each k , \bar{h}^k is defined in the obvious way. We now have that $\bar{p}_{\delta_0}^k$ is essential iff $\bar{p}_{\delta_0}^k \circ \bar{h}^k$ is essential. The latter map is linearly homotopic to \bar{f}^k for the same reason that f_δ was linearly homotopic to $q_\delta \circ h_\delta$. Hence $p_{\delta_0}^k$ is essential iff $q_{\delta_0}^{k+1}$ is. \square

Proof of Theorem 3.10. Since V is semialgebraic, E is semialgebraic and the above lemma applies. In particular, there exists $\delta_0 > 0$ satisfying the conditions given there. By assumption, for each k , there exists δ_k such that Property (1k) holds. Without loss of generality we can assume that $\delta_k \leq \delta_0$. Using Theorem 3.8 and the above lemma, $q_{\delta_0}^k$ is essential for each k . By Theorem 3.7, Property (2) holds. \square

3.2. Sufficiency of Essential Projections. By the results of the previous subsection, checking whether Property (2) holds is equivalent to checking whether p_δ^k or q_δ^k is stably essential, which involves checking the essentiality of an infinity of maps. There is hence the question of whether there exists a k such that the essentiality of the k -th suspension of p_δ implies that p_δ is stably essential. We do not know the answer to this question in general. However, we know from Lemma A.9 that there are conditions when the essentiality of p_δ

implies its stable essentiality. This result, therefore, yields the following theorem, the proof of which is obvious.

Theorem 3.12. *Suppose V is semialgebraic. Let d be the dimension of P_1 . If the dimension of E is less than d , then Property (2) fails to hold. If the dimension of E is d , then Property (2) holds iff p_δ is essential.*

3.3. CKM Perturbations. Hillas et al. [8] introduce the notion of *continuous Kohlberg-Mertens perturbations* (CKM perturbations). A CKM perturbation is a function $g : \Sigma \rightarrow P_1$. Such a function g produces a perturbation φ_g of R defined as follows: $\varphi_g(\sigma)$ is the set of $(1 - g(\bar{\sigma}))\tau + g(\sigma)$ such that $\tau \in R(\sigma)$. Analogous to BR-sets, S is a CKM set if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $g : \Sigma \rightarrow P_\delta$, φ_g has a fixed point within ε of S . Hillas et al. [8] show that the CKM sets are exactly the BR-sets.

The results of the preceding subsection show that one obtains such an equivalence between the two notions. Observe that in the proof of Theorem 3.8, we need only a specific type of perturbation of R^k , which are ‘‘suspensions’’ of CKM perturbations. Specifically, given a function g from $\Sigma \times \Lambda^k$ to P_δ (resp. Q_δ), we can generate a perturbation φ_g^k of R^k by letting $\varphi_g^k(\sigma, \lambda)$ be the set of $(1 - g(\bar{\sigma}, \lambda))\tau + g(\sigma, \lambda)$, where τ is a best reply to σ . For p_δ (resp. q_δ) to be stably essential, it sufficient that for each such g and $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, $\varphi_g^k \times f$ have a fixed point in $V \times \Lambda^k$. In conjunction with Theorem 3.7, we therefore have the following Theorem.

Theorem 3.13. *Property (2) holds iff for each k , and each function $f \times g : \Sigma \times \Lambda^k \rightarrow \Lambda^k \times P_\delta$, $\varphi_g^k \times f$ has a fixed point in $V \times \Lambda^k$.*

3.4. The Small-Worlds Projection Property. We show here that Property (2) is equivalent to a projection property as in the Small Worlds axiom. We begin with a definition.

Definition 3.14 (\mathcal{N} -equivalent game). A finite game \tilde{G} in strategic form is \mathcal{N} -equivalent to G if: (i) the player set of \tilde{G} includes \mathcal{N} ; (ii) for each $n \in \mathcal{N}$, his set of pure strategies is Σ_n° ; and (iii) the payoffs of each $n \in \mathcal{N}$ depend only on the strategy choices of the players in \mathcal{N} and coincide with his payoffs in G .

We refer to those players in \tilde{G} who do not belong to \mathcal{N} as the ‘outsiders.’

Theorem 3.15. *Property (2) iff there exists $\delta > 0$ such that for each game \tilde{G} that is \mathcal{N} -equivalent to G , every δ -perturbation of the best-reply correspondence of \tilde{G} that leaves the best-reply correspondence of the outsiders unchanged has a fixed point whose projection to Σ is contained in V .*

Proof. For an \mathcal{N} -equivalent game \tilde{G} the strategy set of the outsiders is a convex polytope, call it $\tilde{\Lambda}$. Letting k be the dimension of $\tilde{\Lambda}$, any δ -perturbation of the best-reply correspondence of \tilde{G} (even those that perturb the coordinates of the outsiders) is of the form $R^k \times \pi : \Sigma \times \tilde{\Lambda} \rightarrow \Sigma \times \tilde{\Lambda}$. Therefore, the necessity of the condition follows from Remark 3.4.

We now prove the sufficiency part. Fix k and let φ be a δ -perturbation of R^k and let $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ be a function. Suppose $\varphi \times f$ does not have a fixed point in $V \times \Lambda^k$. We now construct an equivalent game and a δ -perturbation of its best-reply correspondence that leaves the best-reply correspondence of the outsiders unchanged and that does not have a fixed point projecting to a point in V . The proof of this last fact is quite involved and is therefore broken into three steps. It uses definitions and results in Appendix B about multisimplices (i.e., products of simplices) and polyhedral complexes.

Step 1—simplicial preliminaries. Let C be the closed set consisting of those points $(\sigma, \lambda) \in V \times \Lambda^k$ such that $\sigma \in \varphi(\sigma, \lambda)$. For each $(\sigma, \lambda) \in C$, $\lambda \neq f(\lambda)$. Therefore, we can choose a number $\alpha > 0$ that is strictly smaller than $\|f(\sigma, \lambda) - \lambda\|$ for all $(\sigma, \lambda) \in C$. Let \mathcal{K} be the simplicial complex obtained by taking a sufficiently fine simplicial subdivision of Λ^k so that each simplex of \mathcal{K} is a convex polyhedron and has diameter at most $\alpha/2$. By the multisimplicial approximation theorem (see Theorem B.3) there exists a barycentric subdivision \mathcal{L} of \mathcal{K} and, for each n , a barycentric subdivision \mathcal{T}_n of Σ_n such that the map $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$ has a multisimplicial approximation g from the complex $\mathcal{T} \times \mathcal{L}$ to \mathcal{K} , where $\mathcal{T} \equiv \prod_n \mathcal{T}_n$. If $(\sigma, \lambda) \in C$, then $\|f(\sigma, \lambda) - \lambda\| > \alpha$, while $\|f(\sigma, \lambda) - g(\sigma, \lambda)\| \leq \alpha/2$ since the diameter of each simplex of \mathcal{K} is at most $\alpha/2$; therefore $\|g(\sigma, \lambda) - \lambda\| > \alpha/2$; i.e., λ and $g^*(\sigma, \lambda)$ belong to different simplices of \mathcal{K} . Finally let \mathcal{Q} be a polyhedral complex that is a refinement of $\mathcal{T} \times \mathcal{L}$ (viewed as a polyhedral complex) such that there exists a convex, piecewise-affine function $\gamma : \Sigma \times \Lambda^k \rightarrow \mathbb{R}$ with the property that the maximal convex domains on which γ is affine are the full-dimensional polyhedra of \mathcal{Q} —see Theorem B.4.

Step 2—constructing an equivalent game \tilde{G} . We are now ready to define a game \tilde{G} that is \mathcal{N} -equivalent to G . The set $\tilde{\mathcal{N}}$ of players in \tilde{G} is $\mathcal{N} \cup O$, where the set O of outsiders comprises three players, denoted o_1, o_2, o_3 .

Step 2A—the strategy sets. The strategy sets of players in \mathcal{N} are as before. The mixed strategy sets Σ_{o_1} and Σ_{o_2} of players o_1 and o_2 have as their vertices (these players' pure strategies) the sets of vertices of \mathcal{L} and \mathcal{K} , respectively. The mixed strategy set Σ_{o_3} of player o_3 has as its vertices the class of full-dimensional polyhedra of \mathcal{Q} . Observe that each pure strategy $s \in \Sigma_{o_1}^\circ$ corresponds to a unique point $\lambda(s)$ in Λ^k ; since Λ^k is a convex

set, there is now an affine function h from the set Σ_{o_1} of his mixed strategies to the set Λ^k that sends each mixed strategy to the corresponding average of the vertices, i.e., for each $\sigma_{o_1} \in \Sigma_{o_1}$, $h(\sigma_{o_1}) = \sum_{s \in S_{o_1}} \sigma_{o_1,s} \lambda(s) \in \Lambda^k$, where $\sigma_{o_1,s}$ is the probability of pure strategy s in σ_{o_1} .

Step 2B—the payoff functions. To complete the description of the game we describe the payoff functions. For players in \mathcal{N} their payoffs depend only on the strategy choices of the players in \mathcal{N} and coincide with their payoffs in G .

Player o_1 's payoffs depend only on his choice and that of player o_2 , and they are defined as follows. For pure strategies $s \in \Sigma_{o_1}^\circ$ and $t \in \Sigma_{o_2}^\circ$, o_1 's payoff $G_{o_1}^*(s, t)$ is the t -th barycentric coordinate of s in the complex \mathcal{K} . (Thus player o_1 tries to mimic o_2 's strategy.)

Player o_2 's payoffs depend on the strategy choices of all players. They are defined as follows. Suppose player o_3 picks a pure strategy s_{o_3} . Let $T \times L$ be the unique multisimplex of $\mathcal{T} \times \mathcal{L}$ that contains the full-dimensional polyhedron corresponding to s_{o_3} . For each pure strategy $s \in \Sigma_{o_2}^\circ$, define a multilinear function $G_{o_2,s,s_{o_3}}^* : T \times L \rightarrow \mathbb{R}$ as follows: for each vertex v of $T \times L$, $G_{o_2,s,s_{o_3}}^*(v) = 1$ if $g(v) = s$ and zero otherwise. Since $T \times L$ is a full-dimensional multisimplex of $\Sigma \times \Lambda^k$, this function extends uniquely to a multilinear function over the whole of $\Sigma \times \Lambda^k$, denoted still as $G_{o_2,s,s_{o_3}}^*$. Given now a mixed strategy profile $\sigma \in \Sigma$ for players in \mathcal{N} , a mixed strategy τ_{o_1} for player o_1 , and the pure strategy s_{o_3} for player o_3 , player o_2 's payoff if he plays a pure strategy s is $G_{o_2,s,s_{o_3}}^*(\sigma, h(\tau_{o_1}))$. Obviously, player o_2 's payoff function is multilinear in the strategies of his opponents.

Player o_3 's payoffs are a linear function of the strategies of players in $\tilde{\mathcal{N}} \setminus \{o_2\}$ and are defined as follows. Let s be a pure strategy of player o_3 . Let $Q \subset \Sigma \times \Lambda^k$ be the full-dimensional polyhedron corresponding to s . The restriction of γ to Q is affine and, since Q is full-dimensional, admits a unique affine extension, call it $G_{o_3,s}^*$, to the whole of $\Sigma \times \Lambda^k$. Player o_3 's payoff function is given by $G_{o_3}^*(s_{o_3}, \sigma_{-o_3}^*) = G_{o_3,s_{o_3}}^*(\sigma, \lambda)$, where $\lambda = h(\sigma_{o_1}^*)$ and σ is the projection of σ^* to Σ .

The description of \tilde{G} is now complete. By construction \tilde{G} is \mathcal{N} -equivalent to G . Let \tilde{R} be the best-reply correspondence of the game \tilde{G} .

Step 3—analyzing \tilde{R} . The perturbation of \tilde{R} we construct below leaves the coordinates of players not in \mathcal{N} the same as in \tilde{R} . Therefore, we first analyze the structure of \tilde{R} . Let σ^* be a mixed strategy profile and let σ be the projection of σ^* to Σ . Let $\lambda = h(\sigma_{o_1}^*)$. The following Lemma summarizes the relevant aspects of \tilde{R} .

Lemma 3.16. *For each player $m \in \tilde{\mathcal{N}}$, let $\Sigma_m^* \subset \Sigma_m^\circ$ be the support of σ_m^* . Then:*

- (1) Suppose that the vertices in $\Sigma_{o_2}^*$ span a simplex K^* in \mathcal{K} . Then each pure optimal reply of player o_1 belongs to K^* .
- (2) Suppose that each $s \in \Sigma_{o_3}^*$ contains (σ, λ) . Each pure optimal reply for o_2 against σ^* is a vertex of the unique simplex of \mathcal{K} that contains $g(\sigma, \lambda)$ in its interior.
- (3) Player o_3 's set of pure optimal replies to σ^* is the class of all polyhedra that contain (σ, λ) .

Proof of Lemma. 1. Player o_1 's payoff if he plays $s_{o_1} \in \Sigma_{o_1}^\circ$, against σ^* is $\sum_{s \in \Sigma_{o_2}^*} \sigma_{o_2,s}^* \times s_{o_1}(s)$, where $\sigma_{o_2,s}$ is the probability of s in $\sigma_{o_2}^*$ and $s_{o_1}(s)$ is the s -th barycentric coordinate of s_{o_1} in the complex \mathcal{K} . By assumption the support of σ_{o_2} spans a simplex in \mathcal{K} , so it follows that a pure strategy s_{o_1} is an optimal reply if and only if $s_{o_1}(s) = 0$ for all $s \notin \Sigma_{o_2}^{**}$, where $\Sigma_{o_2}^{**} \subseteq \Sigma_{o_2}^*$ is the subset of pure strategies of o_2 that are assigned the highest probability under σ_{o_2} . Since the vertices in $\Sigma_{o_2}^{**}$ span a simplex that is a face of K^* , each pure optimal reply for o_1 belongs to K^* .

2. Let $T^* \times L^*$ be the unique multisimplex of $\mathcal{T} \times \mathcal{L}$ that contains (σ, λ) in its interior. Let K' be the unique simplex of \mathcal{K} that contains $g(\sigma, \lambda)$ in its interior. By the construction of player o_2 's payoff function, for each polyhedron $s_{o_3} \in \Sigma_{o_3}^\circ$ that contains (σ, λ) the payoff to player o_2 from playing $s_{o_2} \in \Sigma_{o_2}^\circ$, if player o_3 plays s_{o_3} and all others play according to σ^* , is positive iff s_{o_2} is a vertex of K' . Since, by assumption, each $s_{o_3} \in \Sigma_{o_3}^*$ contains (σ, λ) , each optimal reply for o_2 is a vertex of K' .

3. By construction, for each pure strategy s of player o_3 , $G_{o_3}^*(s, \sigma_{-o_3}^*) \leq \gamma(\sigma, \lambda)$ where the inequality is strict unless the polyhedron s contains (σ, λ) . Thus, player o_3 's set of pure optimal replies is the class of polyhedra that contain (σ, λ) . \square

Step 4—perturbing \tilde{R} . We now construct a δ -perturbation $\tilde{\varphi}$ of \tilde{R} that perturbs only the components of \tilde{R} that pertain to the players in \mathcal{N} , as follows. For each mixed strategy profile σ^* in the game \tilde{G} , the coordinates of the original players in \mathcal{N} under $\tilde{\varphi}$ are given by $\tilde{\varphi}_{\mathcal{N}}(\sigma^*) = \varphi(\sigma, \lambda)$, where σ is the projection of σ^* to Σ and $\lambda = h(\sigma_{o_1}^*)$. Since φ is a δ -perturbation of R^k , $\tilde{\varphi}$ is a δ -perturbation of \tilde{R} .

Step 5—fixed points of $\tilde{\varphi}$. To finish the proof we show that $\tilde{\varphi}$ has no fixed point whose projection is contained in V . Let $\tilde{\sigma}$ be a fixed point of $\tilde{\varphi}$ and let σ be the projection of $\tilde{\sigma}$ into Σ . Also, let $\lambda = h(\tilde{\sigma}_{o_1})$. Since $\tilde{\sigma}$ is a fixed point of $\tilde{\varphi}$, $\sigma \in \varphi(\sigma, \lambda)$. We show that if $\sigma \in V$ then $(\sigma, \lambda) \notin C$, which completes the proof. For each $m \in \tilde{\mathcal{N}}$ let S_m^* be the support of σ_m^* . Also, let K be the unique simplex of \mathcal{K} that contains $g(\sigma, \lambda)$ in its interior. By Property 3 of Lemma 3.16, (σ, λ) belongs to each $s \in \Sigma_{o_3}^*$. Hence, by Property 2, the vertices in $\Sigma_{o_2}^*$ span

a simplex K^* that is a face of K . By Property 1 now, each $s \in \Sigma_{\sigma_1}^*$ belongs to K^* . Hence, $\lambda \in K^* \subseteq K$. By definition, $g(\sigma, \lambda) \in K$. Therefore, $(\sigma, \lambda) \notin C$. (Recall from step 1 that for a point $(\sigma, \lambda) \in C$, λ and $g(\sigma, \lambda)$ belong to different simplices.) Thus, $\tilde{\varphi}$ has no fixed point projecting to a point in V . \square

4. CHARACTERIZATION OF METASTABILITY

We now obtain topological characterizations of metastable sets, and establish the relation of metastable sets to stably essential and \tilde{H} -stable sets.

Lemma 4.1. *S is metastable iff there exists $\delta_0 > 0$ and for each $0 < \delta \leq \delta_0$ there exists a closed semialgebraic subset W_δ of Σ_δ such that:*

- (1) $W_\delta \setminus \partial\Sigma$ is connected and dense in W_δ .
- (2) For each k , each δ -perturbation φ of R^k , and each function $f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, $\varphi \times f$ has a fixed point whose projection to Σ is contained in W_δ .
- (3) $W_{\delta'} \subseteq W_\delta$ if $0 < \delta' < \delta$.
- (4) $\bigcap_\delta W_\delta = S$.

Proof. The sufficiency part of the proof is obvious. We turn now to the necessity of the conditions. Let δ_i be a monotone sequence of positive numbers converging to zero and let V_i be a corresponding sequence satisfying the conditions of Definition 2.1. Take a triangulation of Σ . For each l , let Σ^l be the l -th barycentric subdivision of this triangulation. Let P^l be the simplices of Σ^l that intersect S . For each l let X^l be the closure of $\{(\delta, \sigma) \in [0, 1] \times P^l \mid 0 < \delta < 1, \sigma \in \Sigma_\delta \setminus \partial\Sigma\}$ in $[0, 1] \times P^l$, and let $\partial X^l = \{(\delta, \sigma) \in X^l \mid \delta = 0 \text{ or } \sigma \in \partial\Sigma\}$, and let $g^l : X^l \rightarrow [0, 1]$ be the projection to the first coordinate. By [15, Lemma 2] there exists $0 < \bar{\delta}_l \leq 1$, a finite number of closed semialgebraic subsets $X^{l,1}, \dots, X^{l,j_l}$ of X^l such that for each $0 < \delta \leq \bar{\delta}_l$ and each j , letting $X_\delta^{l,j} = (g^l)^{-1}([0, \delta]) \cap X^{l,j}$, we have:

- (a) $X_\delta^{l,j} \setminus (\partial X^l \cup (g^l)^{-1}(\delta))$ is connected and dense in $X_\delta^{l,j}$;
- (b) $X_\delta^{l,j} \cap X_\delta^{l,j'} \subseteq \partial X^l$ for $j' \neq j$;
- (c) $\bigcup_j X_\delta^{l,j} = (g^l)^{-1}([0, \delta])$.

Suppose for $0 < \delta, \delta' \leq \bar{\delta}_l$ that (δ, σ) and (δ', σ) belong to $X^l \setminus \partial X^l$. Then $((\lambda\delta + (1 - \lambda\delta)'), \sigma)$ belongs to $X^l \setminus \partial X^l$ for all $\lambda \in [0, 1]$, and the above properties imply the property

- (d) (δ, σ) belongs to $X_{\delta_k}^{l,j}$ iff (δ', σ) does.

Since P^l is a neighborhood of S , it is a neighborhood of V_i for large i . Therefore, by the connexity property for V_i , and also by the above properties of X^l , if i is also large enough such that $\delta_i \leq \bar{\delta}_l$ then there exists $1 \leq j_i \leq j_l$ such that $\{\delta_i\} \times (V_i \setminus \partial\Sigma)$, and hence also its

closure $\{\delta_i\} \times V_i$, is contained in $X_{\delta_i}^{l,j_i}$. Along a subsequence of i 's now, j_i is constant, say 1. Obtain by the diagonalization process, a subsequence of i 's such that for each l and $i \geq l$, $V_i \subseteq P^l$, $\delta_i \leq \bar{\delta}_l$ and $\{\delta_i\} \times V_i$ is contained in $X_{\delta_i}^{l,1}$.

For each $0 < \delta \leq \delta_1$ define W_δ to be the projection of $X_\delta^{l,1}$ to Σ , where l is the unique integer such that $\delta_{l+1} < \delta \leq \delta_l$. We show that the W_δ 's satisfy the four enumerated conditions of the theorem. For the first three properties, we fix $\delta_{l+1} < \delta \leq \delta_l$.

Property 1. Property (a) implies that the set of $(\delta', \sigma) \in X_\delta^{l,1}$ such that $\sigma \notin \partial\Sigma$ is connected. Hence $W_\delta \setminus \partial\Sigma$, which is the image of this set under the projection to Σ , is connected. Suppose $\sigma \in W_\delta$. Then there exists $(\delta, \sigma) \in X_\delta^{l,1}$. By Property (a) again, there exists a sequence (δ^i, σ^i) in $X_\delta^{l,1} \setminus \partial X^{l,1}$ converging to (δ, σ) . Obviously the sequence σ^i belongs to $W_\delta \setminus \partial\Sigma$ and converges to σ . Hence $W_\delta \setminus \partial\Sigma$ is dense in W_δ .

Property 2. Since $V_l \subseteq P^l$, if σ belongs to $V_l \cap (\Sigma_\delta \setminus \partial\Sigma)$ then (δ, σ) belongs to X^l . Because $(\delta_l, \sigma) \in X_{\delta_l}^{l,1}$ and $0 < \delta_l \leq \bar{\delta}_l$, property (d) gives us that (δ, σ) belongs to $X_\delta^{l,1} \setminus \partial X^l$. Hence $W_\delta \setminus \partial\Sigma$ contains σ . In other words, $W_\delta \setminus \partial\Sigma \supseteq V_l \cap (\Sigma_\delta \setminus \partial\Sigma)$.

Let $Q_\delta = \{\eta \in P_\delta \mid \bar{\eta} = \delta\}$ and let ∂Q_δ be its relative boundary. Define $F_\delta = \{(\eta, \sigma) \in \mathcal{E} \mid \sigma \in W_\delta, \bar{\eta} = \delta\}$. By Theorem 3.7 it is sufficient to show that the natural projection $q_\delta : (F_\delta, \partial F_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential, where $\partial F_\delta = q_\delta^{-1}(\partial Q_\delta)$. Let $\bar{F}_\delta = \{(\eta, \sigma) \in \mathcal{E} \mid \sigma \in V_l, \bar{\eta} = \delta\}$. By Theorem 3.8 the projection $\bar{q}_\delta : (\bar{F}_\delta, \partial \bar{F}_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential. Let \hat{F}_δ be the closure of $\bar{F}_\delta \setminus \partial \bar{F}_\delta$. By Remark A.7, $\hat{q}_\delta : (\hat{F}_\delta, \partial \hat{F}_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential. Consider now a point $(\eta, \sigma) \in \hat{F}_\delta \setminus \partial \hat{F}_\delta$. Then $\sigma \in V_l \cap (\Sigma_\delta \setminus \partial\Sigma)$. As we saw in the previous paragraph, σ then belongs to W_δ and thus (η, σ) belongs to F_δ . Since $\hat{F}_\delta \setminus \partial \hat{F}_\delta$ is dense in \hat{F}_δ , we have that $(\hat{F}_\delta, \partial \hat{F}_\delta) \subseteq (F_\delta, \partial F_\delta)$. The stable essentiality of \hat{q}_δ now implies that of q_δ .

Property 3. It is sufficient to prove that if $\delta_{l+1} \leq \delta' < \delta$ then $W_{\delta'} \subseteq W_\delta$. If $\delta' > \delta_{l+1}$ then the result follows from the fact that $X_{\delta'}^{l,1} \subseteq X_\delta^{l,1}$. Observe that if $\delta' = \delta_{l+1}$ then property (a) for $X_{\delta_{l+1}}^{l+1,1}$ implies that $X_{\delta_{l+1}}^{l+1,1} \setminus \partial X^{l+1}$ is a connected subset of $(g^l)^{-1}([0, \delta]) \setminus \partial X^l$; moreover, by construction, it contains $\{\delta_{l+1}\} \times (V_{l+1} \setminus \partial\Sigma)$. Since $X_\delta^{l,1} \setminus \partial X^l$ contains this latter set, by properties (a) (b) and (c) above $X_{\delta_{l+1}}^{l+1,1} \setminus \partial X^{l+1} \subseteq X_\delta^{l,1} \setminus \partial X^l$. Using property (a) again, we get that $X_{\delta_{l+1}}^{l+1,1}$ is contained in X_δ^l . Hence $W_{\delta_{l+1}} \subseteq W_\delta$.

Property 4. By property (3) it is sufficient to show that $\cap_l W_{\delta_l} = S$. For each l , since $\{\delta_l\} \times V_l$ is contained in $X_{\delta_l}^{l,1}$, W_{δ_l} contains V_l . Since the V_l 's converge to S , $\cap_l W_{\delta_l}$ contains S . On the other hand, for each l , W_{δ_l} is contained in P^l , and the P^l 's form a basis of neighborhoods of S . Hence $\cap_l W_{\delta_l}$ is contained in S and thus we obtain (4). \square

We now provide a characterization of metastability in terms of subsets of the graph \mathcal{E} of the equilibria of perturbed games. For each $0 < \delta \leq 1$, as in the above proof let $Q_\delta = \{\eta \in P_\delta \mid \bar{\eta} = \delta\}$ and let ∂Q_δ be its relative boundary. Let $(\mathcal{F}_\delta, \partial\mathcal{F}_\delta) = p^{-1}(Q_\delta, \partial Q_\delta)$. We have a well-defined correspondence $\psi_\delta : \Sigma_\delta \rightarrow \mathcal{F}_\delta$ given by $\psi_\delta(\sigma) = \{(\eta, \sigma) \in \mathcal{F}_\delta\}$.

Theorem 4.2. *$S \subseteq \Sigma$ is metastable iff there exists a closed subset E of \mathcal{E} with $E_0 = \{0\} \times S$ and:*

- (1) *Connexity: For every neighborhood V of E_0 in E , the set $V \setminus \partial E_1$ has a connected component whose closure is a neighborhood of E_0 in E .*
- (2) *Stable Essentiality: There exists $\delta_0 > 0$ such for each $0 < \delta \leq \delta_0$, letting $(F_\delta, \partial F_\delta) = p^{-1}(Q_\delta, \partial Q_\delta) \cap E$, the natural projection $q_\delta : (F_\delta, \partial F_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential in homotopy.*

Proof. Given a metastable set S there exists a collection of W_δ 's satisfying the conditions in Lemma 4.1. Define E to be the closure of $\cup_{0 < \delta \leq \delta_0} \psi_\delta(W_\delta \setminus \partial\Sigma) \setminus \partial\mathcal{F}_\delta$. E is obviously a closed subset of \mathcal{E} . Moreover, it is nonempty: indeed, for each $0 < \delta < \delta'$, if $\sigma \in W_{\delta'} \setminus \partial\Sigma$ then $\sigma \in W_\delta \setminus \partial\Sigma$ and $\psi_\delta(\sigma) \setminus \partial\mathcal{F}_\delta$ is nonempty. We prove that it satisfies the other conditions of the theorem.

We show first that $E_0 = \{0\} \times S$. Observe that $(0, \sigma) \in E_0$ iff there exists a sequence of δ_i 's converging to zero, and a corresponding sequence (η^i, σ^i) in $\psi_{\delta_i}(W_{\delta_i} \setminus \partial\Sigma)$ converging to $(0, \sigma)$; this last condition is equivalent to the existence of a sequence $\sigma^i \in W_{\delta_i} \setminus \partial\Sigma$ converging to σ . By Property 4 of Lemma 4.1, therefore, $(0, \sigma) \in E_0$ iff $\sigma \in S$.

Fix $0 < \delta \leq \delta_0$. By the robustness property for W_δ and Theorem 3.8, the projection $\bar{q}_\delta : (\bar{F}_\delta, \partial\bar{F}_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential, where $\bar{F}_\delta = \psi_\delta(W_\delta)$. Let \hat{F}_δ be the closure of $\bar{F}_\delta \setminus \partial\bar{F}_\delta$. Then by Remark A.7, the projection $\hat{q}_\delta : (\hat{F}_\delta, \partial\hat{F}_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is also stably essential. Observe that \hat{F}_δ is contained in F_δ . Indeed, if $(\eta, \sigma) \in \hat{F}_\delta \setminus \partial\hat{F}_\delta$ then $\sigma \in W_\delta \setminus \partial\Sigma$, $\eta \notin \partial Q_\delta$, and $(\eta, \sigma) \in \psi_\delta(\sigma)$. Therefore $\hat{F}_\delta \setminus \partial\hat{F}_\delta$, and hence its closure \hat{F}_δ are contained in F_δ . The stable essentiality of \hat{q}_δ now implies that of q_δ . Hence E satisfies the essentiality condition.

Again fix $0 < \delta \leq \delta_0$. Since $W_\delta \setminus \partial\Sigma$ is connected and ψ_δ is a well-defined correspondence, $\psi_{\delta'}(W_{\delta'} \setminus \partial\Sigma) \setminus \partial\mathcal{F}_{\delta'}$ is connected for all $0 < \delta' < \delta$. Also, for $0 < \delta' < \delta$, if $(\eta', \sigma) \in \psi_{\delta'}(W_{\delta'} \setminus \partial\Sigma) \setminus \partial\mathcal{F}_{\delta'}$ then for all $\delta > \delta'' > \delta'$, $\sigma \in W_{\delta''}$ by Property (3) of Lemma 4.1, and there exists $(\eta'', \sigma) \in \psi_{\delta''}(\sigma) \setminus \partial\mathcal{F}_{\delta''}$. Therefore, for each $\lambda \in [0, 1]$, $(\lambda\eta'' + (1-\lambda)\eta', \sigma) \in \psi_{\lambda\delta'' + (1-\lambda)\delta'}(W_{\lambda\delta'' + (1-\lambda)\delta'} \setminus \partial\Sigma) \setminus \partial\mathcal{F}_{\lambda\delta'' + (1-\lambda)\delta'}$. Hence, $\cup_{0 < \delta' < \delta_0} \psi_{\delta'}(W_{\delta'} \setminus \partial\Sigma) \setminus \partial\mathcal{F}_{\delta'}$ is connected. Since E is obtained by taking the closure of $\cup_{0 < \delta' < \delta_0} \psi_{\delta'}(W_{\delta'} \setminus \partial\Sigma) \setminus \partial\mathcal{F}_{\delta'}$, we have that $E_\delta \setminus \partial E_\delta$ is connected; and its closure is a neighborhood of E_0 in E , since it contains, e.g., $E_{\delta/2}$. The

connexity condition now follows from the fact that the E_δ 's form a basis of neighborhoods of $\{0\} \times S$ in E . This completes the proof of the necessity of the conditions. The sufficiency part follows from Theorems 4.3 and 4.4. \square

Theorem 4.3. *Let E be a subset of \mathcal{E} that satisfies the connexity and essentiality conditions of Theorem 4.2 and let $S = \{\sigma \mid (0, \sigma) \in E\}$. Then S is the Hausdorff limit of a sequence S^l of semialgebraic stably essential sets. Moreover, the sequence can be chosen such that for each l , S^l has a germ E^l that is semialgebraic and satisfies the following stronger version of the connexity requirement: there exists $\delta_l > 0$ such that for each $0 < \delta \leq \delta_l$, $E_\delta^l \setminus \partial E_\delta^l$ is connected and dense in E_δ^l .*

Proof. Triangulate \mathcal{E} such that $\mathcal{E}_0 = p^{-1}(0)$ and $\partial \mathcal{E}_1 = p^{-1}(\partial P_1)$ are full subcomplexes. Let $\tilde{\mathcal{E}}$ be the union of the simplices of \mathcal{E} that do not intersect \mathcal{E}_0 . Denote by \mathcal{E}_0^l the l -th barycentric subdivision of \mathcal{E}_0 . $\tilde{\mathcal{E}}_0^l$ and $\tilde{\mathcal{E}}$ uniquely determine a triangulation \mathcal{E}^l for \mathcal{E} .

By the connexity condition there exists a decreasing sequence V^r of neighborhoods of S in E such that $V^r \setminus \partial E_1$ is connected and dense in V^r . Let $E^{l,r}$ be the union of the simplices of \mathcal{E}^l whose interiors intersect $V^r \setminus \partial E_1$. Obviously V^r is contained in $E^{l,r}$ since $V^r \setminus \partial E_1$ is dense in V^r . For each l , the $E^{l,r}$'s form a decreasing sequence in r . Because \mathcal{E}^l is a finite complex there exists $r(l)$ such that for each $r \geq r(l)$, $E^{l,r}$ is constant, say E^l . If $l' \geq l$ then for each r , $E^{l',r} \supseteq E^{l,r}$ and hence $E^{l'} \supseteq E^l$. For each l , E_0^l contains E_0 , since it contains V^r for large r . Hence, letting $S_0^l = \{\sigma \mid (0, \sigma) \in E^l\}$, we have $S \subseteq \bigcap_l S_0^l$. On the other hand, letting P^l be the set of simplices of \mathcal{E}_0^l that intersect E_0 , we have that $E_0^l \subseteq P^l$: indeed, each principal simplex of E^l intersects V^r for large r and hence intersects E_0 ; since \mathcal{E}_0^l is full subcomplex of \mathcal{E}^l , the intersection of such a simplex with \mathcal{E}_0^l is a face of the simplex and hence belongs to P^l ; thus $E_0^l \subseteq P^l$. The fact that the \mathcal{E}_0^l form a decreasing sequence converging to E_0 therefore implies that $\bigcap_l E_0^l \subseteq E_0$. Consequently, the S_0^l 's converge to S .

Fix l . Both E^l and E_0^l are obviously semialgebraic. To finish the proof, we show that E^l satisfies the stronger form of the connexity condition in the statement of the theorem and also the essentiality condition in Definition 2.6, which then ensures that E_0^l is stably essential. To obtain the connexity condition, we use Theorem 1 of Section 2 of [14]: it is sufficient to show that: (a) E^l is the closure of $E^l \setminus \partial E_1^l$; and (b) for each $0 < \alpha \leq 1$, the set W_α of points in $E^l \setminus \partial E^l$ whose simplicial distance from E_0^l is strictly smaller than α is connected. With regard to (a), since $\partial \mathcal{E}_1^l$ is a full subcomplex, the intersection of every principal simplex of E^l with $\partial \mathcal{E}^l$ is a face of the simplex; moreover, it cannot equal the simplex itself because the simplex intersects $V^r \setminus \partial E_1$ for all large r ; hence E^l is the closure of $E^l \setminus \partial E_1^l$. Now, we turn to (b). \mathcal{E}_0^l and $\partial \mathcal{E}_1^l$ being full subcomplexes, the intersection of a principal simplex of E^l with

$W_\alpha \cap \mathcal{E}_0^l$ and $\partial\mathcal{E}_1^l$ are proper faces of it; hence its intersection with $W_\alpha \setminus \partial E^l$ is connected. But, for r large enough, the connected set $V^r \setminus \partial E_1$ is contained in W_α and intersects every principal simplex of E^l . Therefore, $W_\alpha \setminus \partial E_1^l$ is connected. Thus, we have established the connexity condition for E^l .

There remains to prove the essentiality condition for E^l . By construction E^l contains E_δ for all sufficiently small δ . Therefore, by the Robustness condition for E , $q_\delta : (F_\delta^l, \partial F_\delta^l) \rightarrow (Q_\delta, \partial Q_\delta)$ is stably essential for all small δ , where $(F_\delta^l, \partial F_\delta^l)$ is the inverse image of $(Q_\delta, \partial Q_\delta)$ in E^l under the natural projection. Since E^l is semialgebraic, by Lemma 3.11, $p_\delta : (E_\delta^l, \partial E_\delta^l) \rightarrow (P_\delta, \partial P_\delta)$ is stably essential for some small δ . Hence, E^l satisfies the essentiality condition in Definition 2.6 as well. \square

Theorem 4.4. *The Hausdorff limit of a sequence of stably essential sets is metastable.*

Proof. Let S^l be a sequence of sets converging to a set S such that for each l there exists $E^l \subseteq \mathcal{E}$ such that $E_0^l = \{0\} \times S^l$, and E^l satisfies the essentiality and connexity conditions of Definition 2.6. For each l , E^l satisfies the conditions of Theorem 4.3 and hence we can assume without loss of generality that E^l satisfies the stronger form of connexity: for all small $\delta > 0$, $E_\delta^l \setminus \partial E_\delta$ is connected and dense in E_δ . Since the S^l converge to S , the E_0^l converge to E_0 , and we can now choose a sequence of δ_l 's decreasing to zero such that the $E_{\delta_l}^l$'s converge to E_0 and, for each l , $E_{\delta_l}^l \setminus \partial\mathcal{E}_1^l$ is connected and dense in $E_{\delta_l}^l$, and $p_{\delta_l} : (E_{\delta_l}^l, \partial E_{\delta_l}^l) \rightarrow (P_{\delta_l}, \partial P_{\delta_l})$ is stably essential.

For each l , let V_l be the projection of E_{δ_l} to Σ . Then $V_l \setminus \partial\Sigma$ is connected and dense in V_l . Also, V_l satisfies the Robustness condition of Definition 2.1 by Theorem 3.7. Finally, the V_l 's converge to S since in \mathcal{E} the sets E_{δ_l} converge to E_0 . Thus S is metastable. \square

It is natural to wonder if the Hausdorff limit of stably essential sets is itself stably essential, which would then imply the equivalence between metastability and stable essentiality. In the two theorems above—whose proof techniques were borrowed from Mertens [14, Section 5B] where it is shown that the limit of a sequence of semialgebraic \check{H} -stable sets is itself \check{H} -stable—we could, like Mertens, use the approximations E^l to produce a germ E for S that satisfies the connexity requirement of Definition 2.6. The problem is with the stable essentiality condition. In the case of \check{H} -stability the fact that Čech cohomology is weakly continuous is used to establish the essentiality condition for E . In our case there seems to be no analogue of the following nature. Suppose $(X^l, \partial X^l)$ is a decreasing sequence of compact semialgebraic pairs converging to $(X, \partial X)$ and suppose there is a sequence of stably essential maps $p^l : (X^l, \partial X^l) \rightarrow (B, \partial B)$ where for $l > 1$, p^l is the restriction of the map p^1 to X^l . Is it then necessarily the case that the restriction of p^1 to X is also stably essential? While we

do not have a counter example, the answer to this question appears to be no. In any event, the above three theorems readily imply the following compactness result for metastability.

Theorem 4.5. *The collection of metastable sets is the Hausdorff closure of the collection of stably essential sets.*

Theorem 4.6. *\check{H} -stable sets are stably essential and hence metastable.*

Proof. As is shown in Mertens [14, Section 4E] S is \check{H} -stable iff there exists a sequence of closed d -dimensional semialgebraic subsets E^l of \mathcal{E} , where d is the dimension of P_1 , such that for each l , E^l satisfies the essentiality and Robustness condition in Definition 2.5 and the sequence E_0^l converges to $\{0\} \times S$. Since E^l is d -dimensional, using Lemma A.9 we therefore have that $S_0^l \equiv \{\sigma \mid (0, \sigma) \in E^l\}$ is stably essential and hence metastable. By Theorem 4.5 S is metastable as well. \square

The results of this Section show that, as we asserted earlier, the collection of metastable sets remains the same if we weaken the Robustness condition in Definition 2.1 to the following: For each k there exists $i(k)$ such that for each $i \geq i(k)$, and each correspondence $\varphi^k \times f : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$, where φ^k is a δ_i -perturbation of R^k , there exists a fixed point in $V_i \times \Lambda^k$. We will merely sketch the arguments here. Given a set S and a collection V_i converging to S that satisfy this weak robustness property above and all the other properties in Definition 2.1, the proof of Lemma 4.1 can be modified to show that there exists a nested collection of W_δ that satisfy all the conditions of Lemma 4.1 except for the robustness condition, which now becomes: for each k , there exists $\delta_k > 0$ such that for $0 < \delta \leq \delta_k$, and each correspondence $\varphi^k \times f : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$, where φ^k is a δ -perturbation of R^k , there exists a fixed point in $W_\delta \times \Lambda^k$. The proof of the necessity part of Theorem 4.2 can be used to prove the existence of a set E , with $E_0 = \{0\} \times S$, that satisfies the connectivity condition there and the following essentiality condition: for each k , there exists $\delta(k) > 0$ such that for $0 < \delta \leq \delta_k$, the projection $q_\delta^k : (F_\delta, \partial F_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ is essential. The proof of Theorem 4.3 does not require the stable essentiality of $q_\delta : (E_\delta, \partial E_\delta) \rightarrow (Q_\delta, \partial Q_\delta)$ for some δ , but rather the essentiality for each k of $q_{\delta_k}^k$ for some δ_k : indeed, this follows from the fact the sets E^l constructed there are semialgebraic, coupled with Lemma 3.11. Thus, S can be approximated by a sequence of stably essential sets. Finally, Theorem 4.4 shows that S is indeed metastable in the sense of Definition 2.1. Thus, it is without loss of generality that we imposed the seemingly stronger Robustness condition for metastability.

5. PROPERTIES OF METASTABLE SETS

Kohlberg and Mertens [10] and Mertens [13, 14, 15] list a basic set of game-theoretic properties that they argue any reasonable solution concept should satisfy. In this Section we show that, except for the decomposition property, metastability satisfies all their requirements. Metastability satisfies a slightly weaker version of the decomposition property.

5.1. Basic Properties. Since \check{H} -stable sets exist and are metastable, we get existence for metastability. Also, by definition, metastable sets are connected sets of perfect equilibria. Metastable sets are BR-sets and the proof in Hillas [7] then shows that a metastable set contains a proper equilibrium and thus satisfies the backward induction property. Finally, by Theorem 4.5 the collection of metastable sets is compact in the Hausdorff topology.

5.2. Forward Induction and Iterated Dominance. Kohlberg and Mertens [10] introduce the notion of forward induction by requiring that a solution to the game contain a solution to a game obtained by deleting a strategy that is not a best reply against any equilibrium in the solution of the original game. Mertens [13] strengthens this property by requiring the solution to survive even under deletion of a strategy that, while possibly optimal against some equilibrium in the solution, is nonetheless inferior in any ε -perfect equilibrium close to the set. Here we prove this property for metastability.

Theorem 5.1. *Suppose S is a metastable set of the game G . If there exist a neighborhood V of S , $\delta > 0$, and a pure strategy s_n for some player n such that s_n is an inferior reply against each $\sigma \in V \cap (\Sigma_\delta \setminus \partial\Sigma)$, then S contains a metastable set of the game \bar{G} obtained by deleting the pure strategy s_n .*

If V_i is a sequence as in Definition 2.1 then for large i , by our assumptions, s_n is used with probability at most δ_i at each $\sigma \in V_i \setminus \partial\Sigma$. Hence it is used with zero probability in S . S can thus be viewed as a subset of the strategy space in \bar{G} . And, formally, the theorem states this subset contains a metastable set in \bar{G} .

Proof. Let V_i be a sequence of subsets of Σ converging to S and satisfying the conditions of Definition 2.1. Assume that for each i , V_i is a subset of V and $\delta_i \leq \delta$. By Lemma 4.1, we can further assume that V_i is semialgebraic and contains S . Let $\bar{\Sigma}$ be the face of Σ where s_n is used with zero probability. We can view $\bar{\Sigma}$ as the strategy space of the game \bar{G} obtained by deleting strategy s_n . Since V_i contains S , which as we remarked above is contained in $\bar{\Sigma}$, $\bar{V}_i \equiv V_i \cap \bar{\Sigma}$ is a closed, nonempty, semialgebraic subset of $\bar{\Sigma}$ and \bar{V}_i converges to S . By Statement (3) of Theorem 3.8 and Theorem 4.4, it is sufficient to prove that \bar{V}_i satisfies the Robustness condition of Definition 2.1 in the game \bar{G} .

The set \bar{P}_{δ_i} of δ_i -perturbations for the game \bar{G} can be viewed as the face of P_{δ_i} where the error probability for s_n is zero. Fix k and let $\bar{f} \times \bar{g} : \bar{\Sigma} \times \Lambda^k \rightarrow \Lambda^k \times \bar{P}_{\delta_i}$ be a map. By Theorem 3.13 it is sufficient to show that $\bar{\varphi}_g^k \times \bar{f}$ has a fixed point in $\bar{V}_i \times \Lambda^k$. Extend $\bar{f} \times \bar{g}$ to a map $f \times g$ from $\Sigma \times \Lambda^k$ to $\bar{P}_{\delta_i} \times \Lambda^k$. Suppose (σ, λ) is a fixed point of $\varphi_g^k \times f$ such that $\sigma \in \bar{V}_i$. Then obviously it is also a fixed point of $\bar{\varphi}_g^k \times \bar{f}$. Hence to finish the proof we prove the existence of such a fixed point for $\varphi_g^k \times f$. Choose a sequence of functions $g^l : \Sigma \times \Lambda^k \rightarrow P_{\delta_i}$ converging to g (in the sup norm) such that for each l the image of the map is contained in $P_{\delta_i} \setminus \partial P_{\delta_i}$. For each l there exists a fixed point (σ^l, λ^l) of $\varphi_{g^l}^k \times f$ such that $\sigma^l \in V_i$. Let $\eta^l = g^l(\sigma^l, \lambda^l)$. Since η^l belongs to the interior of P_{δ_i} and σ^l is a perturbed equilibrium of $G(\eta^l)$, σ^l is completely mixed. By assumption, therefore, s_n is used with probability η_{n,s_n}^l under σ_n^l . By passing to a subsequence if necessary, the limit (σ, λ) , which belongs to $V_i \times \Lambda^k$, is a fixed point of $\varphi_g \times f$ where the probability of s_n is zero, i.e. $\sigma \in \bar{V}_i$. \square

The proof actually implies a slightly stronger forward induction property. If s_n is not a best reply to any strategy in the sets $V_i \setminus \partial \Sigma$ then deleting the strategy preserves a metastable set of the smaller game.

5.3. Ordinality and Player-Splitting. Kohlberg and Mertens [10] require that a solution is invariant under the addition or deletion of redundant strategies, i.e. a solution depends only on the reduced strategic form of the game obtained by deleting redundant strategies. Subsequently Mertens [15] provides a formal treatment of this notion, generalizing the idea to the concept of ordinality for solution concepts. Here we show that metastability is ordinal in the sense of Mertens. While Mertens considered the class of strategic-form games—where the strategy sets of the players are arbitrary polytopes and the payoff functions are multiaffine—we restrict ourselves here to games in normal form with finite pure strategy sets. Hence our treatment of ordinality is in the context of normal-form games (even though there is an obvious extension of metastability to this general class and ordinality obtains there as well).

Mertens [15, Theorem 2] gives two sufficient conditions for a solution to be ordinal. The following two theorems establish that metastability satisfies them.

A strategy τ_n is an admissible best reply against a profile σ if there exists a sequence σ^k converging to σ such that τ_n is a best reply against σ^k for all k . A profile τ is an admissible best reply against σ if for each n , τ_n is an admissible best reply against σ . One then obtains an admissible best reply correspondence for the game that assigns to each σ the set of admissible best replies.

Theorem 5.2. *Suppose G and \tilde{G} are two games with the same sets of players and strategies, and they have the same admissible best-reply correspondence. Then they have the same metastable sets.*

Proof. Let S be a metastable set of G and let E be a germ for S satisfying the conditions of Theorem 4.2. We can assume without loss of generality that E is the closure of $E \setminus \partial E_1$; indeed, the connexity condition obviously holds if we do so, while the essentiality condition holds because of Remark A.7. Given $\delta > 0$ and a strategy profile $\tau \in \Sigma \setminus \partial \Sigma$, observe that $G(\delta\tau)$ and $\tilde{G}(\delta\tau)$ have the same set of equilibria, since G and \tilde{G} have the same admissible best-reply correspondence. Therefore, E is also a subset of the graph of the perturbed equilibrium correspondence for the game \tilde{G} . Hence, S is a metastable set of the game \tilde{G} . The result follows from the symmetry between G and \tilde{G} . \square

We now state and prove a theorem that implies that metastability is invariant under addition of redundant strategies and also shows that the player-splitting property holds. Before discussing these properties, we present the theorem.

Suppose \tilde{G} and G are two strategic-form games with strategy spaces $\tilde{\Sigma}$ and Σ respectively. Suppose f is a surjective linear mapping from $\tilde{\Sigma}$ to Σ such that for each $0 \leq \delta \leq 1$ and $\tilde{\tau} \in \tilde{\Sigma}$, $\tilde{\sigma}$ is an equilibrium of the perturbed game $\tilde{G}(\delta\tilde{\tau})$ iff $f(\tilde{\sigma})$ is an equilibrium of $G(\delta f(\tilde{\tau}))$.

Theorem 5.3. *If \tilde{S} is a metastable set of \tilde{G} then $f(\tilde{S})$ is a metastable set of G . If S is a metastable set of G then $f^{-1}(S)$ is a metastable set of \tilde{G} .*

The proof uses the following lemma. It is a version of the Generic Local Triviality Theorem for the case of polyhedra and linear mappings, and because of the formulation it yields a “global triviality” result.

Lemma 5.4. *Suppose $f : X \rightarrow Y$ is a surjective linear mapping where X and Y are compact convex polyhedra, and let $d = \dim(X) - \dim(Y)$. There exists a surjective map $h : Y \times [0, 1]^d \rightarrow X$ such that:*

- (1) $h(\{y\} \times [0, 1]^d) = f^{-1}(y)$ for all $y \in Y$.
- (2) h maps $(Y \setminus \partial Y) \times (0, 1)^d$ homeomorphically onto $(X \setminus \partial X)$, where ∂X and ∂Y are the relative boundaries X and Y , respectively.

Moreover, there exists a continuous selection from f^{-1} .

Proof of Lemma. The existence of a continuous selection from f^{-1} follows from the existence of the function h with the requisite properties, which we now prove. Let k be the dimension of Y . Since the graph of f is a compact convex polyhedron homeomorphic to X , it is

sufficient to prove the lemma for the special case that f is a projection map onto, say, the first k coordinates.

We can further assume that X is a full-dimensional polyhedron in \mathbb{R}^{k+d} and Y is the projection of X onto its first k -coordinates. Indeed, suppose X is a polyhedron in \mathbb{R}^m with Y being the projection of X onto, say, the first k coordinates; then, since X is $(k+d)$ -dimensional and its projection to the first k coordinates is k -dimensional, we can find d additional coordinates such that, after permuting the last $m-k$ coordinates if necessary, the projection from X onto the first $k+d$ coordinates is a homeomorphism between X and its image. Thus, replacing X with its projection onto its first $k+d$ coordinates, we can assume that X is a full-dimensional polyhedron in \mathbb{R}^{k+d} and f is the projection of X onto the first k coordinates.

It is now sufficient to prove the theorem for the case where $d = 1$, since, in general, f can be factored through a series of projections that omit one coordinate at a time. For each $y \in Y$, let $\bar{y}_{k+1}(y)$ (resp. $\underline{y}_{k+1}(y)$) be the maximum (resp. minimum) over $y_{k+1} \in \mathbb{R}$ such that $(y, y_{k+1}) \in X$. Since $f : X \rightarrow Y$ is a linear map, $f^{-1} : Y \rightarrow X$ is a continuous correspondence and, by the Maximum Theorem, $\bar{y}_{k+1}(y)$ and $\underline{y}_{k+1}(y)$ are continuous functions of y . Since X is a full-dimensional polyhedron, $(y, \bar{y}_{k+1}(y))$ and $(y, \underline{y}_{k+1}(y))$ belong to the boundary of X for all $y \in Y$; and $\bar{y}_{k+1}(y) > \underline{y}_{k+1}(y)$ with $\{y\} \times (\underline{y}_{k+1}(y), \bar{y}_{k+1}(y)) \subset X \setminus \partial X$ if y belongs to the interior of Y . Define now $h : Y \times [0, 1] \rightarrow X$ by $h(y, \lambda) = (y, (1-\lambda)\underline{y}_{k+1}(y) + \lambda\bar{y}_{k+1}(y))$. Then h has the required properties. \square

Proof of Theorem 5.3. Because f is a surjective linear map, f^{-1} is a continuous correspondence and therefore by Theorem 4.5 it is sufficient to prove the result for a stably essential set. Also, by Theorem 4.3 we can further assume that the relevant sets have semialgebraic germs and satisfy the stronger connexity condition given there. For simplicity we call a set S with such a semialgebraic germ a semialgebraic stably essential set.

For this proof we view \mathcal{E} and $\tilde{\mathcal{E}}$ as graphs of equilibria (rather than perturbed equilibria) of perturbed games for G and \tilde{G} respectively.

Suppose \tilde{S}^* is a semialgebraic stably essential set with a semialgebraic germ \tilde{E}^* . Let \tilde{E} be the set of $(\tilde{\eta}, \tilde{\sigma})$ such that there exists $(\tilde{\eta}, \tilde{\sigma}') \in \tilde{E}^*$ with $f(\tilde{\sigma}) = f(\tilde{\sigma}')$. Then \tilde{E} is semialgebraic and $f^{-1}(f(\tilde{S})) = \{\tilde{\sigma} \mid (0, \tilde{\sigma}) \in \tilde{E}\}$. For each $(\eta, \sigma') \in \tilde{E}$, $(\eta, \lambda\sigma' + (1-\lambda)\sigma) \in \tilde{E}$, where $(\eta, \sigma) \in \tilde{E}^*$ and $f(\sigma) = f(\sigma')$. Therefore, \tilde{E} satisfies the connexity condition, since \tilde{E}^* does. Obviously \tilde{E} satisfies the essentiality condition since it contains \tilde{E}^* which does. Thus $f^{-1}(f(\tilde{S}))$ is a semialgebraic stably essential set as well. Therefore, to prove the theorem

it is sufficient to show that S is a semialgebraic stably essential set of G iff $f^{-1}(S)$ is a semialgebraic stably essential set in \tilde{G} .

S is a semialgebraic stably essential set in G iff there exists a semialgebraic subset S of \mathcal{E} satisfying the essentiality condition of Definition 2.6 and the connexity condition of Theorem 4.3. For a semialgebraic subset $E \subset \mathcal{E}$, let $\tilde{E} = (\delta\tilde{\tau}, \tilde{\sigma}) \in \tilde{\mathcal{E}}$ be such that $(\delta f(\tilde{\tau}), f(\tilde{\sigma})) \in S$. \tilde{E} is obviously semialgebraic and $S = \{\sigma \mid (0, \sigma) \in E\}$ iff $f^{-1}(S) = \{\tilde{\sigma} \mid (0, \tilde{\sigma}) \in \tilde{E}\}$. Also E satisfies the connexity condition of Theorem 4.3 iff \tilde{E} does. Hence, it is sufficient to show that E satisfies the essentiality condition of Definition 2.6 iff \tilde{E} does too.

Because E and \tilde{E} are semialgebraic, by Lemma 3.11 it is sufficient to prove that E satisfies the essentiality condition of Theorem 4.2 iff \tilde{E} does. Fix $0 < \delta < 1$. We show that the projection q_δ from F_δ to Q_δ is essential iff \tilde{q}_δ from \tilde{F}_δ to \tilde{Q}_δ is. Since Q_δ and \tilde{Q}_δ are homeomorphic to Σ and $\tilde{\Sigma}$, we view F_δ and \tilde{F}_δ as subsets of $\Sigma \times \Sigma$ and $\tilde{\Sigma} \times \tilde{\Sigma}$ respectively. Thus $F_\delta = (\tau, \sigma) \in \Sigma \times \Sigma$ is such that $(\delta\tau, \sigma) \in E_\delta$ and \tilde{F}_δ is the set of $(\tilde{\tau}, \tilde{\sigma})$ such that $(\delta\tilde{\tau}, \tilde{\sigma})$ belongs to \tilde{E}_δ (and therefore $(f(\tilde{\tau}), f(\tilde{\sigma}))$ belongs to F_δ). We view \tilde{q}_δ and q_δ as the projection to the first factor.

Let $k = \dim(\tilde{\Sigma}) - \dim(\Sigma)$. q_δ is stably essential iff $q_\delta^k : (F_\delta, \partial F_\delta) \times (\Lambda^k, \partial\Lambda^k) \rightarrow (\Sigma, \partial\Sigma) \times (\Lambda^k, \partial\Lambda^k)$ is stably essential. Let $\tilde{F}_\delta = (\tilde{\tau}, \sigma) \in \tilde{\Sigma} \times \Sigma$ be such that $(f(\tilde{\tau}), \sigma) \in F_\delta$ and let \tilde{q}_δ be the projection from \tilde{F}_δ to $\tilde{\Sigma}$. By Lemma 5.4 there exists a map $h : (\Sigma, \partial\Sigma) \times (\Lambda^k, \partial\Lambda^k) \rightarrow (\tilde{\Sigma}, \partial\tilde{\Sigma})$ whose restriction to $(\Sigma \setminus \partial\Sigma) \times (\Lambda^k \setminus \partial\Lambda^k)$ is a homeomorphism. Therefore, using Lemma A.5, q_δ^k is stably essential iff \tilde{q}_δ is stably essential.

Observe now that $\tilde{q}_\delta = \tilde{q}_\delta \circ (\text{Id} \times f)$. Therefore, if \tilde{q}_δ is stably essential then so is \tilde{q}_δ and hence also q_δ . On the other hand, suppose q_δ is stably essential, and so too is \tilde{q}_δ . Letting g be a continuous selection from f^{-1} , we have that $\tilde{q}_\delta = \tilde{q}_\delta \circ (\text{Id} \times g)$ and hence \tilde{q}_δ is stably essential. Thus we have shown that the essentiality condition for S is equivalent to that for \tilde{S} . Hence S is a semialgebraic stably essential set iff $f^{-1}(S)$ is. \square

This proof shows that if g is a continuous selection from f^{-1} then S is a metastable set of G iff $g(S)$ is. We are unable to ascertain the following stronger version of this property: \tilde{S} is metastable iff $f(\tilde{S})$ is.

The above theorem applies to invariance and player-splitting as follows. Formally, suppose we have two games \tilde{G} and G with the same player set. Suppose for each player n , there exists a surjective linear map $f_n : \tilde{\Sigma}_n \rightarrow \Sigma_n$ such that if $f : \tilde{\Sigma} \rightarrow \Sigma$ is the corresponding map between the spaces of strategy profiles then for each $\tilde{\sigma} \in \tilde{\Sigma}$ the payoffs of the players in \tilde{G} are their payoffs in G from $f(\tilde{\sigma})$. Since f is surjective, we can actually view Σ_n as a subspace of $\tilde{\Sigma}$ by choosing for each n and each pure strategy s_n in G a pure strategy \tilde{s}_n in

\tilde{G} such that $f_n(\tilde{s}_n) = s_n$. Thus \tilde{G} is obtained from G by adding redundant strategies. The above theorem now relates the solutions of G and \tilde{G} and yields the Invariance property for metastability.

The player-splitting property states the following. In an extensive-form game, if we can partition some player's collection of information sets in such a way that no play of the game intersects more than one element of the partition, then the solution of the game should be the same if we consider the agent-normal form where this player has as many agents as there are elements in the partition. We now formally state this property for metastability.

Suppose one has an N -player extensive-form game in which one can partition some player n 's collection \mathcal{H}_n of information sets into two subcollections \mathcal{H}_{n_1} and \mathcal{H}_{n_2} such that no information set in one subcollection follows an information set in the other. Let \tilde{G} be the strategic form of the game. Consider now a new game G where we 'split' player n into two players n_1 and n_2 , i.e. the player set in G is $(N \setminus \{n\}) \cup \{n_1, n_2\}$. The strategy sets of the players other than n in \tilde{G} are the same as in the two games. Each pure strategy \tilde{s}_n of player n in \tilde{G} prescribes actions at each information set in \mathcal{H}_{n_i} for agent $i = 1, 2$ and thus gives a pure strategy for player n_i in G . Let S_{n_i} be player n_i 's set of pure strategies in G and let Σ_{n_i} be the corresponding set of mixed strategies. We now describe the payoff functions for the players. Observe that a pair (s_{n_1}, s_{n_2}) of pure strategies for the agents defines uniquely a pure strategy for player n in \tilde{G} . Therefore, given a profile of pure strategies in G , the payoffs of the players other than the two agents are the payoffs they get from the corresponding profile in \tilde{G} ; for agent n_i let it be n 's payoff if the outcome induced by the profile follows an information set in \mathcal{H}_{n_i} and let it be arbitrary otherwise.

For each i there is a well-defined affine function f_{n_i} from $\tilde{\Sigma}_n$ to Σ_{n_i} that computes for each $\tilde{\sigma}_n$ the corresponding marginal distribution over S_{n_i} . Let $f : \tilde{\Sigma} \rightarrow \Sigma$ be the map $f(\sigma) = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}, f_{n_1}(\tilde{\sigma}_n), f_{n_2}(\tilde{\sigma}_n), \tilde{\sigma}_{n+1}, \dots, \sigma_N)$. Then f satisfies the conditions of Theorem 5.3 and we get the player-splitting property for metastability, in that it does not matter whether one treats the two agents as one player.

5.4. The Small Worlds and Decomposition Properties. Suppose \tilde{G} is an \mathcal{N} -equivalent game. As specified by Mertens [15] the Small Worlds axiom requires that solutions of G are precisely the projections of solutions of \tilde{G} . Given Theorem 3.15, one might expect metastability to satisfy the Small Worlds axiom. As the following theorem shows, however, we obtain a slightly weaker version.

Theorem 5.5. *Let \tilde{G} be an \mathcal{N} -equivalent game. If \tilde{S} is a metastable set of \tilde{G} then its projection to Σ is a metastable set of G . If S is a metastable set of G then it contains a metastable set that is the projection of a metastable set of \tilde{G} .*

Proof. Let \tilde{S} be a metastable set of \tilde{G} . Let \tilde{V}_i be a sequence of sets satisfying the conditions in Definition 2.1. For each i let V_i be the projection of \tilde{V}_i to Σ . Clearly the V_i 's converge to the projection, call it S , of \tilde{S} . Also, the V_i 's satisfy the connexity condition since the \tilde{V}_i 's do. Finally, as for the Robustness condition, given a correspondence $\varphi^k \times f : \Sigma \times \Lambda^k \rightarrow \Lambda^k$, where φ^k is a δ_i -perturbation of R^k , there is an extension $\tilde{\varphi}^k \times f : \tilde{\Sigma} \times \Lambda^k \rightarrow \tilde{\Sigma} \times \Lambda^k$ given by: $(\tilde{\varphi}^k \times f)(\sigma, \sigma_{-\mathcal{N}}, \lambda)$ is the set of $(\sigma', \sigma'_{-\mathcal{N}}, \lambda')$ such that $\sigma' \in \varphi^k(\sigma, \lambda)$, $\sigma'_{-\mathcal{N}} \in R_{-\mathcal{N}}(\sigma, \sigma_{-\mathcal{N}})$, and $\lambda' \in f(\sigma, \lambda)$. By the Robustness property for \tilde{V}_i there exists a fixed point $(\sigma, \sigma_{-\mathcal{N}}, \lambda)$ of $\tilde{\varphi}^k \times f$ in \tilde{V}_i . Then (σ, λ) is a fixed point of $\varphi^k \times f$ in V_i , which shows that V_i satisfies the Robustness property and hence that S is metastable.

To prove the second statement, let S be a metastable set of G . Let V_i be a sequence of sets converging to S and satisfying the conditions of Definition 2.1. By Lemma 4.1 we can assume that the V_i 's are semialgebraic. Let Σ^o be the mixed strategy space of the outsiders. And let $\tilde{\Sigma} = \Sigma \times \Sigma^o$ be the mixed strategy space in \tilde{G} . By the Robustness condition for metastability of S , for each i , every correspondence $\tilde{\varphi}^k \times f : \tilde{\Sigma} \times \Lambda^k \rightarrow \tilde{\Sigma} \times \Lambda^k$, where $\tilde{\varphi}^k$ is a δ_i -perturbation of the best-reply correspondence in \tilde{G} , has a fixed point in $V_i \times \Sigma^o$. As $V_i \times \Sigma^o$ is semialgebraic, by Theorem 3.8 it contains a stably essential set \tilde{S}_i . By going to an appropriate subsequence, we have that $S \times \Sigma^o$ contains a metastable set. By the first part of this theorem, its projection onto Σ is a metastable set of G , which is obviously contained in S . \square

As shown in [15] the collection of q -stable sets (defined in [14]) satisfy the stronger form of the Small Worlds property, namely that they are precisely the projections of q -stable sets of \mathcal{N} -equivalent games. Since q -stable sets are metastable as well, it would be interesting to know if the collection of metastable sets that satisfy the stronger property for metastability is exactly the collection of q -stable sets.

A property related to the Small Worlds axiom is the Decomposition Property, which states the following. Suppose G^1 and G^2 are two games played by two sets of players in two different rooms. Suppose G is the composite game $G^1 \times G^2$. Then: (D1) the solutions of G project to solutions of G^1 and G^2 ; and, (D2) the product of solutions to G^1 and G^2 are solutions to G . Property (D1) is implied by Theorem 5.5 and hence metastability satisfies it. As for (D2) metastability satisfies the following weaker form.

Theorem 5.6. *Let S^1 be a metastable set of G^1 . There exists a metastable set S^2 of G^2 such that $S^1 \times S^2$ is metastable in $G^1 \times G^2$.*

Proof. By the compactness property for metastability we can assume that S^1 is stably essential. Moreover, by Theorem 4.3 we can assume that S^1 has a semialgebraic germ E^1 : $S_1 = \{\sigma_1 \in \Sigma^1 \mid (0, \sigma) \in E^1\}$; and there exists $\delta_1 > 0$ such that for each $0 < \delta \leq \delta_1$ $E_\delta^1 \setminus \partial E_\delta^1$ is connected and dense in E_δ and the projection from E_δ^1 is stably essential.

Let $E = (\eta^1, \eta^2, \sigma^1, \sigma^2) \in \mathcal{E}$ such that $(\eta^1, \sigma^1) \in E^1$. E is a closed semialgebraic set. For each $0 < \delta \leq \delta_0$, we claim that $p : (E_\delta, \partial E_\delta) \rightarrow (P_\delta, \partial P_\delta)$ is stably essential. To see this, for each $0 < \delta < \delta_1$, let V_δ be the projection of E_δ onto Σ^1 . Then, by Theorem 3.7, V_δ satisfies the robustness condition in the definition of metastability. In particular, for each $0 < \delta \leq \delta_1$, and for each k , each correspondence $\varphi \times f : \Sigma \times \Lambda^k \rightarrow \Sigma \times \Lambda^k$ where φ is a δ -perturbation of R has a fixed point in $V_\delta \times \Sigma^2$. By Theorem 3.8, $p : (E_\delta, \partial E_\delta) \rightarrow (P_\delta, \partial P_\delta)$ is stably essential as claimed.

Let X be the closure of the set of $E \setminus \partial E_1$. Then X is a semialgebraic set as well and by Lemma A.7 the projection from X_δ is stably essential for each $0 < \delta \leq \delta_0$. By [15, Lemma 2] there exists $\delta_0 > 0$, a positive integer k , for each k a semialgebraic set X^k such that for each $0 < \delta \leq \delta_0$: (i) $X_\delta^k \setminus \partial X_\delta^k$ is connected and dense in X_δ^k for each k ; $X_\delta^k \cap X_\delta^l \subseteq \partial X_1$ for $k \neq l$; (iii) $\cup_k X_\delta^k = X_\delta$. Without loss of generality we can assume that $\delta_0 \leq \delta_1$. The stable essentiality of the projection from X_δ along with properties (ii) and (iii) imply that there exists k such that the projection from X_δ^k is stably essential. Let $Y = X^k$. Y satisfies the conditions of Definition 2.6 and the set S given by the projection of Y_0 onto Σ is stably essential set. Let S_2 be the projection of S onto Σ^2 . We will now show that that $S = S^1 \times S^2$. In conjunction with decomposition property (D1) this then proves the result. Choose $\sigma^2 \in S^2$. There exists $\sigma^1 \in S^1$ such that $\sigma \equiv (\sigma^1, \sigma^2) \in S$. Because $Y_{\delta_0} \setminus \partial Y_{\delta_0}$ is a semialgebraic set that is connected and dense in Y_{δ_0} , $(0, \sigma) = \lim_{\delta \rightarrow 0} (\eta(\delta), \sigma(\delta))$ for some path $(\eta(\delta), \sigma(\delta))$ in $Y_{\delta_0} \setminus \partial Y_{\delta_0}$. If $\tilde{\sigma}^1$ is another strategy in S^1 , then by the connexity for property for the semialgebraic set E^1 , there $(0, \tilde{\sigma}^1) = \lim_{\delta \rightarrow 0} (\tilde{\eta}^1(\delta), \tilde{\sigma}^1(\delta))$ for a path in $E^1 \setminus \partial E^1$. Therefore, $(\tilde{\eta}^1(\delta), \eta(\delta), \tilde{\sigma}^1(\delta), \sigma^2(\delta))$ is a path in $X \setminus \partial X_1$ whose limit is $(0, \tilde{\sigma}^1, \sigma^2)$. Moreover because E^1 satisfies the connexity property, it is clear that this path belongs to Y . Thus $(\tilde{\sigma}^1, \sigma^2)$ belongs to S . Since $\tilde{\sigma}^2$ was an arbitrary strategy in S^2 , we thus have that $S = S^1 \times S^2$. \square

6. CONCLUDING REMARKS

The refinements defined in §2 differ chiefly in the formulation of the corresponding version of robustness. As Hillas et al. [8] show, homotopy stability is more restrictive than BR-stability because homotopic essentiality invokes a richer class of perturbations. Stable essentiality is an even stronger requirement because it invokes the Embedding principle, including the axioms of Invariance and the projection property of Small Worlds. (Co)homological essentiality is evidently the strongest criterion—and importantly, unlike homotopy criteria it ensures that essential maps are surjective. It is this difference that accounts for the slightly weaker form (compared to Mertens’ refinement) of the Small Worlds property established in Theorem 5.5, and the possible failure of metastability to satisfy (D2) of the Decomposition property. However, we show in [5] that this difference occurs only for a game whose extensive-form has nongeneric payoffs.

For the foundations of game theory, the development of a canonical refinement of Nash equilibria requires one to choose among these topological criteria. This choice must ultimately be guided by decision-theoretic criteria. The results in this paper imply that the weakest topological criterion that preserves the standard list of decision-theoretic axioms is stable essentiality. Our exposition is cast differently in that we begin straightaway with the definition of metastability and its motivation in terms of the principles of Embedding and Robustness, and then establish that this definition is equivalent to stable essentiality of the projection map from the equilibrium graph. But this is the crux of the matter technically.

Our view is that metastability is a viable substitute for Mertens’ refinements based on (co)homological essentiality of the projection map, since metastability yields basically the same decision-theoretic properties. From a computational viewpoint, the test for metastability (stable essentiality) is more difficult to apply. Its advantage in applications may therefore lie in its conceptual justification and its agreement with Mertens’ stability in generic extensive-form games.

APPENDIX A. MATHEMATICAL BACKGROUND

We first study some properties of a map whose range is homeomorphic to a ball. Throughout, let $(X, \partial X)$ be a compact pair and let $(B, \partial B)$ be homeomorphic to a ball with its boundary. (We are not assuming here that ∂X is the boundary of X .) A map $f : (X, \partial X) \rightarrow (B, \partial B)$ is *essential in homotopy* if it is not homotopic relative to ∂X to a map to ∂B . Mertens [14, Section 4E, Lemma 2] proves the following equivalent characterizations of inessentiality.

Lemma A.1. *The following statements are equivalent.*

- $f : (X, \partial X) \rightarrow (B, \partial B)$ is inessential in homotopy.
- There exists a map $g : X \rightarrow \partial B$ that agrees with f on ∂X .
- There exists a map $g : X \rightarrow \partial B$ such that the restrictions of f and g to ∂X are freely homotopic.

The following Lemma shows that a map that is essential in homotopy has strong fixed-point properties.

Lemma A.2. *Let $f : (X, \partial X) \rightarrow (B, \partial B)$ be a map that is essential in homotopy. Then every function $g : X \rightarrow B$ has a point of coincidence with f , i.e. there exists $x \in X$ such that $f(x) = g(x)$. Moreover, if X is metrizable and B is convex then every correspondence $\varphi : X \rightarrow B$ has a point of coincidence with f .*

Proof. Suppose there exists $g : X \rightarrow B$ that has no point of coincidence with f . Viewing B as a ball, define a map $h : X \rightarrow \partial B$ as follows: for each $x \in X$, $h(x)$ is the unique point in ∂B that is closer to $f(x)$ than $g(x)$ on the line from $g(x)$ through $f(x)$. Clearly h coincides with f on ∂X and hence f is inessential. Assume now the additional hypotheses of the second statement. Using McLennan [12, Proposition 2.25], for each $\varepsilon > 0$ there exists a function $g_\varepsilon : X \rightarrow Y$ whose graph is within ε of the graph of φ . By what we have proved, each g_ε has a point x_ε such that $f(x_\varepsilon) = g_\varepsilon(x_\varepsilon)$. Let x be the limit of a convergent sequence of x_ε as ε goes to zero. Then x is a point of coincidence between f and φ . \square

As the following Lemma shows, the above coincidence property completely characterizes the essentiality of f in some cases.

Lemma A.3. *Suppose $f : (X, \partial X) \rightarrow (B, \partial B)$ is such that $f(X \setminus \partial X) \subseteq B \setminus \partial B$. If f is inessential then there exists a map $g : X \rightarrow B$ with no point of coincidence with f .*

Proof. There is no loss of generality in assuming that B is the unit ball in a Euclidean space. Suppose f is inessential. Then there exists a map $g : X \rightarrow \partial B$ that agrees with f on ∂X .

Define $h : X \rightarrow B$ by letting $h(x)$ be the “antipode” of $g(x)$ in B , i.e. $h(x) = -g(x)$. Clearly, f has no point of coincidence with h . \square

Remark A.4. Let Y be the closure of $X \setminus \partial X$ and let $\partial Y = Y \setminus (X \setminus \partial X)$. Given $f : (X, \partial X) \rightarrow (B, \partial B)$, define g be the restriction of f to Y . If f is inessential then so is g . On the other hand, any map $h : Y \rightarrow \partial B$ that agrees with g on ∂Y extends to a map over X by letting it agree with f on $X \setminus Y$. Hence if g is inessential then f is too. Thus essentiality of f and g are equivalent.

We are often interested in quotient spaces X' and B' of X and B obtained by identifying some points in ∂X and ∂B , respectively, in such a way that the map f induces a map f' from X' to B' . Under some conditions, the essentiality of f is equivalent to the essentiality of f' . Formally, for $Y = X, B$, suppose $q_Y : (Y, \partial Y) \rightarrow (Y', \partial Y')$ is a surjective closed map that sends $Y \setminus \partial Y$ homeomorphically onto $Y' \setminus \partial Y'$. (Since Y and Y' are compact, if Y' is Hausdorff—as it will be in all our intended applications—every surjective map from Y to Y' is a closed map.) Furthermore, suppose that $(B', \partial B')$ is homeomorphic to a ball pair. Let $f : (X, \partial X) \rightarrow (B, \partial B)$ and $f' : (X', \partial X') \rightarrow (B', \partial B')$ be two maps such that: $f' \circ q_X = q_B \circ f$ and $f'(X' \setminus \partial X') \subseteq B' \setminus \partial B'$.

Lemma A.5. *f is essential in homotopy iff f' is.*

Proof. Suppose f is inessential. Let $g : X \rightarrow \partial B$ be a map that agrees with f on ∂X . Define $g' : X' \rightarrow \partial B'$ by $g'(x') = q_B(g(q_X^{-1}(x')))$. Obviously, $g'(x')$ is a singleton set for $x' \in X' \setminus \partial X'$. For $x' \in \partial X'$, $q_X^{-1}(x') \subseteq \partial X$. Therefore, $q_B(g(q_X^{-1}(x'))) = q_B(f(q_X^{-1}(x'))) = f'(q_X(q_X^{-1}(x'))) = f'(x')$. Thus, g' is single-valued and coincides with f' on $\partial X'$. Finally, continuity of g' follows from the fact that q_X is a closed map and from the continuity of q_B and g . Consequently, f' is inessential.

Suppose f' is inessential. By Lemma A.3 there exists a map $g' : X' \rightarrow B'$ that does not have a point of coincidence with f' . Without loss of generality, we can assume that $g'(X) \subseteq B' \setminus \partial B'$. (Indeed, viewing B' as a ball, for a fixed $b' \in B' \setminus \partial B'$, the map sending x' to $(1 - \delta)g'(x') + \delta b'$ has no point of coincidence with f' for sufficiently small $\delta > 0$ and has all its values in $B' \setminus \partial B'$.) Then, the map $q_B^{-1} \circ g' \circ q_X$ is well-defined and has no point of coincidence with f . Thus f is inessential. \square

A.1. Extension of Maps to Suspensions. The (unreduced) suspension SX of X is defined as the quotient space of $X \times [0, 1]$ obtained by identifying $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. One then defines the k -th suspension $S^k X$ of X inductively as follows: $S^0 X = X$ and $S^k X = SS^{k-1} X$ for each $k > 0$.

Given the pair $(X, \partial X)$, $S\partial X$ can be viewed as a subset of SX if for $i = 0, 1$, we identify the “point” $\partial X \times \{i\}$ with $X \times \{i\}$ —thus $S\partial X$ is the set $\partial X \times (0, 1)$ along with the points $X \times \{i\}$ for $i = 0, 1$. One again defines the k -th suspension $S^k(X, \partial X)$ of $(X, \partial X)$ as follows: $S^0(X, \partial X) = (X, \partial X)$ and $S^k(X, \partial X) = SS^k(X, \partial X)$ for each $k > 0$. For example, if B is n -dimensional then SB is an $(n + 1)$ -ball and $S\partial B$ is an n -sphere. Thus, $S(B, \partial B)$ is an $(n + 1)$ -ball pair.

Given a map $f : (X, \partial X) \rightarrow (B, \partial B)$, one defines its extension $Sf : S(X, \partial X) \rightarrow S(B, \partial B)$ to the suspensions of its domain and range as follows: $Sf(x, t) = (f(x), t)$ for $(x, t) \in X \times (0, 1)$, and $Sf(X \times \{i\}) = B \times \{i\}$ for $i = 0, 1$. Then one defines inductively the map $S^k f : S^k(X, \partial X) \rightarrow S^k(B, \partial B)$ as follows: $S^0 f = f$ and $S^k f = SS^{k-1} f$ for each $k > 0$.

Lemma A.6. *Let $f : (X, \partial X) \rightarrow (B, \partial B)$ be such that $f(X \setminus \partial X) \subseteq B \setminus \partial B$. For $k > 0$, let $(B^k, \partial B^k)$ be a pair that is homeomorphic to the k -ball pair and let $f^k : (X, \partial X) \times (B^k, \partial B^k) \rightarrow (B, \partial B) \times (B^k, \partial B^k)$ be the function $f^k(x, b^k) = (f(x), b^k)$. Then $S^k f$ is essential iff f^k is.*

Proof. It is sufficient to prove the Lemma for the case $(B^k, \partial B^k) = ([0, 1]^k, \partial[0, 1]^k)$. Since $f(X \setminus \partial X) \subseteq B \setminus \partial B$, we have that $S^k f(S^k X \setminus S^k \partial X) \subseteq S^k B \setminus S^k \partial B$. $S^k X$ is a quotient space of $S^{k-1} X \times [0, 1]$ and therefore, by Lemma A.5, $S^k f$ is essential iff $(S^{k-1} f)^1 : S^{k-1}(X, \partial X) \times ([0, 1], \{0, 1\}) \rightarrow S^{k-1}(B, \partial B) \times ([0, 1], \{0, 1\})$ given by $(S^{k-1} f)^1(x', t) = (S^{k-1} f(x'), t)$ is essential. Again, using the same Lemma, $(S^{k-1} f)^1$ is essential iff the map $(S^{k-2} f)^2 : S^{k-2}(X, \partial X) \times ([0, 1]^2, \partial[0, 1]^2) \rightarrow S^{k-2}(B, \partial B) \times ([0, 1]^2, \partial[0, 1]^2)$ given by $(S^{k-2} f)^2(x', t_1, t_2) = (S^{k-1} f(x'), t_1, t_2)$ is essential. Continuing this downward induction yields the result since $(S^0 f)^k$ is the map f^k . \square

Remark A.7. The property in Remark A.4 obviously extends to suspensions of f and g as well—a fact that we use later.

Definition A.8 (stably essential). A map f is stably essential if for each k , $S^k f$ is essential in homotopy.

For each $k > 0$, if $S^k f$ is essential in homotopy then so is $S^{k-1} f$. The converse is not true in general—see for instance [14, Section 4F]—but the following Lemma gives a sufficient condition.

Lemma A.9. *Let $(X, \partial X)$ be a CW complex that has the same dimension as $(B, \partial B)$. If $f : (X, \partial X) \rightarrow (B, \partial B)$ is essential then it is stably essential.*

Proof. For $k \geq 0$, suppose the k -th suspension $S^k f$ of f is essential. We show that $S^{k+1} f$ is essential. (Recall that $f^0 = f$.) $S^k(X, \partial X)$ is obviously a CW complex. Also, if n is

the dimension of B then $k + n$ is the dimension of $S^k X$ and $S^k B$. Let (Y, y_0) be the space obtained from $S^k X$ by collapsing $S^k \partial X$ to a point y_0 . Likewise let (C, c_0) be the space obtained by collapsing $S^k \partial B$ to a point c_0 . Let $g : (Y, y_0) \rightarrow (C, c_0)$ be the map induced by $S^k f$. By Mertens [14, Section 4.E, Theorem], because S^k is essential, g is not homotopic to the constant map that sends every $y \in Y$ to c_0 .

Let (Y_1, y_1) be the quotient space of $S(Y, y_0)$ obtained by collapsing Sy_0 to a point, i.e. the quotient space of $Y \times [0, 1]$ obtained by collapsing $(Y \times 0) \cup (Y \times 1) \cup (y_0 \times [0, 1])$ to a point y_1 . Likewise let (C_1, c_1) be the space obtained from (C, c_0) . Let $g_1 : (Y_1, y_1) \rightarrow (C_1, c_1)$ be the map induced by the suspension Sg of g . By Spanier [16, Suspension Theorem 8.5.11], since g is not homotopic to the constant map sending points to c_0 , g_1 is also not homotopic to the constant that sends every $y \in Y_1$ to c_1 . Obviously (Y_1, y_1) is the quotient space of the $(k + n + 1)$ -dimensional CW complex $S^{k+1}(X, \partial X)$ obtained by collapsing $S^{k+1} \partial X$ to a point y_1 . The same is true of (C_1, c_1) . Hence we can again apply Mertens [14, Section 4.E, Theorem] to conclude that $S^{k+1} f$ is essential. \square

If the dimension of $(X, \partial X)$ is smaller than the dimension of $(B, \partial B)$ then the map f is not even surjective, so it is inessential in homotopy. It is when X has higher dimension than B that stable essentiality is possibly stronger than essentiality.

APPENDIX B. MULTISIMPLICIAL AND POLYHEDRAL COMPLEXES

B.1. Multisimplicial Complexes. The material of this subsection is based on [4, Appendix B]. We refer the reader to that article for a proof of the multisimplicial approximation theorem stated below.

A set of points $\{v_0, \dots, v_n\}$ in \mathbb{R}^N is affinely independent if the equations $\sum_{i=0}^n \lambda_i v_i = 0$ and $\sum_i \lambda_i = 0$ imply that $\lambda_0 = \dots = \lambda_n = 0$. An n -simplex K in \mathbb{R}^N is the convex hull of an affinely independent set $\{v_0, \dots, v_n\}$. Each v_i is a vertex of K and the collection of vertices is called the vertex set of K . Each $\sigma \in K$ is expressible as a unique convex combination $\sum_i \lambda_i v_i$; and for each i , $\sigma(v_i) \equiv \lambda_i$ is the v_i -th barycentric coordinate of σ . A face of K is the convex hull of a nonempty subset of the vertex set of K .

A (finite) simplicial complex \mathcal{K} is a finite collection of simplices such that the face of each simplex in \mathcal{K} belongs to \mathcal{K} , and the intersection of two simplices is either empty or a face of each. The set V of 0-dimensional simplices is called the vertex set of \mathcal{K} . The set given by the union of the simplices in \mathcal{K} is called the space of the simplicial complex and is denoted $|\mathcal{K}|$. For each $\sigma \in |\mathcal{K}|$, there exists a unique simplex K of \mathcal{K} containing σ in its interior; define the barycentric coordinate function $\sigma : V \rightarrow [0, 1]$ by letting $\sigma(v) = 0$ if v is not a

vertex of K and otherwise by letting $\sigma(v)$ be the corresponding barycentric coordinate of σ in the simplex K .

A subdivision of a simplicial complex \mathcal{K} is a simplicial complex \mathcal{K}^* such that each simplex of \mathcal{K}^* is contained in a simplex of \mathcal{K} and each simplex of \mathcal{K} is the union of simplices in \mathcal{K}^* . Obviously $|\mathcal{K}| = |\mathcal{K}^*|$.

A multisimplex is a set of the form $K_1 \times \cdots \times K_m$, where for each i , K_i is a simplex. A multisimplicial complex \mathcal{K} is a product $\mathcal{K}_1 \times \cdots \times \mathcal{K}_m$, where for each i , \mathcal{K}_i is a simplicial complex. (The vertex set V of a multisimplicial complex \mathcal{K} is the set of all (v_1, \dots, v_m) for which for each i , v_i is a vertex of \mathcal{K}_i . The space of the multisimplicial complex is $\prod_i |\mathcal{K}_i|$ and is denoted $|\mathcal{K}|$. A subdivision of a multisimplicial complex \mathcal{K} is a multisimplicial complex $\mathcal{K}^* = \prod_i \mathcal{K}_i^*$ where for each i , \mathcal{K}_i^* is a subdivision of \mathcal{K}_i . In the following, $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$ is a fixed multisimplicial complex and \mathcal{L} is a fixed multisimplicial complex.

Definition B.1 (cellular map). A map $f : |\mathcal{K}| \rightarrow |\mathcal{L}|$ is called multisimplicial if for each multisimplex K of \mathcal{K} there exists a simplex L in \mathcal{L} such that:

- (1) f maps each vertex of K to a vertex of L ;
- (2) f is multilinear on $|K|$; i.e., for each $\sigma \in |K|$, $f(\sigma) = \sum_{v \in V} f(v) \times \prod_i \sigma_i(v_i)$.

By Property 1 of the Definition, vertices of K are mapped to vertices of L . Therefore, for each $\sigma \in |K|$, $f(\sigma)$ is an average of the values at the vertices of K . Since the simplex L is a convex set, the image of the multisimplex K is contained in L . If \mathcal{K} is a simplicial complex, then Definition B.1 coincides with the usual definition of a simplicial map. In this case the image of a multisimplex K under f is a simplex of L , but in the multilinear case the image of K could be a strict subset of L .

Definition B.2 (multisimplicial map). Let $g : |\mathcal{K}| \rightarrow |\mathcal{L}|$ be a map. A multisimplicial map $f : |\mathcal{K}| \rightarrow |\mathcal{L}|$ is a multisimplicial approximation to g if for each $\sigma \in |\mathcal{K}|$, $f(\sigma)$ belongs to the simplex that contains $g(\sigma)$ in its interior.

We could equivalently define a multisimplicial approximation by requiring that for each σ , and each simplex L of \mathcal{L} , $g(\sigma) \in L \implies f(\sigma) \in L$. The following theorem is the multisimplicial version of the simplicial approximation theorem.

Theorem B.3. *Let $g : |\mathcal{K}| \rightarrow |\mathcal{L}|$ be a map. There exists $\eta > 0$ such that for each subdivision \mathcal{K}^* of \mathcal{K} with the property that the diameter of each multisimplex is at most η , there exists a multisimplicial approximation $f : |\mathcal{K}^*| \rightarrow |\mathcal{L}|$ of g .*

B.2. Polyhedral Complexes. A polyhedral complex \mathcal{P} is a finite collection of polyhedra such that: (i) each face of a polyhedron in \mathcal{P} belongs to \mathcal{P} ; and (ii) the intersection of two

polyhedra in \mathcal{P} is either empty or a face of each of them. The union of the polyhedra in \mathcal{P} is the space of the polyhedral complex and denoted $|\mathcal{P}|$. Every multisimplicial complex, for example, is a polyhedral complex where the polyhedra are the multisimplices.

A polyhedral complex \mathcal{P}' is a polyhedral subdivision of \mathcal{P} if each polyhedron in \mathcal{P}' is contained in a polyhedron of \mathcal{P} and each polyhedron in \mathcal{P} is the union of polyhedra in \mathcal{P}' . The following Lemma is the basis for defining Player 0's payoff function in Step 2 of the proof of Claim equivalent game.

Theorem B.4. *Let \mathcal{P} be a polyhedral complex such that $|\mathcal{P}|$ is d -dimensional polyhedron in \mathbb{R}^n . There exists a polyhedral subdivision \mathcal{P}' of \mathcal{P} and a convex, piecewise-affine function $\gamma : |\mathcal{P}| \rightarrow \mathbb{R}$ such that the maximal convex domains on which γ is affine are the d -dimensional polyhedra in \mathcal{P}' .*

Proof. The polyhedral complex \mathcal{P}' is derived from \mathcal{P} as follows (Eaves and Lemke, 1981). Let \mathcal{P}_1 be the set of all $(d-1)$ -dimensional polyhedra in \mathcal{P} . For each polyhedron $P \in \mathcal{P}_1$, let $H_P = \{z \in \mathbb{R}^n \mid a'_P z = b_P\}$ be the hyperplane that includes P , and if $d < n$ is orthogonal to $|\mathcal{P}|$. Let \mathcal{P}'_0 be the set of all polyhedra of the form $|\mathcal{P}| \cap [\cap_{P \in \mathcal{P}_1} H_P^i]$ where each $i \in \{+, -\}$ and H_P^+ and H_P^- are the two closed half spaces whose intersection is H_P . \mathcal{P}'_0 is a collection of d -dimensional polyhedra whose union is $|\mathcal{P}|$. Let \mathcal{P}' be the polyhedral complex consisting of all the polyhedra that are faces of some polyhedron in \mathcal{P}'_0 . By construction, \mathcal{P}' is a polyhedral subdivision of \mathcal{P} . Associate with \mathcal{P}' the map $\gamma : |\mathcal{P}| \rightarrow \mathbb{R}_+$ for which $\gamma(\sigma) = \sum_{P \in \mathcal{P}_1} |a'_P \sigma - b_P|$. Then γ is convex and piecewise affine. Moreover, the maximal convex domains on which γ is affine are the polyhedra in \mathcal{P}'_0 , which are the d -dimensional polyhedra of \mathcal{P}' . \square

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