PRICE DYNAMICS ON A STOCK MARKET WITH ASYMMETRIC INFORMATION

By

Bernard De Meyer

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

PRICE DYNAMICS ON A STOCK MARKET WITH ASYMMETRIC INFORMATION

BERNARD DE MEYER

Abstract. The appearance of a Brownian term in the price dynamics on a stock market was interpreted in [De Meyer, Moussa-Saley (2003)] as a consequence of the informational asymmetries between agents. To take benefit of their private information without revealing it to fast, the informed agents have to introduce a noise on their actions, and all these noises introduced in the day after day transactions for strategic reasons will aggregate in a Brownian Motion. We prove in the present paper that this kind of argument leads not only to the appearance of the Brownian motion, but it also narrows the class of the price dynamics: the price process will be, as defined in this paper, a continuous martingale of maximal variation. This class of dynamics contains in particular Black and Scholes' as well as Bachelier's dynamics. The main result in this paper is that this class is quite universal and independent of a particular model: the informed agent can choose the speed of revelation of his private information. He determines in this way the posterior martingale L, where L_q is the expected value of an asset at stage q given the information of the uninformed agents. The payoff of the informed agent at stage q can typically be expressed as a 1-homogeneous function M of $L_{q+1} - L_q$. In a game with n stages, the informed agent will therefore chose the martingale L^n that maximizes the M-variation. Under a mere continuity hypothesis on M, we prove in this paper that L^n will converge to a continuous martingale of maximal variation. This limit is independent of M.

JEL Classification Numbers: G14, C72, C73, D44.

1. Introduction

Brownian motion is omnipresent in finance analysis. Its appearance in the price dynamic is often explained exogenously. Bachelier was the first to use a Brownian motion to model the price evolution even before its precise definition by Einstein and Wiener. The first sentence in his thesis [Bachelier (1900)] illustrates quite well this kind of explanation: "The influences that determines the price variations on the stock market are uncountable, past, present or even future expected events, having often nothing to do with the stock market have repercussion on the prices."

Aside this exogenous explanation, there could also be a strategic reason for its appearance: Institutionals have clearly a better access to information on the market than other agents: they are better skilled to analyze the flow of information and in some cases they are even part of the board of directors of the firms whom they are trading the shares. So, institutionals are better informed and this informational advantage is known publicly. As a consequence, each of their moves on the markets is analyzed by the other agents to extract its informational content. If informed agents act naively, taking moves that depend deterministically on their information, they will completely reveal their information to the other agents, and doing so, they

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will lose their strategic advantage for the future. The only way to take benefit of the information without revealing it to fast is to introduce a noise on their moves: this comes to select random moves with lotteries that depend on their information. The main idea in [De Meyer, Moussa-Saley (2003)] is that the noises introduced by the informed agents in the day after day transactions will generate a Brownian motion. To illustrate this idea, [De Meyer, Moussa-Saley (2003)] analyzes the interactions between two asymmetrically informed market makers.

1.1. **The game $G_n(P)$.** The game considered there is as follows: Two market makers are trading a risky asset R against a numéraire N.

Asymmetry of information: At the beginning of the game, market maker 1, hereafter referred to as player 1 (P1), receives a private message M concerning the risky asset. The message M can be either a good news G or a bad news B. The a priori probability that $M = G$ is P. Market maker 2 (P2) knows that player 1 got the message and and also knows P, but he does not observe M.

Liquidation value: The message M will be publicly revealed at a future date T, say at the next shareholder meeting. The price L of R at that date is called the liquidation value of R. It will depend on M. We considered in [De Meyer, Moussa-Saley (2003)] the case $L(G) = 1$ and $L(B) = 0$. The liquidation value of N is independent of M and is fixed to be 1.

Trading mechanism: In the game $G_n(P)$, there are n trading periods before date T. As market makers, P1 and P2 have to post, at period q ($q = 1, \dots, n$), a price $p^n_{1,q}$ and $p^n_{2,q}$. We suppose that at each period, the choice of $p^n_{1,q}$ and $p^n_{2,q}$ is made simultaneously and independently by the players based on their prior observations and their private information. It is then publicly announced.

Since market regulations stipulates explicitly that the bid ask spread by market makers has to be small, we toke it to be 0 and, therefore, only one price per market maker has to be considered.

The price p posted by a market maker is a commitment to sell or buy a limited amount –say one share– of R in counterpart of p units of N. On the markets, if a trader wants to trade more than this limited amount, it will be at negotiated price. We do not allow for such out of the counter transactions in our model.

Clearly, if $p^n_{1,q} \neq p^n_{2,q}$, a trader will see a possibility of arbitrage, and he will buy at the lowest price the maximal number –one– of shares R to sell it immediately at the highest price. To simplify the analysis– this will lead to a zero sum game—, instead of considering two different transactions with an external trader, we considered only one transaction between the market makers: if $p^n_{1,q} \neq p^n_{2,q}$, one unit of R goes from the lowest pricing market maker to the highest pricing one in counterpart of a common price in Numéraire that was fixed in our initial model to be the maximal price. (Other choices of the common price would have led to the same results).

So, let y^R_q and z^R_q denote the numbers of R shares in P1 and P2's portfolios after the q-th trading period, and similarly, let y^N_q and z^N_q denote the numbers of N shares. Then the above described trading mechanism can be summarized by the following formulas, where $y_q := (y^R_q, y^N_q)$ and $z_q := (z^R_q, z^N_q)$ are the players' portfolios:

$$y_q = y_{q-1} + t(p^n_{1,q}, p^n_{2,q}) \text{ and } z_q = z_{q-1} - t(p^n_{1,q}, p^n_{2,q})$$

with

$$t(p_1, p_2) := 1\!\!1_{p_1 > p_2}(1, -p_1) + 1\!\!1_{p_2 > p_1}(-1, p_2)$$

We assume in this model that the initial endowments y_0 and z_0 are large enough so as to avoid situations where one player ran out of R or of N. The constraints $y_q \geq 0$ and $z_q \geq 0$ are therefore not binding and are ignored in the model.

Players' utility: The players are supposed to be risk neutral, and they aim to maximize the expected liquidation value of their final portfolio. So P1's utility is: $E[y_n^R L(M) + y_n^N]$ and P2's is $E[z_n^R L(M) + z_n^N]$. Since y_0 and z_0 are initially fixed, the liquidation values of the initial portfolios are constants that can be subtracted from player's utilities without affecting their behavior in the game. This turns out to be equivalent to assume $y_0 = z_0 = (0, 0)$, allowing for negative entries in the portfolios. With that hypothesis, we get clearly $y_n = -z_n$ and the game $G_n(P)$ is then a zero sum game.

This conclude the depiction of $G_n(P)$. In [De Meyer, Moussa-Saley (2003)], $G_n(P)$ is proved to have a value $V_n(P)$ and a full description of optimal behavior strategies is given. The main result there concerns the asymptotic of the price dynamic at equilibrium as n goes to ∞. More precisely, let Π^n be the continuous time representation of the process $p_{1,q}^n$, that is for $t \in [0, 1]$: $\Pi_t^n := p_{1,[\![nt]\!]}^n$, where $p_{1,0}^n := P$ and $[\![a]\!]$ is the greatest integer less or equal to a. Then, if the players are using their optimal strategies, Π^n converges in finite dimensional distribution to a process Π (i.e. for all finite set $J \subset [0, 1]$, the random vectors $(\Pi_t^n)_{t\in J}$ converges in distribution to $(\Pi_t)_{t\in J}$). Furthermore, Π is fully described there: it is a martingale on a Brownian filtration referred to hereafter as the continuous martingale of maximal variation corresponding to the distribution μ of the liquidation value L ($L = 1$ with probability P and 0 with probability $1 - P$).

So, the unique exogenous random event in $G_n(P)$ is the toss of a (biased) coin to select the message. To play optimally, the players have then to introduce mixed moves and all these random moves generate in the limit a Brownian motion, which appears therefore for strategic reasons in the price dynamic.

1.2. **The game $G_n(\mu)$ and the continuous martingales of maximal variation.** The model has been generalized in [De Meyer, Moussa-Saley (2002)] to a more general type of messages. Since clearly the important part of the message M is the corresponding liquidation value $L(M)$, we can identify the message with $L(M)$. This leads us to analyze the following game $G_n(\mu)$:

Let Δ be the set of probability distributions on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, where $B_\mathbb{R}$ is the Borel tribe on $\mathbb{R}$. In the sequel, if $\mu \in \Delta$, we will use the notation $L \sim \mu$ to indicate that the random variable L is μ distributed. We also write $\|\mu\|_{L^p}$ for $\|X\|_{L^p}$ where $X \sim \mu$, and we set $\Delta^p := \{\mu \in \Delta : \|\mu\|_{L^p} < \infty\}$ and $\Delta^{1^+} := \cup_{p>1}\Delta^p$.

For $\mu \in \Delta^1$, $G_n(\mu)$ is the game where a lottery selects initially $L \sim \mu$. P1 is then privately informed of L but P2 gets no information: he only knows μ. The game follows then exactly as in $G_n(P)$.

In [De Meyer, Moussa-Saley (2002)], we can prove that, for $\mu \in \Delta^1$, the game $G_n(\mu)$ has a value $V_n(\mu)$ and we also characterize the optimal strategies of both players.

To describe the asymptotic of Π^n, we need to introduce the following notations: Let Z be a gaussian random variable: $Z \sim \mathcal{N}(0, 1)$. For $\mu \in \Delta$, as it is well known there is a unique right continuous increasing function f_μ such that $f_\mu(Z) \sim \mu$. Namely $f_\mu(x) = F_\mu^{-1}(F_\mathcal{N}(x))$, where $F_\mathcal{N}$ and F_μ are the cumulative distribution functions of $\mathcal{N}(0, 1)$ and μ and where $F_\mu^{-1}(y) := \inf\{s : F_\mu(s) > y\}$.

Let B be a standard one dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{G}_t)_{t\geq 0})$. If $\mu \in \Delta^{1^+}$, the martingale $\Pi_t^\mu := E[f_\mu(B_1)|\mathcal{G}_t]$ will be referred to as the continuous martingales of maximal variation of final distribution μ. This terminology will be justified in section 3.8.

The main result in [De Meyer, Moussa-Saley (2002)] is that, at equilibrium in $G_n(\mu)$, Π^n converges in finite dimensional distribution to Π^μ, as $n \to \infty$.

So given the distribution μ, we have the explicit asymptotic distribution of the price dynamic. Observe that since B is Markovian, Π^μ_t is a function $g(B_t,t)$. Since f_μ is increasing, $g(x,t)$ is increasing in x, even strictly increasing for $t<1$ whenever f_μ is not a constant. Finally, Π^μ is a martingale, and therefore g must satisfy the heat equation and with Itô's formula, we get $d\Pi^\mu_t = h(B_t,t)dB_t$, where $h(x,t) := \frac{\partial}{\partial x} g(x,t)$. In turn, since g is strictly increasing in x, B_t is an increasing function of Π^μ_t, and $h(B_t,t)$ is then a function a of (Π^μ_t,t). We get in this way a diffusion equation for Π^μ: $d\Pi^\mu_t = a(\Pi^\mu_t,t)dB_t$

So, the above theory not only justifies the appearance of the Brownian motion in the price dynamics, but it also stipulates a class of dynamics.

My conviction is that this class of dynamics contains the right models for the price evolution on the stock market.

Are there statistical evidences to sustain such an assertion? Well, indirectly, since when μ is a log normal distribution, the corresponding process Π^μ follows then the Black and Scholes dynamic, which is currently one of the most used dynamic in finance for price modeling. Bachelier's dynamic is also included in this class as the particular case corresponding to a normal distribution μ .

Aside this statistical argument, there is also theoretical justifications for this class of dynamics. The appearance of continuous martingales of maximal variation in $G_n(\mu)$ is not accidental, due to some particular details in $G_n(\mu)$. As I aim to prove in this paper, this appearance is on the contrary quite universal and independent of the model.

1.3. **The game $\Gamma_n(\mu)$.** To support this assertion, we analyze in this paper a generalized game $G_n(\mu)$, referred to as $\Gamma_n(\mu)$, where two asymmetrically informed agents, not necessarily market makers, are trading R against N using an abstract trading device T.

An abstract trading mechanism T is simply a game characterized by two action sets endowed with a σ-algebra $(I,\mathcal{I})$ and $(J,\mathcal{J})$, (we will assume that finite sets in I or J are measurable) and by an outcome function

$$T : I\times J \to \mathbb{R}^2 : (i,j) \to T(i,j) := (A_{ij}, B_{ij}),$$

which is measurable from $(\mathcal{I}\times\mathcal{J})$ to the Borelean tribe of $\mathbb{R}^2$. If the players play (i,j), A_{ij} and B_{ij} represent the respective numbers of R and N shares P1 receives from P2. (Typically one is positive an the other negative).

So, in the generalized $\Gamma_n(\mu)$, at trading period q ($q=1,\dots,n$), the players select an action pair (i_q,j_q) independently of each others, based on their prior observations and private information. Actions are then made public and the portfolios are then incremented: $y_q = y_{q-1} + T(i_q,j_q)$ and $z_q = z_{q-1} - T(i_q,j_q)$. T could clearly represent any bargaining mechanism or auction procedure, actions being then strategies in these mechanisms.

A particular example is the bid ask version of our previous game: At each period both market makers post a bid b_q and an ask a_q, with $b_q \le a_q \le b_q(1+\epsilon)$, where the second constraint represent the limit on the spread imposed by the market regulator. In this case, $i_q = (b_{1,q}, a_{1,q})$, $j_q = (b_{2,q}, a_{2,q})$, and

$$A_{i_q,j_q} = 1\!\!1_{b_{1,q}>a_{2,q}} - 1\!\!1_{b_{2,q}>a_{1,q}} \text{ and } B_{i_q,j_q} = 1\!\!1_{b_{2,q}>a_{1,q}} b_{2,q} - 1\!\!1_{b_{1,q}>a_{2,q}} b_{1,q}.$$

In the generalized game $\Gamma_n(\mu)$, players are not posting prices any more, so what is the price process in this case? One possible definition of the price of R at period q could be $-\frac{B_{i_q,j_q}}{A_{i_q,j_q}}$, but this definition would lead to technical problems in case $A_{i_q,j_q} = 0$. Another possibility, adopted in this paper, is to define the price L_q

as the price at which P2 would agree to trade with another uninformed player: $L_q = E[L|i_1, \dots i_q, j_1, \dots, j_q]$.

We prove in theorem 1 that under very general and natural hypotheses on T, the continuous version $\Pi^n_t := L_{[\![nt]\!]}$ of the price process L_q at equilibrium in $\Gamma_n(\mu)$ converges in finite distribution to Π^μ. So the asymptotic distribution of the price process is basically independent of the exchange mechanism.

1.4. **A sketch of the proof.** The proof of this result presented here relies on ingredients that could also appear in many other financial models of incomplete information, even non zero sum games, and this let me hope that the appearance of continuous martingales of maximal variation is not limited to the zero sum games analyzed in this paper.

The first ingredient of the proof is that in an n-period game, the informed player may decide to link or not his actions at stage q with his private information. He controls in this way the martingale $L^n := (L^n_q)_{q=0,\dots n+1}$, with $L^n_{n+1} := L$, and where L^n_q is the expected value of L given the public information $\mathcal{F}_q$ after stage q. The martingale L^n belongs thus to $\mathcal{M}_n(\mu)$: the set of martingales of length $n+1$ with μ-distributed final value. We can then see the maximization problem P1 faces as a two stage problem: He first chooses a martingale $L^n \in \mathcal{M}_n(\mu)$ and he then picks his action i_q at stage q as a function of $(L^n_k)_{k \le q}$.

The second ingredient is that when replying to such a strategy of player 1, the uninformed player is assumed to know the strategy to which he replies. So, to compute the max min of the game, we may assume that, when playing stage q, player 2 has observed L^n_k for $k < q$. Therefore, when choosing $i_q(L^n_1, \dots, L^n_q)$, player 1 has not to care about the information his action will reveal, since in any case, player 2 will observe L^n_q before playing the $q+1$-stage. Player 1 will thus picks the action $i_q(L^n_1, \dots, L^n_q)$ that maximizes his stage payoff: $A_{i_q,j_q} L^n_q + B_{i_q,j_q}$. If $[L^n_q|L^n_1, \dots, L^n_{q-1}]$ denotes the conditional law of L^n_q given $L^n_1, \dots, L^n_{q-1}$, the game he is facing at the q-th stage is thus $\Gamma_1([L^n_q|L^n_1, \dots, L^n_{q-1}])$, and the best he can do is to play optimally in this game, obtaining $V_1([L^n_q|L^n_1, \dots, L^n_{q-1}])$.

As proved in section 2.3, the optimal martingale L^n is thus the one in $\mathcal{M}_n(\mu)$ that maximizes

$$E[\sum_{q=1}^{n} V_1([L^n_q|L^n_1, \dots, L^n_{q-1}])].$$

The conditions on the trading mechanism T are presented in the next section. One of them referred to as Invariance with respect to the risk-less part of the risky asset implies in particular that for all constant β, for all random variable L, $V_1([L+\beta]) = V_1([L])$, where $[L]$ denotes the law of L. This indicates that the optimal martingale L^n will also maximize $\mathcal{V}^{V_1}_n(L^n)$, where, for a function $M : \Delta \to \mathbb{R}$, $\mathcal{V}^M_n(L^n)$ is defined as

$$\mathcal{V}^M_n(L^n) := E[\sum_{q=1}^{n} M([L^n_q - L^n_{q-1}|L^n_1, \dots, L^n_{q-1}])].$$

The second part of the paper is devoted to the analysis of the martingales L^n maximizing $\mathcal{V}^M_n(L^n)$. We prove in theorem 9 that under a continuity and a homogeneity condition on M, any sequence of maximizing martingales L^n converge to the continuous martingale of maximal variation corresponding to μ. This limit is thus independent of M. This result relies on a central limit theorem.

The hypothesis on the trading mechanism called invariance with respect to the Numéraire scale indicates that V_1 fulfills the required homogeneity property and theorem 9 can thus be applied to prove our results on $\Gamma_n(\mu)$.

2. The main result on $\Gamma_n(\mu)$

2.1. **Strategies $\Gamma_n(\mu)$.** Let us first start by defining strategies in $\Gamma_n(\mu)$. A mixed strategy for P2 in $\Gamma_1(\mu)$ is a probability distribution τ on $(J,\mathcal{J})$. However, since A_{ij} and B_{ij} are a priori unbounded, we have to restrict a little bit this definition. Let $\Delta(J)$ be the set of probability distributions τ on $(J,\mathcal{J})$ such that,

$$\forall i \in I : \int_J |A_{ij}| d\tau(j) < \infty \text{ and } \int_J |B_{ij}| d\tau(j) < \infty.$$

For $\tau \in \Delta(J)$, we set: $A_{i\tau} := \int_J A_{ij} d\tau(j)$ and $B_{i\tau} := \int_J B_{ij} d\tau(j)$. In the same way, we define $\Delta(I)$ and, for $\sigma \in \Delta(I)$, $A_{\sigma,j}$ and $B_{\sigma,j}$.

A strategy τ in $\Gamma_n(\mu)$ is a sequence $(\tau_1,\dots,\tau_n)$ of $\Delta(J)$-valued transition probabilities $\tau_q : (H_{q-1},\mathcal{H}_{q-1}) \to (J,\mathcal{J})$, where $(H_q,\mathcal{H}_q) := ((I\times J)^q, (\mathcal{I}\times\mathcal{J})^q)$. In other words: $\forall h_{q-1} \in H_{q-1}$, $\tau_q(h_{q-1}) \in \Delta(J)$ and $\forall A \in \mathcal{J}$: the map $h_{q-1} \to \tau_q(h_{q-1})[A]$ is $\mathcal{H}_{q-1}$-measurable.

In the same way, a strategy σ in $\Gamma_n(\mu)$ is a sequence $(\sigma_1,\dots,\sigma_n)$ of $\Delta(I)$-valued transition probabilities $\sigma_q : (\mathbb{R}\times H_{q-1}, \mathcal{B}_{\mathbb{R}}\times\mathcal{H}_{q-1}) \to (I,\mathcal{I})$. $\mathcal{S}_n$ will denote hereafter the set of P1's strategies.

With Tulcea theorem, a triplet (μ,σ,τ) will induce a unique probability $\pi_{(\mu,\sigma,\tau)}$ on $(\mathbb{R}\times H_n)$.

Still the payoff function could be undefined in general for integrability reasons and we have to restrict our notion of strategy. The intergrability problem can be illustrated as follows: suppose that τ_2 is just a function of j_1 so that $B_{i_2\tau_2(j_1)}$, is a finite function of j_1, but it could fail to be integrable with respect to τ_1.

This leads us to the definition of admissible strategy: A strategy τ is said admissible if for every history $h^1 \in I^n$, the probability $\pi^2_{(h^1,\tau)}$ induced on J^n by (h^1,τ) is such that for all q, the random variables $|A_{i_q,j_q}|$ and $|B_{i_q,j_q}|$ have finite expectation with respect to $\pi^2_{(h^1,\tau)}$. $\mathcal{T}^{adm}_n$ will denote the set of P2's admissible strategies. Observe that $\pi^2_{(h^1,\tau)}$ is just the conditional probability $\pi_{(\mu,\sigma,\tau)}$ on J^n given h^1.

So, $\mathcal{A}^n(h^1,\tau) := E_{\pi^2_{(h^1,\tau)}}[\sum_{q=1}^n A_{i_q,j_q}]$ and $\mathcal{B}^n(h^1,\tau) := E_{\pi^2_{(h^1,\tau)}}[\sum_{q=1}^n B_{i_q,j_q}]$ are the expected R and N quantities in y_n given that player 1 played h^1. These are finite measurable functions of h^1.

Let us now write formally the payoff in $\Gamma_n(\mu)$. Notice that y_n is independent on L conditionally to h^1, since P2's moves depend on h^1 but not on L. Therefore, with expectations taken with respect to $\pi_{(\mu,\sigma,\tau)}$, and assuming the integrability of $Ly^R_n + y^N_n$, we could write:

$$\begin{aligned} E[Ly^R_n + y^N_n] &= E[E[Ly^R_n|h^1]] + E[E[y^N_n|h^1]] \\ &= E[E[L|h^1]\cdot E[y^R_n|h^1]] + E[\mathcal{B}^n(h^1,\tau)] \\ &= E[L\mathcal{A}^n(h^1,\tau) + \mathcal{B}^n(h^1,\tau)] \end{aligned} \tag{1}$$

Observing the last formula, the best player 1 can do to reply to τ is to play a history $h^1(L)$ depending on L, that solves the problem:

$$\phi^n_\tau(L) := \sup_{h^1} L\mathcal{A}^n(h^1,\tau) + \mathcal{B}^n(h^1,\tau) \tag{2}$$

Optimal solution could fail to exist, but measurable ϵ-solution exist. Therefore a strategy τ guarantees $E_\mu[\phi^n_\tau(L)]$ to P2. As supremum of a family of affine functions of L, $\phi^n_\tau(L)$ is a convex l.s.c. function from $\mathbb{R}$ to $\mathbb{R}\cup\{\infty\}$ and is therefore measurable. Since $\mu \in \Delta^1$, we also have: $E_\mu[\phi^n_\tau(L)] > -\infty$.

Notice that there could be integrability problems in equation (1) in general and it could be the case that $E[Ly^R_n + y^N_n]$ is undetermined (meaning that both the positive and the negative part of $Ly^R_n + y^N_n$ have infinite expectation) although

$E[L\mathcal{A}^n(h^1,\tau)+\mathcal{B}^n(h^1,\tau)]$ is finite. This remark leads us to define the payoff function in $\Gamma_n(\mu)$ as $g_n(\mu,\sigma,\tau) := E_{\pi_{(\mu,\sigma,\tau)}}[L\mathcal{A}^n(h^1,\tau)+\mathcal{B}^n(h^1,\tau)]$.

This definition of the payoff could still be undetermined for some pairs of strategies. However, if $E_\mu[\phi^n_\tau(L)] < \infty$, then the payoff function $g_n(\mu,\sigma,\tau)$ is possibly equal to $-\infty$, but there is never indeterminacy, whatever the strategy σ is.

The minimal amount player 2 can guarantee is $\overline{V}_n(\mu) := \inf_{\tau\in\mathcal{T}_n} E_\mu[\phi^n_\tau(L)]$.

A strategy τ of player 2 is optimal in $\Gamma_n(\mu)$ if $\overline{V}_n(\mu) = E_\mu[\phi^n_\tau(L)]$.

A strategy σ is admissible for player 1, if, for all admissible strategy τ:

$$E[\min(L\mathcal{A}^n(h^1,\tau)+\mathcal{B}^n(h^1,\tau),0)] > -\infty,$$

which implies that $g_n(\mu,\sigma,\tau)$ is well defined in $\mathbb{R}\cup\{\infty\}$. Let $\mathcal{S}^{adm}_n$ be the set of admissible strategies for P1.

A strategy $\sigma\in\mathcal{S}^{adm}_n$ guarantees α to P1 if, $\forall\tau\in\mathcal{T}^{adm}_n$: $g_n(\mu,\sigma,\tau)\geq\alpha$.

The maximum amount P1 can guarantee in $\Gamma_n(\mu)$ is

$$\underline{V}_n(\mu) := \sup_{\sigma\in\mathcal{S}^{adm}_n}\inf_{\tau\in\mathcal{T}^{adm}_n} g_n(\mu,\sigma,\tau).$$

A strategy σ is optimal if it guarantees $\underline{V}_n(\mu)$.

It is always true that $\underline{V}_n(\mu)\leq\overline{V}_n(\mu)$. When equality holds, the game $\Gamma_n(\mu)$ is said to have a value $V_n(\mu) := \underline{V}_n(\mu) = \overline{V}_n(\mu)$.

If the game has a value, and if σ^* and τ^* are optimal strategies, then (σ^*,τ^*) is a Nash equilibrium of the game. Conversely, if (σ^*,τ^*) is a Nash equilibrium of the game, then the game has a value and σ^* and τ^* are optimal strategies.

2.2. **The hypotheses on the trading mechanism.** Let us first present the hypotheses we will make on the trading mechanism T. The abstract trading device T could clearly be used to trade other shares than R and N. The idea of the first hypothesis is simply that if T is used to trade R against the dollar, it will lead to the same transactions in value as if was used to trade R against the cent. This means that the same number of R shares would be exchanged in both cases, but the number of cents given in counterpart in the second case would be a hundred times the number of dollars given in the first case. Clearly, the players will not use the same actions when trading in dollars or in cents. Instead, there is a translation rule that maps the the actions in dollar to the actions in cents. This leads us to the following hypothesis:

H1: Invariance with respect with the numéraire scale. $\forall\alpha>0$, *there exist measurable one to one mappings* $\psi_1: I\to I$ *and* $\psi_2: J\to J$ *such that* $\forall i,j$:

$$A_{\psi_1(i),\psi_2(j)} = A_{i,j} \text{ and } B_{\psi_1(i),\psi_2(j)} = \alpha\cdot B_{i,j}.$$

In the original game $G_n(\mu)$, where the actions were prices p_1,p_2, the mappings ψ_1 and ψ_2 corresponding to α would simply be defined by $\psi_1(p)=\psi_2(p)=\alpha p$.

The next hypothesis is also quite natural for the trading device T: consider a risky asset R' which consists of a basket of one share of the risky asset R and one bill of \$100. The hypothesis $H2$ requires that trading R' against the dollar with T will lead to the same trade in value as trading R against the dollar: more precisely, we require that the number a of exchanged R' and R shares is the same in both cases, but the counterpart in dollar for R' is just the counterpart for R plus $a\cdot\$100$. This leads us to the following hypothesis:

H2: Invariance with respect to the risk-less part of the risky asset. $\forall\beta\in\mathbb{R}$, *there exist measurable one to one mappings* $\psi_1: I\to I$ *and* $\psi_2: J\to J$ *such that* $\forall i,j$:

$$A_{\psi_1(i),\psi_2(j)} = A_{i,j} \text{ and } B_{\psi_1(i),\psi_2(j)} = B_{i,j}+\beta\cdot A_{i,j}.$$

In a trading mechanism based on prices such as in $G_n(\mu)$, the translation maps ψ_i consist of increasing prices by β. Clearly the bid ask version of $G_n(\mu)$ also satisfies H1 and H2. We present in section 2.4 a canonical way to obtain trading mechanisms satisfying these hypotheses.

We will not deal in this paper with the technical question of existence of the value for $\Gamma_n(p)$, so that we just assume it in the next hypothesis.

H3: Existence of the value. *For all $\mu \in \Delta^1$, for all n, the game $\Gamma_n(\mu)$ has a value $V_n(\mu)$ and both players have optimal strategies in this game.*

The next hypothesis is that there exists a situation where player 1 can take benefit of his information. This hypothesis is not insignificant: it means in particular that P2 is not free to avoid trading with the more informed player 1.

H4: Positive value of information. $\exists \mu \in \Delta^2 : V_1(\mu) > 0$.

H5: Continuity of V_1. *There exists $p \in [1, 2[$ and $A \in \mathbb{R}$ such that, if X and Y are two random variables on the same probability space with respective distributions μ and ν, then $|V_1(\mu) - V_1(\nu)| \leq A\|X - Y\|_{L^p}$.*

This last hypothesis will be in particular satisfied with $p = 1$ if $\forall i, j : |A_{i,j}| \leq A$.

We are now ready to state our main theorem concerning Γ_n:

Theorem 1. *Under H1, H2, H3, H4, H5, and for a fixed μ in Δ^2, consider a sequence (σ_n, τ_n), where $\forall n$, (σ_n, τ_n) is an equilibrium in $\Gamma_n(\mu)$, and let L^n be the price process in this game, defined as $L^n_q := E_{\pi(\mu,\sigma_n,\tau_n)}[L|i_1, \ldots, i_q, j_1, \ldots, j_q]$. Then the continuous time representation Π^n of L^n defined as $\Pi^n_t := L^n_{[\![nt]\!]}$ converges in finite dimensional distribution to the continuous martingale of maximal variation Π^μ.*

This theorem will be proved in the next sections. As a remark, let us observe that the game $G_n(P)$ where players have to post prices in a discrete grid fails to fulfill H1 since the size of the grid is fixed and independent of the scale of numéraire. This game was analyzed in [De Meyer, Marino (2005)] and indeed it does not display the same asymptotic for the price process.

2.3. **The a posteriori martingale.** In this section we show that the choice of a strategy for player 1 turns out to be a choice of the optimal rate of revelation. The revelation process, represented by the posterior martingale must be optimal in the maximization problem (4) here below.

If Y is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and if $\mathcal{H} \subset \mathcal{A}$ is a σ-algebra, the probability distribution of Y will be denoted $[Y]$, and the conditional distribution of Y given $\mathcal{H}$ will be $[Y|\mathcal{H}]$. We will also write $\Gamma_n[Y]$ and $V_n[Y]$ instead of $\Gamma_n([Y])$ and $V_n([Y])$. In particular $V_1[Y|\mathcal{H}]$ is an $\mathcal{H}$-measurable random variable[1]. Let $\mathcal{W}_n(\mu)$ be the set of pairs $(\mathcal{F}, X)$ where $\mathcal{F} := (\mathcal{F}_q)_{q=0,\ldots,n+1}$ is a filtration on

[1] The set Δ of probability measures on $\mathbb{R}$ may be endowed with the weak*-topology: the weakest topology such that $\phi_g : \mu \to E_\mu[g]$ is continuous, for all continuous bounded function $g : \mathbb{R} \to \mathbb{R}$. If Y is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and if $\mathcal{H} \subset \mathcal{A}$ is a σ-algebra, the conditional distribution $[Y|\mathcal{H}]$ can then be seen as a measurable map from $(\Omega, \mathcal{H})$ to $(\Delta, \mathcal{B}_\Delta)$ where $\mathcal{B}_\Delta$ denotes the Borel σ-algebra on Δ corresponding to the weak*-topology. Let Δ^2_r be the set of $\mu \in \Delta$ such that $\|\mu\|_{L^2} \leq r$. Δ^2_r is a closed subset of Δ. (Indeed $\Delta^2_r = \cap_n \phi^{-1}_{g_n}([0, r^2])$, where $g_n(x) := \min(x^2, n)$.) We next prove that the restriction of V_1 to Δ^2_r is continuous: If $\{\mu_n\}_{n\in\mathbb{N}} \subset \Delta^2_r$ converges weakly to μ, then, according Skorokhod embeding theorem, there exists a sequence X_n of random variables with $X_n \sim \mu_n$ that converges a.s. to $X \sim \mu$. Since $\|X_n - X\|_{L^2} \leq 2r$, we conclude that $|X_n - X|^p$ is a uniformly integrable sequence ($p < 2$), and thus $\|X_n - X\|_{L^p} \to 0$, implying with H5 that $V_1[X_n] \to V_1[X]$. So V_1 is indeed continuous on Δ^2_r. Therefore $V_1 : \Delta^2 \to \mathbb{R}$ is measurable on the trace $\mathcal{B}_{\Delta^2}$ of $\mathcal{B}_\Delta$ on Δ^2. Since if $[Y] \in \Delta^2$, $[Y|\mathcal{H}]$ maps Ω to Δ^2, we get that $V_1[Y|\mathcal{H}]$ is measurable as composition of measurable functions.

a probability space, and $X = (X_q)_{q=0,\dots,n+1}$ is an $\mathcal{F}$-martingale X whose $n+1$-th value X_{n+1} is μ-distibuted. For $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, we define $\mathcal{V}_n(\mathcal{F}, X)$ as:

$$\mathcal{V}_n(\mathcal{F}, X) := E[\sum_{q=0}^{n-1} V_1[X_{q+1}|\mathcal{F}_q]] \tag{3}$$

Let us also define $\overline{\mathcal{V}}_n(\mu)$ as

$$\overline{\mathcal{V}}_n(\mu) := \sup\{\mathcal{V}_n(\mathcal{F}, X) : (\mathcal{F}, X) \in \mathcal{W}_n(\mu)\} \tag{4}$$

Lemma 2. *For all $\mu \in \Delta^1$: $V_n(\mu) \geq \overline{\mathcal{V}}_n(\mu)$.*

Proof: Given $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$ on a probability space $(\Omega, \mathcal{A}, P)$, we have to prove that P1 can guarantee $\mathcal{V}_n(\mathcal{F}, X)$ in $\Gamma_n(\mu)$. At the initial stage of $\Gamma_n(\mu)$, nature selects a μ distributed random variable L and informs P1 of its choice. We can clearly assume that nature uses the probability space $(\Omega, \mathcal{A}, P)$ as lottery, and sets $L = X_{n+1}$, since $X_{n+1} \sim \mu$. We can also assume that P1 observes the whole space $(\Omega, \mathcal{A}, P)$. He can therefore adopt the following strategy in $\Gamma_n[X_{n+1}]$: at stage q he plays an optimal strategy in $\Gamma_1[X_{q+1}|\mathcal{F}_q]$. The payoff at that stage will then be: $E[A_{i_q,\tau_q} X_{n+1} + B_{i_q,\tau_q}]$. Since the distribution of i_q just depends on X_{q+1} and $E[X_{n+1}|\mathcal{F}_{q+1}] = X_{q+1}$, the payoff is also equal to: $E[A_{i_q,\tau_q} X_{q+1} + B_{i_q,\tau_q}]$, and since the move of P1 is optimal in $\Gamma_1[X_{q+1}|\mathcal{F}_q]$, he gets at least $V_1[X_{q+1}|\mathcal{F}_q]$ conditionally to $\mathcal{F}_q$. The result follows then easily. ∎

Lemma 3. *For all $\mu \in \Delta^1$: $V_n(\mu) \leq \overline{\mathcal{V}}_n(\mu)$.*

Proof: We will prove that P1 will not be able to guarantee a higher payoff than $\overline{\mathcal{V}}_n(\mu)$. Indeed, let σ be an optimal strategy of P1 in $\Gamma_n(\mu)$. To reply to σ, P2 may adopt the following strategy: since he knows σ_1, he may compute the distribution of $L_1 := E[L|i_1]$. He plays then an optimal strategy τ_1 in $\Gamma_1[L_1]$. At period q, he computes $[L_q|\mathcal{H}_{q-1}]$, with $\mathcal{H}_q := \sigma(i_1, j_1 \dots i_q, j_q)$, where $L_q := E[L|i_q, \mathcal{H}_{q-1}]$, and plays an optimal strategy τ_q in $\Gamma_1[L_q|\mathcal{H}_{q-1}]$. Clearly, we also have $L_q = E[L|\mathcal{H}_q]$, since conditionally to $\mathcal{H}_{q-1}$, the move j_q is independent of L. Therefore, with $\mathcal{H}_{n+1} := \sigma(L, \mathcal{H}_n)$, $L_{n+1} := L$, $\mathcal{H}_0 := \{\emptyset, (I \times J)^n \times \mathbb{R}\}$, $L_0 := E[L]$, $\overline{L} := (L_q)_{q=0,\dots,n+1}$ and $\mathcal{H} := (\mathcal{H}_q)_{q=0,\dots,n+1}$, the pair $(\mathcal{H}, \overline{L})$ belongs to $\mathcal{W}_n(\mu)$.

With that reply P1's conditional payoff at period q, given $\mathcal{H}_{q-1}$ is the at most $V_1[L_q|\mathcal{H}_{q-1}]$ and the overall payoff in $\Gamma_n(\mu)$ is then less than $\mathcal{V}_n(\mathcal{H}, \overline{L}) \leq \overline{\mathcal{V}}_n(\mu)$. ∎

Theorem 4. *For all $\mu \in \Delta^1$: $V_n(\mu) = \overline{\mathcal{V}}_n(\mu)$. Furthermore, if σ, τ are optimal strategies in $\Gamma_n(\mu)$, if $L_q := E_{\pi(\mu,\sigma,\tau)}[L|\mathcal{H}_q]$, where $\mathcal{H}_q := \sigma(i_1, j_1 \dots i_q, j_q)$, and $\overline{L} := (L_q)_{q=0,\dots,n+1}$, then $(\mathcal{H}, \overline{L})$ solves the maximization problem (4).*

Proof: The first claim follows the two previous lemmas.

Let next assume that the players are playing a pair (σ, τ) of optimal strategies in $\Gamma_n(\mu)$.

Let us then first observe that, for all q, the expectation, conditional to $\mathcal{H}_q$, of the sum of the next $n-q$ stage payoffs must clearly be at least $V_{n-q}[L|\mathcal{H}_q]$. Otherwise P1 could deviate from stage $q+1$ on to an optimal strategy in $\Gamma_{n-q}[L|\mathcal{H}_q]$, obtaining thus a higher payoff against τ than with σ, which is impossible since (σ, τ) is an equilibrium of the game.

At period q, P2 may compute $v_{q-1} := V_1[L_q|\mathcal{H}_{q-1}]$ and $u_{q-1} := E[A_{i_q,j_q} L_q + B_{i_q,j_q}|\mathcal{H}_{q-1}]$. On the event $\{v_{q-1} < u_{q-1}\}$, P2 could then deviate at stage q with an optimal strategy in $\Gamma_1[L_q|\mathcal{H}_{q-1}]$, bringing the expected payoff of that stage to less than v_{q-1}, that is strictly less than the payoff u_{q-1} P1 would obtain with τ.

If P2 then follows with an optimal strategy in $\Gamma_{n-q}[L|\mathcal{H}_q]$, the payoff of P1 in the $n-q$ last stages will be less than $V_{n-q}[L|\mathcal{H}_q]$, which is, as observed above, less than the payoff P1 gets with the pair (σ,τ). Therefore, $\pi_{(\mu,\sigma,\tau)}[v_{q-1} < u_{q-1}] = 0$, since otherwise, P2 would have a profitable deviation.

So for all q, $E[v_{q-1}] \geq E[u_{q-1}]$. Summing up all these inequalities, we get $\mathcal{V}_n(\mathcal{H},\overline{L}) \geq g_n(\mu,\sigma,\tau) = V_n(\mu) = \overline{\mathcal{V}}_n(\mu)$, and the second assertion is proved. ■

2.4. **The canonical representation of $\Gamma_1(\mu)$.** The aim of this section is double: on one hand we will derive the properties of V_1 implied by the hypotheses H1, H2. On the other hand, we will provide a generic way to create trading mechanism having these properties.

In the one shot game, a strategy of player 2 is just an element τ of $\Delta(J)$, the history h^1 reduces to i, and we get $\mathcal{A}^1(h^1,\tau) = A_{i\tau}$, $\mathcal{B}^1(h^1,\tau) = B_{i\tau}$. The function ϕ^1_τ introduced in (2) will simply be denoted ϕ_τ in this section. It becomes then: $\phi_\tau(L) = \sup_{i\in I} LA_{i,\tau} + B_{i\tau}$. We get therefore:

$$V_1(\mu) = \min_{\tau} E_\mu[\phi_\tau(L)] = \min_{\phi\in\Phi} E_\mu[\phi(L)]. \tag{5}$$

where Φ is the set of l.s.c. convex functions ϕ such that $\phi \geq \phi_\tau$ for some $\tau \in \Delta(J)$. Notice that in this formula, we wrote min instead of inf, since the infimum is reached at the optimal strategy τ which exists according to H3.

Lemma 5. *Φ is a convex set of convex functions.*

Proof: Indeed, at fixed L, the map $\tau \to LA_{i,\tau} + B_{i\tau}$ is linear in τ, and therefore, as supremum of linear maps, the map $\tau \to \phi_\tau(L)$ is convex. Therefore, if $\phi,\phi' \in \Phi$, if $\lambda,\lambda' \geq 0$, with $\lambda+\lambda' = 1$, then the function $\phi'' = \lambda\phi+\lambda'\phi'$ is clearly convex u.s.c. and, if $\phi \geq \phi_\tau$ and $\phi' \geq \phi_{\tau'}$, then $\phi'' \geq \lambda\phi_\tau + \lambda'\phi_{\tau'} \geq \phi_{\tau''}$, where $\tau'' = \lambda\tau + \lambda'\tau'$. Hence, ϕ'' belongs to Φ which results to be a convex set. ■

Let us now look at the implications of H1 and H2:

Lemma 6. *If ϕ belongs to Φ, then for all $\alpha > 0$, $\phi_\alpha : L \to \phi_\alpha(L) := \alpha\phi(\frac{L}{\alpha})$ also belongs to Φ.*

If ϕ belongs to Φ, then for all $\beta \in \mathbb{R}$, $\phi^\beta : L \to \phi^\beta(L) := \phi(L+\beta)$ also belongs to Φ.

Proof: For $\alpha > 0$, let ψ_1 and ψ_2 be translation mappings corresponding to α in H1. Let $\tau \in \Delta(J)$, and let τ_α be the τ's image probability on J by ψ_2: τ_α is the probability distribution of $\psi_2(j)$ when j is τ distributed. Then, $\forall i \in I$:

$$\begin{aligned} A_{\psi_1(i),\tau_\alpha} &= E_\tau[A_{\psi_1(i),\psi_2(j)}] = E_\tau[A_{i,j}] = A_{i\tau} \\ B_{\psi_1(i),\tau_\alpha} &= E_\tau[B_{\psi_1(i),\psi_2(j)}] = E_\tau[\alpha B_{i,j}] = \alpha B_{i\tau}. \end{aligned}$$

Therefore, since ψ_1 is one to one,

$$\begin{aligned} \phi_{\tau_\alpha}(L) &= \sup_{i\in I} LA_{i,\tau_\alpha} + B_{i,\tau_\alpha} \\ &= \sup_{i\in I} LA_{\psi_1(i),\tau_\alpha} + B_{\psi_1(i),\tau_\alpha} \\ &= \sup_{i\in I} LA_{i\tau} + \alpha B_{i\tau} \\ &= \alpha \sup_{i\in I} \tfrac{L}{\alpha} A_{i\tau} + B_{i\tau} \\ &= \alpha\phi_\tau(\tfrac{L}{\alpha}). \end{aligned}$$

So, if ϕ belongs to Φ, there exists τ such that $\phi \geq \phi_\tau$. Therefore $\phi_\alpha \geq \phi_{\tau_\alpha}$, and hence ϕ_α also belongs to Φ.

For $\beta \in \mathbb{R}$, let now ψ_1 and ψ_2 be translation mappings corresponding to β in H2. Defining τ^β as the τ's image probability on J by ψ_2, we get now $\forall i \in I$:

$$\begin{aligned} A_{\psi_1(i),\tau^\beta} &= E_\tau[A_{\psi_1(i),\psi_2(j)}] = E_\tau[A_{i,j}] = A_{i\tau} \\ B_{\psi_1(i),\tau^\beta} &= E_\tau[B_{\psi_1(i),\psi_2(j)}] = E_\tau[B_{i,j} + \beta A_{i,j}] = B_{i\tau} + \beta A_{i,\tau}. \end{aligned}$$

Therefore $\phi_{\tau^\beta}(L) = \phi_\tau(L+\beta)$, and, if ϕ belongs to Φ, there exists τ such that $\phi \geq \phi_\tau$. Therefore $\phi^\beta \geq \phi_{\tau^\beta}$, and hence ϕ^β also belongs to Φ. ■

Lemma 7. *The function $V_1[\cdot]$ is 1-homogeneous, positive, and invariant by translation by a constant. In other words, for all random variable X, $\forall \alpha \geq 0$, $\forall \beta \in \mathbb{R}$:*

$$\begin{array}{ll} i) & V_1[X] \geq 0 \\ ii) & V_1[X+\beta] = V_1[X] \\ iii) & V_1[\beta] = 0 \\ iv) & V_1[\alpha X] = \alpha V_1[X]. \end{array}$$

Proof: Let us first prove ii): The map $\phi \in \Phi \to \phi^\beta$ is clearly one to one. Therefore: $V_1[X+\beta] = \inf_{\phi\in\Phi} E[\phi^\beta(X)] = \inf_{\phi\in\Phi} E[\phi(X)] = V_1[X]$

Let us next prove iv) for a strictly positive α. Then the map $\phi \in \Phi \to \phi_\alpha$ is clearly one to one. Therefore: $V_1[\alpha X] = \inf_{\phi\in\Phi} E[\phi_\alpha(\alpha X)] = \alpha \inf_{\phi\in\Phi} E[\phi(X)] = \alpha V_1[X]$.

With $X = 0$ and $\alpha = 2$, we get thus $V_1[0] = 2V_1[0]$ and so $V_1[0] = 0$. This implies in particular that iv) holds also when $\alpha = 0$.

Next, with ii), we get, $\forall \beta \in \mathbb{R} : 0 = V_1[0] = V_1[\beta]$ and iii) is proved.

Finally, due to Jensen's inequality: for all $\phi \in \Phi : E[\phi(X)] \geq \phi(E[X])$. Therefore $V_1[X] \geq V_1[E[X]] = 0$, and i) is proved. ■

One easy consequence of iii) is that Φ is a set of positive functions. Indeed, for all $\phi \in \Phi : \phi(\beta) = E[\phi(\beta)] \geq V_1[\beta] = 0$.

Given a convex set Φ of finite positive convex functions on $\mathbb{R}$, stable for the α and β transforms (i.e. $\forall a > 0, \forall \beta \in \mathbb{R} : \phi_\alpha, \phi^\beta \in \Phi$, whenever $\phi \in \Phi$.), there exists a transaction mechanism T satisfying H1, H2 such that $V_1[X] = \inf_{\phi\in\Phi} E[\phi(X)]$.

In this mechanism $I = \mathbb{R}$, $J = \Phi$, and for all $x \in \mathbb{R}, \phi \in \Phi$, $A_{x,\phi} := \phi'(x)$, $B_{x,\phi} := \phi(x) - \phi'(x)\cdot x$, where the derivative ϕ' is the right continuous selection of the sub gradient of ϕ in case ϕ is not differentiable.

It is then easy to prove that with $\psi_1(x) := \alpha x$ and $\psi_2(\phi) := \phi_\alpha$, the hypothesis H1 is satisfied. Similarly, with $\psi_1(x) := x + \beta$ and $\psi_2(\phi) := \phi^\beta$, H2 is satisfied.

Now observe that $A_{x,\phi}L + B_{x,\phi} = \phi(x) + \phi'(x)(L-x) \leq \phi(L)$, which proves that P2 can guarantee $E[\phi(L)]$ in $\Gamma_1[L]$, and thus also $\inf_{\phi\in\Phi} E[\phi(L)]$. On the other hand, by playing $x(L) = L$ in $\Gamma_1[L]$, P1 guarantees the same amount so that $V_1[L] = \inf_{\phi\in\Phi} E[\phi(L)]$.

Based on the previous properties of V_1, theorem 1 becomes a particular case of theorem 9 hereafter. The proof of this theorem will thus be postponed to section 3.2.

3. Martingales of maximal variation

3.1. **The maximal covariation.** In this section, we solve an auxiliary optimization problem that will be quite useful to deal with the martingale optimization problem analyzed in this part. Let μ be a probability distribution on $\mathbb{R}$ and Z a $\mathcal{N}(0,1)$ random variable on a probability space $(\Omega, \mathcal{A}, P)$. As it is well known there is a unique right continuous increasing function f_μ such that $f_\mu(Z) \sim \mu$. Namely $f_\mu(x) = F_\mu^{-1}(F_\mathcal{N}(x))$, where $F_\mathcal{N}$ and F_μ are the cumulative distribution functions of $\mathcal{N}(0,1)$ and μ and where $= F_\mu^{-1}(y) := \inf\{s : F_\mu(s) > y\}$.

Theorem 8. *For $\mu \in \Delta^{1^+}$, let us define $\alpha(\mu) := \sup\{E[XZ] : X \sim \mu\}$ then*

1) $\alpha(\mu) = E[f_\mu(Z)Z]$.

2) If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of μ-distributed random variables such that $E[X_nZ]$ converges to $\alpha(\mu)$ then X_n converges in L^1-norm to $f_\mu(Z)$.

Proof: Let us first prove the result for a measure μ such that $\mu([0,\infty[) = 1$. Let X be a μ-distributed random variable. Since $\mu \in \Delta^{1^+}$, XZ is in L^1, and with Fubini-Tonelli theorem

$$E[XZ] = E[\int_0^\infty 1\!\!1_{\{c\leq X\}} Zdc] = \int_0^\infty E[1\!\!1_{\{c\leq X\}}Z]dc$$

Therefore

$$E[f_\mu(Z)Z] - E[XZ] = \int_0^\infty E[(1\!\!1_{\{c\leq f_\mu(Z)\}} - 1\!\!1_{\{c\leq X\}})Z]dc$$

Now observe that X and $f_\mu(Z)$ have the same distribution. Therefore

$$\forall c : E[(1\!\!1_{\{c\leq f_\mu(Z)\}} - 1\!\!1_{\{c\leq X\}})] = 0$$

and we infer that

$$E[f_\mu(Z)Z] - E[XZ] = \int_0^\infty E[(1\!\!1_{\{c\leq f_\mu(Z)\}} - 1\!\!1_{\{c\leq X\}}) \cdot (Z - f_\mu^{-1}(c))]dc$$

where f_μ^{-1} is the left continuous inverse of f_μ: $f_\mu^{-1}(c) := \inf\{s : f_\mu(s) \geq c\}$.

An easy computation shows that

$$1\!\!1_{\{c\leq f_\mu(Z)\}} - 1\!\!1_{\{c\leq X\}} = 1\!\!1_{\{X<c\leq f_\mu(Z)\}} - 1\!\!1_{\{f_\mu(Z)<c\leq X\}}$$

Since $c \leq f_\mu(Z)$ if and only if $f_\mu^{-1}(c) \leq Z$, we conclude that

$$E[f_\mu(Z)Z] - E[XZ] = \int_0^\infty E[h(X,Z,c)]dc$$

where

$$h(X,Z,c) := (1\!\!1_{\{X<c\leq f_\mu(Z)\}} + 1\!\!1_{\{f_\mu(Z)<c\leq X\}}) \cdot |Z - f_\mu^{-1}(c)|.$$

Since $h(X,Z,c) \geq 0$, we get $E[f_\mu(Z)Z] \geq E[XZ]$, and assertion 1) follows for μ.

Let next $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of μ-distributed random variables such that $E[X_nZ]$ converges to $\alpha(\mu)$. Then $\int_0^\infty E[h(X_n,Z,c)]dc$ converges to 0, and, since $h(X_n,Z,c) \geq 0$, we conclude that $h(X_n(\omega), Z(\omega), c)$ converges to 0 in $P\otimes\lambda$-measure on the measure space $(\Omega\times\mathbb{R}, \mathcal{A}\otimes\mathcal{B}_\mathbb{R}, P\otimes\lambda)$, where λ is the Lebesgue measure and $\mathcal{B}_\mathbb{R}$ is the Borelean tribe on $\mathbb{R}$.

As a consequence, there exists a subsequence $\{X'_n\}_{n\in\mathbb{N}}$ of $\{X_n\}_{n\in\mathbb{N}}$, such that that $h(X'_n(\omega), Z(\omega), c)$ converges $P\otimes\lambda$-a.e. to 0. Next

$$h(X'_n,Z,c) = l(X'_n,Z,c)\cdot|Z - f_\mu^{-1}(c)|$$

with $l(X'_n,Z,c) := (1\!\!1_{\{X'_n<c\leq f_\mu(Z)\}} + 1\!\!1_{\{f_\mu(Z)<c\leq X'_n\}})$, so that

$$l(X'_n(\omega), Z(\omega), c).1\!\!1_{\{Z(\omega)\neq f_\mu^{-1}(c)\}}$$

converges $P\otimes\lambda$-a.e. to 0.

Since $\forall c$: $P(Z(\omega) = f_\mu^{-1}(c)) = 0$, $l(X'_n(\omega), Z(\omega), c)$ converges also $P\otimes\lambda$-a.e. to 0, and since l is bounded by 2, we conclude with Lebesgue dominated convergence theorem that for all $K < \infty$: $\lim_{n\to\infty}\int_0^K E[l(X'_n,Z,c)]dc = 0$. Now, observe that

$$\begin{aligned} \textstyle\int_0^K E[l(X'_n,Z,c)]dc &= E[\textstyle\int_0^K(1\!\!1_{\{X'_n<c\leq f_\mu(Z)\}} + 1\!\!1_{\{f_\mu(Z)<c\leq X'_n\}})dc] \\ &= E[|X'_n\wedge K - f_\mu(Z)\wedge K|], \end{aligned}$$

where $a\wedge b$ is the minimum of the two numbers a and b. Now

$$\|X'_n - f_\mu(Z)\|_{L^1} \leq \|X'_n - X'_n\wedge K\|_{L^1} + \|X'_n\wedge K - f_\mu(Z)\wedge K\|_{L^1} + \|f_\mu(Z)\wedge K - f_\mu(Z)\|_{L^1}.$$

Since X'_n and $f_\mu(Z)$ are μ-distributed, the first and the third terms are equal and are just a function $g(K)$. So, $\forall K$, $\limsup_{n\to\infty}\|X'_n - f_\mu(Z)\|_{L^1} \leq 2g(K)$. Next, since $X'_n \in L^1$, we get $\lim_{K\to\infty} g(K) = 0$, and we conclude therefore that X'_n converges to $f_\mu(Z)$ in L^1.

We thus have proved that any maximizing sequence $\{X_n\}_{n\in\mathbb{N}}$ (i.e. such that $E[X_n Z] \to \alpha(\mu)$) contains a sub sequence $\{X'_n\}_{n\in\mathbb{N}}$ that converges in L^1 to $f_\mu(Z)$. This implies clearly that any maximizing sequence converges to $f_\mu(Z)$, in L^1: Claim 2) of the theorem is proved for μ.

The previous proof can be generalized to a two sided measures μ, working separately on the positive and negative part of the random variables: Since $E[(f_\mu(Z) - X)Z] = E[(f_\mu^+(Z) - X^+)Z] - E[(f_\mu^-(Z) - X^-)Z]$, the function h to be considered here contains now two non negative terms corresponding respectively to the positive and negative parts of X and $F_\mu(Z)$. ∎

3.2. **The main result for martingales of maximal variation.** L_0^2 will denote hereafter the set of random variables $X \in L^2$ such that $E[X] = 0$, and Δ_0^2 be the set of measure μ on $\mathbb{R}$ such that $X \sim \mu$ implies $X \in L_0^2$. For a function $M : \Delta_0^2 \to \mathbb{R}$, and a random variable X in L^2 with $E[X] = 0$, we will write $M[X]$ instead of $M([X])$. $\mathcal{W}_n(\mu)$ was defined in section 2.3 as the set of pairs $(\mathcal{F}, X)$ where $\mathcal{F} := (\mathcal{F}_q)_{q=0,\ldots,n+1}$ is a filtration on a probability space, and $X = (X_q)_{q=0,\ldots,n+1}$ is an $\mathcal{F}$-martingale X whose $n+1$-th value X_{n+1} is μ-distibuted. Observe that if $\mu \in \Delta^2$ and $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, then $[X_{q+1} - X_q|\mathcal{F}_q] \in \Delta_0^2$, and we may therefore define the M-variation $\mathcal{V}_n^M(\mathcal{F}, X)$ as

$$\mathcal{V}_n^M(\mathcal{F}, X) := E[\sum_{q=0}^{n-1} M[X_{q+1} - X_q|\mathcal{F}_q]]. \tag{6}$$

Since we only will deal in this paper with Lipschitz M in L^p-norm for $p < 2$, we refer to footnote 1 on page 8 for a proof of the measurability of $M[X_{q+1} - X_q|\mathcal{F}_q]$. Let us next define $\overline{\mathcal{V}}_n^M(\mu)$ as

$$\overline{\mathcal{V}}_n^M(\mu) := \sup\{\mathcal{V}_n^M(\mathcal{F}, X) : (\mathcal{F}, X) \in \mathcal{W}_n(\mu)\} \tag{7}$$

The main result of this part is

Theorem 9. *If M satisfies :*
i) For all random variable $X \in L_0^2$, $\forall \alpha \geq 0 : M[\alpha X] = \alpha M[X]$.
ii) There exist $p \in [1, 2[$ and $A \in \mathbb{R}$ such that for all $X, Y \in L_0^2$:

$$|M[X] - M[Y]| \leq A\|X - Y\|_{L^p}$$

Then for all $\mu \in \Delta^2$, with $\rho := \sup\{M(\mu) : \mu \in \Delta_0^2, \|\mu\|_{L^2} \leq 1\}$ and $\alpha(\mu)$ defined as in theorem 8, we have:

$$\lim_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} = \rho \cdot \alpha(\mu).$$

Furthermore, if $\rho > 0$ and if, for all n, $(\mathcal{F}^n, X^n) \in \mathcal{W}_n(\mu)$ satisfies $\mathcal{V}_n^M(\mathcal{F}^n, X^n) = \overline{\mathcal{V}}_n^M(\mu)$, then the continuous time representation Π^n of X^n defined as $\Pi_t^n := X_{[\![n\cdot t]\!]}^n$ converges in finite dimensional distribution to the continuous martingale of maximal variation Π^μ defined in the introduction.

This theorem justifies our terminology when referring to Π^μ as the continuous martingale of maximal variation corresponding to μ.

With $M[X] := \|X\|_{L^1}$, $\mathcal{V}_n^M(\mathcal{F}, X)$ is just the L^1-variation of the martingale X and we recover here Mertens and Zamir's result on the maximal variation of a bounded martingale [Mertens, Zamir (1976)], taking μ such that $\mu(\{1\}) = s$ $\mu(\{0\}) = 1 - s$. With $M[X] := \|X\|_{L^p}$, with $p \in [1, 2[$ we also recover the results of [De Meyer B. (1998)]. The proof presented here is in fact inspired by this last paper.

The last theorem implies also theorem 1.

Proof of theorem 1: Indeed, due to ii) in lemma 7, we have for all variable X: $V_1[X] = V_1[(X - E[X])]$, and thus also, if $\mathcal{F}$ is a σ-algebra: $V_1[X|\mathcal{F}] = V_1[(X - E[X|\mathcal{F}])|\mathcal{F}]$. Therefore, if $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, the function $\mathcal{V}_n(\mathcal{F}, X)$ defined in (3) is just equal to:

$$\mathcal{V}_n(\mathcal{F}, X) = \mathcal{V}_n^{V_1}(\mathcal{F}, X).$$

Since, iv) in lemma 7 indicates that V_1 satisfies the first condition of theorem 9, H5 indicates that V_1 fullfils the second one and H4 indicates that $\rho > 0$, theorem 1 follows then from theorem 4. As a byproduct, we also get that $\lim_{n\to\infty} \frac{V_n(\mu)}{\sqrt{n}} = \rho \cdot \alpha(\mu)$. ■

The six remaining sections of the paper are devoted to the proof of theorem 9. We first provide an upper bound for M in the next section that leads to an upper bound for $\mathcal{V}_n^M$ in the next one. We will then conclude in section 3.6 that $\rho{\cdot}\alpha(\mu)$ dominates the $\limsup$ of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$, using a central limit theorem for martingales presented in section 3.5.

In section 3.7, we prove that $\rho \cdot \alpha(\mu)$ is the $\liminf$ of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$, and in section 3.8, we prove the convergence of the Π^n to Π^μ.

Notice that the case $\rho = 0$ is trivial in theorem 9, since then $M[\mu] \leq 0$ for all μ in Δ_0^2, and thus the constant martingale $X_q = E[X_{n+1}]$, for all $q \leq n$ and $X_{n+1} \sim \mu$ will be optimal in the maximization problem (7), and so $\overline{\mathcal{V}}_n^M(\mu) = 0$. In the sequel, we therefore assume $\rho > 0$.

As a remark, observe that the hypothesis $p < 2$ in ii) of theorem 9 could not be weakened in $p \leq 2$. A counterexample of this is given at the end of section 3.6

3.3. **An upper bound for** M**.** On a probability space $(\Omega, \mathcal{A}, P)$, for $q \geq 1$ and $r > 0$, let $B_r^q(\Omega, \mathcal{A}, P)$ be the set $B_r^q(\Omega, \mathcal{A}, P) := \{X \in L^2(\Omega, \mathcal{A}, P) | \|X\|_{L^q} \leq r\}$. Let next $\mathcal{B}^*(\Omega, \mathcal{A}, P)$ be defined as:

$$\mathcal{B}^*(\Omega, \mathcal{A}, P) := B_\rho^2(\Omega, \mathcal{A}, P) \cap B_{2A}^{p'}(\Omega, \mathcal{A}, P),$$

with A, ρ and p as in theorem 9 and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let us finally define, for $X \in L^2(\Omega, \mathcal{A}, P)$:

$$B(X) := \sup\{E[XY] : Y \in \mathcal{B}^*(\Omega, \mathcal{A}, P)\} \tag{8}$$

Observing that if $Y \in \mathcal{B}^*(\Omega, \mathcal{A}, P)$ then $E[Y|X] \in \mathcal{B}^*(\Omega, \mathcal{A}, P)$, we infer that $B(X) = \sup\{E[Xf(X)] : f(X) \in \mathcal{B}^*(\Omega, \mathcal{A}, P)\}$, and therefore $B(X)$ just depends on the distribution $[X]$. In other words, if $[X] = [X']$, then $B(X) = B(X')$, even if X and X' are defined on different probability spaces. We will therefore abuse the notations and write $B[X]$ instead of $B(X)$.

Lemma 10.

1) For all $X \in L_0^2(\Omega, \mathcal{A}, P) : M[X] \leq \rho \cdot \|X\|_{L^2}$.

2) If $\mathcal{B}(\Omega, \mathcal{A}, P) := \{X \in L^2(\Omega, \mathcal{A}, P) : B[X] \leq 1\}$*, then*

$$\mathcal{B}(\Omega, \mathcal{A}, P) \subset conv(B_{\frac{1}{\rho}}^2(\Omega, \mathcal{A}, P) \cup B_{\frac{1}{2A}}^p(\Omega, \mathcal{A}, P)).$$

3) For all $X \in L_0^2(\Omega, \mathcal{A}, P) : B[X] \geq M[X]$.

Proof: Claim 1) is an obvious consequence of the definition of ρ as $\sup\{M[X] : X \in L_0^2, \|X\|_{L^2} \leq 1\}$ and of the 1-homogeneity of M.

We next prove claim 2): Let $\mathcal{C}$ denote $conv(B^2_{\frac{1}{\rho}}(\Omega,\mathcal{A},P)\cup B^p_{\frac{1}{2A}}(\Omega,\mathcal{A},P))$. Since $B^2_{\frac{1}{\rho}}(\Omega,\mathcal{A},P)$ and $B^p_{\frac{1}{2A}}(\Omega,\mathcal{A},P)$ are closed sets in L^2-norm ($p<2$), so is $\mathcal{C}$. Therefore, if $Z\in L^2_0(\Omega,\mathcal{A},P)$ does not belong to $\mathcal{C}$, we can separate $\{Z\}$ and $\mathcal{C}$ in $L^2(\Omega,\mathcal{A},P)$ by a separating vector Y: $E[YZ]>\alpha:=\sup\{E[YX]:X\in\mathcal{C}\}$. In particular $\alpha\geq\sup\{E[YX]:X\in B^2_{\frac{1}{\rho}}\}=\frac{1}{\rho}\cdot\|Y\|_{L^2}$, and $\alpha\geq\sup\{E[YX]:X\in B^p_{\frac{1}{2A}}\}=\frac{1}{2A}\cdot\|Y\|_{L^{p'}}$. This indicates that $Y':=\frac{Y}{\alpha}\in\mathcal{B}^*(\Omega,\mathcal{A},P)$. Therefore $B[Z]\geq E[Y'Z]>1$ and so $Z\notin\mathcal{B}(\Omega,\mathcal{A},P)$. So the complementary of $\mathcal{C}$ is included in the complementary of $\mathcal{B}(\Omega,\mathcal{A},P)$, or equivalently: $\mathcal{B}(\Omega,\mathcal{A},P)\subset\mathcal{C}$.

To prove claim 3) observe that both M and B are 1-homogeneous on $L^2_0(\Omega,\mathcal{A},P)$. Therefore, we just have to prove that, for all $X\in L^2_0(\Omega,\mathcal{A},P), M[X]\leq 1$ whenever $B[X]\leq 1$. But if $B[X]\leq 1$, then $X\in\mathcal{B}(\Omega,\mathcal{A},P)$. So, by the previous claim $X\in\mathcal{C}$. Since $\mathcal{C}$ is the convex hull of two convex sets, we get that $X=\lambda Y+\lambda' Y'$, with $\lambda,\lambda'\geq 0$, $\lambda+\lambda'=1$, $Y\in B^2_{\frac{1}{\rho}}$ and $Y'\in B^p_{\frac{1}{2A}}$. Since $E[X]=0$, we also have $X=\lambda(Y-E[Y])+\lambda'(Y'-E[Y'])$. Due to property ii) in theorem 9, we get:

$$M[X]\leq M[\lambda(Y-E[Y])]+A\|\lambda'(Y'-E[Y'])\|_{L^p}.$$

Since $\|(Y-E[Y])\|_{L^2}\leq\|Y\|_{L^2}\leq\frac{1}{\rho}$, it follows from claim 1) that the first term is bounded by λ. The second one is bounded by λ' since $\|Y'-E[Y']\|_{L^p}\leq\|Y'\|_{L^p}+\|E[Y']\|_{L^p}\leq 2\|Y'\|_{L^p}\leq\frac{1}{A}$. Thus $M[X]\leq 1$ and the lemma is proved. ■

3.4. **An upper bound for $\overline{\mathcal{V}}^M_n(\mu)$.** For $(\mathcal{F},X)\in\mathcal{W}_n(\mu)$, $\mathcal{V}^M_n(\mathcal{F},X)$ was defined in (6). The term $M[X_{q+1}-X_q|\mathcal{F}_q]$ involved there is then dominated by $B[X_{q+1}-X_q|\mathcal{F}_q]$, and we will therefore concentrate our attention on $\mathcal{V}^B_n(\mathcal{F},X)$.

Next lemma presents $E[B[X_{q+1}-X_q|\mathcal{F}_q]]$ as the result of an optimization problem.

Let $\mathcal{F}_1\subset\mathcal{F}_2$ be two σ-algebras on a probability space $(\Omega,\mathcal{A},P)$. Let $L^2_0(\mathcal{F}_2|\mathcal{F}_1)$ be the set of $X\in L^2(\mathcal{F}_2)$ such that $E[X|\mathcal{F}_1]=0$. Let $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ denote the set of $Y\in L^2(\mathcal{F}_2)$ such that $E[Y^2|\mathcal{F}_1]\leq\rho^2$ and $E[|Y|^{p'}|\mathcal{F}_1]\leq(2A)^{p'}$. Let $\mathcal{B}^*_{(\rho,C)}(\mathcal{F}_2|\mathcal{F}_1)$ denote the set of $Y\in L^2(\mathcal{F}_2)$ such that

1) $E[Y|\mathcal{F}_1]=0$
2) $E[Y^2|\mathcal{F}_1]=\rho^2$
3) $E[|Y|^{p'}|\mathcal{F}_1]\leq C^{p'}$.

Lemma 11. *1) For all $X\in L^2_0(\mathcal{F}_2)$:*

$$E[B[X|\mathcal{F}_1]]=\sup\{E[XY]|Y\in\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)\}.$$

2) If there exists in $L^2(\Omega,\mathcal{F}_2,P))$ a random variable U that is independent of $\sigma(\mathcal{F}_1,X)$ and taking the values 1 and -1 with probability $1/2$ then

$$E[B[X|\mathcal{F}_1]]\leq\sup\{E[XY]|Y\in\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}_2|\mathcal{F}_1)\}.$$

Proof: If $X\in L^2(\mathcal{F}_2)$ and $Y\in\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ then $E[XY]=E[E[XY|\mathcal{F}_1]]$. Since conditionally to $\mathcal{F}_1$, Y belongs to $\mathcal{B}^*$, we get $E[XY|\mathcal{F}_1]\leq B[X|\mathcal{F}_1]$, and thus $E[B[X|\mathcal{F}_1]]\geq\sup\{E[XY]|Y\in\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)\}$.

To prove the reverse inequality, we just have to prove that, $\forall\epsilon>0$, there exists a measurable map $\phi:\Delta^2\times\mathbb{R}\to\mathbb{R}$ such that, $\forall\mu\in\Delta^2$, if $X\sim\mu$, then $E[\phi(\mu,X)^2]\leq\rho^2$, $E[|\phi(\mu,X)|^{p'}]\leq(2A)^{p'}$ and $B(\mu)-\epsilon\leq E[X\phi(\mu,X)]$. Indeed, the random variable $Y:=\phi([X|\mathcal{F}_1],X)$ belongs then to $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ and $E[XY]\geq E[B[X|\mathcal{F}_1]]-\epsilon$. The existence of such a measurable map follows from the fact that the set $\mathcal{D}$ of measure μ with finite support in $\mathbb{Q}$ and rational weights is a countable dense subset of Δ^2 for the L^2-topology: For all $\epsilon>0$, there exists thus a measurable partition

$\{A_n\}_{n\in\mathbb{N}}$ of Δ^2 and a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $\forall \mu \in \Delta_2$, the measure $\mu' := \sum_n \mathbb{1}_{\mu\in A_n}\xi_n$ satisfies $\exists X, X'$ such that $X \sim \mu$ and $X' \sim \mu'$ with $\|X - X'\|_{L^2} \le \delta := \epsilon/(1+2\rho)$. For all n there also exists $\theta_n : \mathbb{R} \to \mathbb{R}$ such that, if $X_n \sim \xi_n$ then $E[|\theta_n(X_n)|^2] \le \rho^2$, $E[|\theta_n(X_n)|^{p'}] \le (2A)^{p'}$ and $E[X_n\theta_n(X_n)] \ge B(\xi_n) - \delta$. Therefore $E[X\sum_n \mathbb{1}_{\mu\in A_n}\theta_n(X_n)] \ge B[X'] - (1+\rho)\delta \ge B[X] - (1+2\rho)\delta = B[X] - \epsilon$. We set then $\phi(\mu, X) := E_\mu[\sum_n \mathbb{1}_{\mu\in A_n}\theta_n(X_n)|X]$.

We next turn to claim 2): Just observe that if $Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$, then $Y' := E[Y|X]$ also belongs to $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ and has thus the property that $E[XY] = E[XY']$. Now, consider $Y'' := Y' - E[Y'|\mathcal{F}_1]$. Since $X \in L^2_0(\mathcal{F}_2|\mathcal{F}_1)$, we have $E[XE[Y'|\mathcal{F}_1]] = 0$ so $E[XY] = E[XY'']$. Now, observe that

$$\theta^2 := E[(Y'')^2|\mathcal{F}_1] = E[(Y')^2|\mathcal{F}_1] - (E[Y'|\mathcal{F}_1])^2 \le E[Y^2|\mathcal{F}_1] \le \rho^2.$$

Finally, let Y''' be defined as $Y''' := Y'' + \sqrt{\rho^2 - \theta^2}U$, since U is independent of $\sigma(\mathcal{F}_1, X)$, and since Y'' is measurable on this σ-algebra, we get obviously

$$E[XY] = E[XY'''],\ E[Y'''|\mathcal{F}_1] = 0 \text{ and } E[(Y''')^2|\mathcal{F}_1] = \rho^2.$$

Observing that $(E[(Y''')^{p'}|\mathcal{F}_1])^{\frac{1}{p'}}$ is just a conditional $L^{p'}$-norm, we get

$$\begin{array}{rcl} (E[|Y'''|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} & \le & (E[|Y''|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} + \rho \\ & \le & (E[|Y'|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} + (E[|E[Y'|\mathcal{F}_1]|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} + \rho \\ & \le & 2(E[|Y'|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} + \rho \\ & \le & 2(E[|Y|^{p'}|\mathcal{F}_1])^{\frac{1}{p'}} + \rho \\ & \le & 4A + \rho. \end{array}$$

Therefore, for all $Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$, there is a $Y''' \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}_2|\mathcal{F}_1)$ such that $E[XY] = E[XY''']$, and claim 2) then follows from claim 1). ■

Let us now use this lemma to compute $\mathcal{V}^M_n(\mathcal{F}, X)$ for a pair $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$ defined on a probability space $(\Omega, \mathcal{A}, P)$. Let us first enlarge this space obtaining a new one $(\Omega', \mathcal{A}', P')$ where $\mathcal{A}$ may be seen as a sub σ-algebra of $\mathcal{A}'$, P' and P coincide on $\mathcal{A}$ and where there is a system of n independent random variables $(U_q)_{q=1,\cdots,n}$, independent of $\mathcal{A}$, with $P'(U_q = 1) = P'(U_q = -1) = 1/2$. Consider next the filtration $\mathcal{F}'$ defined by $\mathcal{F}'_q := \sigma(\mathcal{F}_q, U_k, k \le q)$. X is then also a martingale on $\mathcal{F}'$ and $[X_{q+1} - X_q|\mathcal{F}_q] = [X_{q+1} - X_q|\mathcal{F}'_q]$. Therefore

$$\mathcal{V}^M_n(\mathcal{F}, X) \le \mathcal{V}^B_n(\mathcal{F}, X) = \mathcal{V}^B_n(\mathcal{F}', X)$$

We will denote $\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')$ the set of $\mathcal{F}'$- adapted processes Y such that for all $q = 1, \ldots, n$: $Y_q \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}'_q|\mathcal{F}'_{q-1})$. Then, since $X_q - X_{q-1} \in L^2_0(\mathcal{F}'_q|\mathcal{F}'_{q-1})$, we may apply claim 2) of last lemma to get

$$\begin{array}{rcl} \mathcal{V}^B_n(\mathcal{F}', X) & = & \sum_{q=1}^n E[B[X_q - X_{q-1}|\mathcal{F}'_{q-1}]] \\ & \le & \sup_{Y\in\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')} \sum_{q=1}^n E[(X_q - X_{q-1}) \cdot Y_q] \end{array}$$

Since X is an $\mathcal{F}'$-martingale and $E[Y_q|\mathcal{F}'_{q-1}] = 0$, we get

$$E[(X_q - X_{q-1}) \cdot Y_q] = E[X_q \cdot Y_q] = E[E[X_{n+1}|\mathcal{F}'_q] \cdot Y_q] = E[X_{n+1} \cdot Y_q],$$

and therefore

$$\mathcal{V}^M_n(\mathcal{F}', X) \le \sup_{Y\in\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')} E[X_{n+1} \cdot \sum_{q=1}^n Y_q] \tag{9}$$

For a given $Y \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, let S_q be defined as $S_0 := 0$, $S_q := S_{q-1} + Y_q$. Observe then that S is an $\mathcal{F}'$-martingale. We will denote $\mathcal{S}_{(\rho,4A+\rho)}(\mathcal{F}')$ the set of

$\mathcal{F}'$-martingale S whose increments $S_{q+1} - S_q$ belong to $\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}'_q|\mathcal{F}'_{q-1})$, for all q, and such that $S_0 = 0$. The last formula becomes then

$$\mathcal{V}^M_n(\mathcal{F}', X) \leq \sup_{S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')} E[X_{n+1} \cdot S_n]. \tag{10}$$

Let us make two comments on the last formula: the quantity $\mathcal{V}^M_n(\mathcal{F}', X)$ depends on the laws $[X_{q+1} - X_q|\mathcal{F}_q]$ which are intimately related to the filtration $\mathcal{F}$. The bound we found in the last formula just depends on the laws $[X_{q+1} - X_q|X_1, \ldots, X_q]$. Therefore, if we create a martingale $\tilde{X}$ on another filtration $\mathcal{G}$ with the same law as X — we call this procedure the embedding of X in the filtration $\mathcal{G}$—, the right hand side of last inequality can equivalently be evaluated on $\tilde{X}$.

The second comment is that we will have to deal with $\frac{\mathcal{V}^M_n(\mathcal{F}',X)}{\sqrt{n}}$, and will then have to evaluate $E[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}]$, for $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$. Since the increments of S have a conditional variance equal to ρ^2, $\frac{S_n}{\sqrt{n}}$ will be approximatively normally distributed, due to a central limit theorem for martingales. We need however precise bounds for this approximation. These bounds are provided in the next section which is in fact the crux point of the argument. We embed there both martingales $\frac{S}{\sqrt{n}}$ and X in a Brownian filtration.

3.5. **The embedding in the Brownian filtration.** Let B be a Brownian motion on a probability space $(\Omega_0, \mathcal{A}_0, P_0)$ and let $\mathcal{G}$ be the natural filtration of B. Skorokhod rose the following question: Given a probability distribution $\mu \in \Delta^{p'}$, is there a $\mathcal{G}$-stopping time θ such that $B_\theta \sim \mu$? To avoid trivial uninteresting solutions to this problem, one further require that $E[\theta^{\frac{p'}{2}}] < \infty$. It is well known that Skorokhod's problem has a solution for all $\mu \in \Delta^{p'}_0$ (see for instance [Azéma, Yor (1979)]) and we will denote θ_μ one of these solutions.

We also will need the following fact: For all $p' > 1$, there exist two non negative constants $c_{p'}$ and $C_{p'}$, called the Burkholder Davis Gundy constants (see [Burkholder (1973)]), such that, for all $\mathcal{G}$-stopping times $\tau \geq \tau'$:

$$E[\tau^{\frac{p'}{2}}] < \infty \implies c_{p'} \cdot E[(\tau - \tau')^{\frac{p'}{2}} |\mathcal{G}_{\tau'}] \leq E[|B_\tau - B_{\tau'}|^{p'} |\mathcal{G}_{\tau'}] \leq C_{p'} \cdot E[(\tau - \tau')^{\frac{p'}{2}} |\mathcal{G}_{\tau'}].$$

In the particular case $p' = 2$, we have $c_2 = C_2 = 1$.

Lemma 12. *Let $R = (R_q)_{q:=0,\ldots,n}$ be a martingale with $R_n \in L^{p'}_0$, then there is an increasing sequence of stopping times $\{\tau_q\}_{q:=0,\ldots,n}$ such that $E[\tau_n^{\frac{p'}{2}}] < \infty$ and such that both processes R and $\hat{R}$ have the same distribution where $\hat{R}_q := B_{\tau_q}$.*

Proof: Just take $\tau_0 := \theta_{[R_0]}$ so that $[\hat{R}_0] = [B_{\tau_0}] = [R_0]$. Once τ_q is defined, define τ_{q+1} as follows: $B'_t := B_{\tau_q+t} - B_{\tau_q}$ is another Brownian motion on its natural filtration $\mathcal{G}'$. For all $(r_0, \ldots, r_q) \in \mathbb{R}^{q+1}$ define $\tilde{\theta}(r_0, \ldots, r_q) := \theta'_{[R_{q+1}-R_q|R_0=r_0;\ldots;R_q=r_q]}$, where θ'_μ is a solution of μ-Skorokhod's problem for the Brownian motion B'. This mapping can be chosen measurable from $\mathbb{R}^{q+1}$ to $(\Omega_0, \mathcal{A}_0)$, and define $\tau_{q+1} := \tau_q + \tilde{\theta}(\hat{R}_0, \ldots, \hat{R}_q)$. Then $\hat{R}_{q+1} - \hat{R}_q = B_{\tau_{q+1}} - B_{\tau_q} = B'_{\tilde{\theta}(\hat{R}_0,\ldots,\hat{R}_q)}$. Therefore $[\hat{R}_{q+1} - \hat{R}_q|\hat{R}_0, \ldots, \hat{R}_q] = [R_{q+1} - R_q|R_0, \ldots, R_q]$. We then conclude that R and $\hat{R}$ have the same laws. Next since $c_p \cdot E[\tilde{\theta}(\hat{R}_0, \ldots, \hat{R}_q)^{\frac{p'}{2}} |\mathcal{G}_{\tau_q}] \leq E[(\hat{R}_{q+1} - \hat{R}_q)^{p'} |\mathcal{G}_{\tau_q}]$, we conclude by induction that $E[\tau_n^{\frac{p'}{2}}] < \infty$. ■

We are now ready to start the embedding procedure. Let us consider $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$ and $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, as in the last section.

Let R denote $R := \frac{S}{\rho\sqrt{n}}$. To embed both R and X, we will have to slightly perturb the above procedure: For $\epsilon > 0$ we define $\tau_0 = \tau_{\frac{1}{2}} := \epsilon$ and $\hat{R}_0 := 0$. For $q = 0, \dots n-1$, we then define τ_{q+1}, $\tau_{q+\frac{3}{2}}$ and $\hat{R}_{q+1}$ recursively as follows: Let $\mathcal{G}'$ be the natural σ-algebra of $B'_t := B_{t+\tau_{q+\frac{1}{2}}} - B_{\tau_{q+\frac{1}{2}}}$. Define as above $\tilde{\theta}(r_0, \dots, r_q) := \theta'_{[R_{q+1}-R_q|R_0=r_0;\dots;R_q=r_q]}$ and then set

$$\tau_{q+1} := \tau_{q+\frac{1}{2}} + \tilde{\theta}(\hat{R}_0, \dots, \hat{R}_q),\ \tau_{q+\frac{3}{2}} := \tau_{q+1} + \epsilon,\ \hat{R}_{q+1} := \hat{R}_q + B_{\tau_{q+1}} - B_{\tau_{q+\frac{1}{2}}}.$$

It is convenient to define also τ_{n+1} as $\tau_{n+1} = \tau_{n+\frac{1}{2}} := \tau_n + \epsilon$.

The process $\hat{R}$ has the same distribution as R, is a $\mathcal{G}_{\tau_q}$-martingale and the distribution of $\hat{R}_n - B_{\tau_n}$ is clearly $\mathcal{N}(0, n\epsilon)$, in particular

$$\|\hat{R}_n - B_{\tau_n}\|_{L^2} = \sqrt{\epsilon \cdot n}. \tag{11}$$

For $\epsilon = 0$, we have just the embedding of the lemma. The advantage of introducing $\epsilon > 0$ is that this will allow to embed X also: Let $Z \sim \mathcal{N}(0, \epsilon)$. There exists a measurable function $f_q : \mathbb{R}^{2q+2} \to \mathbb{R}$ such that $\forall (r_0, \dots, r_q, x_0, \dots, x_{q-1}) \in \mathbb{R}^{2q+1}$:

$$[f_q(r_0, \dots, r_q, x_0, \dots, x_{q-1}, Z)] = [X_q | R_0 = r_0, \dots, R_q = r_q, X_0 = x_0, \dots, X_{q-1} = x_{q-1}]$$

Define then $\hat{X}_q := f_q(\hat{R}_0, \dots, \hat{R}_q, \hat{X}_0, \dots, \hat{X}_{q-1}, B_{\tau_{q-1}+\epsilon} - B_{\tau_{q-1}})$. Clearly (R, S) and $(\hat{R}, \hat{X})$ have the same distribution and $(\hat{R}, \hat{X})$ is a $\mathcal{G}_{\tau_q}$-martingale.

In order to obtain our central limit result for $\hat{R}_n$, we will prove hereafter that τ_n is close to be a constant stopping time, which indicates that B_{τ_n} follows approximately a normal distribution.

Lemma 13.

1) For all $t \in [0,1]$: $E[\tau_{[\![nt]\!]}] = ([\![nt]\!] + 1)\epsilon + [\![nt]\!] \cdot \frac{1}{n}$, where $[\![a]\!]$ is the greatest integer less or equal to a.

2) $E[|\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}]|] \le \kappa^2 \cdot n^{\frac{2}{p' \wedge 4} - 1}$, where $\kappa := 2^{\frac{2}{p' \wedge 4}} \frac{4A+\rho}{(c_{p'})^{\frac{1}{p'}} \rho}$

3) $\|B_t - B_{\tau_{[\![nt]\!]}}\|_{L^2} \le \kappa \cdot n^{\frac{1}{p' \wedge 4} - \frac{1}{2}} + \sqrt{t - \frac{[\![nt]\!]}{n} + \epsilon([\![nt]\!] + 1)}$.

Proof: To prove claim 1), observe that $\tau_{[\![nt]\!]} = \tau_0 + \sum_{q=1}^{[\![nt]\!]} (\tau_q - \tau_{q-1})$. Then, since $S \in S^*_{(\rho, 4A+\rho)}(\mathcal{F}')$, we get $\tau_0 = \epsilon$, and

$$\begin{aligned} E[\tau_q - \tau_{q-1} | \mathcal{G}_{\tau_{q-1}}] &= \epsilon + E[\tau_q - \tau_{q-\frac{1}{2}} | \mathcal{G}_{\tau_{q-1}}] \\ &= \epsilon + E[(B_{\tau_q} - B_{\tau_{q-\frac{1}{2}}})^2 | \mathcal{G}_{\tau_{q-1}}] \\ &= \epsilon + E[(\hat{R}_q - \hat{R}_{q-1})^2 | \hat{R}_k, \hat{X}_k, k \le q-1] \\ &= \epsilon + \frac{1}{\rho^2 n} E[(S_q - S_{q-1})^2 | S_k, X_k, k \le q-1] \\ &= \epsilon + \frac{1}{n} \end{aligned}$$

Therefore, $E[\tau_{[\![nt]\!]}] = ([\![nt]\!] + 1)\epsilon + [\![nt]\!] \cdot \frac{1}{n}$, as announced.

We next prove claim 2). Since $E[\tau_q - \tau_{q-1} | \mathcal{G}_{\tau_{q-1}}] = E[\tau_q - \tau_{q-1}]$, we get

$$\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}] = \sum_{q=1}^{[\![nt]\!]} ((\tau_q - \tau_{q-1}) - E[\tau_q - \tau_{q-1}]) = Q_{[\![nt]\!]},$$

where

$$Q_s := \sum_{q=1}^{s} ((\tau_q - \tau_{q-1}) - E[\tau_q - \tau_{q-1} | \mathcal{G}_{\tau_{q-1}}]) = \sum_{q=1}^{s} (\tau_q - \tau_{q-\frac{1}{2}} - \frac{1}{n}).$$

The process $Q = (Q_s)_{s=0,\dots n}$ is clearly a $\mathcal{G}_{\tau_s}$-martingale starting at 0. Since $p \in [1,2[$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we get $p' > 2$, and so, $\tilde{p} := \frac{\min(p',4)}{2} \in]1,2]$. Therefore

$$\|\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}]\|_{L^1} = \|Q_{[\![nt]\!]}\|_{L^1} \leq \|Q_n\|_{L^1} \leq \|Q_n\|_{L^{\tilde{p}}}. \tag{12}$$

We claim next that

$$E[|Q_n|^{\tilde{p}}] \leq 2^{2-\tilde{p}} \sum_{k=0}^{n-1} E[|Q_{k+1} - Q_k|^{\tilde{p}}]. \tag{13}$$

This follows at once from a recursive use of the relation:

$$E[|x+Y|^{\tilde{p}}] \leq |x|^{\tilde{p}} + 2^{2-\tilde{p}} E[|Y|^{\tilde{p}}],$$

that holds for all x in $\mathbb{R}$, whenever Y is a centered random variable: Indeed,

$$|x+Y|^{\tilde{p}} - |x|^{\tilde{p}} = Y \int_0^1 \tilde{p}|x+sY|^{\tilde{p}-1}\mathrm{sgn}(x+sY)ds$$

Thus, since $E[Y] = 0$, we get

$$E[|x+Y|^{\tilde{p}}] - |x|^{\tilde{p}} = E\left[Y \int_0^1 \tilde{p}\left(|x+sY|^{\tilde{p}-1}\mathrm{sgn}(x+sY) - |x|^{\tilde{p}-1}\mathrm{sgn}(x)\right)ds\right].$$

Since $\tilde{p} \leq 2$, straightforward computation indicates that, for fixed a, the function $g(x) := ||x+a|^{\tilde{p}-1}\mathrm{sgn}(x+a) - |x|^{\tilde{p}-1}\mathrm{sgn}(x)|$ reaches its maximum at $x = -a/2$, implying $g(x) \leq 2^{2-\tilde{p}}|a|^{\tilde{p}-1}$.

So, $E[|x+Y|^{\tilde{p}}] - |x|^{\tilde{p}} \leq E\left[|Y| \int_0^1 2^{2-\tilde{p}} p|sY|^{\tilde{p}-1}ds\right] = 2^{2-\tilde{p}}E[|Y|^{\tilde{p}}]$, as announced and inequality (13) follows.

Next $\|Q_{k+1} - Q_k\|_{L^{\tilde{p}}} = \|\tau_{k+1} - \tau_{k+\frac{1}{2}} - \frac{1}{n}\|_{L^{\tilde{p}}} \leq \|\tau_{k+1} - \tau_{k+\frac{1}{2}}\|_{L^{\tilde{p}}} + \frac{1}{n}$. Since $\frac{1}{n} = E[\tau_{k+1} - \tau_{k+\frac{1}{2}}]$, we also have $\frac{1}{n} \leq \|\tau_{k+1} - \tau_{k+\frac{1}{2}}\|_{L^{\tilde{p}}}$, and thus

$$\|Q_{k+1} - Q_k\|_{L^{\tilde{p}}} \leq 2\|\tau_{k+1} - \tau_{k+\frac{1}{2}}\|_{L^{\tilde{p}}} \leq 2\|\tau_{k+1} - \tau_{k+\frac{1}{2}}\|_{L^{\frac{p'}{2}}}.$$

Finally, $\hat{R}_{k+1} - \hat{R}_k = B_{\tau_{k+1}} - B_{\tau_{k+\frac{1}{2}}}$. Therefore, we get with Burkholder Davis Gundy inequality, and since $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$:

$$E[(\tau_{k+1} - \tau_{k+\frac{1}{2}})^{\frac{p'}{2}}] \leq \frac{1}{c_{p'}} E[|\hat{R}_{k+1} - \hat{R}_k|^{p'}] = \frac{1}{c_{p'}\rho^{p'}n^{\frac{p'}{2}}} E[|S_{k+1} - S_k|^{p'}] \leq \frac{(4A+\rho)^{p'}}{c_{p'}\rho^{p'}n^{\frac{p'}{2}}}$$

So: $E[|Q_{k+1} - Q_k|^{\tilde{p}}] \leq 2^{\tilde{p}}(4A+\rho)^{2\tilde{p}}(c_{p'})^{-\frac{2\tilde{p}}{p'}}\rho^{-2\tilde{p}}n^{-\tilde{p}}$, and, with (13), we conclude

$$E[|Q_n|^{\tilde{p}}] \leq 2^2(4A+\rho)^{2\tilde{p}}(c_{p'})^{-\frac{2\tilde{p}}{p'}}\rho^{-2\tilde{p}}n^{1-\tilde{p}}.$$

Therefore, with (12), we get:

$$\|\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}]\|_{L^1} \leq 2^{\frac{2}{\tilde{p}}} \left(\frac{4A+\rho}{(c_{p'})^{\frac{1}{p'}}\rho}\right)^2 n^{\frac{1}{\tilde{p}}-1}$$

and claim 2) is proved.

We next prove claim 3): let $\overline{\tau}$ denote $E[\tau_{[\![nt]\!]}]$. Then

$$\|B_{\tau_{[\![nt]\!]}} - B_t\|_{L^2} \leq \|B_{\tau_{[\![nt]\!]}} - B_{\overline{\tau}}\|_{L^2} + \|B_{\overline{\tau}} - B_t\|_{L^2} = \sqrt{\|\tau_{[\![nt]\!]} - \overline{\tau}\|_{L^1}} + \sqrt{|\overline{\tau} - t|}.$$

The first term is bounded by claim 2), and the second one by claim 1) . ∎

3.6. **An upper bound for** $\limsup \overline{\mathcal{V}}_n^M(\mu)/\sqrt{n}$**.** Let us consider $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$. According to (10), we have:

$$\frac{\mathcal{V}_n^M(\mathcal{F}', X)}{\sqrt{n}} \leq \sup_{S \in \mathcal{S}^*_{(\rho, 4A+\rho)}(\mathcal{F}')} E[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}].$$

For $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, let us define $R_n := \frac{S_n}{\rho\sqrt{n}}$ and let us embed (X, R) in the Brownian filtration, as done in the last section, for $\epsilon > 0$.

Then $E[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}] = \rho \cdot E[X_{n+1} \cdot R_n] = \rho \cdot E[\hat{X}_{n+1} \cdot \hat{R}_n]$.

Claim 3) for $t = 1$ in lemma 13 yields $\|B_1 - B_{\tau_n}\|_{L^2} \leq \kappa \cdot n^{\frac{1}{p' \wedge 4} - \frac{1}{2}} + \sqrt{\epsilon(n+1)}$. With (11), we get then $\|\hat{R}_n - B_1\|_{L^2} \leq \kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}} + \sqrt{\epsilon(n+1)} + \sqrt{\epsilon \cdot n}$.

Therefore, since $\hat{X}_{n+1} \sim \mu$ and $B_1 \sim \mathcal{N}(0,1)$,

$$\begin{aligned} E[\hat{X}_{n+1} \cdot \hat{R}_n] &\leq E[\hat{X}_{n+1} \cdot B_1] + E[\hat{X}_{n+1} \cdot (\hat{R}_n - B_1)] \\ &\leq E[\hat{X}_{n+1} \cdot B_1] + \|\hat{X}_{n+1}\|_{L^2} \cdot \|\hat{R}_n - B_1\|_{L^2} \\ &\leq \alpha(\mu) + \|\mu\|_{L^2} \cdot (\kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}} + \sqrt{\epsilon(n+1)} + \sqrt{\epsilon \cdot n}), \end{aligned}$$

where $\alpha(\mu)$ was defined in theorem 8. Since this holds for all $\epsilon > 0$ and all $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, we conclude that for all $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$:

$$\frac{\mathcal{V}_n^M(\mathcal{F}', X)}{\sqrt{n}} \leq \rho \cdot \alpha(\mu) + \rho \cdot \|\mu\|_{L^2} \cdot \kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}}.$$

Since $p' > 2$, and since the constant κ in lemma 13 is independent of n, we thus have proved:

Theorem 14. *Under the hypotheses of theorem 9,*

$$\limsup_{n \to \infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \leq \rho \cdot \alpha(\mu).$$

We will prove in the next section that $\rho \cdot \alpha(\mu)$ is the limit of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$ as n increases to ∞. To conclude this section, we give here an example to illustrate that p must be strictly less than 2 in hypothesis ii) of theorem 9 in order to get the result: Clearly, the function $M[\mu] := \|\mu\|_{L^2}$ satisfies hypothesis i) of theorem 9, and would also satisfy hypothesis ii) with $A = 1$ if $p = 2$ was allowed. For this M, $\rho = 1$. Let then μ be the probability that assigns a weight $1/2$ to $+1$ and -1. Let X^n be the unique martingale of length $n+1$ such that $\forall q = 0, \ldots, n$, $|X^n_q| = \sqrt{\frac{q}{n}}$ and such that $X^n_{n+1} := X^n_n$. In other words, if, for $q < n$, $X^n_q = \sqrt{\frac{q}{n}}$, then X^n_{q+1} jumps to $\sqrt{\frac{q+1}{n}}$ with probability π or $-\sqrt{\frac{q+1}{n}}$ with probability $1 - \pi$, where $\pi := \frac{1}{2}(1 + \sqrt{\frac{q}{q+1}})$, and symmetric jumps are made if $X^n_q = -\sqrt{\frac{q}{n}}$. An easy computation shows that $E[(X^n_{q+1} - X^n_q)^2 | X^n_1, \ldots, X^n_q] = \frac{1}{n}$, and thus, if $\mathcal{F}^n$ denotes the natural filtration of X^n, we get $\mathcal{V}_n^M(\mathcal{F}^n, X^n) = \sqrt{n}$. Since for all pair $(\mathcal{F}', X) \in \mathcal{M}_n(\mu)$, we can write as in (10): $\mathcal{V}_n^M(\mathcal{F}', X) = E[X_{n+1} S_n]$, where $S_n = \sum_{k=1}^n Y_k$, with $E[Y_k^2] = 1$ we get $E[S_n^2] = n$, and due to Cauchy Swartz inequality, it comes $\mathcal{V}_n^M(\mathcal{F}', X) \leq \|X_{n+1}\|_{L^2}\sqrt{n} = \|\mu\|_{L^2}\sqrt{n} = \sqrt{n}$. Therefore:

$$\sqrt{n} \geq \overline{\mathcal{V}}_n^M(\mu) \geq \mathcal{V}_n^M(\mathcal{F}^n, X^n) = \sqrt{n},$$

and thus

$$\lim_{n \to \infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} = 1 > \rho \cdot \alpha(\mu) = \sqrt{\frac{2}{\pi}}.$$

3.7. **A lower bound for** $\liminf \overline{\mathcal{V}}_n^M(\mu)/\sqrt{n}$**.** Let Y be a random variable in L^4 with $E[Y]=0$, $E[Y^2]=1$. We will provide in this section a sequence $(\mathcal{F}^n, X^n)\in\mathcal{M}_n(\mu)$ such that

$$\liminf_{n\to\infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n,X^n)}{\sqrt{n}} \geq M[Y]\cdot\alpha(\mu).$$

Using lemma 12, we can construct, for each n, an increasing sequence $(\tau_q^n)_{q=0,\ldots,n}$ of stopping times on the Brownian filtration $\mathcal{G}$ such that $Y_q^n := \sqrt{n}\cdot(B_{\tau_q^n}-B_{\tau_{q-1}^n})$ is an i.i.d. sequence with $[Y_q^n]=[Y]$. Observe in particular that $\tau_q^n-\tau_{q-1}^n$ is also an i.i.d. sequence. We also set $\tau_{n+1}^n := \tau_n^n\vee 1$.

The argument of lemma 13 can be applied to this sequence of stopping times, replacing ρ by 1, p' by 4, ϵ by 0 and $4A+\rho$ by $\|Y\|_{L^4}$. We obtain in this way:

Lemma 15.
1) For all $t\in[0,1]$*:* $E[\tau^n_{[\![nt]\!]}]=\frac{[\![nt]\!]}{n}$.
2) $E[|\tau^n_{[\![nt]\!]}-E[\tau^n_{[\![nt]\!]}]|]\leq\gamma^2\cdot n^{-\frac{1}{2}}$*, where* $\gamma:=\frac{\|Y\|_{L^4}}{(c_4)^{\frac{1}{4}}}$
3) $\tau^n_{[\![nt]\!]}$ *converges a.s. to* t *an* n *goes to* ∞.
4) $\|B_1-B_{\tau_n^n}\|_{L^2}\leq\gamma\cdot n^{-\frac{1}{4}}$.

Proof: By construction of the sequence τ_q^n, $\theta_q^n:=\tau_q^n-\tau_{q-1}^n$ is an i.i.d. sequence of random variables and $1=E[(Y_q^n)^2]=n\cdot E[(B_{\tau_q^n}-B_{\tau_{q-1}^n})^2]=n\cdot E[\theta_q^n]$. Burkholder Davis Gundy inequality indicates that

$$c_4\cdot var[\theta_q^n]\leq c_4\cdot E[(\theta_q^n)^2]\leq E[(B_{\tau_q^n}-B_{\tau_{q-1}^n})^4]=E[Y^4]/n^2.$$

Therefore, since $\tau^n_{[\![nt]\!]}=\sum_{q=1}^{[\![nt]\!]}\theta_q^n$, we get $E[\tau^n_{[\![nt]\!]}]=[\![nt]\!]/n$ and

$$\|\tau^n_{[\![nt]\!]}-E[\tau^n_{[\![nt]\!]}]\|_{L^2}^2=var(\tau^n_{[\![nt]\!]})\leq\frac{E[Y^4]\cdot[\![nt]\!]}{c_4\cdot n^2}.\leq\frac{E[Y^4]}{c_4\cdot n}.$$

The strong law of large numbers indicates that $\sum_{q=1}^{[\![nt]\!]}(\theta_q^n-E[\theta_q^n])$ converges a.s. to 0, and point 3) follows then from point 1).

We finally conclude $\|B_{\tau_n^n}-B_1\|_{L^2}^2=E[|\tau_n^n-1|]\leq\frac{\|Y\|_{L^4}^2}{\sqrt{c_4\cdot n}}$, and the lemma is proved. ∎

We define next $\mathcal{F}_q^n:=\mathcal{G}_{\tau_q^n}$ and $X_q^n:=E[f_\mu(B_1)|\mathcal{F}_q^n]$, for $q=0,\ldots,n+1$. Due to the definition of f_μ, we have $X_{n+1}^n=f_\mu(B_1)\sim\mu$ and therefore $(\mathcal{F}^n,X^n)\in\mathcal{M}_n(\mu)$.

We will have to compute $\mathcal{V}_n^M(\mathcal{F}^n,X^n)$. To do so, it is convenient to introduce an approximation $\tilde{X}^n$ of X^n. Due to the Markov property of the Brownian motion, $\Pi_t^\mu := E[f_\mu(B_1)|\mathcal{G}_t]=f(B_t,t)$ where $f(x,t):=E[f_\mu(x+\sqrt{1-t}\cdot Z)]$ with $Z\sim\mathcal{N}(0,1)$. As a convolution with a normal density, f is twice continuously differentiable on $\mathbb{R}\times[0,1[$, and it further satisfies the heat equation, so that $\Pi_t^\mu=f(0,0)+\int_0^t r_s dB_s$, with $r_s=0$ for $s\geq1$ and $r_s=\frac{\partial}{\partial x}f(B_s,s)$ for $s<1$.

Let us observe here that $f(x,t)$ is increasing in x at fixed t since so is $f_\mu(x)$, and thus $r_s\geq0$ for all s. Observe also that r_s is continuous on $[0,1[$ and that $X_q^n=f(0,0)+\int_0^{\tau_q^n}r_s dB_s$. We will then define $\tilde{X}^n$ by:

$$\tilde{X}_q^n=f(0,0)+\int_0^{\tau_q^n}r_s^n dB_s, \tag{14}$$

where $r^n := T_n(r)$ is the image of the process r by the map T_n we now define. Let $\mathcal{H}^2$ be the linear space of $\mathcal{G}$-progressively measurable processes a such that

$$\|a\|^2_{\mathcal{H}^2} := E[\int_0^\infty a_s^2 ds] < \infty.$$

Let also $\mathcal{H}^2_{[0,1]}$ denote the set of $a \in \mathcal{H}^2$ such that $a_s = 0$, for all $s \geq 1$. For $a \in \mathcal{H}^2_{[0,1]}$, we define $T_n(a)$ as the simple process

$$T_n(a)_t := \sum_{q=0}^{n-1} n \cdot E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s ds | \mathcal{G}_{\tau_q^n}] \cdot 1\!\!1_{[\tau_q^n, \tau_{q+1}^n[}(t)$$

Lemma 16. *T_n is a linear mapping from $\mathcal{H}^2_{[0,1]}$ to $\mathcal{H}^2$ and,*

$$\forall a \in \mathcal{H}^2_{[0,1]} : \|T_n(a)\|_{\mathcal{H}^2} \leq \|a\|_{\mathcal{H}^2}.$$

Proof: As a simple process, $T_n(a)$ is progressively measurable and

$$\|T_n(a)\|^2_{\mathcal{H}^2} = E[\sum_{q=0}^{n-1} (n \cdot E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s ds | \mathcal{G}_{\tau_q^n}])^2 \cdot (\tau_{q+1}^n - \tau_q^n)].$$

Since $Y_{q+1}^n := \sqrt{n} \cdot (B_{\tau_{q+1}^n} - B_{\tau_q^n})$ satisfies $[Y_{q+1}^n | \mathcal{G}_{\tau_q^n}] = [Y]$, we get

$$E[\tau_{q+1}^n - \tau_q^n | \mathcal{G}_{\tau_q^n}] = E[(B_{\tau_{q+1}^n} - B_{\tau_q^n})^2 | \mathcal{G}_{\tau_q^n}] = \frac{E[Y^2]}{n} = \frac{1}{n}.$$

Furthermore, with Jensens inequality:

$$(E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s ds | \mathcal{G}_{\tau_q^n}])^2 \leq E[(\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s ds)^2 | \mathcal{G}_{\tau_q^n}],$$

and by Cauchy Schwartz inequality: $(\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s ds)^2 \leq \int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s^2 ds \cdot \frac{1}{n}$. Therefore

$$\|T_n(a)\|^2_{\mathcal{H}^2} \leq E[\sum_{q=0}^{n-1} E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s^2 ds | \mathcal{G}_{\tau_q^n}]] = \|a\|^2_{\mathcal{H}^2},$$

and the lemma is proved. ∎

Lemma 17. $\forall a \in \mathcal{H}^2_{[0,1]} : \lim_{n\to\infty} \|T_n(a) - a\|_{\mathcal{H}^2} = 0.$

Proof: As it follows from the last lemma, the linear maps W_n defined by $W_n(a) := T_n(a) - a$ form an equi-continuous sequence of linear mappings. Therefore, we just have to prove the result for elementary processes a of the form: $a_s := \psi_u \cdot 1\!\!1_{[u,v[}$, where $u < v < 1$ and $\psi_u \in L^\infty(\mathcal{G}_u)$. Indeed, these elementary processes engender a dense subspace of $\mathcal{H}^2_{[0,1]}$. If $\psi_t := E[\psi_u | \mathcal{G}_t]$, the process ψ is a martingale on the Brownian filtration and, as such, has continuous sample paths. It is further uniformly integrable since $\psi_u \in L^\infty(\mathcal{G}_u)$, and with the stopping theorem, we conclude that $E[\psi_u | \mathcal{G}_{\tau_q^n}] = \psi_{\tau_q^n}$. Therefore, when n is high enough for :

$$\begin{aligned} T_n(a) =& \psi_{\tau^n_{[\![nu]\!]}} \cdot ([\![nu]\!] - nu) \cdot 1\!\!1_{[\tau^n_{[\![nu]\!]}, \tau^n_{[\![nu]\!]+1}[} + \psi_{\tau^n_{[\![nv]\!]}} \cdot (nv - [\![nv]\!]) \cdot 1\!\!1_{[\tau^n_{[\![nv]\!]}, \tau^n_{[\![nv]\!]+1}[} \\ &+ \textstyle\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} \psi_{\tau_q^n} \cdot 1\!\!1_{[\tau_q^n, \tau_{q+1}^n[} \end{aligned}$$

The $\mathcal{H}^2$ norm of the two first terms goes to 0 with n since $\|\psi\|_{L^\infty} < \infty$ and $E[\tau_{q+1}^n - \tau_q^n] = 1/n$. We also have

$$\begin{aligned} a =& \psi_u \cdot 1\!\!1_{[u,v[} \\ =& \psi_u \cdot 1\!\!1_{[\tau^n_{[\![nu]\!]}, \tau^n_{[\![nv]\!]}[} \\ &+ \psi_u \cdot 1\!\!1_{[u, u\vee\tau^n_{[\![nu]\!]}[} - \psi_u \cdot 1\!\!1_{[u\wedge\tau^n_{[\![nu]\!]}, u[} + \psi_u \cdot 1\!\!1_{[v\wedge\tau^n_{[\![nv]\!]}, v[} - \psi_u \cdot 1\!\!1_{[v, v\vee\tau^n_{[\![nv]\!]}[} \end{aligned}$$

The terms in the last line go to 0 in $\mathcal{H}^2$ norm, since $\|\psi_u\|_{L^\infty} < \infty$ and $\|v - \tau^n_{[\![nv]\!]}\|_{L^1}$ goes to 0 according to lemma 15. It just remains to prove that η_n converges to 0, where

$$\eta_n := \|\psi_u \cdot 1\!\!1_{[\tau^n_{[\![nu]\!]}, \tau^n_{[\![nv]\!]}[} - \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} \psi_{\tau^n_q} \cdot 1\!\!1_{[\tau^n_q, \tau^n_{q+1}[}\|^2_{\mathcal{H}^2}.$$

Now

$$\eta_n = \|\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} (\psi_u - \psi_{\tau^n_q}) \cdot 1\!\!1_{[\tau^n_q, \tau^n_{q+1}[}\|^2_{\mathcal{H}^2} = \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} E[(\psi_u - \psi_{\tau^n_q})^2 \cdot (\tau^n_{q+1} - \tau^n_q)]$$

It results from the definition of ψ_t that $\psi_t = \psi_u$ if $t \geq u$. Therefore, we infer that: $(\psi_u - \psi_{\tau^n_q})^2 \leq 4\|\psi_u\|^2_\infty 1\!\!1_{\tau^n_q < u}$ and thus

$$\eta_n \leq 4\|\psi_u\|^2_\infty E[\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} 1\!\!1_{\tau^n_q < u} \cdot (\tau^n_{q+1} - \tau^n_q)] \leq 4\|\psi_u\|^2_\infty E[\tau^n_{q^*_n} - \tau^n_{[\![nu]\!]}],$$

where $q^*_n := \inf\{q \geq [\![nu]\!] : \tau^n_q \geq u\}$. Due to claim 3) in lemma 15 we have that $\tau^n_{q^*_n}$ and $\tau^n_{[\![nu]\!]}$ converge a.s. to u. Since $\tau^n_{q^*_n} - \tau^n_{[\![nu]\!]} \leq 2\tau^n_n$ wich is a convergent sequence in L^1, we conclude with Lebesgue's dominated convergence theorem that $E[\tau^n_{q^*_n} - \tau^n_{[\![nu]\!]}] \to 0$, and the lemma is proved. ■

We defined $\tilde{X}^n$ in equation (14) with $r^n := T_n(r)$. We next take benefit of last lemma to prove that $\tilde{X}^n$ is a good approximation of X^n.

Lemma 18.
1) $\lim_{n\to\infty} \|\tilde{X}^n_n - X^n_n\|_{L^2} = 0.$
2) $\lim_{n\to\infty} \frac{|\mathcal{V}^M_n(\mathcal{F}^n, X^n) - \mathcal{V}^M_n(\mathcal{F}^n, \tilde{X}^n)|}{\sqrt{n}} = 0$

Proof: Since Itô's integral is isometric from $\mathcal{H}^2$ to L^2, we get:

$$\|\tilde{X}^n_n - X^n_n\|_{L^2} \leq \|\tilde{X}^n_{n+1} - X^n_{n+1}\|_{L^2} = \|r - r^n\|_{\mathcal{H}^2},$$

and claim 1) then follows from last lemma.

We prove now claim 2). With $\Delta X^n_{q+1} := X^n_{q+1} - X^n_q$ and $\Delta\tilde{X}^n_{q+1} := \tilde{X}^n_{q+1} - \tilde{X}^n_q$, we have, with assumption ii) in theorem 9:

$$\begin{aligned} |\mathcal{V}^M_n(\mathcal{F}^n, X^n) - \mathcal{V}^M_n(\mathcal{F}^n, \tilde{X}^n)| &= |E[\textstyle\sum_{q=0}^{n-1} M[\Delta X^n_{q+1}|\mathcal{F}^n_q] - M[\Delta\tilde{X}^n_{q+1}|\mathcal{F}^n_q]]| \\ &\leq E[\textstyle\sum_{q=0}^{n-1} |M[\Delta X^n_{q+1}|\mathcal{F}^n_q] - M[\Delta\tilde{X}^n_{q+1}|\mathcal{F}^n_q]|] \\ &\leq A \cdot E[\textstyle\sum_{q=0}^{n-1} E[|\Delta X^n_{q+1} - \Delta\tilde{X}^n_{q+1}|^p|\mathcal{F}^n_q]^{\frac{1}{p}}] \\ &\leq A \cdot E[\textstyle\sum_{q=0}^{n-1} E[|\Delta X^n_{q+1} - \Delta\tilde{X}^n_{q+1}|^2|\mathcal{F}^n_q]^{\frac{1}{2}}] \end{aligned}$$

Due to Cauchy Shwartz inequality, we have for all real numbers $x_0, \ldots, x_{n-1}$:

$$\sum_{q=0}^{n-1} x_q \leq \sqrt{n} \cdot \sqrt{\sum_{q=0}^{n-1} x^2_q}.$$

Therefore and since $\sqrt{x}$ is concave in x, we get with Jensens inequality:

$$\begin{aligned} |\mathcal{V}^M_n(\mathcal{F}^n, X^n) - \mathcal{V}^M_n(\mathcal{F}^n, \tilde{X}^n)| &\leq \sqrt{n} \cdot A \cdot E[\sqrt{\textstyle\sum_{q=0}^{n-1} E[|\Delta X^n_{q+1} - \Delta\tilde{X}^n_{q+1}|^2|\mathcal{F}^n_q]}] \\ &\leq \sqrt{n} \cdot A \cdot \sqrt{\textstyle\sum_{q=0}^{n-1} E[|\Delta X^n_{q+1} - \Delta\tilde{X}^n_{q+1}|^2]} \\ &= \sqrt{n} \cdot A \cdot \sqrt{E[|X^n_n - \tilde{X}^n_n|^2]} \end{aligned}$$

Claim 2) follows then from claim 1). ■

We will next compute $\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)$. Defining λ_q^n as: $\lambda_q^n := n \cdot E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} r_s ds|\mathcal{G}_{\tau_q^n}]$, we have $r_t^n := \sum_{q=0}^{n-1} \lambda_q^n \cdot \mathbb{1}_{[\tau_q^n, \tau_{q+1}^n[}(t)$. Since r is a positive process, we clearly have $\lambda_q^n \geq 0$. Next, $\tilde{X}_{q+1}^n - \tilde{X}_q^n = \lambda_q^n \cdot (B_{\tau_{q+1}^n} - B_{\tau_q^n}) = a_q^n \cdot Y_{q+1}^n$, where $a_q^n := \frac{\lambda_q^n}{\sqrt{n}}$. Since r is a positive process, a_q^n is positve and $\mathcal{F}_q^n$-measurable, as it results from the definition of λ_q^n. Therefore, since $M[X]$ is 1-homogeneous in X according to assumption i) in theorem 9, since $[Y_{q+1}^n|\mathcal{F}_q] = [Y]$, and since $E[Y^2] = 1$, $E[Y] = 0$, we get:

$$\begin{array}{rcl} \mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n) & = & E[\sum_{q=0}^{n-1} M[\tilde{X}_{q+1}^n - \tilde{X}_q^n|\mathcal{F}_q^n]] \\ & = & E[\sum_{q=0}^{n-1} M[a_q^n \cdot Y_{q+1}^n|\mathcal{F}_q^n]] \\ & = & E[\sum_{q=0}^{n-1} a_q^n \cdot M[Y]] \\ & = & M[Y] \cdot E[\sum_{q=0}^{n-1} a_q^n \cdot (Y_{q+1}^n)^2] \\ & = & M[Y] \cdot E[(\sum_{q=0}^{n-1} a_q^n \cdot Y_{q+1}^n) \cdot (\sum_{q=0}^{n-1} Y_{q+1}^n)] \\ & = & \sqrt{n} \cdot M[Y] \cdot E[\tilde{X}_n^n \cdot B_{\tau_n^n}] \end{array}.$$

Since $E[B_{\tau_n^n}^2] = 1$, we also have

$$\begin{array}{rcl} \frac{\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)}{M[Y] \cdot \sqrt{n}} & \geq & E[X_n^n \cdot B_{\tau_n^n}] - \|\tilde{X}_n^n - X_n^n\|_{L^2} \\ & = & E[X_{n+1}^n \cdot B_{\tau_n^n}] - \|\tilde{X}_n^n - X_n^n\|_{L^2} \\ & \geq & E[X_{n+1}^n \cdot B_1] - \|X_{n+1}^n\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \|\tilde{X}_n^n - X_n^n\|_{L^2} \\ & = & E[f_\mu(B_1) \cdot B_1] - \|\mu\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \|\tilde{X}_n^n - X_n^n\|_{L^2} \\ & = & \alpha(\mu) - \|\mu\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \|\tilde{X}_n^n - X_n^n\|_{L^2} \end{array}$$

With claim 4) in lemma 15 and claim 1) in lemma 18, we conclude then that:

$$\liminf_{n \to \infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, X^n)}{\sqrt{n}} = \liminf_{n \to \infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)}{\sqrt{n}} \geq M[Y] \cdot \alpha(\mu).$$

Since $\overline{\mathcal{V}}_n^M(\mu) \geq \mathcal{V}_n^M(\mathcal{F}^n, X^n)$, we thus have proved that for all $Y \in L^4$ with $E[Y] = 0$ and $E[Y^2] = 1$:

$$\liminf_{n \to \infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \geq M[Y] \cdot \alpha(\mu).$$

Since $\tilde{\mathcal{D}} := \{\tilde{Y} \in L^4 : E[\tilde{Y}] = 0 \text{ and } E[\tilde{Y}^2] \leq 1\}$ is dense for the L^2-norm in $\mathcal{D} := \{\tilde{Y} \in L^2 : E[\tilde{Y}] = 0 \text{ and } E[\tilde{Y}^2] \leq 1\}$, and since M is continuous for the L^p-norm and thus for the L^2-norm, we infer that there exists a sequence $\{\tilde{Y}_n\}_{n \in \mathbb{N}} \subset \tilde{\mathcal{D}}$ such that

$$\lim_{n\to\infty} M[\tilde{Y}_n] = \rho := \sup\{M[\tilde{Y}] : \tilde{Y} \in \mathcal{D}\} > 0.$$

We may further assume that $M[\tilde{Y}_n] > 0$, so that, since M is 1-homogeneous, we have that $M[Y_n] \geq M[\tilde{Y}_n]$, where $Y_n = \frac{\tilde{Y}_n}{\|\tilde{Y}_n\|_{L^2}}$. Since $Y_n \in L^4$ satisfies $E[Y_n] = 0$ and $E[Y_n^2] = 1$, we thus have proved that

$$\liminf_{n \to \infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \geq \lim_{n\to\infty} M[Y] \cdot \alpha(\mu) = \rho \cdot \alpha(\mu).$$

With theorem 14, we get then

Theorem 19. *Under the hypotheses of theorem 9,*

$$\lim_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} = \rho \cdot \alpha(\mu).$$

The first part of theorem 9 is thus proved. The second part will be proved in the next section.

3.8. **Convergence to the continuous martingale of maximal variation.** Let B be a standard one dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{G}_t)_{t\geq 0})$. If $\mu \in \Delta^{1^+}$, the martingale $\Pi^\mu_t := E[f_\mu(B_1)|\mathcal{G}_t]$ will is referred to in this paper as the continuous martingales of maximal variation of final distribution μ. This terminology is justified by the next result that clearly implies the second part of theorem 9.

If $(\mathcal{F}, X) \in \mathcal{M}_n(\mu)$, we define the continuous time representation $\tilde{X}$ of X as the process $(\tilde{X}_t)_{t\in[0,1]}$ with $\tilde{X}_t := X_{[\![nt]\!]}$, where $[\![a]\!]$ is the greatest integer less or equal to a.

Theorem 20. *Assume that M satisfies the hypotheses i) and ii) of theorem 9, that $\rho > 0$, that $\mu \in \Delta^2$ and that $\{(\mathcal{F}^n, X^n)\}_{n\in\mathbb{N}}$ is a sequence of martingales with for all n $(\mathcal{F}^n, X^n) \in \mathcal{M}_n(\mu)$, that asymptotically maximizes the M-variation, i.e.:*

$$\lim_{n\to\infty} \frac{\mathcal{V}^M_n(\mathcal{F}^n, X^n)}{\sqrt{n}} = \rho \cdot \alpha(\mu)$$

Then $\tilde{X}^n$ converges in finite dimensional distribution to Π^μ: For all finite set $J \subset [0,1]$, $(\tilde{X}^n_t)_{t\in J}$ converges in law to $(\Pi^\mu_t)_{t\in J}$

Proof: Let $\{(\mathcal{F}^n, X^n)\}_{n\in\mathbb{N}}$ be an asymptotically maximizing sequence. Without loss of generality, we may assume that $\mathcal{F}^n$ contains an adapted system $(U_q)_{,q=0,\ldots,n}$ of independent uniform random variables, independent of X^n, (otherwise $\mathcal{F}^n$ could be widened). Therefore, with (10), there exists $S^n \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}^n)$ such that $\mathcal{V}^M_n(\mathcal{F}^n, X^n) - 1 \leq E[X^n_{n+1} \cdot S^n_n]$, and thus

$$\lim_{n\to\infty} \frac{E[X^n_{n+1} \cdot S^n_n]}{\rho \cdot \sqrt{n}} = \alpha(\mu).$$

As in section 3.5, for $\epsilon_n > 0$ to be determined later, we may embed (X^n, R^n) in the Brownian filtration $\mathcal{G}$, where $R^n := \frac{S^n}{\rho\cdot\sqrt{n}}$, obtaining thus an increasing sequence $(\tau^n_q)_{q=0,\ldots,n+1}$ and a pair $(\hat{X}^n, \hat{R}^n)$ of $\hat{\mathcal{F}}^n$ martingales, where $\hat{\mathcal{F}}^n_q := \mathcal{G}_{\tau^n_q}$ such that (X^n, R^n) and $(\hat{X}^n, \hat{R}^n)$ are equally distributed. We then have

$$E[\hat{X}^n_{n+1} B_1] \geq E[\hat{X}^n_{n+1} \hat{R}^n_n] - \|\mu\|_{L^2} \cdot \|B_1 - \hat{R}^n_n\|_{L^2}$$

Since $E[\hat{X}^n_{n+1} \hat{R}^n_n] = E[X^n_{n+1} R^n_n]$, the first term in the right hand side of the last inequality converges to $\alpha(\mu)$. Next, according to (11) and claim 3) in lemma 13 with $t = 1$:

$$\begin{aligned} \|B_1 - \hat{R}^n_n\|_{L^2} &\leq \|B_1 - B_{\tau^n_n}\|_{L^2} + \|B_{\tau^n_n} - \hat{R}^n_n\|_{L^2} \\ &\leq \kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}} + \sqrt{\epsilon_n(n+1)} + \sqrt{\epsilon_n n} \end{aligned}$$

So if ϵ_n is chosen so as to ensure $n\epsilon_n \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{E[\hat{X}^n_{n+1} \cdot B_1]}{\rho \cdot \sqrt{n}} = \alpha(\mu).$$

Since $B_1 \sim \mathcal{N}(0,1)$ and $\hat{X}^n_{n+1} \sim \mu$, we may then apply claim 2) in theorem 8 to infer that $\hat{X}^n_{n+1}$ converges in L^1-norm to $f_\mu(B_1) = \Pi^\mu_1$.

Next observe that $\|\hat{X}^n_{[\![nt]\!]} - \Pi^\mu_t\|_{L^1} \leq \|\hat{X}^n_{[\![nt]\!]} - \Pi^\mu_{\tau^n_{[\![nt]\!]}}\|_{L^1} + \|\Pi^\mu_{\tau^n_{[\![nt]\!]}} - \Pi^\mu_t\|_{L^1}$. But

$$\|\hat{X}^n_{[\![nt]\!]} - \Pi^\mu_{\tau^n_{[\![nt]\!]}}\|_{L^1} = \|E[\hat{X}^n_{n+1} - \Pi^\mu_1|\mathcal{G}_{\tau^n_{[\![nt]\!]}}]\|_{L^1} \leq \|\hat{X}^n_{n+1} - \Pi^\mu_1\|_{L^1}.$$

On the other hand, with claim 1) and 2) in lemma 13 and our choice of ϵ_n we infer that $\tau^n_{[\![nt]\!]} \to t$ in L^1. Since Π^μ is uniformly integrable and, as a martingale on the Brownian filtration, it has continuous sample paths, we then conclude that $\|\Pi^\mu_{\tau^n_{[\![nt]\!]}} - \Pi^\mu_t\|_{L^1} \to 0$ as n increases. Therefore $\hat{X}^n_{[\![nt]\!]}$ converges to Π^μ_t in L^1.

This implies in particular the convergence in finite distribution of the process $(\hat{X}^n_{[\![nt]\!]})_{t\geq 0}$ to Π^μ, and this process has same distribution as $\tilde{X}^n$. ■

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E-mail address: demeyer@univ-paris1.fr