Equilibrium Distributions with Externalities*

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Abstract

This paper presents a general model of an exchange economy with consumption externalities, and establishes the existence of equilibrium in the model, under assumptions comparable to those in classical models. The key aspect of the model is that the economy is described in distributional terms.

1 Introduction

The classical model of competitive markets assumes that each agent cares only about his/her own consumption, but there has long been interest in relaxing that assumption to allow for the possibility that agents care about the consumption of others (consumption externalities) and even about prices (price externalities). For the case of a finite number of agents, a satisfactory model and a proof of existence of equilibrium in that model (with assumptions comparable to the assumptions in classical models) have been known for some time. (See Shafer & Sonnenschein (1975) for a model and existence proof, and Laffont (1977) for discussion and applications.) For the case of a continuum of agents, however, a satisfactory model has proved elusive. The issue has recently received increased attention: Balder (2000) identifies unexpected difficulties with previous models, and Noguchi (2001), Balder (2003), and Cornet & Topuzu (2004) offer models with a continuum of agents and proofs of existence of equilibrium in those models — but only under assumptions that are restrictive, or at least unpleasantly strong. (See below for a discussion and comparison with the assumptions in this paper.)

In this paper, we offer a simple, but quite general, model of an economy with a continuum of agents and consumption externalities, and establish the existence of equilibrium in that model, under assumptions comparable to those in classical models.[1] Our point of departure is that we describe an economy in distributional terms as in Hart, Hildenbrand & Kohlberg (1975), rather than in function-theoretic terms as in Aumann (1964, 1966). Aside from some fussiness required to avoid circularity in the description of agent characteristics and to deal with a space of agent characteristics that is not metrizable, our model and existence proof are surprisingly clean and straightforward.

It is useful to contrast our model and assumptions with those in the papers mentioned above. Out model, and all of those above, view consumer

[1]For simplicity, we focus here on consumption externalities, but there would be no difficulty in allowing for price externalities as well, following Greenberg, Shitovitz & Wieczorek (1979).

preferences as depending on own consumption, taking the consumption of others as a parameter. In the papers cited above, the consumption of others is described by a function from the space consumer names to the space of consumptions. In this paper, the consumption of others is described by a distribution on the product of the space of consumer characteristics with the space of consumptions. In terms of the dependence of preferences on own consumption, Noguchi (2001) and Cornet & Topuzu (2004) require, in addition to the standard assumptions (continuity, anti-symmetry, irreflexivity, transitivity, negative transitivity), that preferences be convex in own consumption, while Balder (2003) assumes a rather general functional form but does not require convexity. In this paper, we make only the standard assumptions about preferences for own consumption. In terms of the dependence of preferences on the consumption of others, Noguchi (2001) requires that consumers care only about the *mean* consumption of others, Cornet & Topuzu (2004) requires that preferences be continuous with respect to the topology of weak convergence of social allocations (as we illustrate in Section 4, this is a very strong requirement when allocations are described as functions from consumer names to consumption bundles), while Balder (2003) requires that the social allocation enters into the preferences of every consumer in exactly the same way. In this paper, we allow preferences to depend rather arbitrarily on the consumption of others, and we allow this dependence to be different for each consumer, and we require only that preferences be continuous with respect to the topology of weak convergence of distributions (as we illustrate in Section 4, this is a rather weak requirement).

However, we should offer a word of caution about our distributional framework: When agents care only about their own consumption, the number of commodities is finite, and commodities are divisible, the function-theoretic description of an economy and the distributional description of an economy are "almost" equivalent, in the sense that every function-theoretic description of an economy gives rise to a distributional description of an economy, any two function-theoretic descriptions that give rise to the same distributional description give rise to sets of Walrasian equilibrium distributions that have the same closure, and every Walrasian equilibrium distribution of an economy arises from some function-theoretic description. Hence, the choice to describe an economy with a finite number of divisible commodities in function-theoretic terms or in distributional terms is largely one of taste and convenience. However, when agents care about the consumption of others, or when the number of indivisible commodities is infinite, the situation is different: although every well-behaved distributional economy admits an equilibrium, not every well-behaved function-theoretic economy admits an equilibrium. (See Mas-Colell (1986) and Gretsky, Ostroy & Zame (1982) for the case of infinitely many indivisible economies and Balder (2003) for the case of externalities.)

Following this Introduction, we present the model in Section 2 and the existence theorem and its proof in Section 3. Section 4 presents several simple examples that illustrate our assumptions.

2 The Economy with Externalities

For X a completely regular topological space[2] we write $\mathcal{B}(X)$ for the family of Borel subsets of X. By a *measure* on X we always mean a finite, positive, countably additive measure σ on $\mathcal{B}(X)$ that is *regular*, in the sense that for each Borel set B and each $\varepsilon > 0$ there are a closed set $F \subset B$ and an open set $U \supset B$ such that $\sigma(U \setminus F) < \varepsilon$. The *norm*, or *total mass* of a measure σ, is $\|\nu\| = \nu(X)$. A *probability measure* or *distribution* on X is a measure having total mass 1. We write $M(X)_+$ for the space of measures on X and $\text{Prob}(X) \subset M(X)_+$ for the subspace of probability measures. We equip both $M(X)_+$ and $\text{Prob}(X)$ with the topology of weak convergence: $\sigma_\alpha \to \sigma$ weakly if and only if

$$\int \varphi d\sigma_\alpha \to \int \varphi d\sigma$$

for all bounded continuous functions $\varphi : X \to \mathbb{R}$. (The assumption of complete regularity of X guarantees that $M(X)_+$ is a Hausdorff space.)

We consider economies with $L \geq 1$ perfectly divisible consumption goods, so the commodity space and price space are both $\mathbb{R}^L$. It is convenient to normalize prices to sum to 1; write

$$\overline{\Delta} \;=\; \{p \in \mathbb{R}^L : \sum p_\ell = 1, p \geq 0\}$$

for the simplex of normalized, positive prices and

$$\Delta \;=\; \{p \in \mathbb{R}^L : \sum p_\ell = 1, p \gg 0\}$$

for the simplex of normalized, strictly positive prices.

We allow agent preferences to depend on the consumption of others. (It would be entirely straightforward to allow for preferences to depend on prices as well.) Because we describe the economy in distributional terms, the most obvious way to formalize this idea is to parametrize agent preferences by a distribution on the product of the space of consumer characteristics with the

[2] Recall that a topological space X is *completely regular* if a) points are closed, and b) for each closed subset $Y \subset X$ and each point $z \in X \setminus Y$ there is a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(z) = 1$ and $\varphi(y) = 0$ for each $y \in Y$.

space of consumptions. However, because consumer characteristics include preferences, this obvious approach leads to a circularity. In order to avoid this circularity, we follow the approach suggested by Mas-Colell (1984): we take as given an abstract space of observable characteristics and parametrize agent preferences by a distribution on the product of the space of observable characteristics with the space of consumptions.

Formally, we take as given a complete separable metric space T and a probability measure τ on T. We view T as the space of *observable characteristics* of agents and τ as the *distribution of observable characteristics* in the actual economy.

Agents care about their own consumption and about the consumptions of others. Because we assume consumptions are non-negative, we can summarize the consumptions of others as a distribution in $\text{Prob}(T \times \mathbb{R}^L_+)$. It is conceivable that agents care about *all* possible distributions of consumptions of others, but it is only necessary for our purposes that agents care about those distributions that involve a finite amount of total resources, shared among the actual population. To identify the relevant distributions, we say that $\sigma \in \text{Prob}(T \times \mathbb{R}^L_+)$ is *integrable* if $\int |x| d\sigma < \infty$. We write $\mathcal{D}$ for the set of integrable distributions and $\mathcal{D}(\tau)$ for the subset of integrable distributions σ for which the marginal of σ on T is τ. With the topology of weak convergence, $\text{Prob}(T \times \mathbb{R}^L_+)$ is a complete metric space; we give $\mathcal{D}(\tau) \subset \text{Prob}(T \times \mathbb{R}^L_+)$ the relative topology.

As in Hildenbrand (1974), we write $\mathcal{P}^*$ for the space of (continuous, anti-symmetric, irreflexive, transitive, negatively transitive) preference relations on $\mathbb{R}^L_+$ and $\mathcal{P}^*_{mo} \subset \mathcal{P}^*$ for the subspace of strictly monotone preference relations. In the topology of closed convergence, $\mathcal{P}^*$ and $\mathcal{P}^*_{mo}$ and are completely metrizable. We shall assume that preferences are strictly monotone in own consumption, so we define a *preference relation with consumption externalities* to be a map

$$R : \mathcal{D}(\tau) \to \mathcal{P}^*_{mo}$$

We use interchangeably the notations $(x, y) \in R(\sigma)$ or $xR(\sigma)y$ to mean that the consumption bundle x is preferred to the consumption bundle y when σ

is the distribution of consumption.[3] As usual, the preference relation R is *continuous* if the set

$$\{(x, y, \sigma) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathcal{D}(\tau) : xR(\sigma)y\}$$

is open. The following Proposition shows that continuity of the preference relation R in this sense is equivalent to continuity of the mapping R; the simple proof is left to the reader.

PROPOSITION *The mapping $R : \mathcal{D}(\tau) \to \mathcal{P}^*_{mo}$ is continuous if and only if the preference relation R is continuous in the sense that the set*

$$\{(x, y, \sigma) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathcal{D}(\tau) : xR(\sigma)y\}$$

is open.[4]

Write $\mathcal{R}^*_{mo}$ for the space of continuous preference relations, and give $\mathcal{R}^*_{mo}$ the topology of uniform convergence on compact sets.[5] A subbase for this topology consists of all sets of the form

$$W(K, U) = \{R \in \mathcal{R}^*_{mo} : R(K) \subset U\}$$

where $K \subset \mathcal{D}(\tau)$ is compact and $U \subset \mathcal{P}^*$ is open.

Agents are characterized by an observable characteristic, a preference relation, and an endowment, so the space of agent characteristics is

$$\mathcal{C} = T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$$

[3]The restriction to integrable distributions of consumption is analogous to the restriction, in the function-theoretic formulations of Noguchi (2003), Balder (2003) and Cornet & Topuzu (2004), to allocations that are integrable with respect to the given population measure.

[4]Strict monotonicity is important here.

[5]Again: we could insist that preferences be defined even for distributions of consumptions that involve an infinite amount of total resources, or are shared among a population different from the actual population, but it is only necessary for our purposes that preferences be defined for distributions that involve a finite amount of total resources, shared among the actual population. Thus, the formulation we have chosen has the advantage of being more general and no less complicated.

(We will show in the following section that $\mathcal{R}^*_{mo}$ and $\mathcal{C}$ are completely regular.) Following Hart, Hildenbrand & Kohlberg (1975), an *economy* is a tight probability measure λ on $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$ whose marginal on T is the given population measure τ and for which aggregate endowment is finite: $\int e_\ell d\lambda < \infty$ for each ℓ. (Recall that a measure λ on a completely regular space X is *tight* if for every $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\lambda(X \setminus K) < \varepsilon$. Billingsley (1968) shows that every measure on a complete, separable metric space is tight. However, the space $\mathcal{R}^*_{mo}$ of preference relations and the space $\mathcal{C}$ of agent characteristics are not metrizable, so we build the requirement of tightness into our description of the economy.)

An *equilibrium* for the economy λ is a price $p \in \Delta$ and a probability measure μ on $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+$ such that

(a) the marginal of μ on $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$ is λ

(b) almost all agents choose in their budget set:

$$\mu\{(t, R, e, x) : p \cdot x > p \cdot e\} = 0$$

(c) markets clear

$$\int x d\mu = \int e d\lambda$$

(Note that (a) and (c) imply that the marginal μ_{14} of μ on $T \times \mathbb{R}^L_+$ (the distribution of consumption) belongs to $\mathcal{D}(\tau)$.)

(d) almost all agents optimize given the price p and the distribution of consumption μ_{14}:

$$\mu\{(t, R, e, x) : y \in R^L_+, yR(\mu_{14})x \Rightarrow p \cdot y > p \cdot e\} = 0$$

3 Existence of Equilibrium

Our main result is:

THEOREM *Let* (T, τ) *be a population and let* $\lambda \in \mathrm{Prob}(T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+)$ *be an economy (i.e, a tight probability measure whose marginal on* T *is* τ *and for which* $\int e_\ell d\lambda < \infty$ *for each* ℓ*). If* $0 < \int e_\ell d\lambda$ *for each* ℓ *(i.e., each good is present in the aggregate), then the economy* λ *admits an equilibrium.*

Before beginning the proof, it is convenient to collect some preliminary results. Recall that a family $S \subset M(X)_+$ of measures is *uniformly tight* if for each $\varepsilon > 0$ there is a compact set K such that $\sigma(X \setminus K) < \varepsilon$ for each $\sigma \in S$. The two lemmas below record extensions to the case of completely regular spaces of familiar facts about separable metric spaces (see Billingsley (1968) for example); we offer proofs here for convenience (but claim no originality).

LEMMA 1 *If* X *is a completely regular topological space and* $\{\sigma_\alpha\}$ *is a uniformly tight net of measures converging weakly to the measure* σ *then*

(i) $\sigma(F) \geq \limsup \sigma_\alpha(F)$ *for every closed set* $F \subset X$

(ii) σ *is tight*

PROOF Fix a closed set F and suppose, for the purpose of obtaining a contradiction, that $\sigma(F) < \limsup \sigma_\alpha(F)$. Choose $\varepsilon > 0$ so that

$$\sigma(F) < \limsup \sigma_\alpha(F) - 3\varepsilon$$

Because $\{\sigma_\alpha\}$ is uniformly tight, there is a compact set K such that $\sigma_\alpha(X \setminus K) < \varepsilon$ for each α. Write $L = F \cap K$. Note that $\sigma_\alpha(L) > \sigma_\alpha(F) - \varepsilon$ for each α. Because σ is positive, $\sigma(F) \geq \sigma(L)$. Use regularity of σ to choose an open set $U \supset L$ such that $\sigma(U) < \sigma(L) + \varepsilon$. Use complete regularity of

X to choose a continuous function $\Phi : X \to [0,1]$ such that $\Phi|L \equiv 1$ and $\Phi|(X \setminus U) \equiv 0$.[6] Weak convergence entails that

$$\int \Phi d\sigma_\alpha \to \int \Phi d\sigma$$

On the other hand

$$\begin{array}{lllll} \sigma_\alpha(F) - \varepsilon & \leq & \sigma_\alpha(L) & \leq & \int \Phi d\sigma_\alpha \\ \int \Phi d\sigma & \leq & \sigma(L) + \varepsilon & \leq & \sigma(F) + \varepsilon \end{array}$$

Putting these together yields

$$\limsup \sigma_\alpha(F) \leq \sigma(F) + 2\varepsilon$$

This is a contradiction, so we obtain (i).

To see that σ is tight, note first that

$$\sigma_\alpha(X) = \int 1 d\sigma_\alpha \to \int 1 d\sigma = \sigma(X)$$

Fix $\varepsilon > 0$ and choose a compact set $K \subset X$ such that

$$\sigma_\alpha(X) - \sigma_\alpha(K) = \sigma_\alpha(X \setminus K) \leq \varepsilon$$

for each α. In view of (i) we have

$$\sigma(K) \geq \limsup \sigma_\alpha(K) \geq \limsup \sigma_\alpha(X) - \varepsilon = \sigma(X) - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that σ is tight. ■

LEMMA 2 *If X is a completely regular topological space then every norm bounded and uniformly tight family of measures $S \subset M(X)_+$ is relatively weakly compact.*

[6] To construct such a function, note first that for each $x \in L$ there is a continuous function $\psi_x : X \to [0,1]$ such that $\psi_x(x) = 0$ and $\psi_x|(X \setminus U) \equiv 1$. For each $x \in L$, set $V_x = \{y \in X : \psi_x(y) < \frac{1}{2}\}$. Compactness of L entails that the covering $\{V_x\}$ of L has a finite subcover, so we can find a finite family of continuous functions $\psi_i : X \to [0,1]$ such that (a) for each $x \in L$ there is some i for which $\psi_i(x) < \frac{1}{2}$, (b) $\psi_i|(X \setminus U) \equiv 1$ for each i. Set $\Psi = \prod \psi_i$, so that $\Psi(x) < \frac{1}{2}$ for each $x \in K$ and $\Psi|F \equiv 1$. Then define $\Phi = 2 - 2\max\{\frac{1}{2}, \Psi\}$.

Proof It suffices to prove that every net in S contains a convergent subnet. To this end, let $\{\sigma_\alpha\}$ be a net in S. By assumption, there is an $M > 0$ such that $\sigma(X) \leq M$ for each $\sigma \in S$. For each integer n, use uniform tightness to choose a compact set $K_n \subset X$ such that $\sigma(X \setminus K_n) < 1/n$ for each $\sigma \in S$. For each n, $\{\sigma_\alpha | K_n\}$ is a net of measures in $M(K_n)^+$ each of which has total mass bounded by M. Because norm bounded balls in $M(K_n)^+$ are weakly compact (see Dunford & Schwartz (1957) for example), the net $\{\sigma_\alpha | K_n\}$ contains a weakly convergent subnet. By a familiar diagonal argument, there is a single subnet $\{\sigma_\beta\}$ of σ_α with the property that $\{\sigma_\beta | K_n\}$ is weakly convergent for each n; say $\sigma_\beta | K_n \to \sigma_n \in M(K_n)_+$. Note that $\sigma_n \leq \sigma_{n+1} | K_n$, so if we view each σ_n as a measure on X, then $\sigma_n \leq \sigma_{n+1}$. Set $\sigma = \sup_n \sigma_n$. That is, for each Borel set $B \subset X$,

$$\sigma(B) = \sup_n \sigma_n(B \cap K_n)$$

Straightforward calculations show that $\sigma \in M(X)_+$ and that $\sigma_\beta \to \sigma$ weakly, so the proof is complete. ■

Finally, we prove, as promised in Section 2, that $\mathcal{R}^*_{mo}$ is a completely regular space. Since metric spaces are completely regular and products of completely regular spaces are completely regular, it follows that the space $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$ of consumer characteristics, and the space $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+$ of characteristics and consumptions are also completely regular.

Lemma 3 *The space $\mathcal{R}^*_{mo}$ of consumer preference relations is a completely regular topological space.*

Proof Fix a metric ρ on $\mathcal{P}^*$. To see that points are closed, fix $R_0 \in \mathcal{R}^*_{mo}$ and let $R_1 \in \mathcal{R}^*_{mo}$ be any other point. Since $R_0 \neq R_1$, there is some $\sigma \in \mathcal{D}(\tau)$ such that $R_0(\sigma) \neq R_1(\sigma)$, whence $\rho(R_0(\sigma), R_1(\sigma)) > 0$. Let U be the ε-ball around $R_1(\sigma)$ in $\mathcal{P}^*$. Then $W(\sigma, U)$ is an open set that contains R_1 and not R_0. Since R_1 is arbitrary, we conclude that R_0 is closed, as desired.

To see that a point and a closed set can be separated by a function, let $F \subset \mathcal{R}^*_{mo}$ be a closed set and let $R_0 \in \mathcal{R}^*_{mo}$ be any point not in F.

By definition, there are compact subsets $K_1, \ldots K_m \subset \mathcal{D}(\tau)$ and open sets $U_1, \ldots, U_m \subset \mathcal{P}^*$ such that

$$R_0 \in W(K_1, U_1) \cap \ldots \cap W(K_m, U_m)$$

and

$$W(K_1, U_1) \cap \ldots \cap W(K_m, U_m) \cap F = \emptyset$$

Let ρ be any metric on $\mathcal{P}^*$ that defines the topology, and use compactness of $R_0(K)$ to choose an $\varepsilon > 0$ sufficiently small that, for each i, U_i contains the ε-ball around every point of $R_0(K_i)$. Set $K = K_1 \cup \ldots \cup K_m$ and define $\varphi : \mathcal{R}^*_{mo} \to [0, 1]$ by

$$\varphi(R) = 1 - \frac{1}{\varepsilon} \min \left\{ \varepsilon, \sup_{\sigma \in K} \rho(R_0(\sigma), R(\sigma)) \right\}$$

It is easily checked that φ is continuous, that $\varphi(R_0) = 1$, and that $\varphi(R) = 0$ for each $R \in F$, as desired. ■

With these preliminaries in hand, we proceed to the proof of the Theorem

Proof of Theorem The proof proceeds in several steps.

Step 1 Tightness of λ entails that there is an increasing sequence $\{H_i\}$ of compact subsets of $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$ such that $\lambda(H_i) > 1 - 2^{-i}$ for each i. For each n, i, set

$$\begin{aligned}
\Delta_n &= \{p \in \Delta : p_\ell \geq \frac{1}{n} \text{ for each } \ell\} \\
K_n &= \{(t, R, e, x) : |x| \leq 2Ln|e|\} \\
K_{ni} &= \{(t, R, e, x) \in K_n : (t, R, e) \in H_i\} \\
\mathcal{F}_n &= \{\mu \in \text{Prob}(K_n) : \mu_{123} = \lambda\}
\end{aligned}$$

(where μ_{123} is the marginal of μ on the first three factors).

We assert that each $\mathcal{F}_n$ is non-empty, convex, and weakly compact. To see that $\mathcal{F}_n$ is non-empty, write

$$\text{proj} : T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \to T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$$

for the projection on the first three factors. For $B \subset K_n$ a Borel set, write

$$B_0 = B \cap \left(T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \{0\}\right)$$

Now define a probability measure $\mu \in \text{Prob}(K_n)$ by $\mu(B) = \lambda\,(\text{proj}(B_0))$, and note that $\mu \in \mathcal{F}_n$, so $\mathcal{F}_n$ is non-empty. It is obvious that $\mathcal{F}_n$ is convex and that it is weakly closed. Moreover, $\mu(K_{ni}) > 1 - 2^{-i}$ for each $\mu \in \mathcal{F}_n$, so $\mathcal{F}_n$ is uniformly tight, hence compact.

Step 2 For $(t, R, e) \in T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$, $\mu \in \text{Prob}(T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+)$ such that $\mu_{14} \in \mathcal{D}(\tau)$ (μ_{14} is the marginal of μ on the product of the first and fourth factors), and $p \in \Delta$, define individual budget and demand sets by:

$$\beta(t, R, e; \mu, p) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot e\}$$
$$d(t, R, e; \mu, p) = \{x \in \beta(t, R, e; \mu; p) : yR(\mu_{14})x \Rightarrow p \cdot y > p \cdot e\}$$

Finally, let $D(\mu, p)$ be the set of agents who choose in their demand set:

$$D(\mu, p) = \{(t, R, e, x) : x \in d(t, R, e; \mu, p)\}$$

For each n, define correspondences

$$\begin{aligned} \phi_n : \Delta_n \times \mathcal{F}_n &\rightarrow \mathcal{F}_n \\ \psi_n : \Delta_n \times \mathcal{F}_n &\rightarrow \Delta_n \\ F_n : \Delta_n \times \mathcal{F}_n &\rightarrow \Delta_n \times \mathcal{F}_n \end{aligned}$$

as follows:

$$\begin{aligned} \phi_n(p, \mu) &= \{\nu \in \mathcal{F}_n : \nu(D(\mu, p)) = 1\} \\ \psi_n(p, \mu) &= \text{argmax}\left\{q \cdot \left(\int x d\mu - \int e d\mu\right) : q \in \Delta_n\right\} \\ F_n(p, \mu) &= \psi_n(p, \mu) \times \phi_n(p, \mu) \end{aligned}$$

We claim that ϕ_n, ψ_n, F_n are upper-hemi-continuous, and have compact, convex, non-empty values.

It is evident that ϕ_n has convex values. Because $\Delta_n, \mathcal{F}_n$ are compact, to see that ϕ_n is upper-hemi-continuous and has compact values, it suffices to show that it has closed graph. To this end, let $\{(p_\alpha, \mu_\alpha)\}$ be a net in $\Delta_n \times \mathcal{F}_n$ converging to (p, μ); for each α, let $\nu_\alpha \in \phi_n(p_\alpha, \mu_\alpha)$ and assume $\nu_\alpha \to \nu$. We must show $\nu \in \phi_n(p, \mu)$. By assumption, $\nu_\alpha(D(\mu_\alpha, p_\alpha)) = 1$. It follows that

$$\nu_\alpha(D(\mu_\alpha, p_\alpha) \cap K_{ni}) = \lambda(H_i)$$

for each i. Set

$$L_i = \limsup \left[D(\mu_\alpha, p_\alpha) \cap K_{ni}\right]$$

If V is a closed neighborhood of L_i then $V \supset D(\mu_\alpha, p_\alpha) \cap K_{ni}$ for sufficiently large α. In view of Lemma 1, it follows that

$$\nu(V) \geq \limsup \nu_\alpha(V) \geq \limsup \nu_\alpha \left[D(\mu_\alpha, p_\alpha) \cap K_{ni}\right] = \lambda(H_i)$$

Because V is arbitrary, it follows that $\nu(L_i) \geq \lambda(H_i)$. On the other hand, the usual argument for upper-hemi-continuity of demand shows that

$$L_i = \limsup \left[D(\mu_\alpha, p_\alpha) \cap K_{ni}\right] \subset D(\mu, p) \cap K_{ni}$$

It follows that $\nu(D(\mu, p) \cap K_{ni}) = \lambda(H_i)$ and hence that $\nu(D(\mu, p)) = 1$, so that $\nu \in \phi_n(p, \mu)$. We conclude that ϕ_n has closed graph, as desired.

To see that ϕ_n has non-empty values, fix (p, μ). Because H_i is compact, $\lambda|_{H_i}$ is the weak limit of a net $\{\zeta^{\alpha i}\}$ of measures with finite support; say

$$\zeta^{\alpha i} = \sum a_k^{\alpha i} \delta_{y_k^{\alpha i}}$$

where

$$y_k^{\alpha i} = (t_k^{\alpha i}, R_k^{\alpha i}, e_k^{\alpha i}) \in H_i$$

For each $y_k^{\alpha i}$, choose $z_k^{\alpha i} \in d(t_k^{\alpha i}, R_k^{\alpha i}, e_k^{\alpha i}; \mu, p)$. Set $w_k^{\alpha i} = (t_k^{\alpha i}, R_k^{\alpha i}, e_k^{\alpha i}, z_k^{\alpha i})$ and

$$\nu^{\alpha i} = \sum a_k^{\alpha i} \delta_{w_k^{\alpha i}}$$

For each n, i, $\nu^{\alpha i}$ is a measure on K_{ni} of total mass equal to $\lambda(H_i)$. The net $\{\nu^{\alpha i}\}$ is uniformly tight, hence has a convergent subnet; the limit of this subnet belongs to $\phi_n(p, \mu)$. Hence $\phi_n(p, \mu)$ is not empty, as asserted.

That ψ_n is upper-hemi-continuous, and has compact, convex, non-empty values follows immediately from the usual argument for Berge's Maximum Theorem.

Finally, F_n is upper-hemi-continuous, and has compact, convex, non-empty values because ϕ_n, ψ_n enjoy these properties.

Step 3 Because $\Delta_n, \mathcal{F}_n$ are compact and convex and F_n is upper-hemi-continuous and has compact, convex, non-empty values, F_n has a fixed point (p_n, μ_n). By definition, p_n maximizes the value of excess demand at p_n, μ_n. However, Walras's Law guarantees that the value of excess demand at p_n, μ_n is 0. Hence, if $q = (\frac{1}{L}, \ldots, \frac{1}{L}) \in \Delta_n$, then

$$q \cdot \left(\int x d\mu_n - \int e d\mu_n \right) \leq 0$$

whence

$$\sum_{j=1}^{L} \int x_j d\mu_n \leq \sum_{j=1}^{L} \int e_j d\mu_n$$

Because consumptions and endowments are non-negative, it follows that:

$$\begin{aligned} \int |x| d\mu_n &= \sum_{j=1}^{L} \int x_j d\mu_n \\ &\leq \sum_{j=1}^{L} \int e_j d\mu_n \\ &= \sum_{j=1}^{L} \int e_j d\lambda \end{aligned}$$

Write $M = \sum \int e_j d\lambda$, so that $\int |x| d\mu_n \leq M$; that is, total demand for all goods (and hence for each good separately) is bounded by M, independently of n.

Step 4 We construct a limit point of some subnet of the sequence $\{(p_n, \mu_n)\}$. To this end, define for each k, i:

$$\begin{aligned} G_k &= \{(t, R, e, x) : |x| > k\} \\ V_i &= \{(t, R, e, x) : (t, R, e) \in H_i\} \\ L_{ki} &= \{(t, R, e, x) : (t, R, e) \in H_i, |x| \leq k\} \end{aligned}$$

Notice that

$$\int_{G_k} |x| d\mu_n \geq k\mu_n(G_k)$$

and hence that

$$\mu_n(G_k) \leq \frac{M}{k}$$

By construction, the marginal of μ_n on $T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+$ is λ, so

$$\mu_n(L_{ki}) \geq 1 - \frac{M}{k} - 2^{-i}$$

for each n, k, i. Because each L_{ki} is compact, this means that $\{\mu_n\}$ is a uniformly tight family, so some subnet of $\{\mu_n\}$ converges. Because prices p_n lie in the closed price simplex $\overline{\Delta}$, some subsequence of prices also converges. Hence some subnet $\{(p_\alpha, \mu_\alpha)\}$ of $\{(p_n, \mu_n)\}$ converges; call the limit (p^*, μ^*).[7] Note that $p^* \in \overline{\Delta}$ and that $\mu^*_{14} \in \mathcal{D}(\tau)$.

Step 5 We claim that $p^* \in \Delta$; that is, no component of p^* is 0. If not, assume without loss that the first component of p^* is strictly positive. Let $E = \{(t, R, e) : e_1 > 0\}$. Because $\int ed\lambda > 0$ it follows that $\lambda(E) > 0$. Because λ is regular and tight, it follows that there is a compact set $J \subset E$ such that $\lambda(J) > 0$.

Define

$$Z \;=\; \{\zeta \in \mathrm{Prob}(T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+) : \zeta_{123} = \lambda, \int xd\zeta \leq \int ed\lambda\}$$

Arguing as above, we see that Z is weakly closed and uniformly tight, hence weakly compact. We claim that as $n \to \infty$ demand is uniformly unbounded on Z. Precisely, we claim: For every $A > 0$ there is an integer n_0 such that if $n \geq n_0$, $(t, R, e) \in J$ and $\zeta \in Z$ and $y \in d(t, R, e; \zeta, p_n)$ then $|y| > A$. To see this, suppose not. Then there is some $A > 0$ such that for every n_0 there is some $n > n_0$, some $(t, R, e) \in J$, some $\zeta \in Z$ and some $y_n \in d(t, R, e; \zeta, p_n)$ such that $|y_n| \leq A$. Letting n_0 tend to infinity, passing to limits of subnets where necessary, and recalling that J, Z are compact,

[7] Because $\mathrm{Prob}(T \times \mathcal{R}^*_{mo} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+)$ is not metrizable, $\{(p_n, \mu_n)\}$ might not contain a convergent *subsequence*.

that preference relations are continuous in the distribution of consumption, and that the topology on $\mathcal{R}^*_{mo}$ is that of uniform convergence on compact sets, and making use of the familiar argument for upper-hemi-continuity of demand, we find $(t^*, R^*, e^*) \in Z$, $\zeta^* \in J$ and $y^* \in d(t^*, R^*, e^*; \zeta^*, p^*)$ such that $|y^*| \leq A$. However, since (t^*, R^*, e^*) has non-zero wealth at prices p^*, the price of the last good is 0, and preferences are strictly monotone, this is absurd. This contradiction establishes the claim.

Now apply the claim with $A = 2 \left| \int e d\lambda \right| / \lambda(J)$ to conclude that there is an n_0 such that for every $(t, R, e) \in J$, each $n \geq n_0$ and every $y \in d(t, R, e; \zeta, p_n)$ we have:

$$|y| > 2 \left| \int e d\lambda \right| / \lambda(J)$$

If follows in particular that

$$\int_J \inf\{|y| : y \in d(t, R, e; \mu_n, p_n)\} d\lambda \geq 2 \left| \int e d\lambda \right|$$

for each n. However, as we have shown above,

$$\int |x| d\mu_n \leq \left| \int e d\lambda \right|$$

so we have obtained a contradiction. We conclude that $p^* \in \Delta$, as asserted.

Step 6 Because $p^* \in \Delta$, it follows that there is some n_0 such that $p_n \in \text{int}\Delta_n$ for all $n \geq n_0$. By construction, p_n maximizes the value of excess demand among all prices in Δ_n. Because the maximizer lies in the interior of Δ_n, it follows that the value of excess demand must be constant for all prices in Δ_n, and hence that excess demand must actually be 0. It is immediate, therefore, that (p_n, μ_n) is actually an equilibrium for the economy λ, provided that $n \geq n_0$, so the proof is complete. ■

4 Examples

Some simple examples may help to illustrate the meaning and generality of our assumptions.

The first example is very much in the spirit of Balder (2003). Take $T = [0,1]$ and let τ be Lebesgue measure. Let

$$\phi : T \times \mathbb{R}^L_+ \to \mathbb{R}^n$$

be a bounded, continuous function, and let

$$v : \mathbb{R}^L_+ \times \mathbb{R}^n \to \mathbb{R}$$

be a continuous function. For $x \in \mathbb{R}^L_+, \mu \in \text{Prob}(T \times \mathbb{R}^L_+)$, define the utility of consuming x, given the consumption distribution μ, by

$$u(x;\mu) = v\left(x, \int \phi(t,y) d\mu(t,y)\right)$$

It is easily checked that u is jointly continuous (giving $\text{Prob}(T \times \mathbb{R}^L_+)$ the topology of weak convergence). If v is strictly increasing in x (own consumption) then so is u. Hence the utility function u induces a preference relation with consumption externalities $R : \mathcal{D}(\tau) \to \mathcal{P}^*_{mo}$ defined by

$$R(\mu) = \{(x,x') : u(x,\mu) > u(x',\mu)\}$$

It is easy to check that the preference relation R satisfies all our assumptions. (Indeed, R is defined and continuous on the entire space of distributions on $T \times \mathbb{R}^L_+$, not just on the subspace $\mathcal{D}(\tau)$ of integrable distributions whose marginal is τ.) Note that utility is a function of own consumption and an average of some function of social consumption.

It may be enlightening to contrast this example with an example in the spirit of Cornet & Topuzu (2004). As above, take $T = [0,1]$ and let τ be Lebesgue measure. Let

$$\psi : T \to \mathbb{R}^L_+$$
$$w : \mathbb{R}^L_+ \times \mathbb{R}^L \to \mathbb{R}$$

be continuous functions. Given $x \in \mathbb{R}^L_+$ (own consumption) and an integrable function $f : T \to \mathbb{R}^L_+$ (the allocation of social consumption), define utility by

$$u(x; f) = w(x, \int \psi(f(t))d\tau$$

Thus utility is again a function of own consumption and an average of a function of social consumption. If w is strictly increasing in x (own consumption) then so is u. However, if w is not a constant function, then in order that u be continuous with respect to the topology of weak convergence of allocations, as required by Cornet & Topuzu (2004), it is necessary and sufficient that ψ be an *affine* function. Requiring that ψ be affine amounts to requiring that utility be a function of own consumption and an average of a weighting of social consumption, rather than an average of an arbitrary function of social consumption, as above.

The examples above share the feature that utility is separable in own consumption and social consumption, but a simple variant allows for non-separability. Again, take $T = [0, 1]$ and let τ be Lebesgue measure. Let

$$\Phi : \mathbb{R}^L_+ \times T \times \mathbb{R}^L_+ \to \mathbb{R}^n$$

be a bounded, smooth function, and let

$$V : \mathbb{R}^L_+ \times \mathbb{R}^n \to \mathbb{R}$$

be a smooth function. For $\mu \in \text{Prob}(T \times \mathbb{R}^L_+)$, define

$$U(x; \mu) = V\left(x, \int \Phi(x, t, y)d\mu(t, y)\right)$$

In order that U be strictly monotone in x (own consumption) it suffices that

$$\frac{\partial V}{\partial x_\ell} + \frac{\partial V}{\partial y_i}\frac{\partial \Phi}{\partial x_\ell} > 0$$

whenever $1 \leq \ell \leq L, n+1 \leq i \leq n+L$. (Note that this condition is satisfied in the first example because $\frac{\partial \phi}{\partial x_\ell} = 0$ for each ℓ.) If U is strictly monotone in x then it induces a preference relation with consumption externalities

$$R : \mathcal{D}(\tau) \to \mathcal{P}^*_{mo}$$

defined by

$$R(\mu) = \{(x, x') : U(x, \mu) > U(x', \mu)\}$$

This preference relation satisfies our assumptions. (Indeed, R is continuous on the whole space of distributions on $T \times \mathbb{R}^L_+$, not just on the space $\mathcal{D}(\tau)$ of integrable distributions whose marginal is τ.)

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