Collusion via Signaling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without Complementarities*

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Abstract

Collusive equilibria exist in simultaneous ascending bid auctions with multiple objects, even with large complementarities in the buyers' utility functions. The bidders collude by dividing the objects among themselves, while keeping the prices low. In the most collusive equilibrium the complementarities are never realized. The scope for collusion however narrows as the ratio between the number of bidders and the number of objects increases.

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1 Introduction

It is well known that the English auction has many desirable properties when a single object is to be sold. For example, with private values it implements the efficient allocation uniquely in weakly dominant strategies, and maximizes the seller's expected revenue within a large class of "simple" selling procedures (Lopomo [30]). However, the properties of generalized versions of the English auction, in situations in which many objects are to be sold, and the buyers have use for more than one object, are yet to be fully understood.

Current research on auctions with multiple objects can be organized into a normative and a positive approach. The first approach consists essentially in looking for mechanisms with equilibria which satisfy some desirable properties, e.g. efficiency (Ausubel and Cramton [5], Dasgupta and Maskin [18], Perry and Reny [38], and Esó and Maskin [22]), or seller's revenue maximization (Armstrong [3], Avery and Hendershott [8], Menicucci [33]). The positive approach instead considers given auction formats, writes them down as games of incomplete information and aims at characterizing their equilibrium sets.

This paper contributes to the second line of research. We study a multi-object version of the English auction, henceforth named "simultaneous ascending bid auction", similar to the one recently used by the Federal Communications Commission (FCC) for the sale of spectrum licences (see McAfee and McMillan [36]). We focus on the claim that this generalization of the one-object English auction is more vulnerable to collusion in the multiple objects case than in the single object case. Concerns about collusive behavior of bidders in the FCC auction have emerged, for example, in an article published in *The Economist* [19]. More recently, Cramton and Schwartz [16] have indicated evidence of collusive behavior in the FCC spectrum auctions, and discussed the effectiveness of various modifications of the auction rules in hindering bidders' collusion.

Our analysis will provide elements to test the veracity of the following conjectures which naturally arise on auctions with multiple objects:

- The presence of multiple objects facilitates collusion by allowing the bidders to signal their willingness to abstain from competing over certain objects, provided they are not challenged on others. In this way, the bidders can allocate the objects among themselves without paying much.

- As the ratio between the number of bidders and the number of objects increases, the room for collusive schemes such as the ones indicated in the previous conjecture becomes smaller.

- Large complementarities in the bidders' utility functions tend to hinder collusion. This is because each bidder is less satisfied with owning only a subset of the objects on sale; she has therefore an incentive to break the collusion and compete for all the objects in order to fully realize the synergies.

In a model with two objects, we show that bidders with private information about their own willingness to pay for each subset of objects can indeed take advantage of the signalling opportunities provided by the sequential nature of open ascending bid auctions with multiple objects, by coordinating on equilibria which generate a high level of expected surplus for them, a low level of revenue for the seller, and socially inefficient allocations of the objects. (Propositions 1 and 2). This kind of coordination however becomes more difficult as the number of bidders increases while the number of objects remains fixed at two. (Propositions 4 and 5). Thus our analysis lends support to the first two conjectures listed above.

With regard to the third conjecture, we show that the sole presence of large complementarities is not sufficient to eliminate the opportunity for the bidders to collude. In fact, in the extreme case in which the levels of synergies are commonly known, and not too different across the bidders, the incentive structure is essentially identical to the case with no complementarities. However, when the complementarities are not only large but also variable, the possibility of collusion is seriously reduced. These results suggest that what is crucial in determining the likelihood of collusion is not whether the complementarities are (on average) 'large', but how variable they are.

It is important to note that the type of collusion considered in this paper requires no side contracts among the bidders. Instead, collusive behavior emerges as a noncooperative equilibrium phenomenon. This is a major difference with the single-object case, in which side contracts, or future interaction, are in general necessary to sustain bidders' collusion[1].

The positive literature on multi-unit auctions has focused mainly on the case of identical objects and non-increasing marginal willingness to pay in the bidders' utility functions. One of the earliest papers on coordination in multiple unit auctions is on procurement auctions, by Anton and Yao [1]. They show that, under a condition which in the monopoly case corresponds

[1]Collusion in the single object case sustained by side contracts has been studied, among others by Campbell [13], Graham and Marshall [23], Mailath and Zemsky [31], McAfee and McMillan [35] and Pesendorfer [39]. An exception to the use of contracts to sustain collusion is in McAfee and McMillan [35]. They show that the bidders can collude in first-price auction in which ties are resolved with equal uniform probabilities. Caillaud and Jéhiel [12] have shown that the presence of negative externalities among the buyers may hinder the effectiveness of collusion. Collusive behavior in repeated single-object auctions has been studied by Hopenhayn and Skrzypacz [25].

to decreasing marginal willingness to pay, sellers who can bid for the entire production as well as for single parts, can coordinate on 'split award' equilibria which generate a low level of surplus for the monopsonist. Viswanathan, Wang and Witelski [42] have characterized equilibrium strategies in sealed-bid discriminatory auctions for the case of two bidders.

The papers which are most closely related to the present paper in terms of the auction rules, are Milgrom [34] and Engelbrecht-Wiggans and Kahn [21]. Milgrom [34] analyzes the simultaneous ascending bid auction, mostly under the assumption that the bidders' utility functions are common knowledge. He discusses issues surrounding the auction's performance in terms of its ability of generating efficient outcomes and its potential for maximizing the seller's expected revenue. In particular, for the case of two bidders, two objects and no private information, Milgrom describes an equilibrium which is similar to the one described in Proposition 1 of this paper: each bidder can buy one object for the minimum price allowed by the rules of the auction. Engelbrecht-Wiggans and Kahn [21] have independently established a result which is essentially identical to our Proposition 1, namely that 'low revenue' equilibria exist under mild conditions on the bidders' information structure. They also show that, without these mild conditions, more limited forms of collusion can be sustained in equilibrium. In the present paper we also show that the bidders can improve upon the equilibrium of Proposition 1 (Proposition 2), and that collusion can also be sustained when complementarities are present (Proposition 7).

Another branch of the literature has analyzed the issue of the so-called 'demand reduction' in auctions with many identical objects. Ausubel and Cramton [4] study sealed-bid auctions of shares of a single divisible asset, under the assumption that each buyer's marginal willingness to pay is non-increasing, and is determined by a privately known one-dimensional parameter. They show that, in the sealed-bid uniform price auction, the buyers have an incentive to bid less than their marginal willingness to pay for each unit, hence no equilibrium can induce an ex-post efficient allocation of the asset. A similar point is made by Engelbrecht-Wiggans and Kahn [20]. When multiple *identical* objects are sold, it is not an equilibrium that bidders bid 'straightforwardly' for each object, i.e. up to the price that equals their valuations. Instead, the bidders want to reduce their demand in order to keep the prices low. This behavior is quite similar to what happens in oligopsonistic markets, the source of inefficiency being the traditional one found in textbook monopoly models: by trying to buy (sell) a marginal unit, the price on inframarginal units increases (decreases). In our paper, the objects for sale are generally perceived as different by each bidder; and the auction rules allow a different price for each object; which eliminates the main rationale for demand reduction. (This is akin to price discrimination in the textbook monopoly model). As a result, 'straightforward' equilibria in which the bidders bid for each object up to their valuation do exist. However, as we show in this paper, these straightforward equilibria are can be used as punishment to form other, more collusive, equilibria.

Our equilibria have features in common with the sub-game perfect equilibria of a multi-market repeated oligopoly model studied by Bernhaim and Winston [10]. They show that two firms with relative cost advantages in producing different goods can collude by creating "spheres of influence": that is, each firm monopolizes the market for the good that it produces most efficiently and withdraws from the other market.

Pesendorfer [39] discusses a "ranking mechanism" which allows bidders to collude in multi-unit auctions. While Pesendorfer analyzes a monopsony situation[2], his ranking mechanism is equivalent to the following mechanism in the monopoly case: each bidder announces a ranking of the objects according to his values, and each object is bought for the minimum price by one of the bidders who rank it the highest, with ties resolved uniformly. This mechanism induces a uniform matching between objects and bidders: for example, with two objects and two bidders, each bidder always buys exactly one object. The allocation induced by the equilibrium described in our Proposition 2 also has this feature, but in Proposition 2, if the bidders rank the two objects in the same way then the preferred object is assigned to the bidder with the highest *difference* of values, and this bidder pays a price equal to her opponent's difference in values. We show that both the ranking mechanism and the outcome of Proposition 2 are *interim*-efficient for the bidders among all incentive compatible outcome functions such that each buyer always buy exactly one object.

Benoît and Krishna [9] analyze a model with complete information, common values, and budget constrained bidders. They show that, with significant complementarities, or with sufficiently different objects' values, a sequence of single-object open ascending auctions yields more revenue than the simultaneous ascending auction. Under different auction rules, but still assuming complete information, Ausubel and Schwartz [6] show that in fact a 'collusive' equilibrium is the unique Nash equilibrium that satisfies backward induction. Finally, under a condition which rules out complementarities in the buyers' utility functions, Gül and Stacchetti [24] have studied a generalized version of the English auction akin to a *tatonnement* process, with emphasis on the relation between its equilibria and the Walrasian equilibria of the underlying economy.

Environments in which the bidders have increasing marginal valuations have been considered in Chakraborty [14], who has studied properties of various sealed-bid auctions formats. His paper

[2]Pesendorfer's paper studies collusion in auctions for school milk contracts in Florida and Texas during the 1980s.

also contains a good survey of existing work on multiple object auctions.

On the experimental side, recent work by Kwasnica and Sherstyuk [28] reports findings which are consistent with the theoretical predictions of the present paper. In particular, collusion occurs in auctions with two bidders and two objects, while it tends to disappear with five bidders.[3]

The present paper proceeds as follows. Section 2 describes the model. In Section 3 we begin the analysis with the benchmark case of purely additive values, i.e. we assume that each bidder obtains no synergies from owning multiple objects, hence her willingness to pay for one object is independent of whether she is also buying other objects. The analysis of this case provides a useful benchmark for the more realistic case in which complementarities are present. In particular, it sheds light on the role played by multiple objects in facilitating collusion among the bidders. We present conditions under which collusion-via-signaling can be sustained in equilibrium. Equilibria in this class can be described for the simple case with only two bidders as follows. Each bidder starts by placing the smallest possible bid on her most valued object, and no bid on the other object. If only one bid is placed on each object, it becomes common knowledge that the objects are ranked differently by the two bidders. In this case the bidders let the auction end in the second round by remaining silent. Each bidder is thus awarded one object for the minimum price. If, instead, the initial bids reveal that both bidders have a higher value for the same object, then the bidding continues according to some equilibrium strategy, which can entail, for example, a reversion to "bidding straightforwardly," i.e. each bidder raising the bid on each object if her value is higher than the current highest bid and she is not assigned the object. (See Proposition 1 below.) Alternatively, the bidders may adopt some other continuation strategy in which they proceed to signal more detailed information about their values in order to try again to coordinate with each other and buy one object each for a relatively low price (Proposition 2). In all equilibria of this kind, the outcome entails socially inefficient allocations in some cases — i.e. the objects are not always assigned in a way that maximizes the sum of the bidders' values — but the bidders end up paying less than they would by bidding straightforwardly throughout the entire auction. The reduced payments make up for the loss of efficiency in assigning the objects, hence each bidder's interim expected surplus is higher.

For these equilibria, however, the probability that the bidders can collude via signaling decreases as the number of bidders increases. More precisely, the probability of assigning each object to the bidder with the highest value increases as the number of bidders increases. These results (Propositions 4 and 5 below) corroborate the conjecture that collusion in multiple unit

[3]Kwasnica [27] has also done experimental work on collusion in multiple object sealed-bid auctions, with additive utility functions.

auctions is a 'low numbers' phenomenon.

In section 4 we consider the case in which the bidders' utility functions exhibit large complementarities, i.e. their willingness to pay for the two objects together is much greater than the sum of the two objects's "stand alone" values. The presence of significant complementarities makes the simultaneous ascending bid auction a natural candidate for allocating multiple objects efficiently, essentially because a bidder's willingness to pay for any given object depends on the probability of winning other complementary objects. As stated in the third conjecture above, immediate intuition may lead one to think that large complementarities hinder collusion by providing each bidder with a strong incentive to buy both objects rather than just one.

We show however that the sole presence of complementarities does not hinder collusion: the bidders can still manage to buy one object each, at low prices. In fact, in the extreme case in which the synergies are commonly known, and not too different across the bidders, the incentive structure for the bidders is essentially identical to the case with no complementarities. The efficiency loss however is much larger in this case because it includes the unrealized complementarities. When the complementarities are not only large but also variable the possibility of collusion is seriously reduced. The final insight is then that not just the presence of large complementarities but also their variability is important in deterring the bidders from colluding. Section 5 contains some concluding remarks, and an appendix collects all the proofs.

2 The Model

Let $N = \{1, ..., n\}$ denote the set of bidders, and $M = \{1, ..., m\}$ the set of objects. Each bidder $i \in N$ has a quasi-linear utility function, with willingness to pay for each bundle $J \subset M$ given by $u_i(J)$. All values $\{u_i(J)\}_{J \in 2^M}$ are privately known to bidder i. It is common knowledge that such values are drawn according to a probability measure with support on a compact subset of $\Re_+^{2^m}$.

The objects are sold with a "simultaneous ascending bid auction", which is a natural extension of the standard one-object English auction to environments with multiple objects. The auction proceeds in rounds. In each round $t = 1, 2, ...$, and for each object $j \in M$, each bidder $i \in N$ can either remain silent or raise the highest bid of the previous round by at least a minimum amount. Formally, i's bid on object j in round t, denoted $b_i^j(t)$, can either be set equal to $-\infty$, which is to be interpreted as "no bid", or must be a real number at least as large as $b^j(t-1) + \varepsilon$, where $\varepsilon > 0$ is the minimum bid increment, and $b^j(t-1)$ is the "current outstanding bid" from the previous round, defined recursively by

$$b^j(0) = 0 \quad \text{and} \quad b^j(t) := \max\left\{b^j(t-1),\ b_i^j(t);\ i \in N\right\}.$$

If at least one bidder places a new bid on some objects, i.e. if $b_i^j(t) \neq -\infty$ for some $i \in N$ and $j \in M$, then for each of these objects j the new highest bid $b^j(t)$ is identified, and a potential winner is selected among the bidders who have made the new highest bid; and the auction moves to the next round, with the potential winners of all other objects unchanged. If instead all bidders remain silent on all objects, i.e. if $b_i^j(t) = -\infty$ for all $i \in N$ and $j \in M$, then the auction ends, with each object $j \in M$ assigned to the winner selected at the end of round $t-1$, who pays his last bid $b^j(t-1)$.

We analyze equilibria of this auction when the minimum increment is negligible, i.e. for $\varepsilon \to 0$. We do not verify explicitly that the equilibria that we find assuming a negligible minimum bid are actually equilibria for ε close enough to zero. This can be done (see Engelbrecht-Wiggans and Kahn [21] for an explicit analysis), but we have decided to omit the formal convergence proofs in order to avoid lengthening the paper.

Furthermore, to keep the analysis as simple as possible, we establish our results for $m = 2$, i.e. with only two objects on sale. To simplify the notation, we define

$$v_i := u_i(\{1\}), \quad \text{and} \quad w_i := u_i(\{2\});$$

and use interchangeably the terms 'object v' (object w) and 'object 1' (object 2).

Also, to model the presence of complementarities in a parsimonious way, we assume that the size of the complementarity is independent of the two objects' 'stand-alone' values. That is, the value to bidder i of having both objects is

$$u_i(\{1,2\}) = v_i + w_i + k_i.$$

We assume that the vectors (v_i, w_i, k_i) are drawn independently across bidders from the same probability distribution with support $[0,1]^2 \times K$, where K is either $\{0\}$ (no complementarities) or the interval $\left[\underline{k}, \overline{k}\right]$ with $\underline{k} > 1$ (large complementarities.) The variables v_i and w_i have identical marginal distributions, with density and c.d.f. denoted by f and F respectively. For later use, we also define the variable $a_i := v_i - w_i$, whose support is the interval $[-1, 1]$. The density function and the c.d.f. of each a_i will be denoted g and G respectively.

We start the analysis in the next section with the case of no complementarities, i.e. $k_i = 0$ for all $i \in N$. Section 4 will be devoted to the case with positive complementarities.

3 Collusive Equilibria with No Complementarities

In this section we assume that the bidders have *purely additive values:*

$$u_i(\{1,2\}) = v_i + w_i, \quad i \in N,$$

or *no complementarities,* $k_i = 0$ for each i. The analysis of the bidders' equilibrium behavior in this case provides a useful benchmark for the more realistic case in which complementarities are present. In particular, it becomes clear that the presence of multiple objects facilitates collusion among the bidders.

With additive values, the problem of allocating the objects efficiently is simple: for example a sequence of one unit objects would assign each object to a buyer with the highest willingness to pay. Work by Armstrong and Avery and Hendershott shows that the efficient allocation may or may not be optimal for a risk neutral revenue maximizing seller.[4] We focus here on the equilibrium set of the simultaneous ascending bid auction.

We begin with a set of three elementary, but important observations. First, with no complementarities, the following 'Separated English Auctions' (SEA) strategy, together with a suitable belief system, forms a perfect Bayesian equilibrium: for bidder i, raise the bid on each object j if the value $u_i(\{j\})$ is higher than the current highest bid, and bidder i is not assigned object j. Clearly, if all other bidders use the SEA strategy, player i's best reply is to do the same. We state this result as Proposition 0, for an arbitrary number of objects and players.

Proposition 0 *With no complementarities, for any n and m, the separated English auctions (SEA) strategy profile forms a perfect Bayesian equilibrium (with some consistent belief system) after any history in the simultaneous ascending bid auction.*

The second observation is that the SEA strategy can be used to form a continuation equilibrium profile after any partial history of the auction. It may then be used as a threat to deter aggressive bidding, and thus sustain collusive outcomes, much like Pareto inferior sub-game perfect equilibria are used to support collusive outcomes in repeated games.

The third observation follows immediately from an extension the well-known Revenue Equivalence Theorem (Myerson [37]), which holds here because the bidders' types are drawn from

[4] Both papers study models in which the values of a buyer who has use for more then one object are drawn from binary distributions. Armstrong [2] analyzes the symmetric case and finds that revenue maximizing auctions are also socially efficient. He also shows however that this result may not hold with more general value distributions.

Avery and Hendershott [8] study a model with one identifiable buyer willing to buy two objects and many other buyers with one-unit demands. They find that the revenue maximizing auctions in this case are often inefficient.

independent and continuous probability distributions:[5] given any objects' allocation rule, the incentive compatibility constraints uniquely determine, up to a constant, both the interim expected payment function and the interim expected surplus function of each bidder, in any equilibrium of any auction game. Thus in particular, any perfect Bayesian equilibrium of the simultaneous ascending bid auction in which each bidder with type $(0, 0)$ expects zero surplus, and all other types are better off than in the SEA equilibrium, must entail a socially inefficient allocation of the objects.[6]

3.1 Two Bidders

We begin with the two bidder case. The next Proposition establishes the existence of a symmetric perfect Bayesian equilibrium which dominates the SEA in terms bidders' interim expected surplus[7]. Recall that F denotes the common marginal c.d.f. of v_i and w_i.

Proposition 1 *Assume that $E(x) := \int_0^1 x\, dF(x) \geq \frac{1}{2}$. Then the following strategy, together with some consistent belief system, forms a symmetric perfect Bayesian equilibrium:*

- *types (v_i, w_i) such that $v_i \geq w_i$ open with $\{b_i^1(1), b_i^2(1)\} = \{0, -\infty\}$;*
- *types (v_i, w_i) such that $v_i < w_i$ open with $\{b_i^1(1), b_i^2(1)\} = \{-\infty, 0\}$;*
- *if the initial bids are different, all types remain silent in round 2, i.e. $b_i^j(1) = -\infty$, for all i and j;*
- *if the initial bids are equal, or if, at any round, any bids differs from the instructions given above, then all types revert to the SEA strategy.*

The equilibrium of Proposition 1 can be described as follows. Each bidder opens by making the minimum bid (zero) only on her most preferred object. If, at the end of the first round, the bidders discover that they rank the objects differently, then they stop bidding, and each bidder is able to buy the preferred object at the lowest possible price. If instead they discover that they rank the two objects in the same way, then they revert to the SEA strategies.

[5] A generalization of Myerson's result which includes our setting has been established, among others, by Krishna and Maenner [26].

[6] Jehiél and Moldovanu [29] use in a similar way the Revenue Equivalence Theorem to point out that with multiple objects the efficient allocation is not in general revenue-maximizing.

[7] The existence of this equilibrium has been established independently by Engelbrecht-Wiggans and Kahn [21].

Why is the condition $E(x) \geq \frac{1}{2}$ needed? Consider any type of bidder 1 with $v_1 \geq w_1$, and suppose that the two bidders have followed the equilibrium strategy and opened with different bids in the first round. Bidder 1 can now obtain v_1 for the minimum price (zero), or compete for both objects, thus obtaining an expected surplus of

$$S(v_1, w_1 \,|\, L) = \int_0^{v_1} (v_1 - v_2)\ dF_V(v_2 \,|\, L) + \int_0^{w_1} (w_1 - w_2)\ dF_W(w_2 \,|\, L),$$

where $F_V(v_2 \,|\, L)$ and $F_W(w_2 \,|\, L)$ denote the c.d.f. of v_2 and w_2 respectively, both conditional on the set $L := \left\{(v_2, w_2) \in [0,1]^2 \mid 0 \leq v_2 \leq w_2\right\}$.

Since for any fixed v_1, $S(v_1, \cdot \,|\, L)$ is increasing, it is enough to check that all types (v_1, v_1) — i.e., all types on the diagonal of the type space — are willing to accept collusion, i.e. that

$$v_1 \geq S(v_1, v_1 \,|\, L) \ \text{ for each } v_1 \in [0,1].$$

Clearly, this inequality holds for $v_1 = 0$; and it is easy to see that the function $S(v_1, v_1 \,|\, L)$ is convex in v_1. Therefore it is enough to check that the inequality holds for the highest type $(1,1)$. This type gets both objects with probability 1 whenever competition is triggered, paying a price equal to the expected value of the two objects for bidder 2, that is $E(v_2 \,|\, L) + E(w_2 \,|\, L)$. Given the symmetry assumption, the expected payment for type $(1,1)$ is therefore $E(v_2 \,|\, L) + E\left(v_2 \,|\, [0,1]^2 \setminus L\right) = 2\,E(x)$. Acceptance of collusion gives a utility of 1. Therefore, the relevant condition becomes:

$$1 \geq 2 - 2E(x) \qquad \text{i.e.} \qquad E(x) \geq \frac{1}{2}.$$

Intuitively, the condition $E(x) \geq \frac{1}{2}$ can be interpreted as requiring that each bidder has to expect a sufficiently high degree of competition from her opponent, should the SEA strategies be triggered. Otherwise there is no point in colluding, since both objects can be obtained at a low expected price.

This point can actually be made even in a simple, complete information framework. Consider the case with two bidders, two objects, and commonly known values $(v_1, w_1) = (h, h)$, and $(v_2, w_2) = (l, l)$, with $0 < l < h$. There is an equilibrium in which the bidders use strategies that are similar to the SEA: bidder 2 bids on both objects up to l, and bidder 1 wins both objects paying l, and receiving a total surplus of $2(h - l)$. However, for some values of l and h there is another equilibrium, similar to the equilibrium characterized by Proposition 1, in which bidder 1 opens offering 0 on object v and nothing on w, bidder 2 opens offering nothing on v and 0 on w, and in the next round the bidders remain silent. If any bidder deviates, each bidder bids on each object up to her valuation. Clearly, bidder 2 has no profitable deviations from this strategy. Bidder 1 will not deviate if $h \geq 2(h - l)$, i.e. if the surplus obtained under collusion is higher than the surplus obtained triggering price competition. This is equivalent to $l \geq \frac{h}{2}$, a condition very similar to the one described for the incomplete information case, which can be interpreted in the same way: in order to accept collusion a bidder with high values has to expect enough competition by the other bidder.

With complete information however the auction can also end with each object sold for the minimum bid, because it is clear at the beginning of the auction that each object will go to the bidder with the highest value. For all other bidders is then optimal to let the first bidder buy the object for the minimum bid (as well as pushing up the price to their values). Thus, in the example above, there is a third equilibrium in which bidder 2 lets bidder 1 buy both objects for the minimum bid (zero). This type of 'collusion' can take place even in the one object case. Instead, the equilibrium similar to the one of Proposition 1 can only exist in the presence of multiple objects.

Remark 1. The equilibrium outcome of Proposition 1 is inefficient whenever the bidders rank the objects differently, and one bidder has higher values for both objects.

Remark 2. In Proposition 1 the SEA strategy is used to support collusion. The SEA strategy is the worst punishment that a bidder can impose to an opponent if we rule out the possibility that a buyer bids above her value on any object. If we allow for bidding above the values, then worse punishments are possible, and the conditions for the existence of collusion become weaker. For example, the following is a possible modification of the strategies and the beliefs described in Proposition 1: if at any point a bidder deviates from the equilibrium path, then the other bidder believes that the deviator's type is $(1,1)$ and raises the bids on both objects up to 1. In this case, a bidder who deviates from the equilibrium strategy receives a payoff equal to 0; hence collusion can be sustained without any condition on the probability distribution. However, this equilibrium relies on the fact that the bidder who observes a deviation is willing to place bids that are above her valuations, confident that she will lose with probability one.

Remark 3. The presence of *ex ante* asymmetries in the valuation of the two objects makes it more difficult to sustain collusion, but not impossible. Consider the same framework as before, but assume that for each i the value w_i is distributed according to a c.d.f. F_W with support on $[0, \beta]$, with $\beta \in (0,1)$. In particular, assume that $F_W(x) = F_V\left(\frac{x}{\beta}\right)$, so that $E(w_i) = \beta E(v_i)$. The type space is now a rectangle with base 1 and height β. We can divide the type space along the diagonal (the line given by the equation $w = \beta v$) and check whether a collusive equilibrium of the type described in Proposition 1 exists. In particular, we now want types such that $w > \beta v$ to open with $\{-\infty, 0\}$, types $w \leq \beta v$ to open with $\{0, -\infty\}$ and the bidding to stop if initial bids are different. Under what conditions on the distribution can collusion be sustained? As in the symmetric case, it is enough to check incentives for types on the diagonal, and it can again be shown that it is enough to make sure that type $(1, \beta)$, that is the type on the upper-right corner of the type space, is willing to collude. Considering first the types in the lower triangle, i.e. with $\beta v_i \geq w_i$, the condition can be written as $1 \geq 1 - E(v) + \beta - E(w)$, or , since $E(w) = \beta E(v)$, as

$$E(x) \geq \frac{\beta}{1+\beta},$$

which is weaker than $E(x) \geq \frac{1}{2}$. It is intuitively clear however that it is harder to convince the types who accept the less valuable object to go along with the collusive strategy instead of triggering the SEA strategies. In order to make sure that all types (v, w) with $w > \beta v$ are willing to accept collusion we have to check that types arbitrarily close to $(1, \beta)$ prefer having w for free to competing for both objects. The condition is therefore $\beta \geq 1 - E(v) + \beta - E(w)$, which leads to:

$$E(x) \geq \frac{1}{1+\beta}.$$

This condition is stronger than $E(x) \geq \frac{1}{2}$. This confirms the intuition that asymmetries tend to hinder collusion. As expected, the difficulty comes from the need to guarantee that the bidder who is assigned the less valuable object is not willing to fight for both objects. The expected payment that this bidder has to make if competition is triggered must now be higher than in the case in which objects are ex-ante symmetric. Using the distance between β and 1 as a measure of the asymmetry, it is also clear that the condition for collusion to be possible becomes increasingly stronger as the asymmetry increases, another intuitive result. As β approaches 1, $\frac{1}{1+\beta}$ tends to $\frac{1}{2}$; hence small deviations from symmetry do not really jeopardize the possibility of collusion. As β approaches zero (strong asymmetry) collusion becomes nearly impossible: no bidder is satisfied with having only the less valuable object.

In the next subsection, we show that the set of perfect Bayesian equilibria of the simultaneous ascending bid auction contains other, "more collusive" equilibria, i.e. equilibria in which each type of both bidders end up with a higher (interim) expected surplus.

3.1.1 Getting More out of Collusion

The equilibrium strategy described in Proposition 1 prescribes that the bidders revert to the SEA strategies when they open with the same bids, i.e. when it becomes common knowledge that their preferred object is the same. It is natural to ask whether, even after learning that they rank the objects in the same way, the bidders can do better than reverting to the SEA strategy, by trying again to coordinate themselves and buy one object each at relatively low prices. The next Proposition establishes that this is indeed possible, if the values' distribution satisfies the following

Condition A *Let F be a c.d.f. satisfying $F(0) = 1 - F(1) = 0$, and let x, y be two independent random variables, each with c.d.f. F. We say that F satisfies Condition A if, for each $\alpha \in [0,1]$, the following inequalities hold:*

$$E[x \mid \alpha \leq x] \ + \ E[y \mid y \leq 1-\alpha] \geq 1, \tag{1}$$

$$E[x \mid x - y \geq \alpha] \ + \ E[y \mid x - y \geq \alpha] \geq 1. \tag{2}$$

Setting $\alpha = 0$ yields $E(x) \geq \frac{1}{2}$ both in (1) and in (2), because by symmetry

$$E[x \mid x \geq y] + E[x \mid y \geq x] = 2\left(\frac{1}{2}E[x \mid x \geq y] + 2E[x \mid y \geq x]\right) = 2E[x].$$

Thus Condition A is stronger than the condition used to sustain the collusive equilibrium of Proposition 1. Condition A is satisfied, for example, by the following family of densities[8]

$$f(x) = 1 + s\left(x - \frac{1}{2}\right), \qquad s \in [0,2].$$

Proposition 2 *Under condition A the following strategy, together with some consistent belief system, forms a symmetric perfect Bayesian equilibrium:*

First round:

- *All types (v_i, w_i) such that $v_i \geq w_i$ open with $\{b_i^1(1), b_i^2(1)\} = \{0, -\infty\}$;*
- *types (v_i, w_i) such that $v_i < w_i$ open with $\{b_i^1(1), b_i^2(1)\} = \{-\infty, 0\}$.*

Subsequent rounds:

[8] With the linear density $f(x) = 1 + s\left(x - \frac{1}{2}\right)$, we have

$$E[x \mid \alpha \leq x] + E[y \mid y \leq 1-\alpha] - 1 = \frac{2s(1-\alpha)^2}{3(4 - s^2\alpha^2)},$$

and

$$E[x + y \mid x - y \geq \alpha] - 1 = \frac{2(1-\alpha)^2 s}{12 - s^2\alpha^2 - 2s^2\alpha}.$$

Both expressions are positive for $\alpha \in [0,1)$ and $s \in [0,2]$.

- *If the initial bids are either* $\{\{0,-\infty\},\{-\infty,0\}\}$ *or* $\{\{-\infty,0\},\{0,-\infty\}\}$, *all types remain silent;*
- *If the initial bids are* $\{\{0,-\infty\},\{0,-\infty\}\}$, *then all types* (v_i,w_i) *such that* $v_i-w_i=a_i$ *keep raising their bid on object* v *while refraining from bidding on* w *until either i) the opponent stops, or ii) the bids reach the value* a_i. *In case i), these types remain silent for the next two rounds; and in case ii) they bid* $\{-\infty,0\}$ *for two consecutive rounds, thus moving the outstanding bid on* w *from* $-\infty$ *to* 0.
- *If the initial bids are* $\{\{-\infty,0\},\{-\infty,0\}\}$, *the strategy is symmetric, with the roles of* v *and* w *switched.*

Out–of–equilibrium paths:

- *If at any round a bid not in accordance to the above described strategy is observed, then each type reverts to the SEA strategy.*

The behavior implied by the equilibrium of Proposition 2 can be described as follows. The bidders open by signalling which object they prefer. If they prefer different objects, the auction ends in the first round, as in the equilibrium of Proposition 1. If instead they prefer the same object, say v, then they keep raising the price on v while abstaining from competing on w, with bidder i prepared to bid up to the difference between her two values $a_i=v_i-w_i$. Therefore, if $a_i>a_{3-i}$, bidder i ends up buying object v at a price equal to the difference between her opponent's values, i.e. a_{3-i}. Her opponent stops competing on v when the price reaches a_{3-i}, and buys w for the minimum bid.

In this equilibrium, the type set of each bidder $[0,1]^2$ is partitioned into lines with slope 1: types on the same line — i.e. with the same difference between the two objects' values — behave identically, hence remain indistinguishable, until the end of the auction.

The role of Condition A is to guarantee that, given the residual pooling of low and high types at any stage, each bidder has no incentive to trigger the SEA strategies, because she assigns a sufficiently high conditional probability to her opponent having high values. To see this, suppose that both bidders have opened signalling that they prefer v to w. Then the equilibrium strategy prescribes that all types (v_1,w_1) such that $v_1-w_1=a$ bid on v until the price reaches a and then yield, obtaining w for free and letting the opponent buy v at price a. When is this an optimal strategy for all types?

It is easy to see that, just as in the case in which the bidders open with different bids, the gain from triggering the SEA strategies is higher for types with higher values of v_1 (and consequently w_1). In particular, suppose that the price for v has reached a, and that bidder 1's type is $(1, 1-a)$, the highest on her iso-difference line. She knows that her opponents' type lies in the set

$$L(a) := \left\{(v_2, w_2) \in [0,1]^2 \mid v_2 - w_2 \geq a\right\}.$$

Therefore she also knows that she will win both objects if the SEA strategies are triggered. Thus she is prepared to buy only object w for free if the following inequality is satisfied:

$$1 - a \geq 1 - E[v_2 \mid L(a)] + (1-a) - E[w_2 \mid L(a)]$$

or, equivalently, if:

$$E[w_2 \mid L(a)] + E[v_2 \mid L(a)] \geq 1.$$

With the appropriate changes in notation, this is exactly inequality (2) in Condition A.

Inequality (1) guarantees, when the price for v has reached a, bidder 2 is willing to buy object v at $a = v - w$, giving up the fight for w. Suppose that bidder 2 observes that bidder 1 stops bidding on v at the price a'. Then it becomes common knowledge that bidder 1's type lies on the line $v = w + a'$. Therefore, conditional on this information, bidder 2 must be better off paying a' for v rather than triggering the SEA strategies. In this case it can be shown again that in order to convince all types on the line $v_1 = w_1 + a$ it is enough to convince the highest type, $(1, 1-a)$. Furthermore, it is enough to show that this type is willing to pay the price a, since at any price $a' < a$ she is better off. After some manipulations, this is shown to be equivalent to inequality (1) in Condition A.

It is worth noting that the bidders' behavior is robust to perturbations in their beliefs about their opponents' values. That is, if the postulated types' distribution F is such that conditions (1) and (2) hold as strict inequalities, then each bidder has no incentive to deviate at any round, even if her beliefs are only approximately described by F.

It is crucial however that no object is assigned before the end of the auction, so that each object can still be bought after any number of rounds in which its outstanding bid has not moved. Thus, in the equilibrium of Proposition 2, for many rounds bidding occurs only on one object, while no activity takes place on the other. It is natural to conjecture then that this equilibrium can be destroyed by introducing some "activity rules", i.e. conditions specifying that if the outstanding bid on an object does not increase by at least a certain amount every given number of rounds, then the object be assigned to a bidder who has made the highest bid.[9] To be effective, these active rules would have to specify sufficiently large minimum bid increments, otherwise it

[9] See for example, Cramton [15].

is easy to circumvent them by raising the bids only slightly, from time to time, on the non-active object. Large minimum bid increments however also work against allocative efficiency: they may prevent a buyer from getting an object when she has the highest value and her opponent's value is not much smaller.[10]

Moreover, even severe activity rules may not be sufficient to eliminate all collusive equilibria. In particular, if "jump bidding"[11] is allowed, equilibrium outcomes that are close to the one of Proposition 2 can be obtained in fewer periods by compressing the competition on the object that they both rank higher (say v), by bidding at each round more than the minimum increment on v and remaining silent (or raising the bid just the minimum increment, if necessary to bypass the activity rules) on w. For a complete analysis of the issue however, it is necessary to specify exactly the activity rules introduced and look at the equilibrium set of the resulting game form, a task beyond the scope of this paper.

The three equilibria that we have identified so far, i.e. the SEA equilibrium and the equilibria described in Propositions 1 and 2, can be ranked both in terms of expected social surplus and bidders' interim expected surplus. The SEA equilibrium outcome is socially efficient, but generates the lowest bidders' surplus. The expected social surplus decreases, while the expected surplus of each bidder, conditional on any realization of her type, increases, as we move to the equilibria of Propositions 1 and then 2. This is because the equilibrium of Proposition 1 entails a lower degree of collusion: if the bidders happen to prefer the same object, they open with the same bids, thus triggering the SEA strategies, which generate efficient allocations. In the equilibrium of Proposition 2 the bidders refrain from using the SEA strategies even after learning that the objects are ranked in the same way. Instead, they continue searching for a way of buying one object each, while keeping the prices as low as possible.

3.1.2 Maximizing Bidders' Surplus.

In this section we provide a partial characterization of the bidder's interim-efficient frontier within the set of all allocations in which each buyer is always awarded exactly one object, and buyers with type $(0, 0)$ receive zero surplus. We interpret the first feature as a 'no regret condition', which may arise when the buyers are trying to implement a collusive scheme in multi-object auction environments: only if a buyer obtains at least one object will she be willing to follow a

[10]For example, suppose that an object is value 0.5 by the first bidder and 0.6 by the second bidder. Suppose also that the minimum increment is 0.15. If the first bidder is currently winning the object with a bid of 0.46, then the second bidder gives up, causing an inefficient allocation.

[11]Jump bidding in one-object English auctions has been studied by Avery [7] and Daniel and Hirshleifer [17].

collusive strategy, and refrain from triggering competition on both objects.

We show that both Pesendorfer's ranking mechanism and the allocation implied by the equilibrium of Proposition 2 can maximize a weighted sum of the expected surplus of all bidder's types among the allocations that satisfy incentive compatibility and the 'no regret' condition requiring that each buyer always obtains one object.

We point out that the question we are tackling in this section is not whether the equilibrium described in Proposition 2 is 'the best' for the bidders within the equilibrium set of the particular simultaneous ascending bid auction that we analyze in this paper. Rather, in this section we provide results that can be used to address the following question: If the bidders were free to design the trading procedure so to implement an incentive compatible, and no regret allocation of the two objects, what allocation would they choose?

It is convenient at this point to reparametrize the model so that buyer i's type becomes the pair (a_i, w_i), where $a_i := v_i - w_i$. The joint density $\xi(a_i, w_i)$, obtained from the joint density of (v_i, w_i), has support

$$\Theta = \Theta^+ \cup \Theta^-$$

where the sets

$$\Theta^+ := \left\{(a, w) \in [0,1]^2 \ \mid w \leq 1 - a\right\}$$

and

$$\Theta^- := \{(a, w) \in [-1, 0] \times [0, 1] \mid \ -a \leq w\}$$

correspond to the triangles below and above the diagonal respectively in the (v, w) space.

A direct mechanism consists of three functions for each buyer $i = 1, 2$, specifying the probability $q_V^i(a_1, w_1, a_2, w_2)$ of obtaining object v, the probability $q_W^i(a_1, w_1, a_2, w_2)$ of obtaining object w and her payment to the seller $m^i(a_1, w_1, a_2, w_2)$, for any type realization (a_1, w_1, a_2, w_2) in Θ^2.

The resulting expected surplus for buyer i conditional on having type (a_i, w_i), and reporting her true type, can be written as

$$U^i(a_i, w_i) \ \equiv \ a_i \ Q_V^i(a_i, w_i) \ + \ w_i \ X^i(a_i, w_i) \ - \ M^i(a_i, w_i)$$

where

$$\begin{aligned}
Q_V^i(a_i, w_i) \ &\equiv \ \int\int_\Theta q_V^i(a_i, w_i, a_{3-i}, w_{3-i}) \ \xi(a_{3-i}, w_{3-i}) \ dw_{3-i} \, da_{3-i}, \\
X^i(a_i, w_i) \ &\equiv \ \int\int_\Theta \left[q_V^i(a_i, w_i, a_{3-i}, w_{3-i}) + q_W^i(a_i, w_i, a_{3-i}, w_{3-i})\right] \ \xi(a_{3-i}, w_{3-i}) \ dw_{3-i} \, da_{3-i}, \\
M^i(a_i, w_i) \ &\equiv \ \int\int_\Theta m^i(a_i, w_i, a_{3-i}, w_{3-i}) \ \xi(a_{3-i}, w_{3-i}) \ dw_{3-i} \, da_{3-i}.
\end{aligned}$$

Restricting attention to allocations in which each buyer is always awarded exactly one object yields the constraint

$$q_V^i\left(a_i, w_i, a_{3-i}, w_{3-i}\right) + q_W^i\left(a_i, w_i, a_{3-i}, w_{3-i}\right) = 1, \quad \text{all } \left(a_i, w_i, a_{3-i}, w_{3-i}\right) \in \Theta^2,$$

hence

$$X^i\left(a_i, w_i\right) = 1, \quad \text{all } \left(a_i, w_i\right),$$

for each i, and the buyer's expected surplus becomes

$$U^i\left(a_i, w_i\right) = a_i\, Q_V^i\left(a_i, w_i\right) + w_i - M^i\left(a_i, w_i\right).$$

Finally, since the buyers are ex ante symmetric, we can focus without additional loss of generality on symmetric mechanisms, hence drop all subscripts and superscripts "i" in the expression for each buyer's expected surplus.

An allocation is interim efficient within a given feasible set if it maximizes a weighted sum of all types' expected surplus; i.e.

$$\int\int_{\Theta} \psi\left(a, w\right)\, U\left(a, w\right)\, da\, dw, \tag{3}$$

for some function $\psi : \Theta \to \mathbb{R}_+$. In particular, if $\psi\left(a, w\right) = \xi\left(a, w\right)$, the joint density of $\left(a, w\right)$, the objective function becomes the ex ante expected surplus.

Let $\psi_A\left(a\right) \equiv \int_0^{1-a} \psi\left(a, w\right)\, dw$, and $\Psi_A\left(a\right) \equiv \int_0^a \psi_A\left(t\right)\, dt$, and without loss of generality normalize the function ψ so that $\Psi_A\left(1\right) = 1$. Recall that g denotes the density of a.

Proposition 3 *Consider the class of all incentive compatible allocation such that each buyer always receives one object, and $U\left(0,0\right) = 0$. The weighted sum in (3) is maximized within this class by the equilibrium of Proposition 2 if the ratio*

$$\frac{1 - \Psi_A\left(a\right)}{g\left(a\right)} \text{ is increasing for } a \geq 0,$$

and by the ranking mechanism if

$$\frac{\int_0^a \left[1 - \Psi_A\left(t\right)\right] dt}{2\int_0^1 \left[1 - \Psi_A\left(t\right)\right] dt} \geq \int_0^a g\left(t\right) dt.$$

Proof. See Appendix. ■

To illustrate, suppose that f is uniform, hence $g\left(a\right) = 1 - a$ for $a \geq 0$, and the weighting function is such that $\psi_A\left(a\right) = 1 + s\left(a - \frac{1}{2}\right)$, with $s \in \left[-2, 2\right]$. Then the equilibrium of Proposition 2 is optimal for $s > 0$, and the ranking mechanism is optimal if $s < 0$.[12]

[12] Proposition 3 is similar to Theorem 1 in McAfee and McMillan [35]. Their result however applies to the maximization of ex-ante expected bidders' surplus, for the single object case.

3.2 More than Two Bidders

The equilibria described in Propositions 1 and 2 may seem to rely heavily on the fact that the number of bidders is equal to the number of objects. However, some degree of collusion is still possible even when there are more bidders than objects. The basic idea is that the bidders can follow the SEA strategy until only 2 players are left, and then adopt the strategies described in Propositions 1 or 2 to divide the objects.

Proposition 4 *If there are $n > 2$ bidders and the c.d.f. $F(x)$ satisfies $E[x|x \geq z] \geq \frac{1+z}{2}$ for each $z \in [0,1]$, then the following strategy, together with some consistent belief system, forms a (symmetric) perfect Bayesian equilibrium:*

- *Round 1: If $v_i \geq w_i$, open with $\{0, -\infty\}$, otherwise open with $\{-\infty, 0\}$;*
- *Round t: if more than two bidders were active at round $t-1$, all types use the SEA strategy.*

  If instead at round $t-1$ only i and $j \neq i$ were active, and bidder j opened with $\{-\infty, 0\}$, then types (v_i, w_i) such that $v_i \geq w_i$ raise the bid on v. Types (v_i, w_i) such that $v_i \leq w_i$ use a symmetric strategy if j opened with $\{0, -\infty\}$.
- *If the observed history of bids is not obtained according to the strategies previously described, then all types revert to the SEA strategy.*

A family of c.d.f.'s which satisfies the condition $E[x|x \geq z] \geq \frac{1}{2}(1+z)$ for each $z \in [0,1]$ is $F(x) = x^{\alpha}$, with $\alpha \geq 1$. In this case we have

$$E(x|x \geq z) = \frac{\alpha}{\alpha+1}\frac{1-z^{\alpha+1}}{1-z^{\alpha}}$$

and the inequality can be written as:

$$\frac{\alpha}{\alpha+1}\frac{1-z^{\alpha+1}}{(1-z^{\alpha})(1+z)} \geq \frac{1}{2} \qquad \text{i.e.} \qquad \frac{\alpha-1}{2\alpha} \geq \frac{(z-z^{\alpha})}{1-z^{\alpha+1}+z-z^{\alpha}}.$$

It can be checked that the RHS is increasing in z, for $z \in [0,1]$, and converging to the LHS as $z \to 1$.

The equilibrium of Proposition 4 works as follows. Each bidder opens signaling how she ranks the two objects. After that, the prices of the two objects start increasing in parallel. If at some point, say when the price of both objects is z, only two players are still active, and they rank the objects differently, then they stop bidding and each buys the preferred object at z. Collusion in this equilibrium becomes less effective as the number of bidders increases, for two reasons. First, the probability of collusion is lower. Second, even if collusion occurs, the price paid is in general higher.

To have an idea of the impact of the number of players on the possibility of collusion, we compute the probability that collusion occurs as a function of the number of bidders. With only two bidders, collusion occurs when the rankings are different, that is:

$$\Pr[\text{collusion}] = \Pr[v_1 < w_1, v_2 > w_2] + \Pr[v_1 > w_1, v_2 < w_2] = \frac{1}{2}$$

With n bidders, collusion occurs when two bidders have the highest two valuations for each objects and they rank the objects differently. For example, the probability that bidders 1 and 2 are able to collude is:

$$\begin{aligned}\Pr[\text{1,2 collude}] &= \Pr[\max\{v_3,\ldots,v_n\} < \min\{v_1,v_2\}] \cdot \Pr[\max\{w_3,\ldots,w_n\} < \min\{w_1,w_2\}] \cdot \frac{1}{2} \\ &= \frac{1}{2}\Pr[\max\{v_3,\ldots,v_n\} < \min\{v_1,v_2\}]^2,\end{aligned}$$

where we have exploited the assumptions of independence and identical distribution among players and objects.

Since the density of $\min\{v_1,v_2\}$ is $2f(y)[1-F(y)]$ and the c.d.f. of $\max\{v_3,\ldots,v_n\}$ is $F^{n-2}(y)$, we have:

$$\begin{aligned}\Pr(\max\{v_3,\ldots,v_n\} < \min\{v_1,v_2\}) &= \int_0^1 F^{n-2}(y)\ 2f(y)[1-F(y)]\,dy \\ &= 2\left[\int_0^1 F^{n-2}(y)\ f(y)\ dy - \int_0^1 F^{n-1}(y)\ f(y)\ dy\right] \\ &= 2\left[\frac{1}{n-1}F^{n-1}(y) - \frac{1}{n}F^n(y)\Big|_0^1\right] = \frac{2}{n(n-1)},\end{aligned}$$

which implies

$$\Pr[\text{1,2 collude}] = \frac{2}{n^2(n-1)^2}.$$

There are $\binom{n}{2}$ possible combinations of 2 bidders, hence with n bidders we have:

$$\Pr[\text{collusion}] = \binom{n}{2}\frac{2}{n^2(n-1)^2} = \frac{1}{n(n-1)}.$$

With $n = 2$, we obtain $\frac{1}{2}$, the same result as before. The probability of collusion decreases rapidly, reaching 5% when $n = 5$. It is worth noting that collusion essentially disappears when there are 5 players in the experiments carried on by Kwasnica and Sherstyuk [28].

The equilibrium of Proposition 2 can also be extended to the case of $n > 2$ bidders.

Proposition 5 *Suppose that there are $n > 2$ bidders and the c.d.f. F is such that for each pair (a, z) such that $z \in [0, 1]$ and $a \in [0, 1 - z]$ the two following conditions are satisfied:*

$$E\left(x\,|z \leq x \leq 1 - a\right) + E\left(y\,|z + a \leq y \leq 1\right) \geq 1 + z \tag{4}$$

$$E\left(x \mid x - y \geq a,\ y \geq z\right) + E\left(y \mid x - y \geq a \geq z,\ x \geq a + z\right) \geq 1 + z. \tag{5}$$

Then the following strategy is part of a symmetric perfect Bayesian equilibrium: Behave as in Proposition 4 except at the following point:

- *If at round $t - 1$ only you and another bidder were active then:*
- *If $v_i \geq w_i$ and you opened with $\{0, -\infty\}$ while the other bidder opened with $\{-\infty, 0\}$ then increase the bid on v and not on w, then stop.*
- *If $v_i < w_i$ and you opened with $\{-\infty, 0\}$ while the other bidder opened with $\{0, -\infty\}$ then increase the bid on w and not on v, then stop.*
- *If both players opened with $\{0, -\infty\}$ and z was the last offer for both objects then increase the bid on v up to $z + a_i$, while keeping the offer for w at z. If the other bidder offers more than $z + a_i$ then get w for z. Otherwise, get v at the price at which competition ends, and leave w to the other bidder.*

Conditions (4) and (5) are also satisfied by the uniform distribution. This equilibrium works as the one of Proposition 4: the bidders start signalling which object they prefer and then push up both prices until only two players are left. The difference is that at that point the same strategies as in Proposition 2 are used: if bidders have opened showing that they rank the two objects in the same way, then they compete only on the top ranked object. The stopping point for each player is $z + a_i$, that is the last bid plus the difference between the two values.

As a final comment to this section, we observe that the equilibria described in Propositions 4 and 5 may be vulnerable to the imposition of anonymity rules. For example, the auction format may specify that at each round only the two best bids are announced, and it is not revealed who made the bid. This obviously reduces the signaling possibilities for the bidders. However, even if anonymity rules are applied it may be possible to find ways to signal the relevant information. Consider the following variant of the equilibrium established in Proposition 4. In the first stage only bidder 1 makes an offer. In the second stage only bidder 2 makes an offer, and so on up to stage n. In this way, in the first n rounds the ranking of the objects of each bidder is made public. At stage $n + 1$ bidder 1 moves again, either increasing the bid on both objects or on a single object. In the first case the bidder signals that she is still available for collusion, while in the second case the signal is that she is dropping out of the race for the other object and is no longer available for collusions. At stage $n+2$ bidder two moves and so on. In this equilibria anonymity rules are completing ineffective in hindering collusion, and the same allocation as in Proposition 4 can be implemented as a perfect Bayesian equilibrium. A similar reasoning applies to the equilibrium of Proposition 5.

4 Collusive Equilibria with Large Complementarities

In this section we consider the case of complementarities, and we restrict the attention to the case of two bidders.[13] When complementarities are present, the value of the bundle is greater than the sum of the 'stand alone' values. As mentioned in Section 2, we define $u_i(1) = v_i$, $u_i(2) = w_i$ and $u_i(\{1,2\}) = v_i + w_i + k_i$, $i = 1, 2$, and we maintain the assumption that v_i and w_i are drawn from a symmetric distribution with support $[0,1]^2$, marginal density f, and marginal c.d.f. F. We also assume that for each player $i = 1, 2$, the value of the complementarity k_i is drawn from a distribution with continuous density f_k and support over an interval $\left[\underline{k}, \overline{k}\right]$. Each random variable k_i is independent of (v_j, w_j, k_j) for each $j \neq i$.

Finding equilibria in the presence of complementarities is complicated by the fact that, at any given round of the auction, a bidder's willingness to pay for a given object depends on the probability of winning the other object. This destroys the 'belief-free' nature of the SEA equilibrium described in Proposition 0. We can show however that, if the complementarities are commonly known to be 'large', in a sense to be made precise, then a "competitive" equilibrium similar to the one found in Proposition 0 can be obtained. Define $\theta_i := v_i + w_i + k_i$, the total value of the bundle for bidder i.

Proposition 6 *With n players, 2 objects, and $\underline{k} > 1$, there exists a perfect Bayesian equilibrium in which the two objects are allocated to the bidder with the highest θ_i, at a price equal to the second highest valuation (i.e. $\max_{j \neq i} \theta_j$).*

The basic intuition here is as follows. Under the assumptions of Proposition 6, if the buyers compete on both objects, then the auction cannot end with each bidder buying just one object. The reason is that if a bidder has won one object then the value of the other object is at least

[13] Menicucci [33] has characterized revenue maximizing auctions in an extension of Armstrong's model which allows for complemetarities in the buyers' utility functions. He finds that in general optimal auctions are not efficient.

$\underline{k} > 1$. This is more than the stand-alone value of any bidder. Therefore, all bidders behave as if they were bidding for a single object, the bundle $\{v, w\}$.

This of course is not true if there are moderate complementarities, i.e. $k_i \in (0, 1)$. In this case it is difficult to characterize even the "competitive" equilibria, similar to the ones of Propositions 0 and 6, which could be used as threat to sustain more collusive outcomes.[14]

The equilibrium of Proposition 6 can be used as a threat to sustain collusive equilibria when large complementarities are present. The next Proposition establishes the existence of an equilibrium which yields a higher expected surplus for both bidders. Define:

$$\Theta_v := \left\{ (v, w, k) \in [0,1]^2 \times \left[\underline{k}, \overline{k}\right] \;\middle|\; v > w \right\}$$

and

$$\Theta_w := \left\{ (v, w, k) \in [0,1]^2 \times \left[\underline{k}, \overline{k}\right] \;\middle|\; v \leq w \right\}.$$

Proposition 7 *There exist two sets $A_v \subset \Theta_v$ and $A_w \subset \Theta_w$ such that the following strategy, together with some belief system, forms a (symmetric) PBE:*

- *Types $(v_i, w_i, k) \in \left\{[0,1]^2 \times \left[\underline{k}, \overline{k}\right]\right\} \setminus \{A_v \cup A_w\}$ open with $\{0, 0\}$ and compete for both objects;*
- *Types $(v_i, w_i) \in A_w$ open with $\{-\infty, 0\}$*
- *Types $(v_i, w_i) \in A_v$ open with $\{0, -\infty\}$.*
- *If the initial bids are $\{\{0, -\infty\}, \{-\infty, 0\}\}$ or $\{\{-\infty, 0\}, \{0, -\infty\}\}$ then bidders do not place any further bid. For all other opening bids having positive probability in accordance to the strategy described above, the bidders play the equilibrium described in Proposition 6.*
- *If, at any stage, a bidder makes a bid which cannot be observed if the strategy above described is followed, then the bidders play the equilibrium described in Proposition 6.*

The sets A_v and A_w are symmetric in the sense that $(v, w, k) \in A_v$ if and only if $(w, v, k) \in A_w$.

[14] To the best of our knowledge, the only results available so far in the moderate complemetatiries case are due to Sherstyuk (2000), who has shown that, for any common complemetarity term $k \geq 0$, there exists a competitive equilibrium, i.e. an allocation of the objects and a price pair (p_v, p_w) such that demand equals supply.

The equilibrium of Proposition 7 is a natural generalization of the equilibrium described in Proposition 1. The set of types of each bidder is partitioned into three subsets. The first subset consists of those types who cannot be induced to collude. These are the types who have very low stand-alone values for each object; hence they are only interested in having the two objects together, and are not interested in having a single object, even at a very low price. To illustrate, suppose that bidder 1's type is $(0, 0, k_1)$, and recall that $\theta_i := v_i + w_i + k_i$, for $i = 1, 2$. If bidder 1 accepts to buy only one object at price zero, her utility is zero. On the other hand, the expected surplus from competing for both objects is $(k_1 - E[\theta_2 \mid \theta_2 \leq k_1]) \Pr(\theta_2 \leq k_1)$, which is positive, although possibly small. It is clear that types like $(\varepsilon_1, \varepsilon_2, k)$, for ε_1 and ε_2 sufficiently small, will also be unwilling to collude.

However, types with a stand-alone value for v sufficiently high are in fact willing to collude. In particular, assume that bidder 1 has type $(v_1, w_1, k_1) \in \Theta_v$, i.e. with $v_1 > w_1$, and suppose that at the first round bidder 1 learns that her opponent's type lies in some subset $A_w \subset \Theta_w$. Then collusion is better than competition if:

$$v_1 \geq \int_{\underline{k}}^{\theta_1} (\theta_1 - \theta_2) \ dH(\theta_2 \,|(v_2, w_2, k_2) \in A_w) \tag{6}$$

where H denotes the conditional c.d.f. of θ_2. In equilibrium, the set A_v will be exactly the set of those types for whom the inequality in (6) is satisfied. A similar inequality will define A_w. In equilibrium the two sets A_v and A_w have to be defined simultaneously. It is intuitive from inequality (6) that the two sets will be symmetric.

The shape of the set A_v is roughly as follows. Suppose that bidder 1 has $v_1 \geq w_1$. Let us summarize the type of bidder 1 by the pair (v_1, θ_1), with $v_1 \in [0, 1]$ and $\theta_1 \in [\underline{k}, 2 + \overline{k}]$. It is clear that if the pair (v_1^*, θ_1^*) satisfies inequality (6) then all pairs (v_1^*, θ_1) with $\theta_1 < \theta_1^*$ will also satisfy the inequality. The inequality is also satisfied by the types characterized by the pair $(0, \underline{k})$. This type has no use for a single object, but is also sure to lose the competition for the two objects. Thus, (6) holds with equality. It is also clear that all types characterized by pairs like $(v_1, v_1 + \underline{k})$ are willing to collude. These are types for whom $w_1 = 0$ and have the lowest possible value for the synergy. If they compete for both objects they pay at least $\underline{k}$ (the lowest possible value for θ_2), and receive less utility than v_1, which is what they would get accepting collusion. In general, for a given v_1 there will be a corresponding value $\theta_1(v_1)$ such that types with $\theta_1 < \theta_1(v_1)$ are willing to accept collusion and types with $\theta_1 > \theta_1(v_1)$ prefer to compete for both objects rather than to accept collusion. The shape of the set A_v is thus similar to the one showed in figure 1.

INSERT FIGURE 1 HERE

One important question that Proposition 7 does *not* answer is how likely is the occurrence of collusion. In the proposed equilibrium, collusion occurs whenever the types belong to the sets A_v and A_w. Figure 1 suggest the shape of these sets, but it is hard to say how large is the area that they represent. In fact, we are only able to prove that the sets A_v and A_w are non-empty, but not they their measure is different from zero[15].

It is clear however that, at least in some cases, the sets A_v and A_w are significant ones, so that collusion is actually a relevant phenomenon even with large complementarities. One particularly simple and striking case is the one in which the extent of the complementarity is known and identical across bidders, i.e. the distribution of k_i is degenerate on some value $k^* \geq 1$. In this case, provided that the condition $E(x) \geq \frac{1}{2}$ holds, the strategies proposed in Proposition 1 are still equilibrium strategies. In other words, the set A_v and A_w described in Proposition 7 can be taken to be Θ_v and Θ_w respectively, when the complementarities k_i are known and identical across bidders. The intuition is straightforward. If k_i is the same for each bidder, then it will be entirely competed away whenever the equilibrium of Proposition 6 is triggered. This makes any attempt to get both objects unattractive, hence even types with very low 'stand-alone' values can be induced to collude.

We conclude this section by reconsidering the conjecture according to which collusion decreases when complementarities are present. We have shown that the presence of complementarities does not destroy collusion. In fact, we have seen that large complementarities which are known and common among the players do not reduce the possibility of collusion at all. What really matters in hindering collusion is the *variability* of the extent of complementarities, rather than their absolute values.

5 Conclusions

When dynamic auctions are used to sell multiple objects, buyers can collude in order to reduce their payments to the seller. A general feature of collusive equilibria in open ascending bid auctions is that each bidder signals to the others which object has the highest value to her. After the signaling rounds, the bidders implicitly promise each other not to compete on the objects that they value less, provided they are not challenged on the objects they value more. We have provided conditions under which this behavior can be made a perfect Bayesian equilibrium. We have also shown that some degree of collusion may still be present when the ratio of bidders to objects is high, and when the bidders' utility functions exhibit high complementarities.

[15] We thank the associate editor for pointing this out.

As a more general point, the set of equilibria in auctions with multiple objects appears to be much richer than in the single object case. In this paper, we have shown some of these equilibria. It is worth pointing out that in all equilibria in which collusion-via-signalling occurs it must be the case that not too much information is revealed by the equilibrium bidding strategy. To see this, suppose, for example, that the bidding strategy were to reveal that one bidder has very low values for both objects. Then the other bidder will decide to compete for both objects, i.e. to revert to the SEA strategies, since her expected payments on both objects will be low. A bidder with high values will accept a collusive outcome only if the information revealed is such that her expected payment in open competition is sufficiently high. But this must imply that there is always some pooling among low and high values. This in turn implies that in general collusion-via-signalling not only reduces the revenue to the seller, but also reduces the efficiency of the final allocation.

Appendix

Propositions 1 and 2 are special cases, with $z = 0$, of Propositions 4 and 5 respectively. The proof is given below, after the proof of Proposition 3.

Proof of Proposition 3.

Since each buyer is ex-ante symmetric, we can restrict attention to symmetric mechanisms, looking only at the 'lower triangle':

$$L := \left\{(v, w) \in [0,1]^2 \mid w \leq v\right\}.$$

In the (a, w) space this corresponds to the set Θ^+.

The following result in mechanism design theory, which we record here as Lemma 0, provides a useful characterization of the (IC) constraints.

Lemma 0 *The functions Q_V, X and M satisfy (IC) if and only if:*

1. *the surplus functions $U(a, w)$ is convex, hence differentiable almost everywhere, and continuous;*

2. $\frac{\partial U(a,w)}{\partial a} = Q_V(a, w)$ *and* $\frac{\partial U(a,w)}{\partial w} = X(a, w)$*, almost everywhere.*

Proof. See Armstrong [2], and Rochet and Choné [40]. ■

The key simplifying step follows from the restriction that each buyer always gets exactly one object.

Lemma 1 *In any mechanism in which each buyer always gets exactly one object the interim assignment function $Q_V(a, w)$ must satisfy:*

$$Q_V(a, w) = Q_V(a, 0)$$

for almost all (a, w).

Proof. By Lemma 0, the difference between the surplus of type (a, w) with $a > 0$ and the surplus of type $(0, 0)$ can be written in two alternative ways:

- the integral along the "backward L" shape path: first from $(0, 0)$ to $(a, 0)$, and then from $(a, 0)$ to (a, w);

- the integral along the path "inverse L" shape: first from $(0,0)$ to $(0,w)$, and then from $(0,w)$ to (a,w)[16].

Integrating along the first path yields

$$\begin{aligned} U(a,w) - U(0,0) &= \int_0^a Q_V(\alpha,0)\, d\alpha + \int_0^w X(a,y)\, dy \\ &= \int_0^a Q_V(\alpha,0)\, d\alpha + w, \end{aligned}$$

while using the second path we obtain

$$\begin{aligned} U(a,w) - U(0,0) &= \int_0^w X(0,y)\, dy + \int_0^a Q_V(\alpha,w)\, d\alpha \\ &= w + \int_0^a Q_V(\alpha,w)\, d\alpha. \end{aligned}$$

From these equalities we have

$$\int_0^a Q_V(\alpha,0)\, d\alpha = \int_0^a Q_V(\alpha,w)\, d\alpha,$$

which in turn immediately implies the result, since the point (a,w) was arbitrary. ■

Writing the surplus function as

$$U(a,w) = \int_0^a Q(\alpha)\, d\alpha + w + U(0,0), \quad \text{all } (a,w),$$

where $Q(\alpha) \equiv Q_V(\alpha,0)$, and substituting into the objective function yields

$$\begin{aligned} \int_0^1 \int_0^{1-a} \psi(a,w)\, U(a,w)\, dw\, da &= \int_0^1 \int_0^{1-a} \psi(a,w) \left(\int_0^a Q(\alpha)\, d\alpha \right) dw\, da \\ &+ \int_0^1 \int_0^{1-a} \psi(a,w)\, w\, dw\, da \\ &+ U(0,0) \left(\int_0^1 \int_0^{1-a} \psi(a,w)\, dw\, da \right). \end{aligned}$$

The first term can be written as:

$$\begin{aligned} \int_0^1 \int_0^{1-a} \psi(a,w) \left(\int_0^a Q(\alpha)\, d\alpha \right) dw\, da &= \int_0^1 \psi_A(a) \left(\int_0^a Q(\alpha)\, d\alpha \right) da \\ &= \int_0^1 [1 - \Psi_A(a)]\, Q(a)\, da. \end{aligned}$$

[16] In the (v,w) space, the first path goes horizontally from $(0,0)$ to $(a,0)$, and then from $(a,0)$ to (v,w), along a 45^0 line. The second path goes from $(0,0)$ to (w,w) along the 45^0 line, and then horizontally from (w,w) to (v,w).

(recall that $\psi_A(a) \equiv \int_0^{1-a} \psi(a,w)\, dw$ and $\Psi_A(a) \equiv \int_0^a \psi_A(\alpha)\, d\alpha$, and the weighting function is normalized so that $\Psi_A(1) = \int_0^1 \psi_A(\alpha)\, d\alpha = 1$).

Setting $U(0,0) = 0$ and ignoring the term $\int_0^1 \int_0^{1-a} \psi(a,w)\, w\, dw\, da$, which does not depend on the control variable, we can focus our attention on the following "relaxed" program:

$$\text{Maximize } \int_0^1 [1 - \Psi_A(a)]\, Q(a)\, da$$

subject to

$$Q(a) \text{ is nondecreasing,} \tag{7}$$

and

$$\int_a^1 Q(\alpha)\, g(\alpha)\, d\alpha \leq \int_a^1 G(\alpha)\, g(\alpha)\, d\alpha, \quad \text{each } a \in [0,1]. \tag{8}$$

The constraint in (7) is implied by Lemma 0, i.e. by the convexity of any incentive compatible surplus function. The constraints in (8) come from the fact that there is only one unit of object v, hence the probability of selling the object to a buyer whose values' difference is above any given threshold a cannot exceed the probability that at least one buyer has difference above a.[17]

The following two steps will conclude the proof. First, we show that the assignment function $Q^{**}(a) \equiv \frac{3}{4}$ induced by the ranking mechanism solves the above program for some positive weighting function. Then we establish that the assignment function $Q^*(a) \equiv G(a)$ induced by the equilibrium of Proposition 2 is optimal, if the ratio $\frac{1-\Psi(a)}{g(a)}$ is increasing.

Optimality of the ranking mechanism. For $a = 0$, the inequality in (8) is

$$\int_0^1 Q(a)\, g(a)\, da \leq \frac{3}{8},$$

since, by symmetry, $G(0) = \frac{1}{2}$ and $\int_0^1 G(\alpha)\, g(\alpha)\, d\alpha = \left[\frac{1}{2}(G(a))^2\right]_0^1 = \frac{1}{2} - \frac{1}{2}[G(0)]^2 = \frac{3}{8}$. For any differentiable function $\mu : [0,1] \to \mathbb{R}_+$, this inequality and the constraints in (7) imply

$$-\int_0^1 \mu(a)\, dQ(a) + \int_0^1 Q(a)\, g(a)\, da \leq \frac{3}{8}$$

or, equivalently

$$\mu(1)\, Q(1) - \mu(0)\, Q(0) + \int_0^1 [\mu'(a) + g(a)]\, Q(a)\, da \leq \frac{3}{8}.$$

[17] See Border [11].

In particular, by the condition stated in Proposition 3, we can choose:

$$\mu(a) = \frac{\int_0^a [1 - \Psi_A(t)]\, dt}{2 \int_0^1 [1 - \Psi_A(a)]\ da} - \int_0^a g(t)\, dt, \quad a \in [0,1], \tag{9}$$

and the previous inequality becomes

$$\int_0^1 [1 - \Psi_A(a)]\ Q(a)\ da \leq \int_0^1 \frac{3}{4}\ [1 - \Psi_A(a)]\ da. \tag{10}$$

The function $Q^{**}(a) = \frac{3}{4}$ is optimal because it maximizes the objective function among all functions which satisfy the weaker condition (10).

Optimality of the equilibrium in Proposition 2. Multiplying each inequality in (8) by a positive weight $\phi(a)$ and integrating over $[0,1]$ yields

$$\int_0^1 \phi(a) \left(\int_a^1 Q(\alpha)\ g(\alpha)\ d\alpha \right) da \leq \int_0^1 \phi(a) \left(\int_a^1 G(\alpha)\ g(\alpha)\ d\alpha \right) da,$$

or, exchanging the order of integration,

$$\int_0^1 \Phi(a)\ Q(a)\, g(a)\, da \leq \int_0^1 \Phi(a)\ G(a)\, g(a)\, da,$$

where $\Phi(a) := \int_0^a \phi(\alpha)\ d\alpha$. Since $\frac{1-\Psi_A(a)}{g(a)}$ is increasing, we can choose the weighting function ϕ to be both positive and such that $\Phi(a) = \frac{1-\Psi_A(a)}{g(a)}$; hence the previous inequality becomes

$$\int_0^1 [1 - \Psi_A(a)]\ Q(a)\ da \leq \int_0^1 [1 - \Psi_A(a)]\ G(a)\, da. \tag{11}$$

It is now immediate to see that $Q(a) \equiv G(a)$ satisfies the feasibility constraints in (8), and maximizes the objective function among all functions which satisfy the weaker condition in (11). Hence it is an optimal solution in the original program.

Proof of Proposition 4. Given the symmetry of the problem, it is enough to check the optimality of the strategy for types having $v \geq w$. We will do this proceeding backward.

Consider the first round, t, at which only two bidders remain, say 1 and 2. Suppose that bidder 1 has $v_1 \geq w_1$ and has opened at round zero with $\{0, -\infty\}$, while bidder 2 has opened with $\{-\infty, 0\}$. Suppose also that the outstanding pair of bids at round $t-1$ is (z, z). Let $F_V(v_2 \,|\, T_z)$ and $F_W(w_2 \,|\, T_z)$ denote the c.d.f.. of v_2 and w_2 respectively, both conditional on the set $T_z := \left\{ (v_2, w_2) \in [0,1]^2 \mid z \leq v_2 \leq w_2 \right\}$.

If bidder 1 changes her bids, then the SEA strategies are triggered, and her expected utility is:

$$S\left(v_1, w_1 \,|\, T_z\right) = \int_z^{v_1} \left(v_1 - v_2\right)\, dF_V\left(v_2 \,|\, T_z\right) + \int_z^{w_1} \left(w_1 - w_2\right)\, dF_W\left(w_2 \,|\, T_z\right).$$

To check that the deviation is unprofitable, we have to verify that

$$v_1 - z \geq S\left(v_1, w_1 \,|\, T_z\right)$$

for each pair (v_1, w_1) such that $v_1 \geq w_1$. Since $S\left(v_1, \cdot \,|\, T_z\right)$ is increasing, it suffices to check the inequality for the types on the diagonal, i.e. types such that $v_1 = w_1$. Defining:

$$\gamma_z\left(v_1\right) \equiv S\left(v_1, v_1 \,|\, T_z\right), \quad v_1 \in [z, 1],$$

the inequalities to be checked are:

$$v_1 - z \geq \gamma_z\left(v_1\right), \text{ for each } v_1 \in [z, 1].$$

We start by noting that this holds at $v_1 = z$, since both sides are zero; and then observe that the derivative of the LHS with respect to v_1 is 1, while the RHS derivative

$$\gamma_z'\left(v_1\right) = \int_z^{v_1} dF_V\left(v_2 \,|\, T_z\right) + \int_z^{v_1} dF_W\left(w_2 \,|\, T_z\right),$$

is zero at $v_1 = z$, and both positive and increasing for each $v_1 \in (z, 1]$. Thus the function $\gamma_z\left(v_1\right)$ is increasing and convex, hence we are done if we can prove that

$$1 - z \geq \gamma_z(1).$$

This can be rewritten as:

$$1 - z \quad \geq \quad E\left[1 - v_2 \,|\, T_z\right] + E\left[1 - w_2 \,|\, T_z\right] \quad = \quad 2 - E\left[v_2 \,|\, T_z\right] - E\left[w_2 \,|\, T_z\right],$$

or, using the symmetry of the joint distribution of v_2 and w_2, as

$$E\left[v_2 \,|\, T_z\right] + E\left[v_2 \,|\, L_z\right] \geq 1 + z, \tag{12}$$

where $L_z := \left\{(v_2, w_2) \in [0,1]^2 \mid z \leq w_2 \leq v_2\right\}$. By symmetry, we have

$$\frac{1}{2} = \Pr\left(T_z \,|\, z \leq v_2, z \leq w_2\right) = \Pr\left(L_z \,|\, z \leq v_2, z \leq w_2\right),$$

hence

$$E\left[v_2 \,|\, T_z\right] + E\left[v_2 \,|\, L_z\right] = 2E\left[v_2 \,|\, z \leq v_2, z \leq w_2\right];$$

and, since v_2 and w_2 are independent, we have $E[v_2 \mid z \le v_2, z \le w_2] = E[v_2 \mid z \le v_2]$, so that the inequality in (12) can be written as:

$$E(v_2 \,| z \le v_2, z \le w_2) \ge \frac{1}{2}(1+z).$$

This is the condition stated in the Proposition, and we can therefore conclude that the bidders will collude when the opportunity arises.

The optimality of the strategies when more than two bidders are left follows from the fact that any other strategy simply destroys the opportunity of collusion should it arise, and does not improve the outcome otherwise.

The only thing which is left to show is that in the first round each bidder is willing to signal truthfully the triangle in which her type is. This is going to matter for bidder i only if she is still bidding after $n-2$ other bidders have dropped out and the only other bidder who is still bidding is competing for both objects. We show that for any given z at which this may happen it is better to have announced the correct triangle at date 0.

If bidder 1 announces the correct triangle, then the expected payoff conditional on being one of the two last bidders, and on z being the last bid for both bidders, is:

$$\frac{1}{2}(v_1 - z) + \frac{1}{2} S(v_1, w_1 \mid T_z) \tag{13}$$

This is because, given the symmetry in the distributions of v and w for each i, with probability $\frac{1}{2}$ the opponent is of type $w_2 \ge v_2$, so that her initial bid is $\{-\infty, 0\}$, and with probability $\frac{1}{2}$ the opponent is of type $v_2 \ge w_2$. In the first case the auction ends immediately, yielding a payoff $v_1 - z$, while in the second case bidders go on playing the SEA equilibrium.

If the bidder opens with $\{-\infty, 0\}$ then the expected payoff conditional on being one of the two last players and both having valuation at least z for both objects is:

$$\frac{1}{2}(w_1 - z) + \frac{1}{2} S(v_1, w_1 \mid L_z) \tag{14}$$

(notice that now S is conditional to $v_2 \ge w_2$ rather than to $v_2 \le w_2$). The expression in (14) does not exceed the one in (13) if

$$v_1 + S(v_1, w_1 \mid T_z) \ge w_1 + S(v_1, w_1 \mid L_z),$$

which holds with equality if $v_1 = w_1$. Moreover, the derivatives with respect to v_1 are

$$1 + F_V(v_1 \mid T_z)$$

for the LHS, and

$$F_V(v_1 \mid L_z)$$

for the RHS. Hence the LHS grows faster than the RHS as v_1 is increased, thus implying that the inequality holds for each $v_1 > w_1$. ■

Proof of Proposition 5. Again, because of symmetry it suffices to check the optimality of the strategy along the equilibrium path for a bidder whose type is in the 'lower triangle.' We proceed backward.

Suppose first that only two players are left, say 1 and 2. If 1 opened with $\{0, -\infty\}$ and 2 opened with $\{-\infty, 0\}$, then the analysis of Proposition 4 applies, since condition (4) implies $E(x\,|z \leq x \leq 1) \geq (1+z)/2$ for $a = 0$, hence deviating to the SEA strategy is not profitable.

If instead both bidders have opened with $\{0, -\infty\}$, then we have to show that bidder 1 with type $v_1 - w_1 = a_1$ is willing to raise the bid on the first object only if she is not assigned object v and the outstanding bids are (p, z) with $p < a_1 + z$. There are two possible deviations from the equilibrium path:

1) Stop bidding on v, and raise the bid on w by a small amount if necessary, i.e. if 1 is not currently assigned w. This deviation yields at most $w_1 - z$. Define

$$L_z(p - z) = \left\{(v_2, w_2) \in [z, 1]^2 \,|\, p - z \leq v_2 - w_2,\; z \leq w_2\right\}.$$

The set $L_z(p)$ is the support of bidder 1's beliefs about 2's values conditional on the last round's bids being (p, z) for each bidder. The expected utility from following the equilibrium strategy is:

$$\begin{aligned} U^*(v_1, w_1|L_z(p-z)) &= \Pr\{a_2 \leq a_1 \,|\, L_z(p-z)\}\,(v_1 - E[a_2 \,|\, a_2 \leq a_1,\, L_z(p-z)]) \\ &\quad + \Pr\{a_2 \geq a_1 \,|\, L_z(p-z)\}\,(w_1 - z), \end{aligned}$$

which can be written as:

$$U^*(v_1, w_1|L_z(p-z)) = w_1 - z + \Pr\{a_2 \leq a_1 \,|\, L_z(p-z)\}\,(a_1 - E[a_2 \,|\, a_2 \leq a_1,\, L_z(p-z)] + z).$$

It is clear that the last expression is higher than $w_1 - z$.

2) Raise the bid on w, without stopping the bidding on v. In this case, the SEA equilibrium is triggered and we have to verify that:

$$U^*(v_1, w_1|L_z(p-z)) \geq S(v_1, w_1|\, L_z(p-z))$$

It is enough to check the inequality at $p = a_1 + z$. Triggering the SEA equilibrium before p reaches that level can only do worse.

Using $v_1 = w_1 + a_1$, the relevant inequality to be checked is therefore:

$$w_1 - z \geq \int_{a_1+z}^{w_1+a_1} (w_1 + a_1 - v_2)\, dF_V(v_2|\, L_z(a_1+z)) + \int_z^{w_1} (w_1 - w_2)\, dF_W(w_2|\, L_z(a_1+z))$$

The inequality is satisfied at $w_1 = z$ and the RHS is increasing and convex. Applying the same reasoning as in Proposition 4 we conclude that it is enough to check the inequality:

$$1 - a_1 - z \geq \int_{a_1+z}^{1} (1 - v_2)\, dF_V \left(v_2 \middle| L_z (a_1 + z) \right) + \int_{z}^{1-a_1} (1 - a_1 - w_2)\, dF_W \left(w_2 \middle| L_z (a_1 + z) \right)$$

where use is made of the fact that the highest possible value for w_1 when $v_1 - w_1 \geq a_1$ is $1 - a_1$. The inequality is equivalent to:

$$E \left[v_2 \middle| v_2 \geq a_1 + w_2, 1 - a_1 \geq w_2 \geq z \right] + E \left[w_2 \middle| v_2 - a_1 \geq w_2 \geq z, v_2 \geq a_1 + z \right] \geq 1 + z$$

or:

$$E \left[x \mid x \geq a + y,\ y \geq z \right] + E \left[y \mid x - y \geq a \geq z,\ x \geq a + z \right] \geq 1 + z$$

which is inequality 5 stated in the Proposition.

Finally, we check that a bidder wants to stop after the other bidder has stopped the bidding, rather than competing for both objects. Suppose that the bidder has $v_1 - w_1 = a$ and the other bidder stopped at $z + a'$ with $a' \leq a$. In this case define:

$$\overline{\Omega}_{a',z} = \left\{ (v_2, w_2) \in [z, 1]^2 \mid v_2 - w_2 = a' \right\}.$$

Then the inequality becomes:

$$v_1 - a' - z \geq \int_{a'+z}^{v_1} (v_1 - v_2)\, dF_V \left(v_2 \middle| \overline{\Omega}_{a',z} \right) + \int_{z}^{w_1} (w_1 - w_2)\, dF_W \left(w_2 \middle| \overline{\Omega}_{a',z} \right)$$

Using $w_1 = v_1 - a$ we can rewrite the inequality as:

$$v_1 - a' - z \geq \int_{a'+z}^{v_1} (v_1 - v_2)\, dF_V \left(v_2 \middle| \overline{\Omega}_{a',z} \right) + \int_{z}^{v_1-a} (v_1 - a - w_2)\, dF_W \left(w_2 \middle| \overline{\Omega}_{a',z} \right)$$

Again, the inequality is satisfied at $v_1 = z + a$, the RHS is increasing and convex and we have only to check:

$$1 - a' - z \geq \int_{a'+z}^{1} (1 - v_2)\, dF_V \left(v_2 \middle| \overline{\Omega}_{a',z} \right) + \int_{z}^{1-a} (v_1 - a - w_2)\, dF_W \left(w_2 \middle| \overline{\Omega}_{a',z} \right)$$

In order to compute the integrals observe:

$$\Pr \left(v_2 \leq x \,| v_2 = w_2 + a', v_2 \geq z, w_2 \geq z \right) = \Pr \left(w_2 \leq x - a' \,| 1 - a' \geq w_2 \geq z \right)$$

$$= \frac{\Pr \left(x - a' \geq w_2 \geq z \right)}{\Pr \left(1 - a' \geq w_2 \geq z \right)} = \begin{cases} \frac{F(x-a') - F(z)}{F(1-a') - F(z)} & \text{if} \quad x \geq z + a' \\ 0 & \text{otherwise} \end{cases}$$

Therefore:

$$f\left(v_2\left|v_2=w_2+a', v_2\geq z, w_2\geq z\right.\right)=\begin{cases}\frac{f(v_2-a')}{F(1-a')-F(z)} & \text{if} \quad v_2\geq z+a' \\ 0 & \text{otherwise}\end{cases}$$

Similar computations lead to:

$$f\left(w_2\left|v_2=w_2+a', v_2\geq z, w_2\geq z\right.\right)=\begin{cases}\frac{f(w_2+a')}{1-F(z+a')} & \text{if} \quad 1-a'\geq w_2\geq z \\ 0 & \text{otherwise}\end{cases}$$

We therefore have:

$$\int_{a'+z}^{1}(1-v_2)\,dF_V\left(v_2\right|\overline{\Omega}_{a',z}\left.\right)=1-\frac{\int_{a'+z}^{1}v_2 f\left(v_2-a'\right)dv_2}{F\left(1-a'\right)-F\left(z\right)}$$

and

$$\frac{\int_{a'+z}^{1}v_2 f\left(v_2-a'\right)dv_2}{F\left(1-a'\right)-F\left(z\right)}=\frac{\int_{z}^{1-a'}\left(y+a'\right)f\left(y\right)dy}{F\left(1-a'\right)-F\left(z\right)}=E\left(x\left|z\leq x\leq 1-a'\right.\right)+a'$$

Similarly, we have:

$$\int_{z}^{1-a}(1-a-w_2)\,dF_W\left(w_2\right|\overline{\Omega}_{a',z}\left.\right)=\frac{\int_{z}^{1-a}\left(1-a-w_2\right)f\left(w_2+a'\right)dw_2}{1-F\left(z+a'\right)}$$

$$=(1-a)\frac{F\left(1-(a-a')\right)-F\left(a'+z\right)}{1-F\left(a'+z\right)}-\frac{\int_{z}^{1-a}w_2 f\left(w_2+a'\right)dw_2}{1-F\left(z+a'\right)}$$

and

$$\frac{\int_{z}^{1-a}w_2 f\left(w_2+a'\right)dw_2}{1-F\left(z+a'\right)}=\frac{\int_{z+a'}^{1-(a-a')}\left(y-a'\right)f\left(y\right)dy}{1-F\left(z+a'\right)}$$

$$\frac{\int_{z+a'}^{1-(a-a')}\left(y-a'\right)f\left(y\right)dy}{1-F\left(z+a'\right)}=\frac{\int_{z+a'}^{1}yf\left(y\right)dy}{1-F\left(z+a\right)}-a$$

Combining these results we obtain the following condition:

$$\begin{aligned}1-a'-z \;\geq\; & 1-E\left(x\left|z\leq x\leq 1-a'\right.\right)-a' \\ & +(1-a)\frac{F\left(1-(a-a')\right)-F\left(a'+z\right)}{1-F\left(a'+z\right)}-\frac{\int_{z}^{1-a}w_2 f\left(w_2+a'\right)dw_2}{1-F\left(z+a'\right)}\end{aligned}$$

The inequality has to hold for each $a\geq a'$. Noticing that the RHS is decreasing in a, the relevant condition is obtained setting $a=a'$. This yields:

$$E\left[x \mid x-y\geq a,\ y\geq z\right]+E\left[y \mid x-y\geq a\geq z,\ x\geq a+z\right]\geq 1+z$$

which is inequality 4 stated in the Proposition.

The argument for optimality when more than three bidders are active is identical to the one of Proposition 4: there is no point in triggering the SEA strategies at the opening, since the decision can always be taken later.

The only thing that remain to be proved is that it is convenient to open in the 'true' triangle. Possible deviations in this case are opening in the 'wrong' triangle or opening bidding on both objects, thus triggering the SEA equilibrium. The initial bid is only relevant if the bidder ends up among the two last bidders. We will show that for every z, and conditional on being one of the two last bidders, opening in the 'true' triangle gives a higher expected utility than any deviation.

The expected utility conditional on being one of the two remaining bidders at z for a type (v_1, w_1) such that $v_1 - w_1 = a_1 \geq 0$ is:

$$U^{\{0,-\infty\}} = \frac{1}{2}v_1 + \frac{1}{2}\left(w_1 + \Pr\left(a_2 \leq a_1\right)\left(a_1 - E\left(a_2 \left| a_2 \leq a_1\right.\right)\right)\right) - z \tag{15}$$

where $a_2 = v_2 - w_2$ and the probability distribution is conditional to $v_2 \geq z, w_2 \geq z$. This is because with probability $\frac{1}{2}$ the other bidder has opened in the upper triangle, so that the auction ends and 1 obtains v_1 at price z, while with probability $\frac{1}{2}$ the other bidder opens in the lower triangle. In the latter case the bidder pays at least z and obtains at least w_1 It additionally obtains a_1 minus the price when the auction is won. Triggering the SEA equilibrium with an opening other than $\{-\infty, 0\}$ or $\{0, -\infty\}$ is obviously dominated, since the SEA equilibrium can be triggered later at no cost. We have therefore only to check that it is not convenient to open in the wrong triangle.

Suppose 1 opens bidding $\{-\infty, 0\}$, i.e. signaling the 'wrong' triangle. If the other bidder also opens with $\{-\infty, 0\}$ then the best strategy is to pretend to have $a_1 = 0$ and get v for z. This is clearly better than getting w for a price greater than z. The other possibility is to trigger the SEA strategies: To show that this cannot be optimal we have to check the inequality:

$$v_1 - z \geq S\left(v_1, w_1 \middle| L_z\right)$$

Under the assumptions stated in the Proposition the inequality is satisfied (the analysis is the same as before).

If the other bidder opens with $\{0, -\infty\}$ then any attempt to compete on good v triggers the SEA equilibrium. The payoff in this case is therefore whatever is best between obtaining w_1 at z and triggering the SEA equilibrium, that is $\max\left\{w_1 - z, S\left(v_1, w_1 \middle| L_z\right)\right\}$. We therefore conclude that the expected payoff, conditional on being one of the two players left at z, when the opening is in the wrong triangle is:

$$U^{\{-\infty,0\}} = \frac{1}{2}\left(v_1 - z\right) + \frac{1}{2}\max\left\{w_1 - z, S\left(v_1, w_1 \middle| L_z\right)\right\} \tag{16}$$

If $w_1 - z \geq S(v_1, w_1 | L_z)$ then this is clearly less that the utility obtained in equilibrium. If $w_1 - z < S(v_1, w_1 | L_z)$ the condition that the deviation be not profitable, that is (15) is greater than (16), can be written as:

$$w_1 + \Pr(a_2 \leq a_1)(a_1 - E(a_2 | a_2 \leq a_1)) - z \geq S(v_1, w_1 | L_z)$$

which is satisfied under the conditions stated in the Proposition because it is equivalent to the condition that it is optimal to follow the equilibrium strategy after opening in the 'true' triangle. ■

Proof of Proposition 6. Recall that $\theta_i := v_i + w_i + k_i$, $i = 1, 2$, and let the bids of each bidder be represented as a pair, with the first element referring to object v and the second to object w. Also, let b_v and b_w denote the highest bids on v and w respectively, and b_v^i and b_w^i the highest bids by bidder i on v and w respectively. The following is a symmetric perfect Bayesian equilibrium yielding the desired outcome:

- Open with the minimum bid on each object. In the following rounds, if you are bidder i, behave as follows:

- If all bidders but you have been silent on v, and at least two bidders have increased the bid on w in the previous round, then stay silent on v, and increase the bid on w by the minimum amount if $b_w < w_i + k_i$; otherwise stay silent;

- If all bidders but you have been silent on w, and at least two bidders have increased the bid on v in the previous round, then stay silent on w, and increase the bid on v by the minimum amount if $b_v < v_i + k_i$;otherwise stay silent;

- In all other cases raise the bid on both v and w by the minimum amount if $b_v + b_w < v_i + w_i + k_i$ and there is a positive probability that $v_i + w_i + k_i > v_j + w_j + k_j$ for all $j \neq i$. Stay silent otherwise.

Beliefs are as follows.

Case 1. The outstanding bid is $b_v = b_w$. The probability distribution on (v_j, w_j, k_j) is defined as follows:

- If $(b_v^j, b_w^j) = (b_v, b_w)$ (that is, bidder j is among the winners of both objects) then it is the conditional probability on (v_j, w_j, k_j) subject to $v_j + w_j + k_j \geq 2b_v$.

- If $(b_v^j, b_w^j) \neq (b_v, b_w)$ then:

– If $b_v^j = b_w^j$ and $2b_v^j > \underline{k}$ then it is the conditional probability on (v_j, w_j, k_j) subject to $v_j + w_j + k_j < 2b_v^j$.

– In all other cases, it is an arbitrary probability distribution with support on $[0, \min\{b_v^j, 1\}] \times [0, \min\{b_w^j, 1\}] \times \{\underline{k}\}$.

Case 2. The outstanding bid is $b_v \neq b_w$. This can only occur if all players have taken an out of equilibrium action. When this happens, we specify that beliefs about any bidder who made an offer (b_v, b_w) with $b_v \neq b_w$ have support on $[0, \min\{b_v^j, 1\}] \times [0, \min\{b_w^j, 1\}] \times \{\underline{k}\}$, while the probability distribution on bidders such that $b_v^j = b_w^j$ is the conditional probability on (v_j, w_j, k_j) subject to $v_j + w_j + k_j < 2b_v^j$.

The outcome of this strategy profile is that each bidder i increases the bids by the minimum amount on both objects up to the point at which the sum of the bids reaches θ_i, and stops bidding afterwards. Therefore, the bidder with the highest θ_i wins the objects paying a price equal to $\max_{j \neq i} \theta_j$.

We now check that there are no profitable deviations. Suppose first that in the previous round all bidders but i have been silent on v but some has increased the bid on w. This can only happen out of equilibrium. Any bidders j who stayed on the equilibrium path must have $\theta_j < b_v + b_w$, so the equilibrium strategy prescribes that she will not bid further. Any bidder j' who was out of the equilibrium is not expected to increase the bid any further, since (given the specified beliefs) with probability 1 their type $\theta_{j'}$ is not the highest. Therefore, bidder i expects no further bids on v. Once v is taken for granted, it is rational to bid on w up to $w_i + k_i$. A symmetric reasoning applies when i is the sole active bidder on w and not on v. Finally, it is obvious that staying silent is i's optimal strategy if all her opponents have been silent on both objects.

Consider now the other cases. By following the equilibrium strategy, bidder i obtains utility $\max\{\theta_i - \max_{j \neq i} \theta_j,\ 0\}$. A deviation can lead to getting both objects, getting a single object and getting no object. In the first case, given the equilibrium strategy of the other bidders, the price paid for the two objects must be at least $\max_{j \neq i} \theta_j$, hence the deviation is not profitable. In the second case, the price paid for the single object is at least $\underline{k}$, thus again the deviation is not profitable. Finally, a deviation is obviously not profitable if it leads to losing both objects. ■

Proof of Proposition 7. Using the arguments of Proposition 6 we have that the strategies described in the last point of the Proposition constitute a perfect Bayesian equilibrium at any given stage. We are left with the task of finding the appropriate sets A_v, A_w, show that the prescribed strategy is optimal for all types at stage 0, and that for types in A_v, A_w it is optimal to stop bidding when the initial bids are $(\{0,-\infty\},\{-\infty,0\})$ or $(\{0,-\infty\},\{-\infty,0\})$.

Let

$$\Theta_v = \left\{ (v,w,k) \middle| \, v \in [0,1], w \in [0,v], k \in \left[\underline{k}, \overline{k}\right] \right\}$$

and

$$\Theta_w = \left\{ (v,w,k) \middle| \, v \in [0,w], w \in [0,1], k \in \left[\underline{k}, \overline{k}\right] \right\}.$$

Define $s \equiv v + w + k$, and let $H(s)$ be the c.d.f. on s, that is:

$$H(x) = \Pr\{v + w + k \leq x\}$$

Given our assumption on the support of v, w and k it is clear that $H(\underline{k}) = 0$ and $H\left(2 + \overline{k}\right) = 1$. Furthermore, given the symmetry of (v, w) and the independence of the distributions of v, w, k we have that $H(s|\,\Theta_v) = H(s|\,\Theta_w)$. Define the sets $A_v^0 = \Theta_v$, $A_w^0 = \Theta_w$, and define:

$$A_v^1 = \left\{ (v,w,k) \in \Theta_v \middle| \, v \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s)\; dH(s|\,\Theta_w) \right\}$$

$$A_w^1 = \left\{ (v,w,k) \in \Theta_w \middle| \, w \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s)\; dH(s|\,\Theta_v) \right\}$$

Thus, A_v^1 is the set of types in Θ_v who prefer to have v for free rather than competing for the bundle when it is known that the type of the other bidder lies in Θ_w. A symmetric interpretation holds for A_w^1. Observe that the sets A_v^1 and A_w^1 are compact and connected.

It is clear that the two sets are symmetric, meaning that if $(a,b,c) \in A_v^1$ then $(b,c,a) \in A_w^1$ Furthermore, it is also clear that $H(s|\,A_w^1) = H(s|\,A_v^1)$. Now, given two symmetric sets A_v^n and A_w^n with the property that $H(s|\,A_w^n) = H(s|\,A_v^n)$ define the sets:

$$A_v^{n+1} = \left\{ (v,w,k) \in \Theta_v \middle| \, v \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s)\, dH(s|\,A_w^n) \right\}$$

$$A_w^{n+1} = \left\{ (v,w,k) \in \Theta_w \middle| \, w \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s)\, dH(s|\,A_v^n) \right\}$$

If A_v^n and A_w^n are compact and connected then A_v^{n+1} and A_w^{n+1} are also compact and connected. We claim that the sequence $\{A_v^n\}$ has a converging subsequence, and that the set A_v to which the subsequence converges is the set we are looking for.

Let $\mathcal{H}(\Theta_v)$ be the set of non-empty compact subsets of Θ_v. For a given set $X \in \mathcal{H}(\Theta_v)$ define the set:

$$B_\varepsilon(X) = \{y \in \Theta_v \,| |y - x| < \varepsilon \text{ for some } x \in X\}$$

The space $\mathcal{H}(\Theta_v)$ is a metric space when endowed with the Hausdorff distance:

$$\rho(X,Y) = \min\{\varepsilon > 0 \,|X \subset B_\varepsilon(Y) \text{ and } Y \subset B_\varepsilon(X)\}$$

Since the set Θ_v is compact, the set $\mathcal{H}(\Theta_v)$ is also compact (see e.g. Mas Colell (1985), Proposition A.5.1). The sequence $\{A_v^n\}$ is a sequence of elements in $\mathcal{H}(\Theta_v)$, and since the set is compact there exists a converging subsequence. Let A_v be the non-empty, compact subset of Θ_v to which the subsequence converge, and observe that since all elements in $\{A_v^n\}$ are connected then A_v is connected too (Mas Colell (1985), Proposition A.5.1). The set A_w can be obtained using exactly the same procedure.

The sets A_v and A_w satisfy the equilibrium conditions. Observe first that for each s and n we have $H(s|\, A_v^n) - H(s|\, A_w^n) = 0$ This implies that for each s:

$$\lim_{n\to\infty} H(s|\, A_v^n) - H(s|\, A_w^n) = H(s|\, A_v) - H(s|\, A_w) = 0 \tag{17}$$

Consider now that a type $(v, w, k) \in A_v$. The equilibrium strategy prescribes:

1. Open with $\{0, -\infty\}$.
2. If the other bidder opens with $\{-\infty, 0\}$ then stop bidding. In all other cases, use the SEA strategy.

Let us first check that the strategy after opening with $\{0, -\infty\}$ and observing $\{-\infty, 0\}$ is optimal. The only possible deviation is to trigger the SEA equilibrium, which yields:

$$S(v_1, w_1, k|\, A_w) = \int_{\underline{k}}^{v_1+w_1+k} (v_1 + w_1 + k - s)\, dH(s|\, A_w)$$

Using (17) and the fact that $(v_1, w_1, k) \in A_v$ we obtain:

$$v \geq S(v_1, w_1, k|\, A_w)$$

We now check optimality at stage 0. It clearly makes no sense to trigger the SEA strategy. The only other possible deviation is to bid $\{-\infty, 0\}$, thus signalling that the type belongs to A_w. It is not profitable to use the SEA equilibrium after the other type signals A_v, since this is equivalent to triggering directly the SEA equilibrium with probability 1, which we know not to be profitable. Suppose now that collusion is accepted. Then we compare the expected utility of the deviation:

$$\Pr(A_v)\, w_1 + (1 - \Pr(A_v))\, S(v_1, w_1, k_1 | \neg A_v)$$

with the expected utility of the equilibrium strategy:

$$\Pr(A_w)\, v_1 + (1 - \Pr(A_w))\, S(v_1, w_1, k_1 | \neg A_w)$$

But now observe that the symmetry of A_v and A_w implies $\Pr(A_v) = \Pr(A_w)$ and:

$$S(v_1, w_1, k_1 | \neg A_v) = S(v_1, w_1, k_1 | \neg A_w)$$

Since $v_1 \geq w_1$ we conclude that the deviation is not profitable.

A symmetric reasoning shows that types $(v_1, w_1, k_1) \notin A_v \cup A_w$ are not better off opening with $\{0, -\infty\}$ or $\{-\infty, 0\}$. In this case the bidder is going to trigger the SEA strategy no matter what the opening bid of the other bidder is, so that announcing $\{0, 0\}$ and triggering the SEA equilibrium from the very beginning is optimal. ■

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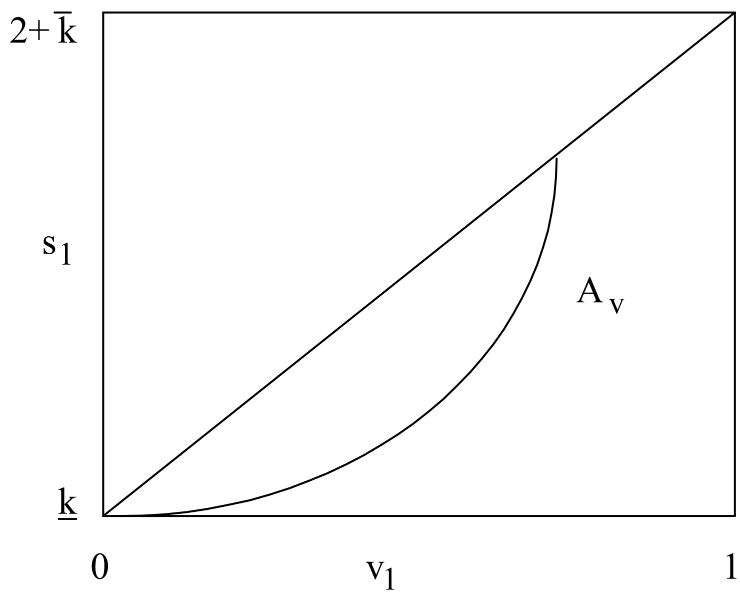

Figure 1: