WEIGHTED SHARING AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study with a weighted sharing method the uniqueness problem of \( f^n P(f) \) and \( g^n P(g) \) sharing one value and obtain some results which extend and improve the results due to Hong-Yan Xu and Ting-Bin Cao.

1. Introduction

Let \( f \) be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

\[ T(r, f), \ m(r, f), \ N(r, f), \ \overline{N}(r, f), \ldots \]

(See Hayman [3], Yang [6] and Yi and Yang [7]). We denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \),

as \( r \to +\infty \), possibly outside of a set with finite measure. For any constant \( 'a' \), we define

\[ \Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \]

Let \( 'a' \) be a finite complex number and \( k \) a positive integer. We denote by \( N_k\left(r, \frac{1}{f-a}\right) \) the counting function for the zeros of \( f(z) - a \) with the multiplicity \( \leq k \), and by \( \overline{N}_k\left(r, \frac{1}{f-a}\right) \) the corresponding one for which the multiplicity is not counted. Let \( N_k\left(r, \frac{1}{f-a}\right) \) be the counting function for the zeros of \( f(z) - a \) with multiplicity at least \( k \), and \( \overline{N}_k\left(r, \frac{1}{f-a}\right) \) be the corresponding one for which the multiplicity is not counted. Set

\[ \sum_{k=1}^{\infty} \overline{N}_k\left(r, \frac{1}{f-a}\right) \]

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We define
\[ \delta_k(a, f) = 1 - \limsup_{r \to \infty} \frac{N_k (r, \frac{1}{f-a})}{T(r, f)}. \]

Let \( g \) be a meromorphic function. If \( f(z) - a \) and \( g(z) - a \), assume the same zeros with the same multiplicities then we say that \( f(z) \) and \( g(z) \) share the value \( 'a' \) CM, where \( 'a' \) is a complex number. Similarly, we say that \( f \) and \( g \) share a IM, provided that \( f(z) - a \) and \( g(z) - a \) have same multiplicities.

In 1996, Fang proved the following result.

**Theorem A**([1]). Let \( f \) and \( g \) be two non-constant entire functions and let \( n, k \) be two positive integers with \( n > 2k + 4 \). If \([f^n]^{(k)}\) and \([g^n]^{(k)}\) share the value \( 1 \) CM, then either \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \) where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k (c_1 c_2)^n n c 2^k = 1 \) or \( f = t g \) for a constant \( t \) such that \( t^n = 1 \).

In 1997, Yang and Hua obtained a unicity theorem corresponding to above result.

**Theorem B**([8]). Let \( f \) and \( g \) be two nonconstant entire functions, \( n \geq 6 \) a positive integer. If \( f^n f' \) and \( g^n g' \) share \( 1 \) CM, then either \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \) where \( c_1, c_2 \) and \( c \) are three constants satisfying \((c_1 c_2)^{n+1} c^2 = 1 \) or \( f = t g \) for a constant \( t \) such that \( t^{n+1} = 1 \).

In 2002, Fang proved the following result.

**Theorem C**([2]). Let \( f \) and \( g \) be two non-constant entire functions and let \( n, k \) be two positive integers with \( n > 2k + 8 \). If \([f^n (f-1)]^{(k)}\) and \([g^n (g-1)]^{(k)}\) share the value \( 1 \) CM, then \( f \equiv g \).

In 2008, Zhang and Lin, Zhang, Chen and Lin extended Theorem C and obtain the following results.

**Theorem D**([10]). Let \( f \) and \( g \) be two non-constant entire functions and let \( n, m \) and \( k \) be three positive integers with \( n > 2k + m + 4 \), and \( \lambda, \mu \) be constants such that \(|\lambda| + |\mu| \neq 0 \). If \([f^n (\mu f^m + \lambda)]^{(k)}\) and \([g^n (\mu g^m + \lambda)]^{(k)}\) share \( 1 \) CM, then

(i) when \( \lambda \mu \neq 0 \), \( f \equiv g \).

(ii) when \( \lambda \mu = 0 \), either \( f \equiv t g \), where \( t \) is a constant satisfying \( t^{n+m} = 1 \), or \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \) where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k \lambda^2 (c_1 c_2)^{n+m} [(n + m) c]^{2k} = 1 \) or \((-1)^k \mu^2 (c_1 c_2)^{n+m} [(n + m) c]^{2k} = 1 \).

**Theorem E**([11]). Let \( f \) and \( g \) be two non-constant entire functions and let \( n, m \) and \( k \) be three positive integers with \( n > 2k + m + 4 \), and let \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 \) or \( P(z) \equiv c_0 \), where \( a_0 \neq 0 \), \( a_1, \ldots, a_{m-1}, a_m \neq 0 \), \( c_0 \neq 0 \) are complex constants. If \([f^n P(f)]^{(k)}\) and \([g^n P(g)]^{(k)}\) share \( 1 \) CM, then
Let $f$ and $g$ be two non-constant entire functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, either $f \equiv t g$ for a constant $t$ such that $t^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_1 \omega_1 + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_1 \omega_2 + a_0)$.

(ii) when $P(z) = c$, either $f(z) = c_1 / \sqrt[n]{c}^n e^{cz}$, $g(z) = c_2 / \sqrt[n]{c}^n e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f = t g$ for a constant $t$ such that $t^n = 1$.

In 2009, H.-Y. Xu and T.-B. Cao proved the following result.

**Theorem F**([5]). Let $f$ and $g$ be two nonconstant entire functions, and let $n$, $m$ and $k$ be three positive integers with $n \geq 5k + 5m + 8$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 0)$, then the conclusion of Theorem E still holds.

**Theorem G**([5]). Let $f$ and $g$ be two nonconstant entire functions, and let $n$, $m$ and $k$ be three positive integers with $n > \frac{5}{2} m + 4k + \frac{9}{2}$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 1)$, then the conclusion of Theorem E still holds.

**Theorem H**([5]). Let $f$ and $g$ be two nonconstant entire functions, and let $n$, $m$ and $k$ be three positive integers with $n \geq 3m + 3k + 5$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 2)$, then the conclusion of Theorem E still holds.

In this paper, by introducing the notion of multiplicity, we reduce and improve Theorems F, G and H. Also we extend these theorems to meromorphic functions and obtain the following results.

**Theorem 1.1.** Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0$, $(a_m \neq 0)$, and $a_1(i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and let $n$, $k$, $m$ be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 1)$ and one of the following conditions holds:

(i) $l \geq 2$ and $s(n + m) > 3k + 10$
(ii) $l = 1$ and $s(n + m) > 5k + 13$
(iii) $l = 0$ and $s(n + m) > 9k + 16$

then either $f = t g$ for a constant $t$ such that $t^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

**Theorem 1.2.** Let $f$ and $g$ be two non-constant entire functions, whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0$, where
(\(a_m \neq 0\)), and \(a_i (i = 0, 1, \ldots, m)\) is the first nonzero coefficient from the right, and let \(n, k, m\) be three positive integers. If \([f^n P(f)]^{(k)}\) and \([g^n P(g)]^{(k)}\) share \((1, 1)\) and one of the following conditions holds:

(i) \(l \geq 2\) and \(s(n + m) > 3k + 5\)
(ii) \(l = 1\) and \(s(n + m) > 4k + 6\)
(iii) \(l = 0\) and \(s(n + m) > 5k + 8\)

then either \(f = t g\) for a constant \(t\) such that \(t d = 1\), where \(d = (n + m, \ldots, n + m - i, \ldots, n)\), \(a_{m-i} \neq 0\) for some \(i = 0, 1, \ldots, m\), or \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) \equiv 0\), where

\[
R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2).
\]

**Remark.** In Theorem 1.2, giving specific values for \(s\), we get the following interesting cases:

(i) If \(s = 1\), then for \(l \geq 2\) we get \(n > 3k + 5 - m\), for \(l = 1\) we get \(n > 4k + 6 - m\) and for \(l = 0\) we get \(n > 5k + 8 - m\).

(ii) If \(s = 2\), then for \(l \geq 2\) we get \(n > \frac{3k + 5}{2} - m\), for \(l = 1\) we get \(n > 2k + 3 - m\) and for \(l = 0\) we get \(n > \frac{5k + 8}{2} - m\).

We conclude that if \(f\) and \(g\) have zeros and poles of higher order multiplicity, then we can reduce the value of \(n\).

2. Some Lemmas

**Lemma 2.1 ([3]).** Let \(f\) be a nonconstant meromorphic function, let \(k\) be a positive integer, and let \(c\) be a nonzero finite complex number. Then

\[
T(r, f) \leq \overline{N}(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{(k)} - c} \right) - N \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f)
\]

\[
\leq \overline{N}(r, f) + N_{k+1} \left( r, \frac{1}{f^{(k+1)}} \right) + N \left( r, \frac{1}{f^{(k)} - c} \right) - N_0 \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f).
\]

where \(N_0 \left( r, \frac{1}{f^{(k+1)}} \right)\) is the counting function which only counts those points such that \(f^{(k+1)} = 0\) but \(f^{(k)} - c \neq 0\).

**Lemma 2.2 ([9]).** Let \(f\) be a nonconstant meromorphic function and \(P(f) = a_0 + a_1 f + \cdots + a_n f^n\), where \(a_0, a_1, \ldots, a_n\) are constants and \(a_n \neq 0\). Then

\[
T(r, P(f)) = n T(r, f) + S(r, f).
\]
Lemma 2.3 ([4, 12]). Let \( f \) be a non-constant meromorphic function and \( k \) be a positive integer, then

\[
N_p \left( r, \frac{1}{f^{(k)}} \right) \leq N_{p+k} \left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f)
\]

\[
\leq (p + k)N \left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f).
\]

This Lemma can be obtained immediately from the proof of Lemma 2.3 in [4] which is the case \( p = 2 \).

Lemma 2.4 ([13]). Let \( F \) and \( G \) be two nonconstant meromorphic functions. If \( F \) and \( G \) share 1 IM, then \( N_L \left( r, \frac{1}{F} \right) \leq N \left( r, \frac{1}{F} \right) + N(r, F) + S(r, F) \).

Lemma 2.5 ([5]). Let \( f \) and \( g \) be two nonconstant entire functions, and let \( k \) be a positive integer. If \( f^{(k)} \) and \( g^{(k)} \) share \((1, l) \) \((l = 0, 1, 2) \). Then

(i) If \( l = 0 \),
\[
\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \Theta(0, g) + \delta_{k+2}(0, f) + \delta_{k+2}(0, g) > 5, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g;
\]

(ii) If \( l = 1 \),
\[
\frac{1}{2} \left[ \Theta(0, f) + \delta_k(0, f) + \delta_{k+2}(0, f) + \delta_{k+2}(0, f) + \Theta(0, g) + \delta_{k+1}(0, g) + \delta_{k+1}(0, g) > \frac{9}{2}, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g;\]

(iii) If \( l = 2 \),
\[
\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+2}(0, g) > 3, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g.
\]

Lemma 2.6. Let \( f \) and \( g \) be two nonconstant meromorphic functions, \( k \geq 1 \) and \( l \geq 0 \) be integers. If \( f^{(k)} \) and \( g^{(k)} \) share \((1, l) \) \((l = 0, 1, 2) \). Then

(i) If \( l \geq 2 \),
\[
(k + 2) \Theta(\infty, f) + 2 \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g;
\]

(ii) If \( l = 1 \),
\[
(2k + 3) \Theta(\infty, f) + 2 \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) > 2k + 9, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g;
\]

(iii) If \( l = 0 \),
\[
(2k + 3) \Theta(\infty, f) + (2k + 4) \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2 \delta_{k+1}(0, f) + 3 \delta_{k+1}(0, g) > 4k + 13, \text{ then either } f^{(k)} g^{(k)} = 1 \text{ or } f \equiv g.
\]

Proof. Let
\[
\Phi(z) = \left( \frac{f^{(k+2)}}{f^{(k+1)}} - 2 \frac{f^{(k+1)}}{f^{(k)} - 1} \right) - \left( \frac{g^{(k+2)}}{g^{(k+1)}} - 2 \frac{g^{(k+1)}}{g^{(k)} - 1} \right).
\]
Suppose that $\Phi(z) \neq 0$. If $z_0$ is a common simple 1-point of $f^{(k)}(z)$ and $g^{(k)}(z)$, substituting their Taylor series at $z_0$ into (2.1), we can get $\Phi(z_0) = 0$. Thus we have,

$$N_{E}^{(1)} \left( r, \frac{1}{f^{(k)} - 1} \right) = N_{E}^{(1)} \left( r, \frac{1}{g^{(k)} - 1} \right) \leq N \left( r, \frac{1}{\Phi} \right) \leq T(r, \Phi) + O(1)$$

$$\leq N(r, \Phi) + S(r, f) + S(r, g),$$

where $N_{E}^{(1)} \left( r, \frac{1}{f^{(k)} - 1} \right)$ denotes the counting function of common 1-points of $f^{(k)}$ and $g^{(k)}$.

According to our assumption, $\Phi(z)$ has simple poles only at zeros of $f^{(k+1)}$, $f^{(k)} - 1$ and $g^{(k+1)}$, $g^{(k)} - 1$ as well as poles of $f$ and $g$.

From Lemma 2.1, we have

$$T(r, f) + T(r, g) \leq N(r, f) + N(r, g) + N_{k+1} \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{g} \right)$$

$$+ N \left( r, \frac{1}{f^{(k)} - 1} \right) + N \left( r, \frac{1}{g^{(k)} - 1} \right) - N \left( r, \frac{1}{f^{(k+1)}} \right)$$

$$- N_{0} \left( r, \frac{1}{g^{(k+1)}} \right) + S(r, f) + S(r, g).$$

(2.3)

Obviously,

$$N \left( r, \frac{1}{f^{(k)} - 1} \right) \leq T(r, f^{(k)}) + O(1) \leq T(r, f) + kN(r, f) + S(r, f).$$

(2.4)

If $l \geq 2$, we have

$$N(r, \Phi) \leq N(r, f) + N \left( r, \frac{1}{f} \right) + N(r, g) + N \left( r, \frac{1}{g} \right) + N_{l+1} \left( r, \frac{1}{f^{(k)} - 1} \right)$$

$$+ N_{0} \left( r, \frac{1}{f^{(k+1)}} \right) + N_{0} \left( r, \frac{1}{g^{(k+1)}} \right).$$

(2.5)

and

$$N_{l+1} \left( r, \frac{1}{f^{(k)} - 1} \right) + N \left( r, \frac{1}{f^{(k)} - 1} \right) + N \left( r, \frac{1}{g^{(k)} - 1} \right)$$

$$\leq N \left( r, \frac{1}{f^{(k)} - 1} \right) + N \left( r, \frac{1}{g^{(k)} - 1} \right).$$

(2.6)

From (2.2)–(2.6) we deduce that

$$T(r, g) \leq (k + 2)N(r, f) + 2N(r, g) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) + N_{k+1} \left( r, \frac{1}{f} \right)$$

$$+ N_{k+1} \left( r, \frac{1}{g} \right) + S(r, f) + S(r, g).$$

Without loss of generality, we suppose that there exists a set $I$ with infinite linear measure such that $T(r, F) \leq T(r, G)$ for $r \in I$. Hence

$$T(r, g) \leq ([k + 2](1 - \Theta(\infty, f)) + 2(1 - \Theta(\infty, g)) + (1 - \Theta(0, f))$$
+ (1 − Θ(0, g)) + (1 − δ_{k+1}(0, f)) + (1 − δ_{k+1}(0, g)) + ε] T(r, g) + S(r, g),

for \( r \in I \) and \( 0 < ε < Δ_1 - (k + 7) \), that is \[ Δ_1 - (k + 7) - ε] T(r, g) \leq S(r, g).\]

ie.,

\[
Δ_1 \leq (k + 7),
\]

If \( l = 1 \), then

\[
N(r, Φ) \leq \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}(r, g) + \overline{N}\left( r, \frac{1}{g} \right) + \overline{N}_{(2)}\left( r, \frac{1}{f(k) - 1} \right) + N_0\left( r, \frac{1}{f(k+1)} \right).
\]

Obviously,

\[
\overline{N}\left( r, \frac{1}{f(k) - 1} \right) + \overline{N}\left( r, \frac{1}{g(k) - 1} \right) \leq N^{(1)}_{1}\left( r, \frac{1}{f(k) - 1} \right) + \overline{N}\left( r, \frac{1}{f(k) - 1} \right).
\]

Thus, we deduce from (2.2)−(2.4), (2.8) and (2.9) that

\[
T(r, g) \leq (k + 2)\overline{N}(r, f) + 2\overline{N}(r, g) + \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{g} \right) + N_{k+1}\left( r, \frac{1}{f} \right)
\]

\[
+ N_{k+1}\left( r, \frac{1}{g} \right) + \overline{N}_{(2)}\left( r, \frac{1}{f(k) - 1} \right) + S(r, f) + S(r, g).
\]

Note that \( l = 1 \), from Lemma 2.3, we have

\[
\overline{N}_{(2)}\left( r, \frac{1}{f(k) - 1} \right) \leq \overline{N}\left( r, \frac{1}{f(k+1)} \right) = N_{1}\left( r, \frac{1}{f(k+1)} \right)
\]

\[
\leq N_{k+2}\left( r, \frac{1}{f} \right) + (k + 1)\overline{N}(r, f) + S(r, f).
\]

The inequality (2.10) together with (2.11) yields

\[
T(r, g) \leq (2k + 3)\overline{N}(r, f) + 2\overline{N}(r, g) + \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{g} \right) + N_{k+1}\left( r, \frac{1}{f} \right)
\]

\[
+ N_{k+1}\left( r, \frac{1}{g} \right) + N_{k+2}\left( r, \frac{1}{f} \right) + S(r, f) + S(r, g).
\]

Hence

\[
T(r, g) \leq [(2k + 3)(1 − Θ(∞, f)) + 2(1 − Θ(∞, g)) + (1 − Θ(0, f))
\]

\[
+(1 − Θ(0, g)) + (1 − δ_{k+1}(0, f)) + (1 − δ_{k+1}(0, g)) + (1 − δ_{k+2}(0, f))
\]

\[
+ ε] T(r, g) + S(r, g),
\]

for \( r \in I \) and \( 0 < ε < Δ_2 - (2k + 9) \), that is \[ Δ_2 - (2k + 9) - ε] T(r, g) \leq S(r, g),\]

ie.,

\[
Δ_2 \leq (2k + 9).
\]
If \( l = 0 \), i.e., \( f^{(k)} \) and \( g^{(k)} \) share 1 IM, at this circumstance, we have
\[
N(r, \Phi) \leq \overline{N}(r, f) + \overline{N} \left( r, \frac{1}{f} \right) + \overline{N}(r, g) + \overline{N} \left( r, \frac{1}{g} \right) + \overline{N}_L \left( r, \frac{1}{f^{(k)} - 1} \right) \\
+ \overline{N}_L \left( r, \frac{1}{g^{(k)} - 1} \right) + N_0 \left( r, \frac{1}{f^{(k+1)}} \right) + N_0 \left( r, \frac{1}{g^{(k+1)}} \right). \tag{2.13}
\]
From Lemma 2.4, we have
\[
\overline{N}_L \left( r, \frac{1}{f^{(k)} - 1} \right) + 2\overline{N}_L \left( r, \frac{1}{g^{(k)} - 1} \right) \leq \overline{N}(r, f) + 2\overline{N}(r, g) + \overline{N} \left( r, \frac{1}{f} \right) \\
+ 2\overline{N} \left( r, \frac{1}{g} \right) + S(r, f) + S(r, g). \tag{2.14}
\]
From Lemma 2.3, we can deduce that
\[
\overline{N} \left( r, \frac{1}{f^{(k)}} \right) + 2\overline{N} \left( r, \frac{1}{g^{(k)}} \right) = N_1 \left( r, \frac{1}{f^{(k)}} \right) + 2N_1 \left( r, \frac{1}{g^{(k)}} \right) \\
\leq N_{k+1} \left( r, \frac{1}{f} \right) + 2N_{k+1} \left( r, \frac{1}{g} \right) + k\overline{N}(r, f) + 2k\overline{N}(r, g) + S(r, f) + S(r, g). \tag{2.15}
\]
When \( l = 0 \), we can get
\[
\overline{N} \left( r, \frac{1}{f^{(k)} - 1} \right) + \overline{N} \left( r, \frac{1}{g^{(k)} - 1} \right) \leq N_{E}^{(1)} \left( r, \frac{1}{f^{(k)} - 1} \right) + \overline{N}_L \left( r, \frac{1}{g^{(k)} - 1} \right) + N \left( r, \frac{1}{f^{(k)} - 1} \right). \tag{2.16}
\]
From (2.2)–(2.4) and (2.13)–(2.15) and the above inequality, we can obtain
\[
T(r, g) \leq (2k + 3)\overline{N}(r, f) + (2k + 4)\overline{N}(r, g) \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{g} \right) \\
+ 2N_{k+1} \left( r, \frac{1}{f} \right) + 3N_{k+1} \left( r, \frac{1}{g} \right) + S(r, f) + S(r, g). \tag{2.17}
\]
In the same way, we can also get
\[
T(r, g) \leq [(2k + 3)(1 - \Theta(\infty, f)) + (2k + 4)(1 - \Theta(\infty, g)) + (1 - \Theta(0, f)) \\
+ (1 - \Theta(0, g)) + 2(1 - \delta_{k+1}(0, f)) + 3(1 - \delta_{k+1}(0, g)) + \epsilon] T(r, g) + S(r, g),
\]
for \( r \in I \) and \( 0 < \epsilon < \Delta_3 - (4k + 13) \), that is \( |\Delta_3 - (4k + 13) - \epsilon| T(r, g) \leq S(r, g) \), ie.,
\[
\Delta_3 \leq (4k + 13), \tag{2.17}
\]
Hence, we get \( \Phi(z) \equiv 0 \), i.e.,
\[
\frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} = \frac{g^{(k+2)}}{g^{(k+1)}} - 2\frac{g^{(k+1)}}{g^{(k)} - 1}.
\]
Integration yields
\[
\frac{1}{f^{(k)} - 1} \equiv \frac{b g^{(k)} + a - b}{g^{(k+1)} - 1},
\]
where \( a \) and \( b \) are two constants and \( a \neq 0 \). By using the same argument as in [13], we can obtain \( f^{(k)} g^{(k)} \equiv 1 \) or \( f \equiv g \), we here omit the detail. The proof of Lemma 2.6 is completed.
**Lemma 2.7.** Let $f$ and $g$ be two non-constant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be integers. Then

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \neq 1.$$  

**Proof.** Let

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1. \tag{2.18}$$

Let $z_0$ be a zero of $f$ of order $p_0$. From (2.18) we get $z_0$ is a pole of $g$. Suppose that $z_0$ is a pole of $g$ of order $q_0$. Again by (2.18), we obtain $np_0 - k = nq_0 + mq_0 + k$,

i.e., $n(p_0 - q_0) = mq_0 + 2k.$

which implies that $q_0 \geq \frac{n - 2k}{m}$ and so we have $p_0 \geq \frac{n + m - 2k}{m}$.

Let $z_1$ be a zero of $f$ of order $p_1$, then $z_1$ is a zero of $[f^n P(f)]^{(k)}$ of order $p_1 - k$. Therefore from (2.18) we obtain $p_1 - k = nq_1 + mq_1 + k$

i.e., $p_1 \geq (n + m)s + 2k.$

Let $z_2$ be a zero of $f'$ of order $p_2$ that is not a zero of $fP(f)$, then from (2.18) $z_2$ is a pole of $g$ of order $q_2$. Again by (2.18) we get $p_2 - (k - 1) = nq_2 + mq_2 + k$

i.e., $p_2 \geq (n + m)s + 2k - 1.$

In the same manner as above, we have similar results for the zeros of $[g^n P(g)]^{(k)}$.

On other hand, suppose that $z_3$ is a pole of $f$. From (2.18), we get that $z_3$ is the zero of $[g^n P(g)]^{(k)}$.

Thus

$$\overline{N}(r, f) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - 1}\right) + \overline{N}\left(r, \frac{1}{g}\right)$$

$$\leq \frac{1}{p_0} N\left(r, \frac{1}{g}\right) + \frac{1}{p_1} N\left(r, \frac{1}{g - 1}\right) + \frac{1}{p_2} N\left(r, \frac{1}{g}\right)$$

$$\leq \left[\frac{m}{n + m - 2k} + \frac{1}{(n + m)s + 2k} + \frac{2}{(n + m)s + 2k - 1}\right] T(r, g) + S(r, g). \tag{2.19}$$

By second fundamental theorem and equation (2.19), we have

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - 1}\right) + \overline{N}(r, f)$$

$$\leq \frac{m}{n + m - 2k} N\left(r, \frac{1}{f}\right) + \frac{1}{(n + m)s + 2k} N\left(r, \frac{1}{f - 1}\right)$$

$$+ \left[\frac{m}{n + m - 2k} + \frac{1}{(n + m)s + 2k} + \frac{2}{(n + m)s + 2k - 1}\right] T(r, g) + S(r, g) + S(r, f).$$

$$T(r, f) \leq \left[\frac{m}{n + m - 2k} + \frac{1}{(n + m)s + 2k}\right] T(r, f)$$
$$T(r, g) \leq \left[ \frac{m}{n + m - 2k} + \frac{1}{(n + m)s + 2k} \right] T(r, g)$$

Similarly, we have

$$T(r, f) \leq \left[ \frac{m}{n + m - 2k} + \frac{1}{(n + m)s + 2k} \right] T(r, f)$$

Adding (2.20) and (2.21) we get

$$T(r, f) + T(r, g) \leq \left[ \frac{2m}{n + m - 2k} + \frac{2}{(n + m)s + 2k} + \frac{2}{(n + m)s + 2k - 1} \right] \{T(r, f) + T(r, g)\} + S(r, g) + S(r, f).$$

which is a contradiction. Thus Lemma proved.

3. Proofs of the Theorems

In this section we present the proofs of the main results.

**Proof of Theorem 1.1.** Let $F = f^n P(f)$ and $G = g^n P(g)$.

Consider

$$N_k(\frac{1}{F}) = N\left(r, \frac{1}{f^n P(f)} \right) \leq \frac{1}{s(n + m)} N\left(\frac{1}{F} \right) \leq \frac{2}{s(n + m)} [T(r, F) + O(1)].$$

$$\Theta(0, F) = 1 - \limsup_{r \to \infty} \frac{N\left(\frac{1}{F} \right)}{T(r, F)} \geq 1 - \frac{2}{s(n + m)}. \quad (3.1)$$

Similarly,

$$\Theta(0, G) \geq 1 - \frac{2}{s(n + m)}. \quad (3.2)$$

$$\Theta(\infty, F) = 1 - \limsup_{r \to \infty} \frac{N(r, F)}{T(r, F)} \geq 1 - \frac{1}{s(n + m)}. \quad (3.3)$$

Similarly,

$$\Theta(\infty, G) \geq 1 - \frac{1}{s(n + m)}. \quad (3.4)$$

Consider

$$N_{k+1}\left(\frac{1}{F} \right) = N_{k+1}\left(\frac{1}{f^n P(f)} \right) = (k + 1)\frac{1}{s(n + m)} \left[ T(r, F) + O(1) \right].$$

Next, we have

$$\delta_{k+1}(0, F) = 1 - \limsup_{r \to \infty} \frac{N_{k+1}\left(\frac{1}{F} \right)}{T(r, F)} \geq 1 - \frac{(k + 1)}{s(n + m)}. \quad (3.5)$$
Similarly,
\[ \delta_{k+1}(0, G) \geq 1 - \frac{(k+1)}{s(n + m)}. \] (3.6)

Case(i) If \( l \geq 2 \) and from (3.1) to (3.6) and also from Lemma 2.6, we get
\[ \Delta_1 = (k+2)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) \]
\[ > (k+8) - \frac{3k+10}{s(n+m)} \]
Since \( s(n+m) > 3k+10 \), we get \( \Delta_1 > k+7 \).

Therefore, by Lemma 2.6, we deduce that either \( F^{(k)}G^{(k)} \equiv 1 \) or \( F \equiv G \).

If \( F^{(k)}G^{(k)} \equiv 1 \), that is
\[ [f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0)]^{(k)}[g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0)]^{(k)} \equiv 1, \] (3.7)
then by Lemma 2.7 we can get a contradiction.

Hence, we deduce that \( F \equiv G \), that is
\[ f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0). \] (3.8)

Let \( h = \frac{f}{g} \). If \( h \) is a constant, then substituting \( f = gh \) in (3.8) we obtain
\[ a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m+1} - h^{n+m+1} - 1 + \cdots + a_0 g^n(h^n - 1) = 0, \]
which implies \( h^d = 1 \), where \( d = (n+m, \ldots, n+m-i, \ldots, n) \), \( a_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \).
Thus \( f \equiv tg \) for a constant \( t \) such that \( t^d = 1 \), where \( d = (n+m, \ldots, n+m-i, \ldots, n) \), \( a_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \).

If \( h \) is not a constant, then we know (3.8) that \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2) \).

Case(ii) If \( l = 1 \) and from (3.1) to (3.6) and also from Lemma 2.6, we get
\[ \Delta_2 = (2k+3)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) \]
\[ > (2k+10) - \frac{5k+13}{s(n+m)} \]
Since \( s(n+m) > 5k+13 \), we get \( \Delta_2 > 2k+9 \).

By continuing as in case(i), we get case(ii).

Case(iii) If \( l = 0 \) and from (3.1) to (3.6) and also from Lemma 2.6, we get
\[ \Delta_3 = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) \]
\[ > (4k+14) - \frac{9k+16}{s(n+m)} \]
Since $s(n + m) > 9k + 16$, we get $\Delta_2 > 4k + 13$.

By continuing as in case(i), we get case(iii).

This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Since $f$ and $g$ are entire functions we have $N(r, f) = \overline{N}(r, g) = 0$. Proceeding as in the proof of Theorem 1.1 we can easily prove Theorem 1.2.

**References**


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