Optimality and Robustness of the English Auction

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Abstract

In Milgrom and Weber's (1982) "general symmetric model," under few additional regularity conditions, the English auction maximizes the seller's expected profit within the class of all posterior-implementable trading procedures, and fails to do so among all interim incentive-compatible procedures in which 'losers do not pay.' These results suggest that appropriate notions of robustness and simplicity which imply the optimality of the English auction for a risk neutral seller must impose "bargaining-like" features to the set of feasible trading mechanisms. *Journal of Economic Literature* Classification Numbers: D44, D82.

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1 Introduction

Under what conditions is the English auction optimal for a seller who aims at maximizing her expected profit? Is there an appealing characterization of the feasible set of trading mechanisms which implies the optimality of the English auction for a risk neutral seller?

The motivation for this type of inquiry is two-fold: on one hand the English auction is often the chosen trading procedure by the owner of an indivisible object facing a number of potential buyers; on the other hand, it is known from the theory of mechanism design that, under generic assumptions on the buyers' preferences and information structure, one can construct incentive-compatible and individually rational trading procedures which dominate the English auction in terms of seller's expected revenue. In particular, if the buyers are risk averse, the seller earns more on average with mechanisms in which risk is used as a screening device than with any of the "standard" auctions — i.e. Dutch, English, first-price and second-price;[1] and, if all buyers are risk neutral, then any nonzero degree of correlation, no matter how small, among their private information enables the seller to extract all expected gains from trade.[2]

This discrepancy between theory and common practice has led researches to argue that a seller's set of feasible trading mechanisms should be restricted according to a set of simplicity and robustness criteria in order to rule out mechanisms which, like the "full extraction" mechanisms mentioned above, rely heavily on the common knowledge of fine details of the model, e.g. the buyers' beliefs conditional on their private information, or the curvature of their utility functions. Indeed, in all "standard" auctions, the terms of trade are determined by a small set of simple rules. [3]

In this paper we consider two nested classes of equilibrium outcomes of trading mechanisms, which correspond to alternative sets of simplicity and robustness. First, attention is restricted to the class of all equilibrium outcomes of selling procedures which satisfy a no-regret condition: each buyer has no incentive to revise his decisions after observing his opponents' behavior. This no-regret property is implied by the traders' inability to commit to their actions before observing their opponents' choices, and results in outcome functions — i.e. functions which specify how the object

[1]For a description of the rules of these four auction formats see for example McAfee and McMillan (1987). A classic reference documenting the widespread use of auctions is Cassady (1967).

[2]The optimality of the all standard auctions has been established by Myerson (1981) and Riley and Samuelson (1981). Optimal selling mechanisms with risk averse buyers and independent values have been characterized by Matthews (1983) and Maskin and Riley (1984). 'Full extraction' results with risk neutral buyers have been established by Crèmer and McLean (1988) for the case of discrete probability distributions. With continuous distributions McAfee and Reny (1992) have provided a nearly-full extraction result.

[3]Here is a representative quote: "A reasonable question for the mechanism design literature is how to capture the importance of robustness. Specifically, we think the answer to questions like 'under what circumstances are English auctions used?' has to do with the need for an institution perform well in a variety of circumstances." [McAfee and McMillan (1987).] See also Milgrom (1985) and (1987).

is allocated, and how much each buyer pays to the seller, for each realization of the buyers' private information — that are *posterior-implementable,* as defined by Green and Laffont (1987). We show that, in Milgrom and Weber's (1982) "general symmetric model", the symmetric equilibrium[4] of an English auction in which the reserve price is set after all but one buyers have dropped out maximizes the seller's expected profit among all posterior-implementable outcome functions.

It is worth noting that this optimality result does not generalize to environments with asymmetric distributions. To see this, consider the case with private and independent values. As shown by Bulow and Roberts (1989), the optimal auction in this case, which entails awarding the object to the buyer with the highest "virtual utility", can be implemented in dominant strategies: the winner pays the lowest value that he could have reported without losing the object. With asymmetric distributions, it can happen that the buyer with the highest virtual utility does not have the highest value. Thus the object may not always be awarded to the buyer with the highest value, as in the English auction. By Myerson's Revenue Equivalence Theorem, this implies that the optimal auction generates a higher seller's expected revenue than the English auction.

The second subset of equilibrium outcomes of selling procedures considered in this paper is identified by the sole additional restriction that "losers do not pay" (LDNP), i.e. only the buyer who is awarded the object makes a payment to the seller. This class of outcome functions includes all equilibrium outcomes of the four standard auctions, as well as all posterior-implementable outcome functions.

In light of the following two observations, one may conjecture that, in a symmetric model with risk neutral buyers, no LDNP outcome function generates more seller's expected revenue than the symmetric equilibrium outcome of the English auction: first, in any 'full extraction' mechanism the losers must make payments to the seller for some realizations of their opponents' signals; second, in an English auction the seller can set an optimal reserve price for the last active bidder based on all losers' private signals which are revealed by their quitting times.

Proposition 2 of this paper establishes however that the set of all LDNP selling mechanisms includes a large class of sealed-bid auction, named *b-composite auctions,* which dominate the English auction in terms of seller's expected revenue. This result demonstrates that, to improve upon the English auction, the seller does not have to resort to procedures in which the losers pay, such as "all-pay" auctions or mechanisms with entry fees.[5]

A third interesting class of trading procedures among which the English auction maximizes the

[4]In the common value case, Bikhchandani and J. Riley (1991) have shown that the "irrevocable exit" English auction also has many asymmetric equilibria.

[5]Krishna and Morgan (1996) have shown that, under conditions which guarantee the existence of a symmetric equilibrium, the "second-price all-pay auction" generates dominates the standard second-price auction in terms of seller's expected profit.

seller's expected profit has been considered in Lopomo (1998). The seller's feasible set in that paper consists of a large family of dynamic bidding procedures called "Simple Sequential Auctions," whose essential feature is that each buyer chooses his *actual* payment conditional on being awarded the object from a given set. The ability for each buyer to determine his payment conditional on receiving the object is a "bargaining-like" property that also characterizes the posterior-implementable outcome functions considered in this paper. The buyer who receives the object is guaranteed an amount of information rent which cannot be lower than the difference between his value and the highest among his opponents' values.

The rest of this paper is organized as follows. Section 2 reviews the assumptions of Milgrom and Weber's general symmetric model (1982) and states a few additional regularity conditions under which the optimality of the English auction among all posterior-implementable mechanisms will be established. In Section 3 the optimality of the English auction among all posterior-implementable mechanisms is established. Section 4 shows that the b-composite auctions dominate the English auction in terms of seller's expected revenue, and sheds light on the key idea underlying the construction of *any* mechanisms which does better than the English auction in terms of seller's expected revenue. Section 5 provides some conclusive remarks.

2 The Model

This section reviews the assumptions of Milgrom and Weber's "general symmetric model", and introduces some additional regularity conditions that will be used to establish the optimality of the English auction among all posterior-implementable outcome functions.

The owner of an indivisible object faces n risk-neutral potential buyers. Let $N := \{1, \dots, n\}$ denote the set of buyers. Each buyer $i \in N$ observes privately the realization of a random variable θ_i, i.e. his 'signal', drawn jointly with the other $n-1$ signals $\theta_{-i} := (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$ from a symmetric distribution with density f, which is strictly positive on its support $\Theta := [0,1]^n$. The signals $(\theta_1, \dots, \theta_n)$ are *affiliated,* i.e.

$$f\left(\theta \vee \theta'\right) \cdot f\left(\theta \wedge \theta'\right) \geq f\left(\theta\right) \cdot f\left(\theta'\right) \quad \text{for all } \theta, \theta' \in \Theta,$$

where $\theta \vee \theta'$ and $\theta \wedge \theta'$ denote the component-wise maximum and minimum of θ and θ'. The affiliation property has the following two useful implications. First, for any decomposition of the vector θ into a k-dimensional vector θ_K and the vector θ_{-K} containing its remaining $n-k$ elements, the ratio of the conditional densities

$$\frac{f_{|-K}\left(\theta_K | \theta_{-K}\right)}{f_{|-K}\left(\theta'_K | \theta_{-K}\right)}$$

is nondecreasing in θ_{-K} whenever $\theta_K > \theta'_K$. Second, the function

$$h\left(a_1, ..., a_n, b_1, ..., b_n\right) \equiv E\left[g\left(\theta\right) | a_i \leq \theta_i \leq b_i; i \in N\right]$$

is nondecreasing for any nondecreasing function $g : \Theta \to \mathbb{R}$.

The amount that each buyer $i \in N$ is willing to pay for the object is determined by a function u_i of the realization of his signal θ_i, and possibly of the other $n-1$ signals θ_{-i}. Moreover, there exists a 'valuation function' u,

$$u : \Theta_i \times \Theta_{-i} \to \mathbb{R},$$

where $\Theta_i := [0,1]$ and $\Theta_{-i} := [0,1]^{n-1}$, strictly increasing in its first argument, and weakly increasing and symmetric in its last $n-1$ arguments, such that

$$u(\theta_i, \theta_{-i}) \equiv u_i(\theta_1, ..., \theta_n) \text{ for each } i \in N.$$

The overall payoff function of buyer i is

$$u(\theta_i, \theta_{-i})\ Q^i - M^i$$

where Q^i denotes the probability that he is awarded the object and M^i denotes his expected payment to the seller.

This general symmetric model includes as special cases both the "private values" case, where $u(\theta_i, \theta_{-i}) \equiv \overline{u}(\theta_i)$ for some function $\overline{u} : [0,1] \to \mathbb{R}$; and the "common value" case, in which $u(\theta_i, \theta_{-i}) = u(\theta_j, \theta_{-j})$ for any two permutations (θ_i, θ_{-i}) and (θ_j, θ_{-j}) of any given realization $\theta \in \Theta$.

To establish the optimality result in Section 3, we will use the following additional assumptions. On the valuation function u:

A1: Fix any $(\theta_1, ..., \theta_n) \in \Theta$, pick two elements θ_i and θ_j, and let $\theta_{-ij} \in [0,1]^{n-2}$ denote the vector containing the remaining $n-2$ signals. Then, $\theta_i > \theta_j$ implies $u(\theta_i, \theta_j, \theta_{-ij}) \geq u(\theta_j, \theta_i, \theta_{-ij})$;

A2: $u_{11} \leq 0$, (where the subscripts denote partial derivatives in the usual way);

A3: $u_{1j} \geq 0$, $j = 2, ..., n$;

and on the signals' distribution F:

A4: all conditional hazard ratios $\dfrac{f_{|-i}(\theta_i \mid \theta_{-i})}{1 - F_{|-i}(\theta_i \mid \theta_{-i})}$, where $F_{|-i}$ and $f_{|-i}$ denote the c.d.f. and the density of θ_i conditional on $\theta_{-i} \in \Theta_{-i}$, are nondecreasing in θ_i;

A5: The derivative $\frac{\partial}{\partial \theta_i} f_{|i}(\theta_{-i}|\theta_i)$, where $f_{|i}(\cdot)$ denotes the density of θ_{-i} conditional on θ_i, exists for all $\theta \in \Theta$;

Assumption A1 is made to guarantee that buyers with higher signals have higher values. Assumptions A2 and A3 imply that buyers with higher signals have a lower sensitivity of their value to their own signal. Assumption A4 extends the standard 'monotone hazard ratio' condition to the distribution of each signal conditional on all other signals. Finally, A5 is a smoothness assumption made to simply the proof of Lemma 1 in Section 3.

3 The English Auction is Optimal among Posterior Implementable Mechanisms

Any trading mechanism can be represented as a set of $2n$ functions

$$p^i : B_1 \times ... \times B_n \to \mathbb{R}, \quad i \in N,$$

and

$$x^i : B_1 \times ... \times B_n \to [0,1], \quad i \in N,$$

such that

$$\sum_{i \in N} x^i (b_1, .., b_n) \leq 1, \quad \text{for each } (b_1, .., b_n) \in B_1 \times ... \times B_n,$$

where, B_i denotes the set of feasible actions, i.e. "messages," for buyer i, the function p^i determines his payment to the seller[6] and the function x^i determines the probability that he is awarded the object, for any n-tuple of messages $(b_1, ..., b_n) \in B_1 \times ... \times B_n$.

Any trading mechanism $(p, x) := \left(p^1, ..., p^n, x^1, ..., x^n\right)$ induces an incomplete information game with payoff functions

$$U^i (b_1, ..., b_i, ..., b_n; \theta_i, \theta_{-i}) := u (\theta_i, \theta_{-i}) \cdot x^i (b_1, ..., b_i, ..., b_n) - p^i (b_1, ..., b_i, ..., b_n), \quad i \in N,$$

and (mixed) strategy sets

$$\Sigma_i := \left\{\sigma^i (\cdot|\theta_i) \in \Delta (B_i) : \theta_i \in \Theta_i\right\}, \quad i \in N.$$

Any strategy profile $\sigma := \left(\sigma^1, ..., \sigma^n\right) \in \Sigma_1 \times ... \times \Sigma_n$ determines a probability distribution μ^σ over the set $B_1 \times ... \times B_n \times \Theta_i^n$, and implements an *outcome function,* i.e. a set of $2n$ functions $(q, m) = \left(q^1, ..., q^n, m^1, ..., m^n\right)$ which assign a probability vector

$$q^i (\theta_1, ..., \theta_i, ..., \theta_n) := \int_B x^i (b_1, ..., b_i, ..., b_n) \cdot \sigma^1 (db_1|\theta_1) \cdot ... \cdot \sigma^n (db_n|\theta_n), \quad i \in N,$$

and an expected payment vector

$$m^i (\theta_1, ..., \theta_i, ..., \theta_n) := \int_B p^i (b_1, ..., b_i, ..., b_n) \cdot \sigma^1 (db_1|\theta_1) \cdot ... \cdot \sigma^n (db_n|\theta_n), \quad i \in N,$$

to each type profile $\theta \in \Theta$.

In this section attention is restricted to outcome functions which are *posterior-implementable*, i.e., which can be implemented by a profile σ such that, for μ^σ-almost every $(b_1, ..., b_n, \theta_1, ..., \theta_n)$,

$$b_i \in \arg\max_{b_i' \in B_i} w^i (\theta_i, b_{-i}) \cdot x^i \left(b_1, ..., b_i', ..., b_n\right) - p^i \left(b_1, ..., b_i', ..., b_n\right) \quad \text{all } i \in N,$$

[6] Since the buyers are risk neutral attention can be restricted to deterministic payment functions without loss of generality.

where

$$w^{i}\left(\theta_{i},b_{-i}\right):=\int_{\Theta_{-i}}u\left(\theta_{i},\theta_{-i}\right)\ \frac{\sigma^{-i}\left(b_{-i}|\theta_{-i}\right)\ f_{|i}\left(\theta_{-i}|\theta_{i}\right)}{\int_{\Theta_{-i}}\sigma^{-i}\left(b_{-i}|\tau_{-i}\right)\ f_{|i}\left(\tau_{-i}|\theta_{i}\right)d\tau_{-i}}\,d\theta_{-i}$$

denotes bidder i's willingness to pay for the object, conditional on his type θ_i and the information revealed by his opponents' actions b_{-i}.

We are now ready to state and prove the main result of this section.

Proposition 1 *Define the "ex-post virtual utility function" $v:[0,1]\times[0,1]^{n-1}\rightarrow R$ by*

$$v\left(\theta_{i},\theta_{-i}\right)\equiv u\left(\theta_{i},\theta_{-i}\right)-\frac{1-F_{|-i}\left(\theta_{i}|\theta_{-i}\right)}{f_{|-i}\left(\theta_{i}|\theta_{-i}\right)}\ u_{1}\left(\theta_{i},\theta_{-i}\right). \tag{1}$$

In Milgrom and Weber's general symmetric model, and under assumptions A1-A5, the seller's expected revenue is maximized, among all posterior-implementable and individually rational outcome functions, by the symmetric equilibrium outcome of the irrevocable-exit English auction in which, after $n-1$ buyers drop out, the auctioneer sets the reserve price at

$$r\left(\theta_{-i}\right)\equiv u\left(t_{0}\left(\theta_{-i}\right),\theta_{-i}\right),$$

where the function $t_0\left(\theta_{-i}\right)$ is defined by the equation $v\left(t_0\left(\theta_{-i}\right),\theta_{-i}\right)=0$.

Proof: The proof is broken in four Lemmas. Lemma 1 establishes a revenue equivalence result for posterior-implementable outcome functions with affiliated types' distributions, similar to Myerson's (1981) Revenue Equivalence Theorem: it derives an 'envelope condition' (equation (2) below) akin to the standard 'interim' envelope condition in mechanism design, which determines the payment of each type of buyer i once the strategies used by his opponents, the assignment function q^i, and the expected surplus of the lowest type are given. Lemma 2 shows that the seller's expected revenue is maximized, among all posterior-implementable outcome functions which have the same assignment functions $q^1,...,q^n$, by the function in which the buyer who wins the object learns his opponents' type perfectly. Lemma 3 finds a revenue maximizing outcome function among all the functions that are 'fully revealing' for the winner. Finally, Lemma 4 shows that the equilibrium outcome of the English auction with optimal ex-post reserve prices coincides almost everywhere with the optimal function found in Lemma 3, hence is optimal for the seller among all posterior-implementable and individually rational outcome functions.

Lemma 1. *If the outcome function (q,m) is posterior-implementable, then, for all $i\in N$, all $(\theta_1,...,\theta_i,...,\theta_n)\in\Theta$, and all $b_{-i}\in B_{-i}$, we have*

$$m^{i}\left(\theta_{1},...,\theta_{i},...,\theta_{n}\right)\ =\ w^{i}\left(\theta_{i},b_{-i}\right)q^{i}\left(\theta_{1},...,\theta_{i},...,\theta_{n}\right)-\int_{0}^{\theta_{i}}q^{i}\left(\theta_{1},...,\tau,...,\theta_{n}\right)w_{1}^{i}\left(\tau,b_{-i}\right)d\tau$$

$$-U^i(0,b_{-i}), \tag{2}$$

where $U^i(0,b_{-i})$ denotes the surplus of buyer i's lowest type, given his opponents messages.

Proof. Fix a posterior equilibrium profile σ of a selling mechanisms (p,x) and a buyer $i \in N$. Take any selection $\beta^i(\cdot)$ from the correspondence which assigns the support of $\sigma^i(\cdot|\theta_i)$ to each type $\theta_i \in \Theta_i$, i.e. $\beta^i(\theta_i) \in Supp\, \sigma^i(\cdot|\theta_i)$ for all $\theta_i \in \Theta_i$. Define

$$\chi^i\left(\widehat{\theta}_i, b_{-i}\right) := x^i\left(b_1, ..., \beta^i\left(\widehat{\theta}_i\right), ..., b_n\right),$$

$$\pi^i\left(\widehat{\theta}_i, b_{-i}\right) := p^i\left(b_1, ..., \beta^i\left(\widehat{\theta}_i\right), ..., b_n\right),$$

and

$$U^i(\theta_i, b_{-i}) := \max_{\widehat{\theta}_i}\left\{w^i(\theta_i,b_{-i})\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right) - \pi^i\left(\widehat{\theta}_i, b_{-i}\right)\right\}.$$

By posterior implementability, we have, for μ^σ-almost all b_{-i},

$$\begin{aligned} U^i(\theta_i,b_{-i}) &\geq w^i(\theta_i,b_{-i})\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right) - \pi^i\left(\widehat{\theta}_i, b_{-i}\right) \\ &= w^i\left(\widehat{\theta}_i, b_{-i}\right)\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right) - \pi^i\left(\widehat{\theta}_i, b_{-i}\right) \\ &\quad + \left[w^i(\theta_i,b_{-i}) - w^i\left(\widehat{\theta}_i, b_{-i}\right)\right]\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right) \\ &= U^i\left(\widehat{\theta}_i, b_{-i}\right) + \left[w^i(\theta_i,b_{-i}) - w^i\left(\widehat{\theta}_i, b_{-i}\right)\right]\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right), \end{aligned} \tag{3}$$

and interchanging $\widehat{\theta}_i$ with θ_i we obtain

$$U^i\left(\widehat{\theta}_i, b_{-i}\right) \geq U^i(\theta_i,b_{-i}) + \left[w^i\left(\widehat{\theta}_i, b_{-i}\right) - w^i(\theta_i,b_{-i})\right]\chi^i(\theta_i,b_{-i}). \tag{4}$$

Combining (3) and (4) yields

$$\begin{aligned} \left[w^i(\theta_i,b_{-i}) - w^i\left(\widehat{\theta}_i, b_{-i}\right)\right]\chi^i(\theta_i,b_{-i}) &\geq U^i(\theta_i,b_{-i}) - U^i\left(\widehat{\theta}_i, b_{-i}\right) \\ &\geq \left[w^i(\theta_i,b_{-i}) - w^i\left(\widehat{\theta}_i, b_{-i}\right)\right]\ \chi^i\left(\widehat{\theta}_i, b_{-i}\right). \end{aligned} \tag{5}$$

Since by assumption A5 the derivative $w^i_1(\theta_i,b_{-i})$ exists everywhere and $\chi^i(\cdot,b_{-i}) \in [0,1]$, (5) implies that $U^i(\cdot,b_{-i})$ is Lipschitz continuous, hence absolutely continuous. Therefore $U^i(\cdot,b_{-i})$ can be written as the integral of its derivative (which exists almost everywhere[7]), i.e.

$$U^i(\theta_i, b_{-i}) = U^i(\theta_i', b_{-i}) + \int_{\theta_i'}^{\theta_i} U_1^i(\tau, b_{-i})\, d\tau \quad \text{ for any } \theta_i, \theta_i' \in \Theta_i.$$

Moreover, by affiliation $w^i(\cdot, b_{-i})$ is nondecreasing, thus (5) also implies that $\chi^i(\cdot, b_{-i})$ is nondecreasing, hence continuous almost everywhere. Therefore, choosing $\theta_i > \widehat{\theta}_i$, dividing through (5) by $\theta_i - \widehat{\theta}_i$, and taking the limit as $\widehat{\theta}_i \to \theta_i$ yields

$$U_1^i(\theta_i, b_{-i}) \;=\; w_1^i(\theta_i, b_{-i})\, \chi^i(\theta_i, b_{-i}) \;\; \text{almost everywhere.} \tag{6}$$

By (6), for each bidder $i \in N$, the probability $\chi^i(\theta_i, b_{-i})$ of being assigned the object is unique for almost all types $\theta_i \in \Theta_i$. Thus we can define

$$q^i(\theta_1, ..., \theta_i, ..., \theta_n) \equiv \chi^i\left(\beta^1(\theta_1), ..., \beta^i(\theta_i), ..., \beta^n(\theta_n)\right) \text{ for almost all } \theta \in \Theta,$$

where each β^j is any selection from the support of σ^j, all $j \in N$. Integrating both sides of (6) yields

$$U^i(\theta_i, b_{-i}) = U^i(0, b_{-i}) + \int_0^{\theta_i} w_1^i(\tau, b_{-i})\, q^i(\theta_1, ..., \tau, ..., \theta_n)\, d\tau,$$

which is equivalent to (2). □

Equation (2) shows that, in any posterior-implementable outcome function, the payment of type θ_i of bidder i conditional on his opponents' actions b_{-i} depends on two things: i) the probability $q^i(\theta_1, ..., \tau, ..., \theta_n)$ that his and each of his lower types $\tau \le \theta_i$ receives the object, and ii) how much information about his opponents' signals θ_{-i} their actions b_{-i} reveal. The next Lemma shows that, among all posterior-implementable outcome functions with the same q^i, $i \in N$, the seller's expected revenue is maximized by the outcome function in which the winner learns *all* his opponents' private information.

Lemma 2. *If an outcome function* $\left\{q^i, m^i; i \in N\right\}$ *is posterior-implementable, then the outcome function* $\left\{q^i, m_*^i(\cdot|q); i \in N\right\}$, *where* $m_*^i(\cdot|q)$ *is defined by*

$$\begin{aligned} m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q) \;\equiv\;& u(\theta_i, \theta_{-i})\, q^i(\theta_1, ..., \theta_i, ..., \theta_n) - \int_0^{\theta_i} q^i(\theta_1, ..., \tau, ..., \theta_n)\, u_1(\tau, \theta_{-i})\, d\tau \\ & - U^i(0, b_{-i}) \end{aligned} \tag{7}$$

for each $i \in N$, *is also posterior-implementable. Moreover* $\left\{q^i, m_*^i(\cdot|q); i \in N\right\}$ *generates at least as much seller's expected revenue as* (q, m).

[7] For a proof of this, see, for example, Kolmogorov and Fomin (1970), Theorem 6, p. 340.

Proof. The proof of the first claim is standard, and is reported here for completeness. Define

$$U_*^i(\theta_i, b_{-i}|q) := u(\theta_i, \theta_{-i})\; q^i(\theta_1, ..., \theta_i, ..., \theta_n) - m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q),$$

and take $\theta_i, \widehat{\theta}_i \in \Theta_i$, assuming $\theta_i > \widehat{\theta}_i$ without loss of generality. Then, by the definition of $m_*^i(\cdot|q)$ and since q^i is nondecreasing in θ_i, we have

$$\begin{aligned}
U_*^i(\theta_i, b_{-i}|q) - U_*^i\left(\widehat{\theta}_i, b_{-i}|q\right) &= \int_{\widehat{\theta}_i}^{\theta_i} u_1(\tau, \theta_{-i})\; q^i(\theta_1, ..., \tau, ..., \theta_n)\; d\tau \\
&\geq \int_{\widehat{\theta}_i}^{\theta_i} u_1(\tau, \theta_{-i})\; q^i\left(\theta_1, ..., \widehat{\theta}_i, ..., \theta_n\right)\; d\tau \\
&= \left[u(\theta_i, \theta_{-i}) - u\left(\widehat{\theta}_i, \theta_{-i}\right)\right]\; q^i\left(\theta_1, ..., \widehat{\theta}_i, ..., \theta_n\right)
\end{aligned}$$

or equivalently

$$\begin{aligned}
U_*^i(\theta_i, b_{-i}|q) &\geq U_*^i\left(\widehat{\theta}_i, b_{-i}|q\right) + \left[u(\theta_i, \theta_{-i}) - u\left(\widehat{\theta}_i, \theta_{-i}\right)\right]\; q^i\left(\theta_1, ..., \widehat{\theta}_i, ..., \theta_n\right) \\
&= u(\theta_i, \theta_{-i})\; q^i\left(\theta_1, ..., \widehat{\theta}_i, ..., \theta_n\right) - m_*^i\left(\theta_1, ..., \widehat{\theta}_i, ..., \theta_n|q\right)
\end{aligned}$$

Similarly, we have

$$U_*^i(\theta_i, b_{-i}|q) - U_*^i\left(\widehat{\theta}_i, b_{-i}|q\right) \leq \left[u(\theta_i, \theta_{-i}) - u\left(\widehat{\theta}_i, \theta_{-i}\right)\right]\; q^i(\theta_1, ..., \theta_i, ..., \theta_n),$$

i.e.

$$U_*^i\left(\widehat{\theta}_i, b_{-i}|q\right) \geq u\left(\widehat{\theta}_i, \theta_{-i}\right)\; q^i(\theta_1, ..., \theta_i, ..., \theta_n) - m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q).$$

To establish the revenue inequality, it is sufficient to show that

$$\int_{\Theta_{-i}} m^i(\theta_1, ..., \theta_i, ..., \theta_n)\; f_{|i}(\theta_{-i}|\theta_i)\, d\theta_{-i} \leq \int_{\Theta_{-i}} m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q)\; f_{|i}(\theta_{-i}|\theta_i)\, d\theta_{-i} \tag{8}$$

for each $\theta_i \in \Theta_i$. Integrating (2) and (7) by parts yields

$$m^i(\theta_1, ..., \theta_i, ..., \theta_n) = \int_0^{\theta_i} w^i(\tau, b_{-i})\;\; \chi_1^i(\tau, b_{-i})\; d\tau\; - U^i(0, b_{-i}), \tag{9}$$

and

$$m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q) = \int_0^{\theta_i} u(\tau, \theta_{-i})\;\; \chi_1^i(\tau, b_{-i})\; d\tau\; - U_*^i(0, b_{-i}),$$

where as usual $\chi_1^i(\cdot)$ denotes the partial derivative of χ^i with respect to the first variable, and integrating both expressions over the set

$$T_{-i}(b_{-i}) := \left\{\theta_{-i} \in \Theta_{-i} | \beta^{-i}(\theta_{-i}) = b_{-i}\right\},$$

with respect to the density $f_{|i}(\theta_{-i}|\theta_i)$ yields

$$\int_{T_{-i}(b_{-i})} m^i(\theta_1, ..., \theta_i, ..., \theta_n)\ f_{|i}(\theta_{-i}|\theta_i)\ d\theta_{-i} = \int_0^{\theta_i} w^i(\tau, b_{-i})\ \chi_1^i(\tau, b_{-i})\ d\tau \cdot \int_{T_{-i}(b_{-i})} f_{|i}(\theta_{-i}|\theta_i)\, d\theta_{-i}$$

and

$$\begin{aligned} \int_{T_{-i}(b_{-i})} m_*^i(\theta_1, ..., \theta_i, ..., \theta_n|q)\ f_{|i}(\theta_{-i}|\theta_i)\ d\theta_{-i} &= \int_0^{\theta_i} \int_{T_{-i}(b_{-i})} u(\tau, \theta_{-i})\ f_{|i}(\theta_{-i}|\theta_i)\ d\theta_{-i}\, \chi_1^i(\tau, b_{-i})\ d\tau \\ &= \int_0^{\theta_i} E\left[u(\tau, \theta_{-i})\,|\theta_i,\ b_{-i}\right]\ \chi_1^i(\tau, b_{-i})\ d\tau \\ &\quad \cdot \int_{T_{-i}(b_{-i})} f_{|i}(\theta_{-i}|\theta_i)\ d\theta_{-i}. \end{aligned}$$

By affiliation we have $w^i(\tau, b_{-i}) = E\left[u(\tau, \theta_{-i})\,|\tau,\ b_{-i}\right] \leq E\left[u(\tau, \theta_{-i})\,|\theta_i,\ b_{-i}\right]$, hence the inequality in (8) holds. □

In light of Lemma 2, we can restrict the search for an optimal posterior-implementable outcome function without loss of generality to the class of functions in which the winner learns his opponent's types perfectly. That is, we can restrict attention to outcome functions in which the payments satisfy (7).

Taking the expected value with respect to the distribution of θ_i conditional on θ_{-i} in both sides in (7), and integrating the right-hand side by parts, yields the following expression for the expected payment by bidder i, conditional on his opponents' types θ_{-i},

$$\begin{aligned} \int_0^1 m_*^i(\theta|q)\ dF_{|-i}(\theta_i|\theta_{-i}) &= \int_0^1 v(\theta_i, \theta_{-i})\ q^i(\theta)\ dF_{|-i}(\theta_i|\theta_{-i}) \\ &\quad -U^i(0, b_{-i}), \end{aligned}$$

where the "ex-post virtual utility" function v is defined in (1). Integrating over Θ_{-i}, and summing over all buyers $i \in N$, yields

$$\int_\Theta \sum_{i\in N} m^i(\theta|q)\ dF(\theta)\ = \int_\Theta \left[\sum_{i\in N} v(\theta_i, \theta_{-i})\ q^i(\theta)\right] dF(\theta) - \sum_{i\in N} \overline{U}^i(0)\,, \tag{10}$$

where $\overline{U}^i(0)$ denotes the expected surplus of the lowest type of bidder i. By individual rationality, each $\overline{U}^i(0)$ $i \in N$, cannot be negative, and it is optimally set equal to zero.

Next, Lemma 3 shows that, under the assumptions A1-A4 , it is optimal for the seller to assign the object to a buyer with the highest ex-post virtual utility, if this is positive.

Lemma 3. *Under assumptions A1-A4, the first term in the objective function (10) is maximized, subject to the feasibility constraint $\sum_{i \in N} q^i(\theta) \leq 1$ all $\theta \in \Theta$, by the following assignment function*

$$q_*^i(\theta_1, ..., \theta_i, ..., \theta_n) := \mathbf{1}_{[\theta_i > \max\{\theta_1, ..., \theta_{i-1}, t_0(\theta_{-i}), \theta_{i+1}, ..., \theta_n\}]}, \quad i \in N, \tag{11}$$

where $\mathbf{1}_{[\cdot]}$ denotes the indicator function i.e. $\mathbf{1}_{[A]} = 1$ if and only if A is true.

Proof. As in the statement of Assumption A1, fix an arbitrary $(\theta_1, ..., \theta_n) \in \Theta$, pick two elements θ_i and θ_j such that $\theta_i > \theta_j$, and let θ_{-ij} denote the vector containing the remaining $n-2$ signals. Assumption A1 immediately implies $u(\theta_i, \theta_j, \theta_{-ij}) \geq u(\theta_j, \theta_i, \theta_{-ij})$. Moreover we have

$$\begin{aligned} u_1(\theta_i, \theta_j, \theta_{-ij}) &\leq u_1(\theta_j, \theta_j, \theta_{-ij}) && \text{(by A2)} \\ &\leq u_1(\theta_j, \theta_i, \theta_{-ij}), && \text{(by A3)} \end{aligned}$$

and

$$\begin{aligned} \frac{1 - F_{|-i}(\theta_i|\theta_j, \theta_{-ij})}{f_{|-i}(\theta_i|\theta_j, \theta_{-ij})} &\leq \frac{1 - F_{|-i}(\theta_j|\theta_j, \theta_{-ij})}{f_{|-i}(\theta_j|\theta_j, \theta_{-ij})} && \text{(by A4)} \\ &\leq \frac{1 - F_{|-i}(\theta_j|\theta_i, \theta_{-ij})}{f_{|-i}(\theta_j|\theta_i, \theta_{-ij})}. && \text{(by affiliation)} \end{aligned}$$

These inequalities immediately imply

$$v(\theta_i, \theta_j, \theta_{-ij}) \geq v(\theta_j, \theta_i, \theta_{-ij}), \tag{12}$$

hence the statement in the lemma is immediate. □

By the envelope condition (7) in Lemma 1, the payment function corresponding to the optimal assignment function q_*^i is

$$\begin{aligned} m_*^i(\theta_1, ..., \theta_i, ..., \theta_n | q_*) &\equiv u(\theta_i, \theta_{-i})\, q_*^i(\theta_1, ..., \theta_i, ..., \theta_n) - \int_0^{\theta_i} q_*^i(\theta_1, ..., \tau, ..., \theta_n)\, u_1(\tau, \theta_{-i})\, d\tau \\ &= \max\left\{ u\left(\theta_{-i}^{(1)}, \theta_{-i}^{(1)}, \theta_{-i}^{(2)}, ..., \theta_{-i}^{(n-1)}\right), u\left(t_0(\theta_{-i}), \theta_{-i}^{(1)}, \theta_{-i}^{(2)}, ..., \theta_{-i}^{(n-1)}\right) \right\} \end{aligned} \tag{13}$$

where the last equality is obtained by integrating by parts, and $\theta_{-i}^{(j)}$ denotes the j-th order statistic among the components of θ_{-i}.

Lemma 4. *The optimal outcome function defined in (11) and (13) coincides almost everywhere with the symmetric equilibrium outcome of the irrevocable exit English auction in which, after $n-1$ buyers drop out, the auctioneer sets the reserve price at $r(\theta_{-i})$ as defined in the statement of Proposition 1.*

Proof. The key step of the proof consists in verifying that the introduction of the seller's reserve price strategy does not alter each bidder's equilibrium strategy until all other bidders have dropped out. As shown in Milgrom and Weber (1982) (Theorem 10), the symmetric equilibrium in the English auction without any reserve price is $s \equiv \left(s^0, ..., s^{n-2}\right)$ defined recursively as follows

$$\begin{aligned} s^0(\theta_i) &= u(\theta_i, \theta_i, ..., \theta_i), \\ s^j(\theta_i | p_1, ..., p_j) &= u\left(\theta_i, \theta_i, ..., \theta_i, \left(s^{j-1}\right)^{-1}(\theta_i | p_1, ..., p_{j-1}), ..., \left(s^0\right)^{-1}(p_1)\right), \quad j = 1, ..., n-2, \end{aligned}$$

where p_j denotes the price at which the j-th buyer has dropped out. The auctioneer acts a buyer who, when the price reaches p_{n-1}, drops out if $p_{n-1} > r^0(\theta_{-i})$, and remains active until the price reaches $r^0(\theta_{-i})$ otherwise. If his opponents use strategy s, buyer i's payoff is

$$u\left(\theta_i, \theta_{-i}^{(1)}, \theta_{-i}^{(2)}, ..., \theta_{-i}^{(n-1)}\right) - \max\left\{u\left(\theta_{-i}^{(1)}, \theta_{-i}^{(1)}, \theta_{-i}^{(2)}, ..., \theta_{-i}^{(n-1)}\right), u\left(t_0(\theta_{-i}), \theta_{-i}^{(1)}, \theta_{-i}^{(2)}, ..., \theta_{-i}^{(n-1)}\right)\right\},$$

if he wins the auction, and zero otherwise. By assumption A1, this is nonnegative if

$$\theta_i \geq \max\left\{\theta_{-i}^{(1)}, t_0(\theta_{-i})\right\},$$

hence it is optimal for buyer i to use s and buy the object if and only if $\theta_i \geq t_0(\theta_{-i})$. □

4 "Losers Do not Pay" and the "b-composite" Auctions

The main objective of this section is to show that the English auction fails to maximize the seller's expected revenue among all selling mechanisms in which the losers do not pay.

For simplicity, assume that there are only two bidders with private values: $u(\theta_i, \theta_{-i}) \equiv \theta_i$, $i = 1, 2$. Consider the following family of direct revelation mechanisms, parametrized by $b \in (0, 1]$: for any pair of reports $\left(\widehat{\theta}_i, \widehat{\theta}_{-i}\right)$, bidder i is awarded the object with probability

$$q^i\left(\widehat{\theta}_i, \widehat{\theta}_{-i}\right) = \mathbf{1}_{[\widehat{\theta}_i > \widehat{\theta}_{-i}]}, \tag{14}$$

and pays

$$m^i\left(\widehat{\theta}_i,\widehat{\theta}_{-i};b\right)=\begin{cases}\widehat{\theta}_i, & \text{if } \ b\le\widehat{\theta}_{-i}<\widehat{\theta}_i;\\[2ex] \widehat{\theta}_{-i}-B\left(\widehat{\theta}_i;b\right), & \text{if } \widehat{\theta}_{-i}<b<\widehat{\theta}_i;\\[2ex] \widehat{\theta}_{-i}, & \text{if } \widehat{\theta}_{-i}<\widehat{\theta}_i\le b;\\[2ex] 0, & \text{if } \widehat{\theta}_i\le\widehat{\theta}_{-i};\end{cases} \tag{15}$$

where

$$B\left(\widehat{\theta}_i;b\right):=\int_b^{\widehat{\theta}_i}\left(\frac{G(\tau\mid\tau)}{G(b\mid\tau)}-1\right)d\tau,$$

and G denotes the c.d.f. of the type of buyer i's opponent θ_{-i} conditional on his own type θ_i. To complete the specification of the mechanism, we stipulate that in the (zero-probability) event of a tie, i.e. $\theta_1=\theta_2$, no bidder is awarded the object and each pays zero.

Any mechanism in this class can be interpreted as a "b-composite," sealed-bid auction, by taking the report $\widehat{\theta}_i$ of each buyer $i=1,2$ as his bid. The object's allocation and winner's payment are determined by the rules of the first-price auction if both bids are higher than b, and by the rules of the second-price auction if both bids are lower than b. If $\widehat{\theta}_{-i}<b<\widehat{\theta}_i$, then buyer i is awarded the object and pays his opponent's bid $\widehat{\theta}_{-i}$ minus a bonus $B\left(\widehat{\theta}_i;b\right)$. See Figure 1.

We now show that, under a condition on the values' distribution F, 'truth-telling' is an equilibrium in the direct mechanism described above, for any $b\in(0,1]$. In other words, the strategy pair in which each buyer i bids his value θ_i is a Bayesian equilibrium in any b-composite auction, $b\in(0,1]$. For a fixed $b\in(0,1]$, the condition on the values' distribution is:

$$\frac{\partial}{\partial\tau}\left(\frac{G(y|\tau)}{G(b|\tau)}\right)\le\frac{G(y|x)}{G(b|x)},\qquad \text{for all } x\in[0,1],\ y\in(b,1] \text{ and } \tau\in(y,1]. \tag{16}$$

Condition (16) is satisfied, for example, by following family of affiliated densities[8]

$$f(x,y)=\frac{1}{\gamma}\left(\alpha+x^\beta y^\beta\right),$$

where $\gamma=\frac{(\beta+1)^2}{\alpha(\beta+1)^2+1}$, $\alpha>0$, and β is not too large. Since

$$\lim_{\beta\to0^+}\left(\frac{\partial}{\partial\tau}\left(\frac{G(y|\tau)}{G(b|\tau)}\right)\right)=\lim_{\beta\to0^+}\frac{\beta y\tau^\beta\alpha(\beta+1)}{\tau\left(y\alpha\beta+\alpha y+\tau^\beta b^{\beta+1}\right)^2}\left(y^{\beta+1}-b^{\beta+1}\right)=0,$$

[8]The example is due to Riley (1988), p. 418. To verify that each density in this family is affiliated, we can use the fact that for twice continuously differentiable densities (Milgrom and Weber (1982), Theorem 1) affiliation is equivalent to $\frac{\partial^2}{\partial\theta_i\,\partial\theta_j}\ln f(\theta)\ge0$ for each $i\ne j$, and find $\frac{\partial^2}{\partial x\,\partial y}\left(\ln\frac{1}{\gamma}\left(\alpha+x^\beta y^\beta\right)\right)=\frac{\alpha\beta^2x^{\beta-1}y^{\beta-1}}{\left(\alpha+x^\beta y^\beta\right)^2}>0.$

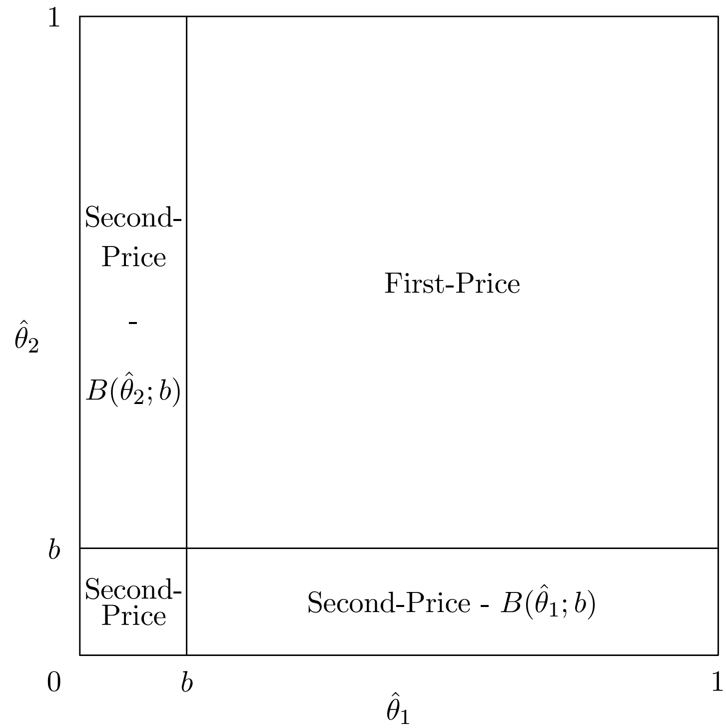

Figure 1: A b-composite auction

and

$$\lim_{\beta \to 0^+} \left(\frac{G(y|x)}{G(b|x)} \right) = \lim_{\beta \to 0^+} \left(\frac{\alpha(\beta+1)\, y + x^\beta y^{\beta+1}}{\alpha(\beta+1)\, y + x^\beta b^{\beta+1}} \right) = \frac{(\alpha+1)\, y}{\alpha y + b} > 0, \text{ for all } y \in (b, 1],$$

there must exist a β^* such that condition (16) holds for all densities with $\beta \in (0, \beta^*)$.

Lemma 1 *For any $b \in (0, 1]$, if condition (16) holds, the direct mechanism described in (14) and (15) is incentive compatible.*

Proof. To verify that truth-telling is a best reply to itself, suppose that buyer 2 reports his true type. There are two cases.

Case 1: Buyer 1's type is at least as high as b, i.e. $\theta_1 \in [b, 1]$. Bidding above b, i.e. bidding $\widehat{\theta}_1 \in (b, 1]$, yields

$$\begin{aligned} \overline{S}\left(\widehat{\theta}_1, \theta_1; b\right) &\equiv \left[\theta_1 - E[\theta_2 \,|\, \theta_2 < b] + B\left(\widehat{\theta}_1; b\right)\right] \Pr[\theta_2 < b \,|\, \theta_1] + (\theta_1 - \widehat{\theta}_1) \Pr\left[b < \theta_2 < \widehat{\theta}_1 \,|\, \theta_1\right] \\ &= \left[\theta_1 - E[\theta_2 \,|\, \theta_2 < b] + B\left(\widehat{\theta}_1; b\right)\right] G(b|\theta_1) + (\theta_1 - \widehat{\theta}_1) \left[G\left(\widehat{\theta}_1 | \theta_1\right) - G(b|\theta_1)\right]. \end{aligned}$$

The first derivative with respect to $\widehat{\theta}_1$ is

$$\begin{aligned}
\frac{\partial \overline{S}\left(\widehat{\theta}_1,\theta_1;b\right)}{\partial \widehat{\theta}_1} &= G\left(b|\theta_1\right)\,\frac{\partial B\left(\widehat{\theta}_1;b\right)}{\partial \widehat{\theta}_1} - \left[G(\widehat{\theta}_1|\theta_1)-G(b|\theta_1)\right] + \left(\theta_1-\widehat{\theta}_1\right)\,g\left(\widehat{\theta}_1|\theta_1\right) \\
&= G(b|\theta_1)\left(\frac{G(\widehat{\theta}_1|\widehat{\theta}_1)}{G(b|\widehat{\theta}_1)}-1\right) - \left[G(\widehat{\theta}_1|\theta_1)-G(b|\theta_1)\right] + \left(\theta_1-\widehat{\theta}_1\right)\,g\left(\widehat{\theta}_1|\theta_1\right) \\
&= \left(\theta_1-\widehat{\theta}_1\right)\,g\left(\widehat{\theta}_1|\theta_1\right) - G(\widehat{\theta}_1|\theta_1) + G(b|\theta_1)\frac{G(\widehat{\theta}_1|\widehat{\theta}_1)}{G(b|\widehat{\theta}_1)} \\
&= (\theta_1-\widehat{\theta}_1)\,g(\widehat{\theta}_1|\theta_1) - G(b|\theta_1)\left(\frac{G(\widehat{\theta}_1|\theta_1)}{G(b|\theta_1)}-\frac{G(\widehat{\theta}_1|\widehat{\theta}_1)}{G(b|\widehat{\theta}_1)}\right) \\
&= \int_{\widehat{\theta}_1}^{\theta_1}\left[g(\widehat{\theta}_1|\theta_1) - G(b|\theta_1)\,\frac{\partial}{\partial\tau}\left(\frac{G(\widehat{\theta}_1|\tau)}{G(b|\tau)}\right)\right]d\tau,
\end{aligned}$$

where g' denotes the density corresponding to g. By condition (16), the last expression is nonnegative, for $\widehat{\theta}_1<\theta_1$, and nonpositive for $\widehat{\theta}_1>\theta_1$. Thus $\overline{S}\left(\cdot,\theta_1;b\right)$ is weakly increasing in the interval (b,θ_1), and weakly decreasing in the interval $(\theta_1,1]$. Since $\overline{S}\left(\cdot,\theta_1;b\right)$ is also continuous, we have that bidding θ_1 maximizes $\overline{S}(\cdot,\theta_1;b)$ in the interval $(b,1]$.

Bidding b or less, i.e. $\widehat{\theta}_1\in[0,b]$, yields

$$\underline{S}\left(\widehat{\theta}_1,\theta_1\right)\equiv\int_0^{\widehat{\theta}_1}(\theta_1-y)\;dG\left(y|\theta_1\right).$$

Since

$$\begin{aligned}
\int_0^{\widehat{\theta}_1}(\theta_1-y)\;dG\left(y|\theta_1\right) &\le \int_0^{b}(\theta_1-y)\;dG(y|\theta_1) \\
&\le \int_0^{b}(\theta_1-y)\;dG(y|\theta_1)+G(b|\theta_1)\,B(\theta_1;b) \\
&= \overline{S}\left(\theta_1,\theta_1;b\right),
\end{aligned}$$

buyer 1 has no incentive to bid below b either. Thus bidding θ_1 is optimal for buyer 1 whenever his type is in the interval $[b,1]$.

Case 2: Buyer 1's type is lower than b, i.e. $\theta_1\in[0,b)$. Bidding $\widehat{\theta}_1\in[0,b]$ yields $\underline{S}\left(\widehat{\theta}_1,\theta_1\right)=\int_0^{\widehat{\theta}_1}(\theta_1-y)\;dG\left(y|\theta_1\right)$. Since the derivative

$$\frac{\partial \underline{S}\left(\widehat{\theta}_1,\theta_1\right)}{\partial\widehat{\theta}_1}=\left(\theta_1-\widehat{\theta}_1\right)\,g\left(\widehat{\theta}_1|\theta_1\right)$$

has the same sign of $\theta_1 - \widehat{\theta}_1$, the payoff is increasing for $\widehat{\theta}_1 < \theta_1$ and decreasing for $\theta_1 < \widehat{\theta}_1$. As in the first half of Case 1 above, by continuity $\underline{S}$, we have that $\underline{S}(\cdot, \theta_1)$ is maximized by θ_1 in the interval $[0, b]$, i.e.

$$\underline{S}\left(\widehat{\theta}_1, \theta_1\right) \leq \underline{S}(\theta_1, \theta_1), \quad \text{for all } \widehat{\theta}_1 \in [0, b]. \tag{17}$$

Finally, reporting $\widehat{\theta}_1 \in (b, 1]$ yields

$$\begin{aligned}
\overline{S}\left(\widehat{\theta}_1, \theta_1; b\right) &= \left[\theta_1 - E\left[\theta_2 \,|\, \theta_2 < b\right] + B\left(\widehat{\theta}_1; b\right)\right] G\left(b|\theta_1\right) + \left(\theta_1 - \widehat{\theta}_1\right)\left[G\left(\widehat{\theta}_1|\theta_1\right) - G\left(b|\theta_1\right)\right] \\
&\leq \left(\theta_1 - E\left[\theta_2 \,|\, \theta_2 < b\right]\right) G\left(b|\theta_1\right) \\
&= \underline{S}(b, \theta_1) \\
&\leq \underline{S}(\theta_1, \theta_1)
\end{aligned}$$

where the last inequality is implied by (17) above. □

We are now ready to show that, if the buyers' values are strictly affiliated and condition (16) holds, the b-composite auctions can be ranked strictly in terms of seller's expected revenue: the lower the parameter b, the higher the seller's expected revenue. Therefore the English auction, which is equivalent to the "1-composite" auction (we are assuming private values), generates a strictly lower expected revenue for the seller than any b-composite auction, for $b \in (0, 1)$.

Proposition 2 *Let* $\Pi(b)$, $b \in (0, 1]$, *denote the seller's expected revenue generated by the* b*-composite auction. If the signals' distribution satisfies both strict affiliation and condition (16), then* Π *is strictly decreasing in* b.

Proof. Each buyer's *ex ante* equilibrium payoff in a b-composite auction is:

$$\begin{aligned}
S(b) &= \int_0^b \left(\int_0^x (x - y)\; g(y|x)\; dy\right) dF_i(x) \\
&\quad + \int_b^1 \left(\int_0^b \left[x - y + B(x; b)\right]\; g(y|x)\; dy\right) dF_i(x),
\end{aligned}$$

where F_i denotes the marginal c.d.f. of buyer i's type. It is sufficient to show that $S(b)$ is strictly increasing in b, because in any b-composite auction the object is always sold to a buyer with the highest value, hence the same (maximum) expected social surplus is realized. Differentiating yields:

$$\frac{dS(b)}{db} = \int_b^1 \left(\left[x - b + B(x; b)\right]\; g(b|x) + \int_0^b \frac{\partial B(x; b)}{\partial b}\; g(y|x)\; dy\right) dF_i(x)$$

$$= \int_b^1 \left[(x-b)\ g(b|x) + B(x;b)\ g(b|x) + \frac{\partial B(x;b)}{\partial b} G(b|x) \right] dF_i(x)$$

$$= \int_b^1 \left[(x-b)\ g(b|x) + g(b|x) \int_b^x \left(\frac{G(\tau|\tau)}{G(b|\tau)} - 1 \right) d\tau \right.$$

$$\left. -G(b|x) \int_b^x \left(\frac{G(\tau|\tau)}{[G(b|\tau)]^2} g(b|\tau) \right) d\tau \right] dF_i(x)$$

$$= \int_b^1 \left[g(b|x) \int_b^x \frac{G(\tau|\tau)}{G(b|\tau)} d\tau - G(b|x) \int_b^x \frac{G(\tau|\tau)}{G(b|\tau)} \frac{g(b|\tau)}{G(b|\tau)}\ d\tau \right] dF_i(x)$$

$$= \int_b^1 \left[G(b|x) \int_b^x \frac{G(\tau|\tau)}{G(b|\tau)} \left(\frac{g(b|x)}{G(b|x)} - \frac{g(b|\tau)}{G(b|\tau)} \right) d\tau \right] dF_i(x).$$

Strict affiliation implies $\frac{g(y|\tau)}{g(b|\tau)} > \frac{g(y|x)}{g(b|x)}$ for all τ, x, y and b such that $\tau < x$ and $y < b$. Thus we have $\frac{\int_0^b g(y|\tau)dy}{g(b|\tau)} > \frac{\int_0^b g(y|x)dy}{g(b|x)}$, hence $\frac{g(b|\tau)}{G(b|\tau)} < \frac{g(b|x)}{G(b|x)}$ whenever $\tau < x$, for any b. Since the inside integral in the last line of the expression above is taken for $\tau \in [b, x]$, the derivative $\frac{dS(b)}{db}$ is strictly positive. □

Corollary 1 *Any b-composite auction, $b \in (0,1)$ (with two bidders), generates a strictly higher seller's expected revenue than the English auction.*

Proof. The result follows from proposition 2 and the fact that $S(1) = \int_0^1 \int_0^x (x-\tau)\ g(\tau|x)\, d\tau$ is also each buyer's *ex ante* expected surplus in the English auction. □

The rest of this section is devoted to illustrate the key idea behind the results of Proposition 2 and Corollary 1 and clarify the role of condition (16).[9] Consider a model in which the two buyers' signals have a discrete distribution. The table in Figure 2 (last page) represents the payoffs of buyer 1 when his signal (i.e., his 'type') is θ_1 and he reports the row's type, in a mechanism that mimics the symmetric equilibrium of the English auction[10], except when $(\theta_1, \theta_2) \in \{(3,1), (3,2), (4,1)\}$. If $\theta_1 = 3$, the buyer's *expected* payoff is as in the English auction, but he pays ε more if $\theta_2 = 2$, and $\varepsilon\ p(2|3)/p(1|3)$ less if $\theta_2 = 1$, where $p(j|i)$ denotes the probability that $\theta_2 = j$ conditional on $\theta_1 = i$. This difference from the English auction relaxes type 4's "downward-adjacent" incentive constraint:

[9] The idea is the same that allows the construction of full extraction mechanisms. A brief explanation can be found in Myerson (1981) p. 71.

[10] Ties are resolved in favor of bidder 1.

if he reports $\widehat{\theta}_1 = 3$, his expected payoff is

$$\left(u\left(4,1\right) - u\left(1,1\right) + \varepsilon \frac{p\left(2|3\right)}{p\left(1|3\right)} \right) p\left(1|4\right) + \left(u\left(4,2\right) - u\left(2,2\right) - \varepsilon\right) p\left(2|4\right) + \left(u\left(4,3\right) - u\left(3,3\right)\right) p(3|4),$$

while his payoff in the English auction would be

$$(u(4,1) - u(1,1))\ p(1|4) + (u(4,2) - u(2,2))\ p(2|4) + (u(4,3) - u(3,3)\,)\ p(3|4).$$

The difference

$$\delta(\varepsilon) = \left(\frac{p\left(2|3\right)}{p\left(1|3\right)} p\left(1|4\right) - p\left(2|4\right) \right)\ \varepsilon$$

is negative for any $\varepsilon > 0$, since $p\left(2|3\right)/p\left(1|3\right) < p\left(2|4\right)/p\left(1|4\right)$, by the monotone likelihood ratio property, which is implied by the affiliation hypothesis. Thus type 4's expected payment can be made higher than the English auction's

$$\sum_{j=1}^{4} u\left(j,j\right)\ p\left(j|4\right)$$

by $\delta\left(\varepsilon\right)$; and so can the expected payments of all types above 4. To summarize the above discussion in one sentence, the difference in preferences over lotteries among the various types of each buyer can be exploited to increase their expected payments.

In any b-composite auction the bonus function B changes each type's equilibrium payment function in the same way: it induces a higher payment when his opponent's type is above b, and a lower payment otherwise. To see why condition (16) is needed, note that in the example of Figure 2 the expected payoff of type 2 from reporting $\widehat{\theta}_1 = 3$,

$$\left(u(2,1) - u(1,1) + \frac{p\left(2|3\right)}{p\left(1|3\right)} \varepsilon \right) p\left(1|2\right) + \left(u(2,2) - u(2,2) - \varepsilon\right)\ p(2|2) + \left(u(2,3) - u(3,3)\,\right)\ p(3|2),$$

is *increasing* in ε, since $\frac{p(2|3)}{p(1|3)} > \frac{p(1|2)}{p(2|2)}$ by affiliation. Thus, for any ε, his "upward-adjacent" constraint will be violated if the degree of affiliation is high enough. Condition (16) limits the degree of affiliation, so that no 'upward' incentive constraint is violated by the changes in the payment functions that the bonus function B induces. It should be clear however that condition (16) is not crucial for the results of this section: even if it does not hold, it is still possible to construct LDNP selling mechanisms which generate a higher seller's expected revenue that the EA.

5 Conclusive Remarks

The comparison between the widespread use of the English auction in reality, and the fact that optimal selling mechanisms, as engineered by the theory of mechanism design, are never observed in practice, poses a puzzle to which the results of this paper provide partial answers. The main insight, which also emerges from the results established in Lopomo (1998), appears to be that the English auction can only maximize the seller's expected profit if the rules of all feasible trading mechanisms allow each buyer to determine his actual payment conditional on the allocation of the object. The central role played by this "bargaining-like" feature deserves further investigation. The reward could be a significant step toward a unified theory of auctions and bargaining.

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	1	2	3	4
4	$u(\theta_1, 1) - u(1, 1)$ $-\delta(\varepsilon)$	$u(\theta_1, 2) - u(2, 2)$	$u(\theta_1, 3) - u(3, 3)$	$u(\theta_1, 4) - u(4, 4)$
3	$u(\theta_1, 1) - u(1, 1)$ $+\frac{p(2\|3)}{p(1\|3)}\varepsilon$	$u(\theta_1, 2) - u(2, 2)$ $-\varepsilon$	$u(\theta_1, 3) - u(3, 3)$	0
2	$u(\theta_1, 1) - u(1, 1)$	$u(\theta_1, 2) - u(2, 2)$	0	0
1	$u(\theta_1, 1) - u(1, 1)$	0	0	0

Figure 2: An LDNP mechanism generating more seller's expected profit than the English auction