JOB MARKET SIGNALING OF RELATIVE POSITION, OR BECKER MARRIED TO SPENCE

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Abstract
This paper considers a matching model of the labor market where workers, who have private information on their quality, signal to firms that also differ in quality. Signals allow assortative matching in which the highest-quality workers send the highest signals and are hired by the best firms. Matching is considered both when wages are rigid (nontransferable utility) and when they are fully flexible (transferable utility). In both cases, equilibrium strategies and payoffs depend on the distributions of worker and firm types. This is in contrast to separating equilibria of the standard model, which do not respond to changes in supply or demand. With sticky wages, despite incomplete information, equilibrium investment in education by low-ability workers can be inefficiently low, and this distortion can become worse in a more competitive environment. In contrast, with flexible wages, greater competition improves efficiency. (JEL: C72, C78, D82)

1. Introduction

It is now more than 30 years since Spence (1973) introduced the idea that investment in education could be undertaken as a signal to prospective employers. Although Spence's classic work provided many important insights, it arguably provides an incomplete picture of the labor market phenomena it set out to model. In particular, whereas workers differ in quality, firms are in effect identical and offer the same wages. This ignores the role of educational achievement in effectively allocating heterogeneous workers to heterogeneous jobs. A modified model with differentiated employers could also resolve another known problem with the Spence model. There, in any separating equilibrium, strategies and outcomes (such as wages) do not respond to changes in the relative frequency of high- and low-quality workers. That is, the wages of skilled workers are oddly unresponsive to changes in the supply of either

skilled or unskilled labor. In contrast, in the model presented in this paper, workers compete for high-quality jobs that are in scarce supply. As a result, the quality and quantity of workers and the relative supply of jobs all matter for equilibrium outcomes.

This type of model is important because many real-world labor markets, particularly for professionals, clearly have vertical differentiation on both sides of the market. There is careful effort devoted by both sides to ensuring a good match between employer and employee, and intense competition for high-ranked employers and for star candidates. This strongly resembles Becker's (1973) marriage matching model, which allows for differences in quality on both sides of the market. However, the literature on matching has assumed that employment decisions are made on the basis of the intrinsic characteristics of candidates. This ignores the clear competition between candidates in educational achievement in order to achieve top jobs. Equally, while the literature on tournaments in labor economics goes back to Lazear and Rosen (1981), tournaments are usually assumed to be internal to a firm. In contrast, here I look at what I call a *matching tournament* model of a labor market in which agents make an investment decision before participating in a frictionless matching market. If that investment is a signal of otherwise unobservable ability, then matching tournaments marry aspects of both Spence's and Becker's models. When combined with the comparative statics methodology developed here, matching tournaments allow one to analyze how matching considerations affect the level of competition in signaling activity.

This paper investigates matching tournaments under incomplete information in two situations: when wages are sticky and when they are fully flexible. Workers undertake visible investment in education to signal underlying heterogeneous ability. Employers are also vertically differentiated, but this is observable. In a separating equilibrium, there is positive assortative matching in which high-quality workers send high signals and are matched with high-quality firms. Equilibrium strategies and payoffs depend on the distributions of characteristics of both firms and workers. This is shown to hold even when wages are determined by bargaining between workers and firms (transferable utility), provided the stronger assumption is imposed that workers and firms are complements in production. That is, there is a dependence on demand and supply that is absent in Spence's original model. An increase in the quantity or quality of labor or a decrease in the quality of jobs increases the competitiveness of the labor market. Under sticky wages, this leads to higher investment by high types, but low types are discouraged and invest less. In a more difficult situation, low-ranking workers have little prospect of a good job and so have a lower incentive to compete. With flexible wages, both investment and wages fall at each level of ability and wages fall for each level of investment.

Signaling models are well known for generating inefficiently high investment in signals. But in the current matching tournament environment, this is married with a hold-up problem: workers may not take into account the benefit of their investment to potential partners and hence invest inefficiently little. When wages are flexible, the signaling effect is stronger and investment is everywhere too high. However, when

wages are not flexible, the hold-up effect dominates for low-ability workers, who will invest below the social optimum. In contrast, high achievers can overinvest.

Even more interesting is how the level of efficiency responds to changes, such as an increase in quantity or quality of labor supply. Increased competition leads to lower and thus more efficient investment under flexible wages. But when wages are not fully flexible, increased competition makes investment less efficient as it discourages low-ability workers, who already invest too little. If one takes the US labor market to be closer to the flexible model and European markets to be closer to the less flexible, the two different forms of matching tournament offer a way of comparing the differing effect of demand and supply shocks on these different labor markets. This analysis suggests that, when wages are not fully flexible, there is a serious poverty trap at the bottom end of the distribution. Greater competition discourages rather than encourages low-ability workers to invest in human capital, and decreases rather than increases efficiency.

This is not the case in the classic signaling model of Spence (1973), or its generalization to a continuous type space (see Spence 1974; Mailath 1987). Specifically, in a separating equilibrium of the classic model, a signaler's equilibrium payoffs are determined by the absolute level of her productivity—for example, she ends up being paid her marginal product—and this is independent of market conditions. Further, the type of signal sent does not determine which kind of job is obtained with which type of employer. This is despite the fact that educational achievement clearly has this allocative function in real-world labor markets.[1] Of course, there are other equilibria in Spence's model. For instance, in a pooling equilibrium, the wage paid is equal to average worker quality and depends on the distribution of ability. This implies that if worker quality rises then so do wages. However, it is shown that, in the separating outcome considered here, an increase in worker quality causes wages to fall at each ability level and for each level of investment. Pooling equilibria also exist in the current model but are effectively no different from pooling outcomes in the Spence model.

In the separating equilibrium of the model presented here, a signaler's payoff will depend on which type of job she is able to obtain. High-quality jobs are in fixed supply and are allocated to the most successful candidates. An agent's job market outcome will therefore depend on her rank in the distribution of types in the population.[2] Furthermore, changes in the distribution of jobs available or in quality of other workers will change her competitive position and hence how much she must signal to communicate successfully that she is a leading candidate. The classic model can be derived as a special case of our model simply by setting the two distributions on either side of the market to be identical. Cole, Mailath, and Postlewaite (1992, 2001)

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1.   Strangely, more has been written on how conspicuous consumption signals suitability to potential social partners (Pesendorfer 1995; Rege 2008).

2.   An intriguing aspect of this model is that, because equilibrium utility will be increasing in one's rank, an outside observer might conclude that workers had an intrinsic concern for relative position.

pioneered the analysis of matching tournaments but concentrated on this case.[3] The comparative statics results in this paper are obtained by changing one distribution while holding the other constant, something clearly not possible when they are constrained to be equal.

This paper looks at labor market matching under two differing assumptions: first with constraints on wage setting and second when wages are flexible. This difference corresponds to the distinction between nontransferable utility (NTU) and transferable utility (TU) found in the matching literature. It is assumed that any match between a worker and a firm produces a surplus. In the NTU case, there are exogenous constraints on how this surplus can be divided. Wages can still depend on the value of the match but are not fully flexible as in TU. Although TU might seem the more natural assumption to many economists, the NTU matching tournament may prove a better fit for some real-world matching markets. For example, Bulow and Levin (2006) observe that wages paid for medical residents are *impersonal*. That is, they are set before the labor market commences and are not sensitive to the individual characteristics of the candidate who is hired (but vary across hiring institutions). In Europe the situation is often even more extreme: in most equivalent professional labor markets, wages are set at the national level and so there is no variation at all. For example, within most European countries, each university will pay new faculty the same starting salary.

One might well think that wage rigidity is the source of the positional effects found in matching tournaments, where high-ranking workers earn rents from their high relative position. Imagine that the quality of workers is poor; the best of that poor bunch would get the best job even if of low quality in absolute terms. In contrast, if wages were flexible (the TU case), then one might think that any such positional rents would be bargained away: low-quality workers would be offered low wages. In the end, just as in the classical models, workers would be paid their product. It is shown in Section 3 that this is not the case. If the additional assumption is made that the attributes of workers and firms are strict complements in production, then equilibrium wages, signaling, and welfare all depend on the distributions of characteristics of both firms and workers. Even with fully flexible wages, earnings and investment in education by workers of a given ability level will depend on the level of competition from other workers.

Matching tournaments were introduced by Cole, Mailath, and Postlewaite (1992, 1995). This paper is one of the first to look at comparative statics in such models. Hoppe, Moldovanu, and Sela (2009) recently and independently have also examined the comparative statics of matching tournaments. Their work is innovative in its development of techniques to deal with a finite number of participants (and taking the limit as the number approaches infinity). However, importantly, they assume that signaling is a pure waste and therefore always exceeds the socially optimal level. In contrast, one focus of the analysis here is the possibility of underinvestment that can be compounded by greater competition. Further, here there is a treatment of both TU and

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3.   They also concentrate on situations of complete information, although a brief treatment of signaling appears in Cole, Mailath, and Postlewaite (1995).

NTU in the context of a labor market, which permits the analysis of how differences in wage flexibility affect signaling behaviour. In contrast, Hoppe et al. concentrate on NTU matching in a relatively abstract environment.

Finally, while allocation is the primary concern of the literatures on the assignment of workers to jobs and on matching, it has always been assumed that matching occurs on the basis of intrinsic characteristics rather than on choices made by participants. That is why in pure matching models, although similar comparative statics exercises are possible (Costrell and Loury 2004), they have nothing to say about competition for matches or workers' investment decisions and whether the resulting investment is efficient. In summary, standard signaling models capture competition between workers but lack sensible comparative statics; assignment and matching models allow comparative statics but lack competition. This paper shows that matching tournaments incorporate the best features of both classes of model.

2. Matching Tournaments

This section outlines a model of a matching tournament, where the prizes of a standard tournament are replaced by matching opportunities. I have in mind three prime examples: students competing for places at college, the marriage market, and the market for jobs. For example, students in the final year of graduate school seek faculty positions at universities. I will use the terminology of this last case and talk about workers and firms. I also make the simplifying assumption that workers have a common ordering over potential jobs. That is, in (say) the academic job market, all graduating students agree on the best university position, the second best, and so on. Employers all agree that they would like to hire the most able candidate, but the ability of candidates is not observable. Hence, potential employers can only infer the ability of workers from an investment decision in education, made before matching. We shall look at equilibria in which all employers rank all workers in terms of this investment. In this paper, the employers have no investment decision of their own to make. Indeed, one can also consider, as a special case of the model, situations such as sports tournaments where the *firms* are only inanimate monetary prizes, which are assigned to candidates according to their performance.

The model can be viewed as an incomplete information version of the model introduced by Cole, Mailath, and Postlewaite (1992, 1995, 1998, 2001), hereafter CMP. However, we generalize their model to allow for different distributions of characteristics on the two sides of the market. This will allow both for a richer model and for comparative statics analysis of the effect of changes in those distributions. This generalization also distinguishes our model from standard signaling models.

There are two populations of agents: workers and firms. Each population is taken to be a continuum and each is differentiated in quality, with a worker's type or ability z being distributed on $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$ according to the distribution $G(z)$. This distribution is twice differentiable with strictly positive bounded density $g(z)$. Firms are also differentiated in quality s, which has the twice differentiable distribution function

$H(s)$ on $[\underline{s}, \bar{s}]$ and has strictly positive bounded density $h(s)$ (in the case of a sports tournament, $H(s)$ is just the distribution of prize money).

There are two principal differences between firms and workers. First, the type of a worker is her private information, whereas the types of firms are common knowledge. Second, workers must make an investment decision before attempting to match with firms. In particular, they must choose a visible level of investment x from the positive real line $[0, \infty)$. Following Spence, this could be a choice of education level. A worker's type z is usual interpreted as her ability and is positively related to the worker's productivity. After the choice of investment, matching will take place, with one worker matching with each firm. A match between a worker of type z investing x with a firm of type s will produce output $\pi(z, s, x)$, where $\pi(\cdot)$ is a smooth increasing function. As we shall see, stable matching is positive and assortative. That is, workers with high z will match with firms with high s.

In this matching tournament, an equilibrium will have two components: a strategy for the workers $x(z)$ that gives the choice of investment as a function of worker type, and a matching scheme that assigns workers to firms. For an equilibrium, the matching scheme must be stable given observable investment and the strategy $x(z)$.[4] Second, no worker can have an incentive to deviate given the strategies of her fellow workers and the matching scheme in place during the matching phase. The equilibrium is like that of CMP (2001), a hybrid. The second stage of the tournament is treated as a cooperative game, in that it requires stability in the matching process.[5] However, the choice of investments in the first stage is non-cooperative. We call such an equilibrium *symmetric* if all workers use the same strategy, that is, the same mapping $x(z)$ from type to investment.

The equilibrium will be constructed as follows. First, I show that, if there is complete separation in choice of investment, then the only stable matching in the second stage is positive and assortative. Second, moving backwards to the first stage, I show that there exists a unique separating strategy from which no worker wishes to deviate, given the behavior of other workers and the anticipated positive assortative matching.

Recall the assumption that the product from a match is $\pi(z, s, x)$. Assume further that (a) $\pi(\cdot)$ is twice continuously differentiable, and (b) $\pi(\cdot)$ is (weakly) increasing in all its arguments. Within this general framework, we can consider two special cases.

Story A: Valueless Signaling. Here the observable action x is itself of no use to firms. But, it may serve as a signal of a worker's type z, and the profit of firms is increasing in the type of their match. For example, as in Spence's (1973) classic model, education may signal ability. The product of a match is strictly increasing in the worker's type: $\pi_z(z, s, x) > 0$ and $\pi_x(z, s, x) = 0$.

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4.   In a fully separating equilibrium, the matching will also be stable ex post. That is, firms will not regret their match once the type of the worker has been revealed.

5.   Following the literature on cooperative matching, I look at stable matchings, and not the procedure that produces them. However, there are ways of modeling the matching process non-cooperatively, admittedly assuming away frictions, that would generate the same results; see CMP (1998) or Niederle and Yariv (2008).

Story B: Constructive Signaling. Here the observable action x increases the product of a match. However, the product also depends on a worker's unobservable type z. For example, education may increase human capital as well as signal ability. The product of a match is strictly increasing in both a worker's type and her investment: $\pi_x(z, s, x) > 0$ and $\pi_z(z, s, x) > 0$.

In this section, I consider preferences under nontransferable utility (transferable utility is analyzed in Section 3).[6] In general, NTU means that the benefits arising from the match between firm and worker are in some way indivisible and/or non-excludable; under TU the product of the match is freely divisible. The interpretation in our context is that under TU wages are fully flexible, but under NTU have some rigidity. The increased flexibility under TU makes stable matching harder to achieve, because alternative matches are easier to negotiate. Consequently, for positive assortative matching to hold, stronger assumptions are required than with NTU.

Let us assume, similarly to Spence (1974), that utility for workers is of the following form:

$$U(z, s, x) = b(z, s, x) - c(z, x), \qquad (1)$$

where b denotes benefits and c costs. Costs will be increasing in investment x but decreasing in a worker's ability z. More precisely, assume the following properties:

   (i) $b(\cdot)$ and $c(\cdot)$ are twice continuously differentiable;
  (ii) $b_s(z, s, x) > 0$, $b_z(z, s, x) \geq 0$, and $b_x(z, s, x) \geq 0$; (monotonicity of benefits);
 (iii) $c_x(z, x) > 0$ and $c_z(z, x) < 0$ (monotonicity of costs);
  (iv) $b_{zx}(z, s, x) \geq 0$, $b_{zs}(z, s, x) \geq 0$, and $c_{zx}(z, x) < 0$ (complementarity);
   (v) $b_{xx}(z, s, x) \leq 0$ and $c_{xx}(z, x) > 0$ (concavity).

These assumptions are compatible with the two principal stories used within the NTU literature. First, if we interpret the firm's type s as its level of prestige, we could have $b(z, s, x) = \hat{w} + s$. That is, the benefit from the job is a fixed wage $\hat{w}$ plus its prestige s.[7] Second, taking s to be a productive asset, such as the firm's existing human or physical capital, some form of wage inflexibility would justify $b(z, s, x) = \alpha \pi(z, s, x)$ for some $\alpha \in (0, 1)$. Social conventions or legal constraints fix wages as a fixed proportion of the product of a match. Firms receive the product $\pi(\cdot)$ from a match, less wages.[8] Thus firms, in their choice of worker, simply prefer the one who generates the highest product. That is, the profits of firms are strictly increasing in z (Story A) or in both x and z (Story B).

Following CMP (1992, 1998), a *matching* is a function $\phi: [0, 1] \to [0, 1]$ such that a worker with rank $G(z_i)$ is matched with a firm of rank $H(s_j)$, $\phi(G(z_i)) = H(s_j)$

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6. Legros and Newman (2007) consider matching in the intermediate case where the degree of transferability is a variable.

7. For example, academic wages in some European countries are fixed by national agreement. Since all universities pay the same, candidates simply prefer to be employed by the most prestigious institution.

8. Implicitly it is also assumed that money wages paid to a worker by a firm are always less than the firm and worker's joint product $\pi(z, s, x)$. That is, the firms' participation constraint is always met.

$\in [0, 1]$. The matching function ϕ is measure-preserving and one-to-one on $\phi([0, 1])$. That is, for all measurable subsets $A \subset [0, 1]$, $\phi^{-1}(A)$ is measurable and $\lambda(\phi^{-1}(A)) = \lambda(A)$, where λ denotes Lebesgue measure. A matching is *stable* if there does not exist $i \neq i' \in [0, 1]$ such that $\phi(i')P_i\phi(i)$ and $i P_{\phi(i')}i'$, where P denotes "preferred to", and both preferences hold strictly.

The first condition is the equivalent in a continuum to the assumption with finite numbers that exactly one worker is matched to one firm. It is stated in terms of the types—firm quality s and worker ability z—to ensure that matching is feasible. The second is the stability condition, standard in most matching problems, that requires that matches made are not subject to unraveling in the sense that it should not be possible to find a worker and a firm who would prefer to match with each other in place of their current matches. The main issue here is that a firm must choose a worker on the basis of visible investment x, because her type z is hidden. We make the standard assumption that firms' beliefs must be consistent with equilibrium strategies. The case on which we focus is where the workers' equilibrium strategy $x(z)$ is strictly increasing and hence separating. Then, firms must believe that a higher investment implies higher ability. Thus, under Story A or Story B, a firm will prefer worker i over worker j if and only if $x_i > x_j$. Lastly, workers' utility is strictly increasing in firm type s which is public information, thus a worker prefers firm i over firm j if and only if $s_i > s_j$.

Suppose for the moment that there exists a symmetric equilibrium strategy $x(z)$ that is differentiable and strictly increasing (we will go on to show that such an equilibrium exists). Let us aggregate all the investment decisions of the workers into a distribution summarized by a distribution function $F(x)$. A strictly increasing symmetric strategy implies that, in equilibrium, an agent of type z_i who produces $x(z_i)$ would have a position in the distribution of investment $F(x(z_i))$ equal to his rank $G(z_i)$ in the distribution of ability. This enables the firms to infer which worker is in fact the most able. This in turn allows the matches to be made through the following assortative matching mechanism such that workers with high (respectively low) x are matched with firms with high (respectively low) s. More specifically, a worker's rank in level of investment determines the rank of his match. That is, a worker who chooses x_i will achieve a match of value $s_i = H^{-1}(F(x_i))$ or $F(x_i) = H(s_i)$. Then, we can show that the assortative scheme outlined here is stable. That is, we can find no worker and firm who would both prefer each other in place of their current match.[9]

LEMMA 1. *Suppose all workers adopt a symmetric strictly increasing strategy $x(z)$. Then the only stable matching the assortative matching, such that a worker of type $z_i \in [\underline{z}, \bar{z}]$ with investment $x_i = x(z_i)$ has a match of type s_i, where*

$$G(z_i) = F(x_i) = \phi(G(z_i)) = H(s_i). \tag{2}$$

9.   Results of this type go back to Becker (1973). See Cole, Mailath, and Postlewaite (1995), Fernández and Galí (1999) for a tournament approach similar to that employed here. Eeckhout (2000) and Legros and Newman (2007) find conditions under which positive assortative matching is the *only* stable matching scheme.

We now derive a symmetric equilibrium strategy for the workers. Suppose that all agents adopt a strictly increasing differentiable strategy $x(z)$. Then the equilibrium relationship (2) implies that we can define the function

$$S(z) = H^{-1}(G(z)); \tag{3}$$

this gives for a worker of type z his equilibrium match, which depends on both G and H. Note that we have $S'(z) = g(z)/h(S(z))$. This implies an equilibrium utility of the form

$$U(z, S(z), x(z)) = b(z, S(z), x(z)) - c(z, x(z)). \tag{4}$$

Note that utility, through $S(z)$, now depends on both the distribution $G(z)$ of workers' types and the distribution $H(s)$ of firms' characteristics. Furthermore, each worker's utility is increasing in her rank $G(z)$ in the workforce.

Suppose positive assortative matching was assigned by a central planner, rather than determined by the workers' competitively chosen investments. In this case, what level of investment would workers choose? Because workers can gain from their own investment x, their choice will be greater than zero if indeed $b_x > 0$. The level of investment that is optimal in the absence of matching considerations will be useful as a point of comparison with the Nash equilibrium level of investment that will eventually be derived.

DEFINITION 1. Let $x = N(z)$ maximize $U(z, S(z), x)$; that is, let the complementary slackness condition

$$N(z)\left[b_x(z, S(z), N(z)) - c_x(z, N(z))\right] = 0$$

hold at every level of $z \in [\underline{z}, \bar{z}]$. The function $N(z)$ is the privately optimal level of investment x under NTU.

Suppose now one agent produces $x(\hat{z})$ in place of her equilibrium choice $x(z)$ and then chooses $\hat{z}$ to maximize her payoff. Her reduced form utility is $U(z, S(\hat{z}), x(\hat{z}))$, which yields the following first-order condition:

$$(b_x(z, S(\hat{z}), x(\hat{z})) - c_x(z, x(\hat{z}))) x'(\hat{z}) + b_s(z, S(\hat{z}), x(\hat{z}))S'(\hat{z}) = 0. \tag{5}$$

In a symmetric equilibrium we must have $\hat{z} = z$. Using this and rearranging the resulting first-order condition, we obtain the differential equation

$$x'(z) = \frac{b_s(z, S(z), x)}{c_x(z, x) - b_x(z, S(z), x)} S'(z). \tag{6}$$

This differential equation will give us our equilibrium strategy, when combined with the boundary condition $x(\underline{z}) = N(\underline{z})$. This boundary condition implies that the lowest ranked worker acts as though matching considerations did not matter. This reflects the equilibrium competitive response to the expectation that one is going to come last.

PROPOSITION 1. *There exists a unique solution to the differential equation (6), with the boundary condition $x(\underline{z}) = N(\underline{z})$. This solution, together with the assortative matching scheme (2), constitutes the unique symmetric separating equilibrium to the tournament matching game under NTU. In equilibrium, investment is greater and workers' equilibrium utility is lower than with privately optimum investment; that is, $x(z) > N(z)$ and $U(z, S(z), x(z)) < U(z, S(z), N(z))$ everywhere on $(\underline{z}, \bar{z}]$.*

The proof follows (see the Appendix) from the results of Mailath (1987) on the existence of separating equilibria in standard signaling models. Because it is a separating equilibrium, beliefs held by firms about worker quality will be accurate. Specifically, a firm observing a worker who invests x_i correctly believes that her quality is $z_i = x^{-1}(x_i)$. On the other hand, it is possible, as in standard signaling models, to construct other equilibria based on different beliefs. For example, a pooling equilibrium where all workers choose the same investment level $\hat{x}$ can be supported if all firms believe that levels of investment other than $\hat{x}$ indicate a low-quality worker. Since all firms consequently view all workers as being equally able (and all have the same investment), any matching, including random matching, would be stable.

An important question will be whether separating equilibria are efficient. Compare $x(z)$ with $N(z)$, the amount invested if the assortative matching scheme $S(z)$ was imposed. Proposition 1 establishes that, from the point of view of workers, they are Pareto ranked. Workers obtain the same match in both cases, but with higher effort in the separating equilibrium. All workers (except the lowest type $\underline{z}$) would be better off under $N(z)$. To be clear, this does not imply that $N(z)$ is socially optimal. When investment is productive and enters into the profits of firms (Story B), welfare is a more complex issue. We discuss this further in Section 5.

3. Transferable Utility

Suppose in contrast to what we have assumed so far that the surplus created by matching is fully divisible between the two partners. In the labor market we consider, this means that workers and firms must bargain over wages. However, even with flexible wages, the job that a worker obtains and her equilibrium utility will still depend on her rank in the distribution of workers. Furthermore, the wage she is paid in equilibrium will depend on both the distribution of worker ability $G(z)$ and the distribution of firm quality $H(s)$.

Becker (1973) discovered that in this case of transferable utility (TU), assortative matching is only stable if the two attributes, here z and s, are complements in a joint production process. This is in contrast to the case of nontransferable utility assumed up to now, where stability requires only that workers' utility be increasing in s and firms' profits be increasing in z. In this section, we therefore need some additional assumptions on the production function $\pi(z, s, x)$: (c) $\pi_{zs}(z, s, x) > 0$ and $\pi_{zx}(z, s, x) \geq 0$ (complementarity); (d) $\pi_{sx}(z, s, x) = 0$ (partial separability); (e) $\pi_{xx}(z, s, x) \leq 0$ (concavity in investment). The complementarity condition (c) will be needed to ensure stability of positive assortative matching. Condition (d) is a convenient simplification

that will allow integrability of the wage function. Condition (e) will help to define the privately optimal investment.

Denote the share of this product that goes to the worker as a wage w, and share of the firm, a profit $r = \pi(z, s, x) - w$. We now replace the original form of the worker's utility with

$$U(z, w, x) = w - c(z, x), \tag{7}$$

which is exactly the form assumed by Spence (1974). That is, now the worker only values a match in terms of the wages she will receive from that job. The assumptions on the cost function $c(z, x)$ remain the same.

What I again assume is that workers make their choice of investment non-cooperatively. Then, matching between firms and workers takes place cooperatively in that the outcome is assumed to be stable. In equilibrium, no worker has an incentive to deviate given others' choices and the matching scheme. With transferable utility, positive assortative matching will not be stable unless the product of the match $\pi(z, s, x)$ can be divided in such a way that no worker–firm pair of differing ranks has the incentive to match with each other rather than with a partner of the same rank. To determine this, first, we examine what the conditions for stable assortative matching would be under complete information. Second, we find that, because of the partial separability assumption (d), the stability condition implies a wage schedule $w(z, s, x)$ which does not depend on the functional form of the workers' strategy $x(z)$. This means that the wage schedule is also applicable in the separating equilibrium of the game of incomplete information.[10] This is possible because in a separating equilibrium, workers' actions fully reveal their underlying type.

The first step in determining the appropriate level of wages under complete information is taken from Becker's (1973) observation that stability requires that the payment to each partner should be related to her marginal productivity. Given that here a worker's output depends both on her type z and her investment x, assumption (d) will be important. It implies that a worker's choice of investment x does not affect her match. This is because, while condition (c) ensures complementarity in types, condition (d) means that there is no complementarity between firm quality and worker investment. Together they ensure that, under complete information, firms only care about a worker's ability z in choosing whom to hire. Thus, we consider positive assortative matching only in terms of workers' ability. Specifically, again we look at the case where a worker of ability z will be matched with a firm of type $S(z)$. For this matching to be stable, for a given level of investment x, we must have

$$w(z + \varepsilon, S(z + \varepsilon), x) + \pi(z, S(z), x) - w(z, S(z), x) \geq \pi(z + \varepsilon, S(z), x). \tag{8}$$

That is, the total payoff to a worker of type $z + \varepsilon$ and a firm of type $S(z)$ must be greater under the current matching arrangements than the output from a matching between

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10.   That is, the functional form of the wage schedule is the same under complete and incomplete information. However, it will be shown that workers' choices of investment will be higher with incomplete information. Feeding this higher investment into the same wage schedule implies that wages at a given ability level will be higher than under complete information. See Example 1 at the end of this section.

each other. Otherwise, the worker of type $z + \varepsilon$ could strike a bargain with the firm of type $S(z)$ whereby both would be better off.

Turning to investment, if we fix the type of worker at z, for stability given two workers producing investment levels $x + \varepsilon$ and x, it must be that

$$w(z, S(z), x + \varepsilon) + \pi(z, S(z), x) - w(z, S(z), x) \geq \pi(z, S(z), x + \varepsilon). \qquad (9)$$

It is possible to derive equilibrium marginal conditions on the wage function by taking the limit ε to zero. Specifically, in the proof of Proposition 2, it is shown that from (8) and (9) one can derive

$$w_z(z, S(z), x) + w_s(z, S(z), x)S'(z) = \pi_z(z, S(z), x),$$

$$w_x(z, S(z), x) = \pi_x(z, S(z), x). \qquad (10)$$

Finally, we need a boundary condition for wage bargaining. Suppose that the lowest wage is exogenously fixed at $C \geq 0$. It is then possible to integrate the marginal conditions (10) to derive the wage function $w(z, S(z), x)$ given in what follows. This in turn will support positive assortative matching as a stable outcome for the matching tournament.

PROPOSITION 2. *Let C be an arbitrary constant satisfying $0 \leq C \leq \pi(\underline{z}, \underline{s}, 0)$. Assume complete information and that the workers' strategy $x(z)$ is increasing. Then positive assortative matching that satisfies the relation (2) is stable given the bargaining solution,*

$$w(z, S(z), x) = \int_{\underline{z}}^{z} \pi_z(t, S(t), 0)\, dt + \int_0^x \pi_x(z, S(z), t)\, dt + C; \qquad (11)$$

and this is the only stable matching.

Cole, Mailath, and Postlewaite (2001) offer a much more detailed treatment of a similar problem, but where only the investment x but not the type z matters for the surplus π.[11] In our setup, the bargaining solution is complicated by wages potentially depending on both type z and action x. In this context, assumption (d) is helpful as it implies that the marginal conditions (10) are integrable. To see this, differentiate the right-hand side of the wage equation (11) with respect to z and observe that the derivative matches the first differential equation in (10) only if $\pi_{sx} = 0$.

This means that it is possible to construct the wage function $w(z, S(z), x)$ from the matching function $S(z)$ and the exogenous functions π_z, π_x, without knowing the equilibrium investment function $x(z)$. Alternatively, one could combine the two differential equations (10) into a single one in terms of observable investment, or

$$w'(x) = w_x(\gamma(x), S(z), x)$$

$$+ \frac{w_z(\gamma(x), S(\gamma(x)), x) + w_s(\gamma(x), S(\gamma(x)), x)S'(\gamma(x))}{x'(\gamma(x))}, \qquad (12)$$

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11.   In particular, they show that the bargaining solution, here $w(\cdot)$, can have a finite number of discontinuities or jumps, although, completely continuous solutions are not excluded. For simplicity, I concentrate on continuous solutions.

where $\gamma(x) = x^{-1}(x)$, the inverse of the workers' strategy. But clearly such an approach depends on the strategy $x(z)$. This in turn would require simultaneous solution of the wage schedule and investment strategy, that is solution of the simultaneous differential equations (12) and (14). Assumption (d) thus allows the current simpler method at a relatively small cost in loss of generality.[12]

I now turn to incomplete information. Matches will be made and wage bargains struck on the basis of the perceived type of the workers. However, in any separating outcome, the wage equation (11) is still the condition for stability. Firms can observe investment, from investment they can accurately deduce ability. More precisely, if all workers adopt the strictly increasing strategy $x(z)$, then, in equilibrium, firms must believe that a worker choosing investment $x(\hat{z})$ is of type $\hat{z}$. Note that positive assortative matching is stable under the wage schedule (11) for any positive relationship between investment x and z, regardless of its exact functional form. Thus, whatever the separating strategy $x(z)$, if firms offer a wage $w(\hat{z}, S(\hat{z}), x(\hat{z}))$ to a worker with visible investment $x(\hat{z})$, then positive assortive matching will be stable. Further, this is the only stable matching.

We now turn to the prior stage where workers choose investment non-cooperatively. I assume that a separating strategy $x(z)$ is adopted by all workers and that consequently wages are determined by the wage equation (11) and see if workers have an incentive to deviate. If one worker contemplates a deviation to $x(\hat{z})$, that is, investing as if he were of type $\hat{z}$, then he would be perceived to be of type $\hat{z}$ and expect a match with a firm of type $S(\hat{z})$ and a payment of $w(\hat{z}, S(\hat{z}), x(\hat{z}))$, even though the actual product of the match will be $\pi(z, S(\hat{z}), x(\hat{z}))$. This gives a reduced-form utility of $U = w(\hat{z}, S(\hat{z}), x(\hat{z})) - c(z, x(\hat{z}))$. To find the condition for such a deviation not to be profitable, differentiate this with respect to $\hat{z}$ and set the resulting derivative to zero. For a symmetric equilibrium, set $\hat{z} = z$ in the first-order condition to obtain

$$w_z(z, S(z), x(z)) + w_s(z, S(z), x(z))S'(z)$$
$$+ (w_x(z, S(z), x(z)) - c_x(z, x(z)))x'(z) = 0. \qquad (13)$$

Then substituting from (10), one obtains the following differential equation:

$$x'(z) = \frac{\pi_z(z, S(z), x)}{c_x(z, x) - \pi_x(z, S(z), x)}. \qquad (14)$$

In order to provide a boundary condition for this equation, we must define a level of investment x which is privately optimal, that is, one that is independent of matching considerations. Assume that the positive assortative matching scheme $S(z)$ is exogenously imposed. This implies that an increase in x can only increase wages by increasing output, not by achieving a more favorable match. Or in other words, in the absence of matching considerations we need only consider the partial derivative of wages with respect to investment $w_x(z, S(z), x) = \pi_x$. This enables the following definition.

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12.   I do not believe that assumption (d) affects the qualitative results obtained. While this assumption is essential for Proposition 2, it could be replaced by a (weaker) complementarity assumption $\pi_{sx} \geq 0$ in the proof of the comparative statics result of Proposition 5.

DEFINITION 2. Let $x = T(z)$ maximize $U = w(z, S(z), x) - c(z, x)$; that is, the complementary slackness condition

$$T(z)\left[w_x(z, S(z), T(z)) - c_x(z, T(z))\right]$$

$$= T(z)\left[\pi_x(z, S(z), T(z)) - c_x(z, T(z))\right] = 0 \qquad (15)$$

holds at every level of $z \in [\underline{z}, \bar{z}]$. The function $T(z)$ is called the privately optimal level of investment x under TU.

Equation (15) has a unique solution by assumption (e) on $\pi(\cdot)$ and the assumption that $c(\cdot)$ is convex in x. Note that when investment is productive (Story B, $\pi_x > 0$), then it can be that $T(\underline{z}) > 0$. That is, the equilibrium investment by the least able worker may be greater than zero (but when $\pi_x = 0$, then $T(\underline{z}) = 0$). This is because, with productive investment and transferable utility, additional investment is worthwhile even to the least able as it increases wages. In any case, this privately optimal level of investment will give us the appropriate boundary condition, $x(\underline{z}) = T(\underline{z})$ for the equilibrium differential equation. That is, again the lowest-ranked worker chooses the privately optimal investment. This, together with the earlier Proposition 1, leads to the next result.

PROPOSITION 3. *There exists a unique solution to the differential equation (14). Together with the boundary condition $x(\underline{z}) = T(\underline{z})$ and with the assortative matching scheme (2) and the wage function (11), this solution constitutes a symmetric equilibrium of the tournament matching game with transferable utility. In equilibrium, investment is greater and workers' equilibrium utility is lower than with privately optimum investment; that is, $x(z) > T(z)$ and*

$$w(z, S(z), x(z)) - c(z, x(z)) < w(z, S(z), T(z)) - c(z, T(z))$$

everywhere on $(\underline{z}, \bar{z}]$.

Our equilibrium differential equation (14), while clearly not identical to the differential equation (6) that arose in the NTU case, does depend on the distributions $G(z)$ and $H(s)$ through the function $S(z)$. Hence, both equilibrium payments $w(z, S(z), x)$ and the equilibrium strategy $x(z)$ will respond to changes in either in the distribution of ability $G(z)$ or of jobs $H(z)$. As well as the above-mentioned separating equilibrium, there will also exist pooling equilibria in which wages would reflect average productivity. In that sense, it is true that pooling equilibria, of the current and classic model, do respond to changes in demand and supply that affect average productivity. In particular, an increase in the quality of workers would lead to an increase in wages while investment would remain fixed across all workers. In contrast, in the separating equilibrium we examine, we will see that investment would change, and wages would fall at each level of ability.

In any case, just as for NTU, in the separating equilibrium investment is excessive from the point of view of workers. All workers (except with the lowest ability $\underline{z}$) will invest more than what is privately optimal. Wages can be higher but the increase in costs is greater so that workers are definitely worse off than with assortative matching

and the privately optimal level of investment. Here is a specific example of such an equilibrium.

Example. Assume that the production function is $\pi(z, s, x) = zs + x$ and that the cost function is $c(z, x) = x^2 - xz + x$. Assume further $G(\cdot) = H(\cdot)$ with $\underline{z} = \underline{s} = 0$ so that $S(z) = z$. Thus, $\pi_z(z, S(z), x) = S(z)$ and $\pi_x(z, S(z), x) = 1$. If $w(0, 0, 0) = 0$, then by the previous analysis, $w(z, S(z), x) = z^2/2 + x$. In this case, solving (15), the privately optimal investment is $T(z) = z/2$. In contrast, for the noncooperative equilibrium the differential equation (14) is now $x'(z) = z/(2x - z)$ with $x(0) = 0$, which has the solution $x(z) = z$. Given the privately optimal investment, wages would be $w(z, S(z), T(z)) = z^2/2 + z/2$ and will be lower at a given level of ability z than in the non-cooperative equilibrium, $w(z, S(z), x(z)) = z^2/2 + z$, as there workers invest more. However, given the additional costs incurred, workers would be better-off if all made the privately optimal investment:

$$w(z, S(z), T(z)) - c(z, T(z)) = 3z^2/4 > z^2/2 = w(z, S(z), x(z)) - c(z, x(z)).$$

4. Comparative Statics

We will now consider the effect on equilibrium utility and strategies of changes in the distribution of workers' abilities $G(z)$ and of firms or jobs $H(s)$. In doing this, we consider only separating equilibria. We saw in Sections 2 and 3 that equilibrium behavior depends on the matching function S which is jointly determined by G and H. Our first question is what are the effects of changes in the underlying distributions on the matching function $S(z)$. We will then be better placed to answer questions about changes in equilibrium behavior.

In what follows we assume two economies A, B that are identical apart from having different distributions of workers or different distributions of jobs. Importantly, we interpret an increase in quality of workers or a decrease in the quality of jobs, in the form respectively of a stochastically higher or stochastically lower distribution, as increases in the competitiveness of the labor market. Since in the matching tournaments considered here workers compete in quality, an increase in the quality of other workers will worsen the competitive situation of a given worker. Equally, as workers compete to obtain high-quality jobs, if high-quality jobs become relatively scarce, competition will become more intense. Changes in the relative quantity of jobs and workers are considered in Section 6 and give similar results. For investigation of the effect of changes in the degree of inequality amongst workers in a similar framework, see Hopkins and Kornienko (2004). For analysis of the effect of changes in the distribution of prizes in tournaments, see Moldovanu and Sela (2006), an analysis that is expanded to matching tournaments with a finite number of participants in Hoppe, Moldovanu, and Sela (2009). Hopkins and Kornienko (2009, 2010) propose a different methodology for comparative statics, which makes comparisons at a constant rank (here $r = G(z)$) rather than at a constant ability z.

When we change one distribution we hold the other constant. For example, suppose we considered two distributions of workers $G_A(z)$ and $G_B(z)$, then the distribution of
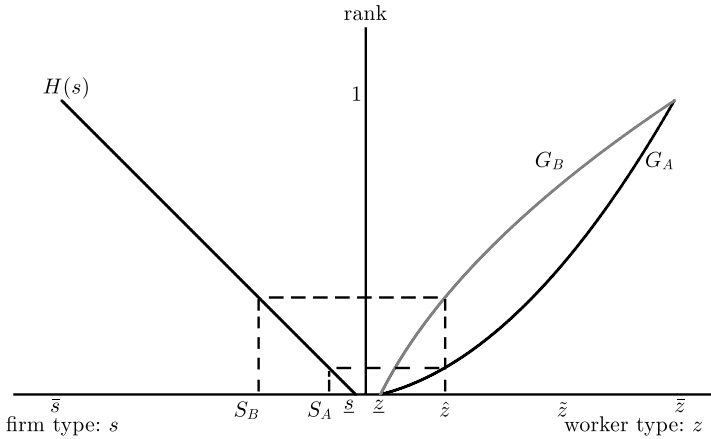
FIGURE 1. A worker with ability $\hat{z}$ has a match S_A under the stochastically higher distribution of ability G_A that is worse than the match S_B under the lower distribution of ability G_B.

jobs $H(s)$ is fixed and is the same in both economy A and economy B. Further, I assume that the support of the distribution of workers remains $[\underline{z}, \bar{z}]$ and the support of the distribution of jobs is $[\underline{s}, \bar{s}]$. Stochastic dominance is often used to order different distributions. One says that one distribution G_A stochastically dominates, or is stochastically higher than, another distribution G_B if $G_A(z) \leq G_B(z)$ for all z. Here, I employ a modest refinement of stochastic dominance and write $G_A >_{st} G_B$ if $G_A(z) < G_B(z)$ for all $z \in (\underline{z}, \bar{z})$ and $G'_A(\underline{z}) < G'_B(\underline{z})$.[13]

LEMMA 2. *If either $G_A >_{st} G_B$ or if $H_B >_{st} H_A$, then $S_A(z) < S_B(z)$ for all $z \in (\underline{z}, \bar{z})$ and $S'_A(\underline{z}) < S'_B(\underline{z})$.*

That is, if the distribution of workers becomes stochastically higher, or if the distribution of job quality becomes stochastically worse, then the job assigned becomes worse at almost every ability. Note that the comparative statics from changes in H are the reverse to those from changes in G. These results are illustrated in Figures 1 and 2. Notice that if we constrain the distributions so that $G(\cdot) = H(\cdot)$, then $S(z) = z$, and any dependence of matching on the distributions disappear. As changes in the two distributions have opposite effects, if the two distributions are constrained to be equal to each other, a movement of one distribution is cancelled out by the movement of the other.

4.1. Nontransferable Utility

We have just seen how matching responds to changes in the distribution of jobs $H(s)$ or workers $G(z)$: in a more competitive environment, for a given level of ability z a

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13.   Note that if, for example, $G_A(z)$ first-order stochastically dominates $G_B(z)$ then $G'_A(\underline{z}) > G'_B(\underline{z})$ is not possible, and if the two distributions G_A and G_B are distinct then $G'_A(\underline{z}) = G'_B(\underline{z})$ is not generic.
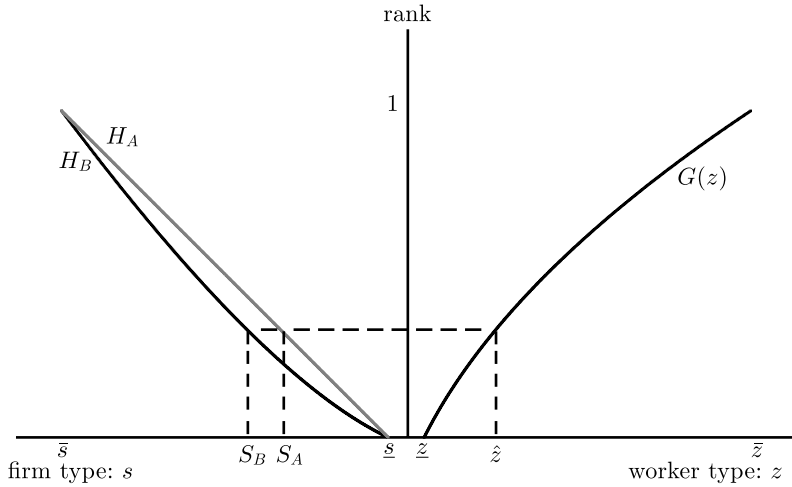
FIGURE 2. A worker with ability $\hat{z}$ has a match S_A under the stochastically lower distribution of jobs H_A that is worse than the match S_B under the higher distribution of jobs H_B.

worker is matched with a worse job. We now see how equilibrium investment and utility reacts to such changes. The effects are not obvious as we will see that the effect on investment is not monotonic. Low-ability workers will invest an amount that is closer to their privately optimal amount in a more competitive economy, and yet they are still worse-off.

This nonmonotonicity of investment is easy to explain. Imagine a foot race where a new, very fast runner is added to the field. Those runners who are not very fast realize that the prospects of placing high up the field are even lower and therefore try less hard. Those runners who are competing for the top places realize that competition is now fiercer and respond accordingly.

Let $U(z) = b(z, S(z), x(z)) - c(z, x(z))$ be a worker's equilibrium utility under NTU. We have by the envelope theorem, $U'(z) = b_z(z, S(z), x(z)) - c_z(z, x(z))$. I show that, despite the ambiguous effect on investment, equilibrium utility will be lower in a more competitive environment, that is, when $G(z)$ is stochastically higher or $H(s)$ is stochastically lower.

In effect, the fall in $S(z)$ at each level of z has a first-order effect on workers' utility through the benefit function $b(z, S(z), x(z))$. The fact that costs may also fall for low-ability workers as their investment falls is not enough to compensate. The assumption (iv) on complementarity in benefits and costs is crucial for this. These results are illustrated in Figure 3.

PROPOSITION 4. *Suppose that either $G_A(z) >_{st} G_B(z)$, or $H_B(z) >_{st} H_A(z)$. Let $U_A(z)$ and $U_B(z)$ and $x_A(z)$ and $x_B(z)$ be the corresponding NTU equilibrium utility and equilibrium investment strategies respectively. Then, first, $U_A(z) < U_B(z)$ for all z in $(\underline{z}, \bar{z})$. Second, there is a point $\tilde{z} \in (\underline{z}, \bar{z})$ such that $x_B(z) > x_A(z)$ on the interval $(\underline{z}, \tilde{z})$ and $x_A(z) > x_B(z)$ on $(\tilde{z}, \bar{z})$.*
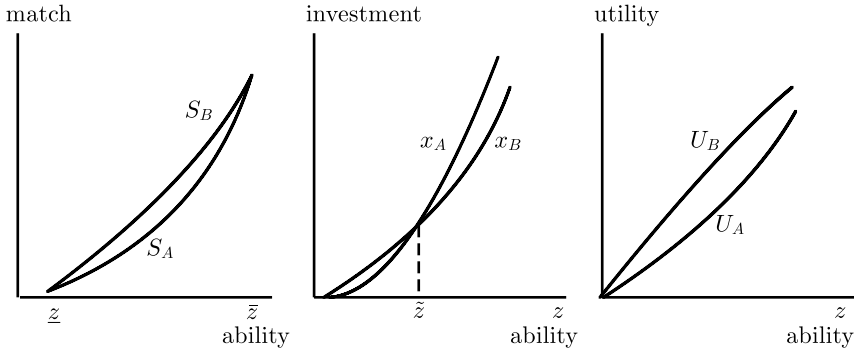
FIGURE 3. Comparative statics under NTU. A stochastically higher distribution of ability or a stochastically lower distribution of job quality leads to a lower matching function $S_A(z)$ and hence lower utility $U_A(z)$ and lower (higher) investment by workers of low (high) ability.

4.2. Transferable Utility

It is also possible to obtain comparative statics for the model under TU, including what happens to wages when the distributions change. In particular, in a more competitive economy there is lower investment by all types of worker. Investment falls because in a more competitive situation, a worker of given ability obtains a worse job. Because firm quality is a complementary factor to workers' marginal product $\pi_z(z, S(z), x)$, the marginal product falls leading to lower incentives and less effort. These results on investment in turn imply a similar result on workers' utility. Let $U(z) = w(z, S(z), x(z)) - c(z, x(z))$ be workers' equilibrium utility under TU. As in NTU, the effect of a lower match value $S(z)$ dominates the effect of lower costs due to lower investment. Thus, equilibrium utility will be lower at every level of ability in a more competitive economy. These results are illustrated in Figure 4.
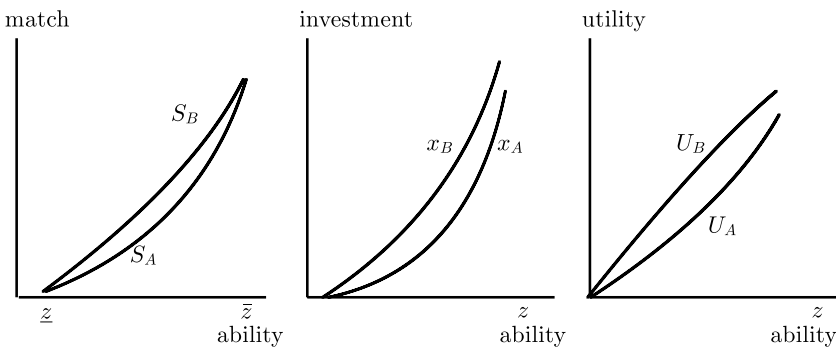

FIGURE 4. Comparative statics under TU. A stochastically higher distribution of ability or a stochastically lower distribution of job quality leads to a lower matching function $S_A(z)$ and hence, lower utility $U_A(z)$ and lower investment $x_A(z)$ by workers.

PROPOSITION 5. *Suppose that either $G_A(z) >_{st} G_B(z)$ or $H_B(z) >_{st} H_A(z)$. Let $U_A(z)$ and $U_B(z)$ and $x_A(z)$ and $x_B(z)$ be the corresponding TU equilibrium utility and equilibrium investment strategies respectively. Then, $x_B(z) > x_A(z)$ on $(\underline{z}, \bar{z}]$. Second, $U_A(z) < U_B(z)$ for all z in $(\underline{z}, \bar{z}]$.*

Define $w(z) = w(z, S(z), x(z))$ as the equilibrium wage schedule. One can then show that, when the distribution of ability is stochastically higher, the equilibrium wage is lower at each level of ability. In Spence's (1973, 1974) original work, the wage schedule was written in terms of visible investment or education x. Here, we have $w'(x) = w'(\gamma(x))/x'(\gamma(x))$ where $z = \gamma(x) = x^{-1}(x)$, the inverse of the equilibrium strategy $x(z)$. The wage schedule will be a solution to this differential equation with boundary condition $w(x(\underline{z})) = C$. This can be used to show that wages will also be lower at each level of investment.

PROPOSITION 6. *Suppose that either $G_A(z) >_{st} G_B(z)$ or $H_B(z) >_{st} H_A(z)$. Let $w_A(z)$ and $w_B(z)$ be the corresponding equilibrium wage schedules. Then, $w_B(z) > w_A(z)$ on $(\underline{z}, \bar{z}]$ and $w_B(x) > w_A(x)$ on $(x(\underline{z}), x_A(\bar{z})]$.*

4.3. Comparing NTU and TU

We have seen that there are different comparative statics when wages are flexible (TU) than when they are not (NTU). In particular, a more competitive environment, such as a stochastically higher distribution of worker ability, produces uniformly lower investment under TU but higher investment by the high-ability under NTU. Let us look at what is driving this difference.

Suppose we take very simple formulations. For the NTU case, let $b(z, s, x) = s$, the benefit of holding a job is simply equal to the quality of the employer. Then, we can write the equilibrium differential equation (6) as

$$c_x(z, x(z)) = S'(z)/x'(z). \tag{16}$$

Similarly, for expository purposes, suppose simply that $\pi(z, s, x) = zs$, then the differential equation for the TU case (14) can be written

$$c_x(z, x(z)) = S(z)/x'(z). \tag{17}$$

In both cases, we have the marginal cost of increasing investment x on the left-hand side, and the marginal benefit on the right.

In the TU case, the marginal benefit is proportional to $S(z)$ (or more generally $\pi_z(z, S(z), x)$), and will be lower everywhere in a more competitive environment (see, for example, Figure 1). Under TU, incentives are driven by the wage which is determined by a worker's marginal product. In a more competitive environment, a worker of a given ability gains a lower value match. Since the employer type is a strict complement, the worker's marginal product falls, lowering the incentive to invest at all levels of ability.

However, in an NTU world, the marginal return depends on $S'(z)$, the marginal return to moving up in terms of one's match. Again using the metaphor of a foot race,

replacing low-quality runners with high-quality runners increases the relative density of competitors at high ability levels and reduces it at low levels. The incentive to invest is increased for the high-ability and reduced for the low-ability.

5. Efficiency

This section addresses the efficiency of equilibrium investment decisions. Given the presence of incomplete information, it is no surprise that full social efficiency is not obtained. However, what is interesting is the application of the novel comparative statics techniques introduced in the previous section, to see whether efficiency increases or decreases as the distributions of workers and jobs change.

If there are complementarities between firms and workers then, from the results of Becker (1973), the maximization of total output demands the positive assortative matching scheme $S(z)$. Since matching is efficient, this allows us to concentrate on a different issue: whether, for each pair formed under this scheme, the worker chooses a level of investment that is optimal from the point of view of joint welfare. Whether this will be the case is not obvious as there are two factors that work in opposite directions. First, workers are unlikely to internalize the benefit of the effect of additional investment on the profits of firms, leading to too little investment. That is, there is a form of hold-up problem. Second, competition between workers for matching opportunities can push investment up, possibly to excessive levels. In the case of complete information, CMP (2001) find that as investment raises one's marginal product, which in a TU framework leads to higher wages, this solves the first problem. Thus, efficient investment is possible even without enforceable contracts (see also Peters and Siow 2002). However, Peters (2007) finds that, in a NTU framework, again under complete information, the second factor is stronger than the first, and investment is inefficiently high.

Turning to the tournaments under incomplete information considered here, we have already seen that investment is excessive from the point of view of workers. This leads immediately to the result that when such investment is not useful for firms (Story A), its equilibrium level is also socially excessive. But it is possible to show that, under TU, investment is still too high even when investment is productive. In contrast, under NTU, I find productive investment under some circumstances will be too low.

This difference arises because what is privately optimal is also socially optimal under TU, but is too low under NTU. The lowest ranked individual in both cases will choose what is privately optimal. Thus, under NTU we start too low, and under TU we start just right. As non-cooperative investment for higher levels of ability rises above the privately optimal level, under TU investment will be too high everywhere, while under NTU we may eventually rise above the social optimum at high levels of ability.

As the tournament becomes more competitive, the two cases diverge. As we have seen in the previous section, greater competition lowers the quality of job obtained for a given level of ability. This lowers a worker's marginal product, leading to reduced investment under TU. Thus, investment will become closer to the optimum. Under NTU, greater competition discourages the low-ability and their investment falls, even

though it is too low already. However, high-ability workers will invest more under greater competition even though their investment may already be excessive. Thus, under NTU, greater competition can lead to greater inefficiency. Notice, however, we have already seen that under both TU and NTU, this increase in competition makes workers worse off (Propositions 4 and 5). So, any gains in efficiency in the TU case do not go to workers.

There is another potential point of comparison. Rege (2008) and Hoppe, Moldavanu, and Sela (2009), in matching tournament models similar to the current one, compare a separating outcome that supports positive assortative matching not with the social optimum but with a pooling outcome and consequent random matching. The welfare comparisons are in general ambiguous as when there are complements in production, there is a trade-off between the costs of signaling and the benefits of assortative matching that it permits. However, Hoppe et al. find that for plausible-type distributions total welfare is higher in the pooling equilibrium with zero signals than in the separating equilibrium. An important difference is that in their analysis, signals are entirely wasteful (Story A). Here, the most interesting results are where signals take the form of productive investment. Clearly, a pooling equilibrium with zero or minimal investment will be less attractive in such an environment than in the model of Hoppe et al. But achieving clear welfare comparisons would not be possible in this more complex environment without making very strong assumptions on preferences and costs.

5.1. Nontransferable Utility

Under NTU, as we have already seen (Proposition 1), the lowest ability worker has no incentive to invest more than is privately optimal. As we will now see, as under Story B, such investment would benefit her employer, this level of investment is inadequate from a social point of view. In contrast, for higher-ability workers competition for matches raises investment above privately optimal levels, and possibly above socially optimal levels too.

Let us assume that the total welfare of an individual match between a firm and worker is given by a weighted sum of the worker's utility and the firm's profit:

$$W = U(z, S(z), x) + \beta \pi(z, S(z), x), \tag{18}$$

for some $\beta > 0$. For example, if the benefits to a worker are a fixed proportion of the product $b(z, s, x) = \alpha \pi(z, s, x)$ with $\alpha \in (0, 1)$, then $\beta = 1 - \alpha$. Then the first-order conditions for an interior solution to a social planner's choice of investment are

$$\frac{dW}{dx} = b_x(z, S(z), x) - c_x(z, x) + \beta \pi_x(z, S(z), x) = 0. \tag{19}$$

Note that if π_x is zero, so that x is nonproductive, the social optimum requires x to be equal to the privately optimal level $N(z)$ (which may be zero). Then, immediately by Proposition 1, we have the unsurprising result that since investment is socially

unproductive, in equilibrium its supply is excessive. However, if we make the assumption that investment is productive or $\pi_x > 0$, then comparison of (5) and (19) leads directly to the next result.

PROPOSITION 7. *Assume conditions (c)–(e) on $\pi(\cdot)$ and that $\pi_x(z, s, x) > 0$. Then there exists a unique solution $x = N^*(z) > 0$ to equation (19) at each level of z. For at least some types, the non-cooperative level of investment $x(z)$ is less than the social optimum $N^*(z)$. That is, there is an $z_1 \in (\underline{z}, \bar{z}]$ such that $x(z) < N^*(z)$ on $[\underline{z}, z_1)$. To compare two economies A, B assume either $G_A(z) >_{st} G_B(z)$ or $H_B(z) >_{st} H_A(z)$. Then, there is an $z_2 \in (\underline{z}, \bar{z})$ such that $x_A(z) < x_B(z) < N^*(z)$ on $(\underline{z}, z_2)$.*

That is, at least some low types invest too little as their low prospects give no incentive to do more than that which is privately optimal. However, it is impossible to determine for higher types whether investment is too high or too low. While it is possible for all types to invest too little, one would imagine that typically high types will invest too much. Particularly, if the production function is strictly concave, then as the marginal product of investment falls, the socially optimal investment will approach the privately optimal level for high x and cross the non-cooperative level.

The results on investment do have a striking conclusion, as illustrated in Figure 5 . In the more competitive environment, which has a matching function $S_A(z)$ that is worse from the point of view of workers, distortions from the socially optimal are larger. In particular, the low-type workers who in any case invest too little will invest even less. And the high-ability workers who may put in too much effort will do even more. Note that Figure 5 illustrates only one particular scenario: in general, there is no assurance that N^* will cross x_A and x_B only once or that it will cross at all.
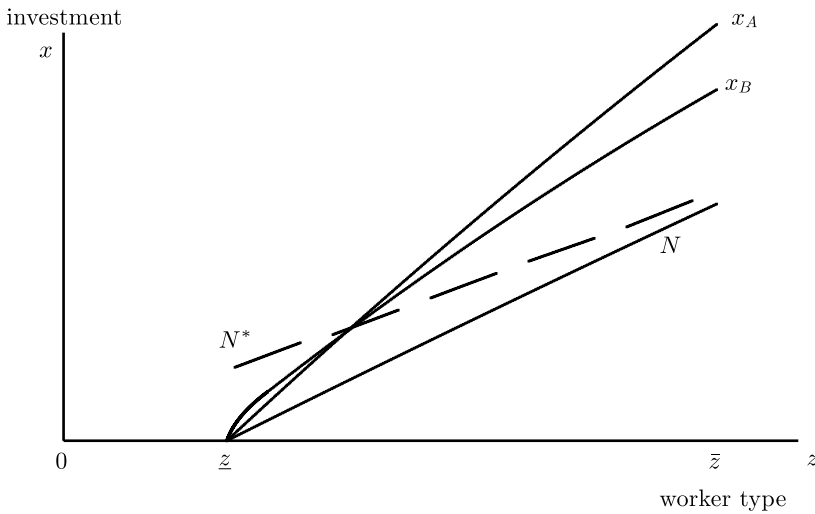
FIGURE 5. Under NTU, investment in the more competitive environment x_A may be further from the social optimum N^* for low- and high-ability workers than investment in the less-competitive environment x_B.

5.2. Transferable Utility

Under TU, the welfare results are quite different. The fact that wages respond to productive investment gives a natural incentive to invest. Unfortunately, under incomplete information, investment also serves to gain an improved match, which leads to excess investment. The total payoff of a match is given by

$$W = \pi(z, S(z), x) - c(z, x) \tag{20}$$

as the sum of wages and profits must equal output $\pi(\cdot)$. Thus the complementary slackness condition for a social optimum, conditional on positive assortative matching, is

$$x \frac{dW}{dx} = x[\pi_x(z, s, x) - c_x(z, x)] = 0. \tag{21}$$

That is, when x is a productive investment, the social optimum equates the marginal cost of investment to the worker c_x and the investment's marginal product π_x. Note that under TU, this condition is the same as for the privately optimal level of investment $T(z)$. This reflects the results of CMP (2001), who find that with complete information, a matching tournament can induce the efficient amount of investment.

However, under incomplete information there is a gap between private incentives and the social optimum. This is because each individual has an additional private return from increasing investment as it permits a better match. However, a change in either the distribution of jobs or workers that induces greater competition pushes investment closer to the socially optimal level.

PROPOSITION 8. *In the matching tournament with incomplete information and under TU, the equilibrium level of investment $x(z)$ exceeds the socially optimal level $T^*(z)$ almost everywhere. To compare two economies A, B assume either $G_A(z) >_{st} G_B(z)$ or $H_B(z) >_{st} H_A(z)$. Then, $x_B(z) > x_A(z) > T^*(z)$ on $(\underline{z}, \bar{z}]$.*

6. Unemployment

Until now, we have assumed that all workers are matched to jobs. Obviously, it is a characteristic of many real-world labor markets that the least successful candidates fail to attract any offers as there are more candidates than there are job openings. It is relatively easy to modify the basic matching tournament model to allow for this. We find again that the model delivers sensible comparative statics. For example, a decrease in the number of jobs available relative to the number of workers will, in the TU case, lower wages at every level of ability.

Assume now that that the measure of firms relative to that of workers is $1 - \mu$, so that a proportion $1 > \mu > 0$ of workers will not find employment. Under assortative matching, these will be the least able, so that those having ability on the range $[\underline{z}, \hat{z})$, where $G(\hat{z}) = \mu$, will be unemployed and have no match, which I write as $s = s_0$ with $s_0 < \underline{s}$, that is, the value of being unmatched is worse than being matched to the

worst job. The benefits from being unemployed are assumed to be different in TU and NTU and will be specified in what follows. But in either case I assume the following assortative matching scheme:

$$
S(z) = \begin{cases} s_0 & \text{for } z \in [\underline{z}, \hat{z}), \\ H^{-1}\left(\dfrac{G(z) - \mu}{1 - \mu}\right) & \text{for } z \in [\hat{z}, \bar{z}]. \end{cases} \tag{22}
$$

This implies that $S'(z)$ is equal to zero on $[\underline{z}, \hat{z}]$ and to $g(z)/(h(S(z))(1 - \mu))$ on $[\hat{z}, \bar{z}]$.

Our main interest is the effect of increased scarcity of jobs on equilibrium outcomes. An increase in μ in this framework is like a proportional increase in the population of workers at every level of ability, while keeping the distribution of available jobs fixed. Thus, an increase in unemployment increases competition for jobs. Given the matching rule (22), an increase in μ, the proportion unemployed, will lower the quality of job for those who find employment for a given ability level (that is, $S(z)$ falls on the range $[\hat{z}, \bar{z}]$). Given that this effect is similar to the one generated by a decrease in quality in jobs as analyzed in Section 4, perhaps not surprisingly the consequent comparative static effects are also similar.

6.1. Nontransferable Utility

Again it is possible to construct a symmetric separating equilibrium based on assortative matching. Given our earlier assumption that a worker's benefit is increasing in the quality of her match, it must be that $b(z, s_0, x) < b(z, \underline{s}, x)$. That is, the benefit of unemployment is worse than that of the worst job. Further, since the match they obtain does not vary with investment, those workers who anticipate unemployment will not invest any more than the privately optimal level. A problem is that if the worst job is strictly better than unemployment, there must be a jump in the equilibrium strategy $x(z)$ at $\hat{z}$ to prevent unemployed workers imitating the investment levels of those who are successful. It is still possible for there to be a pure strategy equilibrium, provided one provides suitable off-equilibrium beliefs.

PROPOSITION 9. *Let $x(z) = N(z)$ on $[\underline{z}, \hat{z})$ where $\mu = G(\hat{z})$. Let $\hat{x} \geq N(\hat{z})$ solve $U(\hat{z}, \underline{s}, \hat{x}) = U(\hat{z}, s_0, N(\hat{z}))$. Let $x(z)$ be the solution to (6) on $[\hat{z}, \bar{z}]$ with boundary condition $x(\hat{z}) = \hat{x}$. Then $x(z)$, together with the matching scheme (22), is a symmetric equilibrium strategy of the matching tournament under NTU.*

The obvious question is, what happens if the ratio of workers to jobs increases? Clearly, unemployment goes up, but we can also show that worker utility falls as the job market becomes more competitive. Further, it discourages effort for the lowly ranked, but increases investment by the highly ranked. Define $G(\hat{z}_i) = \mu_i$ for $i = A$, B. With higher μ, the ability level of the lowest-ranked worker to find employment will be also be higher, so that $\hat{z}_A > \hat{z}_B$ if $\mu_A > \mu_B$. In fact, it is relatively easy to obtain appropriate results from the earlier Proposition 4, replacing $\underline{z}$ with $\hat{z}_B$ at each

point of the proof. So, the comparative statics with unemployment are stated here as a corollary.

COROLLARY 1. *Suppose $\mu_A > \mu_B$. Let $x_A(z), U_A(z)$ and $x_B(z), U_B(z)$ be the equilibrium strategy and utility respectively under the two respective values of μ. Then, $U_A(z) < U_B(z)$ for all $z \in (\hat{z}_B, \bar{z}]$. Further, there is a point $\tilde{z} \in (\hat{z}_B, \bar{z})$ such that $x_A(z) < x_B(z)$ on $(\hat{z}_B, \tilde{z})$ but $x_A(z) > x_B(z)$ on $(\tilde{z}, \bar{z}]$.*

6.2. Transferable Utility

Under TU, assume that the μ unmatched workers are paid a fixed wage or benefit so that $w(z, s_0, x) = \underline{w}$ for $z \in [\underline{z}, \hat{z}]$. Clearly, again the unemployed choose the privately optimal investment, as investment affects neither their wage nor match. As in the NTU case, there may be a problem with a discontinuity at $\hat{z}$ as there $S(z)$ jumps from s_0 to $\underline{s}$. Particularly, if the worst job pays more than the unemployment benefit, then some unemployed workers might be happy to raise investment above the privately optimal level in order to get that job.

Thus, for a pure strategy equilibrium we need continuity in the wage schedule. This is possible, if the worst firm appropriates the entire surplus of the match with the worst employed worker, so that the worst employed worker is paid no more than the unemployed. This is technically convenient but the presence of an excess supply of labor makes it reasonable economically. The way to implement this is to set the value of the constant C in the wage equation (11) to be equal to a particular value, in this case the unemployment benefit $\underline{w}$. Thus, for the employed, that is for $z \in [\hat{z}, \bar{z}]$, let

$$w(z, S(z), x) = \int_{\hat{z}}^{z} \pi_z(t, S(t), 0) \, dt + \int_{T(\hat{z})}^{x} \pi_x(z, S(z), t) \, dt + \underline{w}. \qquad (23)$$

So, the wage of the lowest ranked employed worker is equal to the unemployment benefit, or $w(\hat{z}, \underline{s}, T(\hat{z})) = \underline{w}$. We then have the following equilibrium.

PROPOSITION 10. *Let $w(z, S(z), x) = \underline{w}$ on $[\underline{z}, \hat{z}]$ and let $w(z, S(z), x)$be given by equation (23) on the interval $[\hat{z}, \bar{z}]$, where $S(z)$ is as given in equation (22). Let $x(z) = T(z)$ on $[\underline{z}, \hat{z})$, where $\mu = G(\hat{z})$ and, on $[\hat{z}, \bar{z}]$, let $x(z)$ be the solution to equation (14) with boundary condition $x(\hat{z}) = T(\hat{z})$. Then $x(z)$ is a symmetric equilibrium strategy of the matching tournament under TU.*

It is also possible to show that an increase in unemployment will lower equilibrium wages and utility. Investment falls at each ability level as well. Again the result follows from the earlier comparative statics results of Propositions 5 and 6.

COROLLARY 2. *Suppose $\mu_A > \mu_B$ and let $G(\hat{z}_i) = \mu_i$ for $i = A, B$. Let $w_A(z), x_A(z), U_A(z)$ and $w_B(z), x_B(z), U_B(z)$ be, respectively the equilibrium wage, strategy and utility under the two respective values of μ. Then $w_A(z) < w_B(z), x_A(z) < x_B(z)$ and $U_A(z) < U_B(z)$ on $(\hat{z}_B, \bar{z}]$.*

7. Conclusions

This paper has introduced a model of relative signaling in a tournament-like labor market. By allowing for vertical differentiation among employers as well as workers, it generalizes the classic model of Spence (1973). Competition for good jobs generates competition for relative position, implying that the outcome for any individual worker depends on the distribution of characteristics of all firms and all workers. It is true that these relative effects are known to exist in matching and assignment models—see, for example, Costrell and Loury (2004). However, in the assignment literature, matching is based on the intrinsic characteristics of workers and jobs. Therefore there is no explicit competition between workers, no comparative statics on workers' education decisions, and no welfare analysis, as everything is efficient. Here, matching is based on investment decisions by workers that are driven by the matching opportunities available. This feature differentiates the model from traditional signaling models, as here changes in either the distribution of firms and workers, representing changes in the demand and supply of labor respectively, affect equilibrium strategies and welfare.

The equilibria in this model, as is common under imperfect information, are not efficient. Workers may overinvest in education because it serves as a signal of ability as well as increasing productivity. The innovation here is to show that, when the job market is a tournament, this inefficiency depends on the distribution of ability. As Frank (1997) observes, such positional competition has externalities which could potentially be lessened by taxation. That is, suitable labor taxes could increase rather than decrease labor market efficiency. One of the contributions of this paper is to refine previous arguments that have focused on the case where position is signaled by wasteful activities, such as conspicuous consumption. In fact, it is when signaling is in the form of a productive activity such as education that the current model of positional competition gives the greatest support for redistributive taxation.[14] This is because, in the equilibrium of the matching tournament analyzed here, low-ability workers can underinvest in developing useful skills, and high-ability workers may overinvest with respect to the social optimum.

I also hope that matching tournaments will provide a useful framework for the analysis of a number of labor market issues. It has been found (Oyer 2006) that the returns to participating in a matching labor market can be procyclical, with graduates obtaining better matches in good years than in bad. However, there is a further question of whether such variations in the level of competition are anticipated by market participants, leading to changes in their investment decisions in a way predicted by the model presented here. Two recent empirical studies provide fascinating supporting evidence. Ramey and Ramey (2009) report that US parents are investing more time in their children's education in response to increased competition in university admissions. Similarly, Wei and Zhang (2009) argue that parents in China anticipate the

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14.   According to earlier work (Hopkins and Kornienko 2004), taxes that correct the externality from wasteful expenditure are as likely to be regressive as progressive.

competition that their sons will face in the marriage market due to unbalanced gender ratios, and so increase saving to enhance their sons' relative wealth.

This paper already has shown that greater competition can induce greater dispersion in educational investment with those at the bottom end of the labor market investing less and the high-ability investing more. It has also demonstrated that the labor market response will depend on the degree of wage flexibility. For example, if wages are not fully flexible then this increase in competition will actually worsen efficiency. This suggests that the effects of greater competition on investment decisions will be different in countries that have flexible labor markets than in those where labor markets are more regulated. In order to address these issues more fully, further work is necessary to develop the analysis of the simultaneous determination of education and wages, an issue that has only received an initial treatment here.

Appendix

Proof of Lemma 1. In a symmetric equilibrium with a strictly increasing strategy $x(z)$, the probability that an agent of type z_i has higher investment than another of type z_j will be

$$F(x(z_i)) = \Pr[x(z_i) > x(z_j)] = \Pr[x^{-1}(x(z_i)) > z_j] = G(z_i).$$

Then the positive assortative matching ϕ that assigns an agent with investment x_i to a firm of type $s_i = H^{-1}(F(x_i)) = H^{-1}(G(z_i))$ is clearly stable as while any worker with rank $G(z_i)$ would prefer a match with any firm with $s > H^{-1}(G(z_i))$, such a firm would prefer its current match whose z, say $\hat{z}$, would be greater than z_i (and since $x(z)$ is strictly increasing, $\hat{x} = x(\hat{z}) > x_i$). Suppose there is another matching $\tilde{\phi}$, such that a set of workers Z with positive measure are matched differently than under the positive assortative matching ϕ. Then, there must exist $\hat{z} \in Z$, such that $\tilde{\phi}(G(\hat{z})) > G(\hat{z})$, that is, there must be a positive measure of workers who are matched strictly higher than under ϕ. For this matching to be stable, all workers with ability higher than $\hat{z}$ must be matched with firms whose s is greater than $\tilde{\phi}(G(\hat{z}))$. If not, then firm $s = \tilde{\phi}(G(\hat{z}))$ could propose a match with a worker of type $\tilde{z}$ where $\tilde{z} > \hat{z}$ and the worker $\tilde{z}$ would find it acceptable. But the measure of workers with z higher than $\hat{z}$, $\lambda(z \geq \hat{z})$, is strictly larger than the measure of firms with s greater than $\tilde{\phi}(G(\hat{z}))$, $\lambda(s \geq \tilde{\phi}(G(\hat{z})))$. But this implies that $\tilde{\phi}$ is not measure-preserving. □

Proof of Proposition 1. That the only separating equilibrium is a solution to the differential equation (6) with boundary condition $x(\underline{z}) = N(\underline{z})$ follows from Theorems 1 and 2 of Mailath (1987, p. 1353). It then follows by Proposition 3 of Mailath (1987, p. 1362) that $x(z) > N(z)$ on $(\underline{z}, \bar{z})$. The task is to show that the current model fits into Mailath's framework. First, the boundary condition must hold as in a separating equilibrium, an individual with ability $\underline{z}$ has perceived rank 0 and utility $U(z, s, x) = U(\underline{z}, \underline{s}, x(\underline{z}))$ that does not depend on the agent's rank. Therefore, in equilibrium she chooses x to maximize $U(\underline{z}, \underline{s}, x)$. That is, she must choose $N(\underline{z})$, or

there would be a profitable deviation. Second, in Mailath's paper, the signaler's utility is of the form (in current notation) $V(z, \hat{z}, x)$ where V is a smooth utility function and $\hat{z}$ is the perceived type, so that in a separating equilibrium the signaler has utility $V(z, z, x)$. To apply this here, first, fix $G(z)$ and $H(s)$. Now, clearly, one can define the function $V(\cdot)$ such that $V(z, \hat{z}, x) = b(z, S(\hat{z}), x) - c(z, x)$ everywhere on $[\underline{z}, \bar{z}] \times [\underline{s}, \bar{s}] \times \mathbb{R}_+$. One can then verify that the conditions (i)–(v) imposed on $U(\cdot)$ imply conditions (1)–(5) of Mailath (1987, p. 1352) on V.[15] Condition (1) is that V is C^2, condition (2) is that V_2 is always nonzero, and here $V_2 = b_z > 0$. Condition (3) is that V_{13} is never zero and here $V_{13} = b_{zx} - c_{zx} > 0$. Since here $V_{33} < 0$, Mailath's condition (5) that V_3 is bounded when $V_{33} \geq 0$ is automatically satisfied.

Mailath's condition (4) requires that $V_3(z, z, x) = 0$ has a unique solution in x which maximizes $V(z, z, x)$. If $b_x > 0$, then this follows from the assumptions that $b_{xx} \leq 0$, and the assumption that c is convex in x. However, if $b_x = 0$, then the maximizer of $V(z, z, x)$ with respect to x is $x = N(z) = 0$. This actually makes equivalent results to Mailath's easier. In particular, Mailath's Proposition 3 that establishes that, in current notation, $x(z) \neq N(z)$ on $(\underline{z}, \bar{z}]$ becomes easy as if $b_x = 0$ then $x'(z) = b_s(z, S(z), x)/c_x(z, x) > 0 = N'(z)$. This also obviates the need for Mailath's Proposition 5 as the simpler differential equation $x'(z) = b_s(z, S(z), x)/c_x(z, x)$ is Lipschitz continuous even at $\underline{z}$.

Finally, to show that workers are worse off than under privately optimal investment, let $U(z) = b(z, S(z), x(z)) - c(z, x(z))$ and $U_N(z) = b(z, S(z), N(z)) - c(z, N(z))$ denote equilibrium utility and utility under privately optimal investment $N(z)$ respectively. Since $N(z)$ maximizes a worker's utility given match $S(z)$ and $x(z) > N(z)$ for all $z \in (\underline{z}, \bar{z})$, the result follows. □

Proof of Proposition 2. Following Becker (1973) (see also Sattinger 1979; CMP 2001), one obtains from (8),

$$w(z + \varepsilon, S(z + \varepsilon), x) - w(z, S(z), x) \geq \pi(z + \varepsilon, S(z), x) - \pi(z, S(z), x).$$

Dividing both sides by ϵ and taking the limit of ϵ to zero, one finds that

$$w_z(z, S(z), x) + w_s(z, S(z), x)S'(z) \geq \pi_z(z, S(z), x). \tag{A.1}$$

Similarly from (9), one obtains

$$w_x(z, S(z), x) \geq \pi_x(z, S(z), x). \tag{A.2}$$

This also give us a bound on the total derivative $dw(z, S(z), x)/dz \geq \pi_z + \pi_x x'(z)$. A similar analysis finds that the share of the firm satisfies

$$dr(z, S(z), x)/dz \geq \pi_s(z, S(z), x)S'(z). \tag{A.3}$$

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15.   Mailath, in proving the intermediate result Proposition 5 (1987, p. 1364), also assumes that $\partial V / \partial \hat{z}$ is bounded. Here, if we assume that both b_s and $S'(z)$ are bounded (the latter requires $g(\cdot)$ is bounded and $h(\cdot)$ is nonzero), this result will also hold.

But since

$$\frac{dw(z, S(z), x)}{dz} + \frac{dr(z, S(z), x)}{dz} = \frac{d\pi(z, S(z), x)}{dz} = \pi_x x'(z) + \pi_z + S'(z)\pi_s,$$

the preceding conditions hold with equality. The choice of the boundary condition $C = w(\underline{z}, \underline{s}, 0)$ is arbitrary, except that it must be feasible, that is $0 \leq w(\underline{z}, 0) \leq \pi(\underline{z}, \underline{s}, 0)$.

We check that these marginal conditions imply general as well as local stability (that is, it is not possible to construct a blocking pair even when one can choose any type, and not just within a radius of ε). Take any two types of worker z_1, z_2 with $z_2 > z_1$. The stability condition (8) can be rewritten as

$$w(z_2, S(z_2), x) - w(z_1, S(z_1), x) \geq \pi(z_2, S(z_1), x) - \pi(z_1, S(z_1), x).$$

Using $w_z(z, S(z), x) + w_s(z, S(z), x)S'(z) = \pi_z(z, S(z), x)$, yields

$$\int_{z_1}^{z_2} \pi_z(z, S(z), x) \, dz \geq \int_{z_1}^{z_2} \pi_z(z, S(z_1), x) \, dz. \tag{A.4}$$

Now, since matching is positive and assortative, the matching function $S(z)$ is increasing and $S(z) > S(z_1)$ for any $z \in (z_1, z_2]$. If, as assumed, $\pi_{zs} > 0$ then the inequality (A.4) must hold for any pair $z_2 > z_1$.

To prove that positive assortative matching is the only stable form of matching, suppose that instead there is a matching $\tilde{\phi}$ that is not positive assortative. Let $\tilde{S}(z) = \tilde{\phi}(G(z))$. Then, one can choose $z_2 > z_1$ such that $\tilde{S}(z_1) > \tilde{S}(z)$ for all z on the interval (z_1, z_2). For this matching to be stable, given $z_2 > z_1$, wages must satisfy the inequality (8) and hence the inequality (A4) with $\tilde{S}(\cdot)$ replacing $S(\cdot)$. But as $\tilde{S}(z) < \tilde{S}(z_1)$ on (z_1, z_2) and given the complementarity assumptions (c) on $\pi(\cdot)$, this inequality is clearly violated. Thus, positive assortative matching again is the only stable matching. □

Proof of Proposition 3. The aim is again to apply the results of Mailath (1987). Fix $S(z)$, given the two exogenous distributions $G(z)$ and $H(s)$. Then, from the exogenous partial derivatives given in (10), one can integrate using formula (11) to obtain $w(z, S(z), x)$ as a smooth increasing function $[\underline{z}, \bar{z}] \times \mathbb{R} \mapsto \mathbb{R}$ (on integrability, see, for example, Varian 1992, pp. 483-484). Given this wage function, a worker of perceived type $\hat{z}$ will have utility $U = w(\hat{z}, S(\hat{z}), x) - c(z, x)$. In a symmetric equilibrium, the lowest-type worker has a match $\underline{s}$ and therefore should choose x to maximize $U(\underline{z}, w(\underline{z}, \underline{s}, x), x)$, that is choose $T(\underline{z})$, which confirms the boundary condition. Furthermore, define utility function $V(\cdot)$ as $V(z, \hat{z}, x) = w(\hat{z}, S(\hat{z}), x) - c(z, x)$. Then it is easy to verify that our assumptions on $c(\cdot)$ and $\pi(\cdot)$ imply Mailath's (1987, p. 1352) conditions (1)–(5) on V. In particular, note that $V_2 = w_z = \pi_z > 0$, $V_{13} = -c_{zx} > 0$ and that $V_3 = \pi_x - c_x$. Mailath's condition (4) requires that $V_3(z, z, x) = 0$ has a unique solution. If $\pi_x > 0$ then this follows from the assumptions that $\pi_{xx} \leq 0$, and the assumption that c is convex in x.[16] However, if $\pi_x = 0$ then $V_3(z, z, x) < 0$ and $T(z) = 0$. But as established in

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16.   Mailath also assumes that $\partial V / \partial \hat{z}$ is bounded above (see the previous footnote). This here is ensured if π_z is bounded above.

the proof of Proposition 1, it is easy to adapt Mailath's proofs to this slightly different case. Existence of an incentive compatible signaling equilibrium then follows from Theorems 1 and 2 of Mailath. It is then easy to adapt the proof of Proposition 1 and Proposition 3 of Mailath (1987, p. 1362) to show that $x(z) > T(z)$ on $(\underline{z}, \bar{z})$.

Finally, to show that workers are worse off in equilibrium than under privately optimal investment, denote $U(z) = w(z, S(z), x(z)) - c(z, x(z))$ and $U_T(z) = w(z, S(z), T(z)) - c(z, T(z))$ to be equilibrium utility and utility under privately optimal investment $T(z)$ respectively. Since $T(z)$ maximizes a worker's utility given the wage schedule $w(z, x)$ as defined in (11) and $x(z) > N(z)$ for all $z \in (\underline{z}, \bar{z})$, the result follows. $\qquad \square$

Proof of Lemma 2. The first claim follows as since $H(\cdot)$ is a strictly increasing function so is $H^{-1}(\cdot)$. Therefore, if for any z, $G_A(z) < G_B(z)$ then $S_A(z) < S_B(z)$. For the second claim note that if $H_B >_{\text{st}} H_A$, then we have $H_A(s) > H_B(s)$ for all $s \in (\underline{s}, \bar{s})$. This implies that if $G(z) = H_A(s^-) = H_B(s^+)$, then $s^+ > s^-$. But then $s^- = H_A^{-1}(G(z)) < s^+ = H_B^{-1}(G(z))$. Finally, $S'(z) = G'(z)/H'(S(z))$, thus given our assumption that if $G_A >_{\text{st}} G_B$ then $G'_A(\underline{z}) < G'_B(\underline{z})$, it clearly follows that $S'_A(\underline{z}) < S'_B(\underline{z})$, and similarly if $H_B >_{\text{st}} H_A$. $\qquad \square$

Proof of Proposition 4. The function $U(z)$ is continuously differentiable because $x(z)$ and $S(z)$ are continuously differentiable. Given that the boundary condition $x(\underline{z}) = N(\underline{z})$ is the same in any equilibrium, we have $U_A(\underline{z}) = U_B(\underline{z})$. In equilibrium, $x(z) > N(z)$ (except at $\underline{z}$). It then follows from assumption (v) that $U_x(z, S(z), x(z)) = b_x(z, S(z), x(z)) - c_x(z, x(z))$ is strictly negative.

Note that any point where $U_A(z) = U_B(z)$ on $(\underline{z}, \bar{z})$, given that by Lemma 2 one has $S_A(z) < S_B(z)$, it must be that $x_A(z) < x_B(z)$. But then as $U'(z)$ is strictly increasing in $x(z)$ and increasing in $S(z)$ by assumption (iv), it must be that $U'_A(z) < U'_B(z)$ at any point of crossing. Therefore, there can be only one crossing of $U_A(z)$ by $U_B(z)$ and that must be from below. We now rule out the remaining possibility that $U_A(z) \geq U_B(z)$ on an interval $[\underline{z}, z_1]$ with $z_1 \leq \bar{z}$. On the interior of the interval, again we have $S_A(z) < S_B(z)$ and, therefore, to make it possible that $U_A(z) \geq U_B(z)$, it must be that $x_A(z) < x_B(z)$. This implies that $U'_A(z) < U'_B(z)$ on $(\underline{z}, z_1)$. This, together with $U_A(\underline{z}) = U_B(\underline{z})$ implies $U_A(z) < U_B(z)$ on that interval which is a contradiction.

I turn to investment. First, as by Lemma 2 $S'_B(\underline{z}) > S'_A(\underline{z})$, but $x_A(\underline{z}) = x_B(\underline{z})$ by the common boundary condition, then $x'_B(\underline{z}) > x'_A(\underline{z})$. So, $x_B(z) > x_A(z)$ holds immediately to the right of $\underline{z}$. Suppose there is no crossing on $(\underline{z}, \bar{z}]$, so that $x_A(\bar{z}) < x_B(\bar{z})$ which implies that, as $S_A(\bar{z}) = S_B(\bar{z}) = \bar{s}$, the utility for the highest type must be ranked $U_A(\bar{z}) > U_B(\bar{z})$, which is a contradiction to our earlier result, Proposition 4. Now, suppose $x_A(\bar{z}) = x_B(\bar{z})$, then it follows that $U_A(\bar{z}) = U_B(\bar{z})$. But as $U_A(z) < U_B(z)$ on $(\underline{z}, \bar{z})$ then (generically) one has $U'_A(\bar{z}) > U'_B(\bar{z})$. For this last inequality to hold, given $S_A(\bar{z}) = S_B(\bar{z})$, then, as $U'(z)$ is strictly increasing in $x(z)$ by assumption (iv), it must be that $x_A(\bar{z}) > x_B(\bar{z})$, a contradiction. Thus, $x_A(\bar{z}) > x_B(\bar{z})$. This implies one final result, that $U_A(\bar{z}) < U_B(\bar{z})$. $\qquad \square$

Proof of Proposition 5. Given the common boundary condition that $x_A(\underline{z}) = x_B(\underline{z}) = T(\underline{z})$ and that $S_A(\underline{z}) = S_B(\underline{z}) = \underline{s}$, evaluating the differential equation (14) at $\underline{z}$, we find that $x_A'(\underline{z}) = x_B'(\underline{z})$. However, given Lemma 2, we have

$$x_A''(\underline{z}) - x_B''(\underline{z}) = \frac{\pi_{zs}(\underline{z}, \underline{s}, x(\underline{z})) \left(S_A'(\underline{z}) - S_B'(\underline{z}) \right)}{c_x(\underline{z}, x(\underline{z})) - \pi_x(\underline{z}, \underline{s}, x(\underline{z}))} < 0.$$

This implies that $x_A'(z) < x_B'(z)$ immediately to the right of $\underline{z}$ and hence $x_A(z) < x_B(z)$ also holds in a neighborhood of $\underline{z}$. Consider any possible subsequent crossing of $x_A(z)$ and $x_B(z)$. At any point where $x_A(z) = x_B(z)$ we would have

$$x_A'(z) = \frac{\pi_z(z, S_A(z), x)}{c_x(z, x) - \pi_x(z, S_A(z), x)} < \frac{\pi_z(z, S_B(z), x)}{c_x(z, x) - \pi_x(z, S_B(z), x)} = x_B'(z)$$

by assumptions (c) and (d) on π and as $S_A(z) < S_B(z)$ on $(\underline{z}, \bar{z})$. Hence $x_A(z)$ cannot cross $x_B(z)$ from below on $(\underline{z}, \bar{z})$ (generically also not at $\bar{z}$) and thus we have $x_A(z) < x_B(z)$ on $(\underline{z}, \bar{z}]$.

   Given the original definition of worker's utility (7), the envelope theorem implies that $U'(z) = -c_z(z, x(z)) > 0$. It has just been shown that $x_A(z) < x_B(z)$ for all $z \in (\underline{z}, \bar{z}]$. By assumption, it holds that $c_{zx} < 0$. Therefore, $U_A'(z) = -c_z(z, x_A(z)) < -c_z(z, x_B(z)) = U_B'(z)$ for all $z \in (\underline{z}, \bar{z})$. As $U_A(\underline{z}) = U_B(\underline{z})$ by the common boundary condition $x(\underline{z}) = T(\underline{z})$, the result clearly follows. $\square$

Proof of Proposition 6. The earlier Proposition 5 established that $x_A(z) < x_B(z)$ on $(\underline{z}, \bar{z})$. This implies that the second integral on the right-hand side of the wage equation (11) is strictly lower in case A than in case B. But as by Lemma 2 we have $S_A(z) < S_B(z)$ on $(\underline{z}, \bar{z})$, the first integral is lower by assumption (c) on $\pi(\cdot)$. Turning to wages in terms of investment, or $w(x)$, with substitution from the differential equation (14) we have $w'(x) = c_x(\gamma(x), x)$. Note that from the earlier Proposition 5, we have $x_A(z) < x_B(z)$ on $(\underline{z}, \bar{z})$ with $x_A(\underline{z}) = x_B(\underline{z}) = x(\underline{z})$. This implies that $\gamma_A(x) > \gamma_B(x)$ on $(x(\underline{z}), x_A(\bar{z}))$. Now, since $c_{zx} < 0$ by assumption, we have that $w_B'(x) > w_A'(x)$ on $(x(\underline{z}), x_A(\bar{z}))$ and the result follows. $\square$

Proof of Proposition 7. The concavity of U in x and the concavity of π together ensure the first order conditions (19) define a maximum. We have, for the lowest type, $x(\underline{z}) = N(\underline{z})$ by Proposition 1. However, at $\underline{z}$, as $\pi_x > 0$, for a social optimum from (19), the lowest type should produce more than $N(\underline{z})$. The final result follows from application of Proposition 4. $\square$

Proof of Proposition 8. Existence of a unique solution $T^*(z)$ to (21) at every level of z follows from assumption (e) on $\pi(\cdot)$ and the convexity of $c(z, x)$ in x. The first result then follows directly from comparison of (13) and (21). The comparative static result follows from Proposition 5. $\square$

Proof of Proposition 9. Observe that, in the proposed equilibrium, investment levels on the interval $(N(\hat{z}), \hat{x})$ are off the equilibrium path. Assume that if any worker deviates and chooses x on that interval, firms believe with probability 1 that her type z

is strictly less than $\hat{z}$. Then any deviation by any unemployed worker to any level of $x \in [0, \hat{x})$ will not result in a job offer, so there is no incentive to make such a deviation. Deviation to a level of x above $\hat{x}$ is unprofitable by the definition of $\hat{x}$. For workers of type $z \in [\hat{z}, \bar{z}]$, the equilibrium is the same as in the case of full employment. $\square$

Proof of Proposition 10. The proposed equilibrium strategy $x(z)$ and wage schedule $w(z) = w(z, S(z), x(z))$ are continuous at $\hat{z}$. Workers with ability on the interval $[\hat{z}, \bar{z}]$ play the equilibrium identified in Proposition 3. Workers with ability on the interval $[\underline{z}, \hat{z})$ have no incentive to deviate to $x(\hat{z})$, given the definition of $T(z)$ and that the resulting wage $\underline{w}$ is no higher. Deviation to x above $x(\hat{z})$ cannot profitable for a worker with ability in $[\underline{z}, \hat{z})$ since it is not profitable for a worker of ability $\hat{z}$. $\square$

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