RISK SHARING AND MARKET INCOMPLETENESS

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1. Introduction

Does incompleteness of financial markets impede risk sharing? This paper presents a simple model suggesting that it may not, provided consumers are patient, risk is purely idiosyncratic, and bond markets are open.

To make this point, we consider a one-good, infinite horizon exchange economy. Intertemporal trade is accomplished through short-lived real assets, one of which is a riskless real bond. The population consists of a finite number of infinitely lived consumers, who maximize discounted expected utility relative to stationary period utility functions. Consumers share common probability assessments and a common subjective discount factor ρ. Risk is purely idiosyncratic; that is, each consumer's endowment follows an iid process, but the social endowment is constant. Our conclusion is that, when the discount factor is close to 1 (that is, when consumers are sufficiently patient), equilibrium utilities are close to the utilities of perfect risk sharing.

Of course the idea that patient consumers can self-insure is not a new one. Yaari (1976) for example, considers a perfectly patient consumer who lives a long but finite lifetime, faces an uncertain endowment stream, and can borrow and save at a zero interest rate. Yaari shows that the optimal plan for such a consumer has the property that, as the consumer's lifetime tends to infinity, the per period average utility converges to the utility of constant average consumption. Our work differs fundamentally from Yaari's however, because we treat an equilibrium problem, not just an individual optimization problem. In particular, we *derive* the equilibrium interest rate. Moreover, although this rate cannot be much above 0, it might be quite negative. Because saving is difficult when the interest rate is quite negative, an argument like Yaari's cannot be made in our environment. Indeed, our argument rests on the ability of consumers to self-insure by borrowing alone, *without ever saving*.

The questions we ask here are reminiscent of what Friedman (1957) called the *permanent income hypothesis*: that consumers behave in such a way to maintain a constant marginal utility of income. See Yaari (1976) and Bewley (1980) for theoretical formulations and analysis of Friedman's idea. Our work parallels simulations carried out by Telmer (1993) and Lucas (1994), who found that market incompleteness is not sufficient to explain observed large variances in riskless interest rates (the "riskless rate puzzle") or the observed large premium over equities over riskless securities (the "equity premium puzzle").

Some of our assumptions are quite strong. Levine and Zame (1999) examines similar questions under weaker assumptions. Roughly speaking, that paper concludes that market incompleteness is compatible with perfect risk sharing even if endowments are recurrent Markov (rather than iid) and there is aggregate risk, provided options on the social endowment are traded. On the other hand, riskless bonds alone do *not* provide perfect risk sharing when there is aggregate risk. Moreover, if there is more than one consumption good, then price risk — introduced endogenously through the action of the market — may interfere further with perfect risk sharing.

2. The Economy

2.1. **Time and Uncertainty.** Time and uncertainty are represented by a countably infinite tree S. Each node on the tree represents a date-event. The initial date-event (the root of the tree) is denoted by $0 \in S$. For date-events $s, s' \in S$, we write $s \leq s'$ to mean that s' follows s (and s precedes s'). For each date-event $s \in S$ other than 0, we write s^- for the (unique) date-event that immediately precedes s and s^+ for the set of date-events that immediately follow s. For simplicity, we assume s^+ is finite.

Each $s \in S$ is a finite history of exogenous events; we denote the length of that history by $\tau(s)$. Thus $\tau(s^-) = \tau(s) - 1$ and $\tau(0) = 0$. A complete path through the tree S is a complete history of exogenous events; write H for the set of all such infinite histories. Given a history $h \in H$ and a date t, write h_t for the history up to and including time t. Thus $h_t \in S$ and $\tau(h_t) = t$. In our notation, S is the set of finite histories and H is the set of infinite histories.

2.2. **Commodities.** There is a single consumption good available at each date-event. The *commodity space* is the space $\ell^\infty(S)$ of bounded functions $x : S \to \mathbf{R}$. For $x \in \ell^\infty(S)$, we write $x_s \in \mathbf{R}$ for the bundle specified at node s. A *consumption plan* is an element of $\ell^\infty(S)_+$; that is, a bounded function $x : S \to \mathbf{R}_+$. Since there is a single consumption good, we normalize so that its spot price is 1 at each date event $s \in S$, and henceforward suppress spot prices.

2.3. **Securities.** Intertemporal trades takes place through the exchange of securities. For simplicity, we assume that J securities are available at each date-event, that security returns are denominated in units of the consumption good, and that each security is *short-lived*, yielding returns only at the immediate successor nodes. The portfolio $\theta \in \mathbf{R}^J$ of securities acquired at date-event $s \in S$ yields as dividends $\mathrm{div}_\sigma \theta$ units of the numeraire commodity at the date-event $\sigma \in s^+$. (Note that $\mathrm{div}_\sigma : \mathbf{R}^J \to \mathbf{R}$ is a linear operator.) We assume that a riskless bond (numbered A^1) is traded at each node; $A^1_s(\sigma) = 1$ for each $\sigma \in s^+$.

2.4. **Utilities.** There are N infinitely lived traders $i = 1, \ldots, I$, having utility functions $U^i : \ell^\infty(S)_+ \to \mathbf{R}$. We assume traders maximize the discounted sum of expected utility, according to a stationary period utility function u^i. Thus

$$U^i_\rho(x) = (1-\rho)\sum_{t=1}^{\infty} \rho^t \sum_{\tau(s)=t} \pi_s u^i(x_s).$$

We assume that u^i is a smooth (C^3) strictly concave function, with a strictly positive first derivative. We frequently write U^i_ρ in order to emphasize the dependence on the discount factor ρ, which we think of as a parameter. The leading factor $(1-\rho)$ normalizes so that the discounted utility of the constant consumption stream c is $u^i(c)$, independent of the discount factor ρ.

2.5. **Endowments.** We asssume individual endowments are iid, and that range of the endowment process is finite.

2.6. **Budget Sets and Debt Constraints.** Given security prices q, trader i chooses a consumption plan $x^i : S \to \mathbf{R}_+$ and a portfolio trading plan $\theta^i : S \to \mathbf{R}^J$. At each date-event s, trader i faces a *spot budget constraint* which may be written:

$$x^i_s + q_s \cdot \theta^i_s \leq e^i_s + \mathrm{div}_s \theta^i_{s^-}$$

That is, expenditure to purchase consumption and to purchase securities does not exceed income from sale of endowment and from dividends on securities acquired at the previous date-event. In our infinite horizon setting, these spot constraints are not sufficient to rule out Ponzi schemes (doubling strategies) and hence unlimited amounts of borrowing. As we show in Levine and Zame (1996), the additional constraints necessary to rule out Ponzi schemes may be formalized in any of a number of ways, each of which leads to an equivalent notion of equilibrium.[1] Here we find it convenient to formalize the constraints by requiring that it should be possible to repay *almost all* the debt in finite time.

To this end, fix prices q, a consumption plan x^i and a portfolio plan θ^i for trader i that satisfies the spot budget constraint at each date-event s. Define trader i's *debt* at date event s as his obligation to repay on securities he holds entering date event s:

$$d_s = -\mathrm{div}_s \theta^i_{s^-}$$

If this quantity is positive, trader i is in debt. To meet this debt, trader i must raise income from the sale of endowment and/or securities (selling securities is borrowing). We constrain debt at date-event s by prescribing

[1] See also Magill and Quinzii (1994).

a positive upper bound on d_s.[2] (Prescribing a negative upper bound would require traders to save.) We say that the debt $d_s \geq 0$ can be *repaid in T periods from s* if there are consumption and portfolio plans y, φ such that:

- y, φ satisfy the spot budget constraint at every date event
- if $\sigma < s$ then $y_\sigma = x^i_\sigma$ and $\varphi_\sigma = \theta^i_\sigma$
- if $s \leq \sigma$ and $t(\sigma) - t(s) \geq T$ then $d_\sigma \leq 0$

That is, the plans y, φ meet the spot budget constraints at every date-event, agree with x^i, θ^i prior to the date-event s, and leave no debt at any date-event following s by T or more periods. The debt $d_s \geq 0$ can be *repaid in finite time from s* if it can be repaid in T periods for some T. Define the *finitely effective debt constraints* as:

$$D^i_s = \inf\{d : d \text{ can be paid in finite time from } s\}$$

Finally, define the *budget set* for trader i at prices q as:

$$B^i(q) = \left\{\, x^i, \theta^i : \forall s \in S, \forall \sigma \in s^+, x^i_s + q_s \cdot \theta^i_s \leq e^i_s + \text{div}_s \theta^i_{s^-}\ , d_\sigma \leq D^i_s \,\right\}$$

Note that we constrain behavior at date event s by limits on debt at succeeding date events $\sigma \in s^+$.

2.7. **Equilibrium.** An *equilibrium* consists of security prices q, consumption plans (x^i) and portfolio plans (θ^i) such that

- for each s:

$$\sum_i x^i_s = \sum_i e^i_s$$

- for each s:

$$\sum_i \theta^i_s = 0$$

- for each i:

$$(x^i, \theta^i) \in B^i(q) \text{ and } (y^i, \varphi^i) \in B^i(q) \Rightarrow U^i(x^i) \geq U^i(y^i)$$

That is, commodity markets clear, security markets clear, traders optimize in their budget sets. Levine and Zame (1996) show that (with assumptions weaker than those made here) an equilibrium exists.

[2]The reader familiar with Levine and Zame (1996) will note that we use here the opposite sign convention for debt and debt constraints.

3. PERFECT RISK SHARING

We make 2 additional assumptions:
Assumption 1 The social endowment $e = \sum_h e^h_s$ is constant across states and time (no aggregate risk).[3]
Assumption 2 For each h, Du^h is convex.

The latter assumption will be satisfied if absolute risk aversion is non-increasing. To see this, differentiate absolute risk aversion:

$$0 \geq D\left[-\frac{D^2u^h}{D^2u^h}\right] = -\frac{(D^3u^h)(Du^h)-(D^2u^h)^2}{(D^2u^h)^2}$$

Simplifying and transposing yields

$$(D^3u^h)(Du^h) \geq (D^2u^h)^2$$

We have assumed that $Du^h > 0$, so we conclude that $D^3u^h > 0$ so that Du^h is convex as asserted.

We are interested in the nature of equilibrium for discount factors ρ close to 1. It is convenient therefore to fix securities, endowments and period utility functions u^i. For each discount factor $\rho < 1$, write $\mathcal{E}_\rho$ for the economy with the securities, endowments and period utility functions, in which traders use the common discount factor ρ, and write E^ρ for the set of equilibria of $\mathcal{E}_\rho$.

Because individual endowments are iid with finite range, they each possess a long run average; write $\bar{e}^i$ for the long run average of e^i. Our assumptions imply that, for every ρ, Pareto optimal allocations of $\mathcal{E}_\rho$ consist of constant shares of the constant social endowment. In particular, the perfect risk-sharing allocation $\bar{e} = (\bar{e}^1, \ldots, \bar{e}^N)$ at which each trader consumes a constant amount, equal to his long run average endowment, is Pareto optimal (for every ρ).

Our main result below asserts that when ρ is sufficiently close to 1 (that is, when consumers are sufficiently patient), equilibrium utilities are close to the utilities of the perfect risk sharing allocation.

Theorem *If Assumptions 1, 2 are satisfied then for every trader i:*

$$\lim_{\rho \to 1} \sup_{E^\rho} \left| U^i_\rho(x^i) - u^i(\bar{e}^i) \right| = 0$$

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[3]Because the social endowment is constant and the number of consumers is finite, individual endowments must necessarily be correlated with each other. However, this correlation is an artifact of the finiteness of our model; a model with a continuum of consumers would permit us to assume a constant social endowment and independent individual endowments. We prefer the model with a finite number of consumers because we can rely on Levine and Zame (1996) to guarantee that equilibrium exists.

Before beginning the proof, we record two useful lemmas. The first is simply a convenient version of Kolmogorov's generalization of Chebyshev's inequality; see Feller (1971, p. 242).

Lemma 1 *Let (z_t) be an iid sequence of bounded random variables with mean 0 and variance V. Write*

$$Z_T = \sum_{t=0}^{T} z_t$$

Then

$$Prob\left\{\max_{T<T_0} |Z_T| > A{T_0}^{1/2}\right\} < \frac{V}{A^2}$$

for every $A > 0$.

The second lemma provides a lower bound for the price of the riskless bond (and hence an upper bound for the riskless interest rate).

Lemma 2 *If q, x^i, θ^i is an equilibrium, s is a date event and A^1_s is a riskless bond then $q^1_s \geq \rho$.*

Proof Let $K \leq N$ be the number of traders whose equilibrium consumptions at s are strictly positive. Re-numbering if necessary, assume that $x^k_s > 0$ for $k = 1, \ldots, K$ and that $x^i_s = 0$ for $i > K$.

Let M be the set of K-tuples $\mu = (\mu^1, \ldots, \mu^K) \in \mathbf{R}^K_+$ for which there are consumptions $c^1, \ldots, c^K$ such that $\sum_{k=1}^K c^k \leq e_s$ and $\mu^k \geq Du^k(c^k)$ for all $k = 1, \ldots, K$. We assert that M is a convex set. Let $\mu_a, \mu_b \in M$. Then by the definition of M there is c_a, c_b such that

$$\mu^k_i \geq Du^k_i(c^k_i) \text{ for } i = a, b \text{ and all } k = 1, \ldots, K$$

Let $c^k_\lambda \equiv \lambda c^k_a + (1-\lambda)c^k_b$. Clearly $\sum_{k=1}^K c^k_\lambda \leq e_s$. Moreover since Du^k is convex

$$Du^k(c^k_\lambda) \leq \lambda Du^k(c^k_a) + (1-\lambda)Du^k(c^k_b) \leq \lambda\mu_a + (1-\lambda)\mu_b$$

which proves that $\lambda\mu_a + (1-\lambda)\mu_b$ is in M which shows that M is convex.

By assumption, at the date-event s each of the traders $k \leq K$ has strictly positive consumption. Because A^1_s is riskless, the first order condition for an equilibrium implies that, for each $k \leq K$,

$$q^0_s Du^k(x^k_s) \geq \rho \sum_{\sigma \in s^+} \frac{\pi_\sigma}{\pi_s} Du^k(x^k_\sigma)$$

The definition of M guarantees that for each $\sigma \in s^+$ the K-tuple $\left(Du^k(x^k_\sigma)\right)$ belongs to M. Because $\sum \pi_\sigma/\pi_s = 1$, convexity of M guarantees that the K-tuple $\left(\sum_\sigma(\pi_\sigma/\pi_s)Du^k(x^k_\sigma)\right)$ also belongs to M. Hence the K-tuple $\left((q^1_s/\rho)Du^k(x^k_s)\right)$ belongs to M. The definition of M guarantees that there are consumptions

(c^k) such that $\sum c^k \leq e$ and $(q_s^1/\rho)Du^k(x_s^k) \geq Du^k(c^k)$ for each k. Because each u^k is concave, Du^k is decreasing. If $q_s^1/\rho < 1$ then it would follow that $x_s^k \leq c^k$ for each k, contradicting the fact that $\sum c^k \leq e = \sum x_s^k$. We conclude that $q_s^1/\rho \geq 1$, and hence that $q_s^1 \geq \rho$, as asserted. ■

With these lemmas in hand, we turn to the proof of the Theorem.

Proof of Theorem Fix a discount factor ρ, a trader i and a small real number $\varepsilon > 0$. We show that equilibrium utility $U_\rho^i(x^i)$ cannot be much less than $u^i(\bar{e}^i)$, provided ρ is sufficiently close to 1. To accomplish this, we construct alternative feasible consumption and portfolio plans y^i, φ^i so that $U_\rho^i(y^i) \approx u^i(\bar{e}^i)$ for ρ close to 1. Individual optimization will guarantee that equilibrium utilities are at least as large as $U_\rho^i(y^i)$; the nature of the Pareto set will guarantee that equilibrium utilities cannot be much larger than this.

The alternative consumption and portfolio plans involve consumption and buying and selling the riskless bond (only). The consumption plan prescribes consumption level almost equal to $\bar{e}^i - \varepsilon$ until the debt exceeds a predetermined limit; the portfolio plan prescribes buying and selling the riskless bond in order to maintain this consumption level. Debt will be repaid when endowment is high and additional debt will be incurred when endowment is low. The quantity ε represents the interest required to service the debt.

There is no loss in assuming that $u^i(0) = 0$. Set $m = \min_s e_s^i$, and fix a real number ε with $0 < \varepsilon < m$. Set $d^* = \varepsilon/(1-\rho)$ and $d = d^* - \bar{e}^i$. We use d as a debt limit and ε as a set-aside to pay interest on the debt.

For each date event s, write y_s for consumption and b_s for the sales of the riskless bond. No other securities will be bought or sold, so debt at date event s is $d_s = b_{s^-}$. We prescribe consumption and portfolio choices y_s, b_s at date event s in the following way:

(1) If $d_\sigma \leq d$ for all $\sigma \leq s$ and $e_s^i \leq \bar{e}^i$, set $y_s = \bar{e}^i - \varepsilon$ and

$$b_s = \frac{1}{\rho}[d_s - \varepsilon + \bar{e}^i - e_s^i]$$

That is: if the debt limit has not been reached and $e_s^i < \bar{e}^i$, consume $\bar{e}^i - \varepsilon$, repay ε of the outstanding debt, and roll over the remaining debt.

(2) If $d_\sigma \leq d$ for all $\sigma \leq s$ and $e_s^i > \bar{e}^i$, set $y_s = \bar{e}^i - \varepsilon$ and

$$b_s = \frac{1}{\rho}\max\left\{[d_s - \varepsilon + \bar{e}^i - e_s^i], 0\right\}$$

That is: if the debt limit has not been reached and $e_s^i \geq \bar{e}^i$, consume $\bar{e}^i - \varepsilon$, repay $\varepsilon + (e_s^i - \bar{e}^i)$ of the outstanding debt (but never repay more than the outstanding debt; i.e., never save), and roll over the remaining debt.

(3) If $d_\sigma > d$ for some $\sigma \leq s$, set $y_s = e_s - \varepsilon$ and

$$b_s = \frac{1}{\rho}[d^* - m]$$

That is: if the debt limit has been reached, consume $e^i_s - \varepsilon$, use ε to service the existing debt, and roll the remaining debt.

By construction, this consumption/portfolio plan satisfies the spot budget constraints at every date event. To see that it satisfies the debt constraints, note first that, because $\varepsilon < m$, a debt of d^* can be carried forever. (Use ε of the endowment to repay part of the debt and and sell $(1/q^1_s)(d^* - \varepsilon)$ units of the riskless bond, leaving a debt of $(1/q^1_s)(d^* - \varepsilon)$ next period. Because $q^1_s \geq \rho$, the next period's debt will not exceed d^*.) Hence any debt less than d^* can be repaid in finite time. In particular, the specified consumption/portfolio plan, which never attains a debt as large as d^* at any date event, satisfies the debt constraint.

To obtain a lower bound for $U^i_\rho(y^i)$ we estimate how long the consumption/portfolio plan is likely to continue before hitting the debt constraint. To this end, write $M = \max e^i_s$, and set $z = \bar{e}^i - e^i$; z is an iid process with mean 0 and variance at most M. If the debt limit has not been exceeded at the date event s, then the change in debt from s to s^+ is $(1/\rho)z_s$ at the date event s (debt increases if $z_s > 0$ and decreases if $z_s < 0$), except that debt is never allowed to become negative. Thus the debt limit d will not be reached before $|\sum_{t\leq T} z_{h_t}| \geq d/2$.

Set

$$\begin{aligned} A &= M^{1/2}(1-\rho)^{-1/4} \\ T_0 &= \frac{1}{4}\left(\frac{\varepsilon}{M^{1/2}(1-\rho)^{3/4}} - \frac{(1-\rho)^{1/4}}{M^{1/2}}\bar{e}^i\right)^2 \end{aligned}$$

Recall that H is the set of all infinite histories. For $h \in H$, write

$$Z_T(h) = \sum_{t\leq T} z_{h_t}$$

Let H_0 be the set of histories $h \in H$ such that $|Z_T(h)| \leq d/2$ for every $T < T_0$. If trader i follows the plan y^i, φ^i in the history $h \in H_0$, he will consume at least $\bar{e}^i - \varepsilon$ at every date $T < T_0$ and at least 0 thereafter, so his utility in history h will be at least

$$(1-\rho)\sum_{t=0}^{T_0-1} \rho^t u^i(\bar{e}^i - \varepsilon) = (1-\rho^{T_0})u^i(\bar{e}^i - \varepsilon)$$

(Recall that $u^i(0) = 0$.) Our specifications of A, T_0 imply that $AT_0^{1/2} = d/2$ and $M/A^2 = (1-\rho)^{1/2}$, so Lemma 1 guarantees that

$$\text{Prob}\{H : \max_{T<T_0} |Z_T(h)| > \frac{d}{2}\} < (1-\rho)^{1/2}$$

Hence $\text{Prob}(H_0) \geq 1 - (1-\rho)^{1/2}$, so consumer i's expected utility if he follows the plan y^i, φ^i will be at least

$$U^i_\rho(y^i) \geq \left[1 - (1-\rho)^{1/2}\right]\left[1 - \rho^{T_0}\right] u^i(\bar{e}^i - \varepsilon)$$

From the definition of T_0, we see that as $\rho \to 1$, $T_0/\varepsilon^2/4M(1-\rho)^{3/2} \to 1$. Taking logarithms and applying L'Hospital's rule, we see that

$$\lim_{\rho \to 1} \rho^{\frac{\varepsilon^2}{4M(1-\rho)^{3/2}}} = 0$$

Hence

$$\lim_{\rho \to 1} \inf_{E_\rho} U^i_\rho(x^i) \geq u^i(\bar{e}^i - \varepsilon)$$

for each i. Because $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{\rho \to 1} \inf_{E_\rho} U^i_\rho(x^i) \geq u^i(\bar{e}^i)$$

for each i.

As we have already noted, the constant allocation $(\bar{e}^i)$ is Pareto optimal. Strict concavity of utility functions implies that the Pareto set is strictly convex. It follows that

$$\lim_{\rho \to 1} \sup_{E_\rho} \left| U^i_\rho(x^i) - u^i(\bar{e}^i) \right| = 0$$

for each i, which is the desired result. ■

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