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# Bubbles, collateral and monetary equilibrium

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# BUBBLES, COLLATERAL AND MONETARY EQUILIBRIUM

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Abstract. Consider an economy where infinite-lived agents trade assets collateralized by durable goods. We obtain results that rule out bubbles when the additional endowments of durable goods are uniformly bounded away from zero, regardless of whether the asset's net supply is positive or zero. However, bubbles may occur, even for state-price processes that generate finite present value of aggregate wealth. First, under complete markets, if the net supply is being endogenously reduced to zero as a result of collateral repossession. Secondly, under incomplete markets, for a persistent positive net supply, under the general conditions guaranteeing existence of equilibrium. Examples of monetary equilibria are provided.

Keywords. Monetary Equilibrium, Bubbles, Default, Collateral, Durable Goods.

## 1. Introduction

Sequential economies with infinite-lived assets have been studied for quite a long time in finance and macroeconomics. In the overlapping generations models by Samuelson (1958) and Gale (1973), as well as in the infinite-lived consumers model by Bewley (1980), money has a positive price, although its fundamental value (the discounted stream of future returns) is zero. This excess in the price of an asset over its fundamental value is known as a bubble. However, more recent general equilibrium models, with infinitely lived agents and incomplete markets, showed that the general requirements for existence of equilibrium end up limiting severely the occurrence of price bubbles for assets in positive net supply (such as money, in the above models, and also stocks).

Generic existence of equilibrium, with borrowing constraints (or *a priori* transversality restrictions), was established for uniformly impatient preferences and endowments uniformly bounded away from zero (see Magill and Quinzii (1996) and also Hernandez and Santos (1996) for related results). Together, these assumptions imply a joint uniform impatience condition which rules out speculation for the Kuhn-Tucker deflator processes (as in Magill and Quinzii ( 1996)) or if the deflator determines a finite present value of aggregate wealth (as in Santos and Woodford (1997)). Moreover, it was also argued that, in economies with sufficiently productive financial sectors, present values of wealth should be finite, since aggregate resources can be easily super-replicated using a finite-cost portfolio plan.

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Hence, examples of price bubbles for assets in positive net supply and infinite-lived agents, presented by Santos and Woodford (1997), were regarded as being very fragile, by the authors themselves. All examples dealt with the very special case of borrowing constraints precluding short-sales, as in this case existence of equilibrium dispensed with the above requirements. There were two types of examples: one (Example 4.5) where joint uniform impatience was violated and another (Examples 4.2 and 4.4) admitting, for some deflators, infinite present values of aggregate wealth. The former had very special endowments (zero beyond the initial date) and did not seem to accommodate the case of money. The latter were not robust to adding sufficiently productive assets.

When assets are collateralized by durable goods (directly or through other assets), bubbles may occur, both for zero and positive net supply assets, under the general conditions ensuring existence of equilibrium. As in the case of short-lived assets, studied by Araujo, Páscoa and Torres-Martinez (2002), Ponzi schemes can never occur, even for non-uniformly impatient preferences,[1] and the successive additional endowments of durable goods do not have to be bounded away from zero. This allows for non-summable deflated commodity prices and for asset price bubbles (as illustrated in our examples, where the Inada effect cancels out the discounting factor). Moreover, adding assets with sufficiently productive promises might not end speculation, as these assets might default and fail to super-replicate aggregate resources.

However, in the case of assets in positive net supply, the occurrence of price bubbles requires more than just the absence of uniform positive lower bounds for additional endowments. Under complete markets, price bubbles can only occur when collateral repossession makes the asset's positive net supply not persistent. If default occurs or even in the borderline case, when agents are indifferent between honoring the promise or not, the collateral (or part of it) is surrendered, the asset's positions decrease and its positive net supply is reduced.

In the incomplete markets case, bubbles are compatible with persistent positive net supply. The diversity in agents' Kuhn-Tucker multipliers processes may allow for the positive net supply to change hands in such a way that the asset can be priced at infinity without violating individual optimality. More precisely, each agent perceives a bubble, as the limiting deflated price of the asset will be nonzero, but is not a lender at infinity, since the deflated value of her long position tends to zero.

In our examples we focus on the case of a single asset that pays no dividends. In these examples, some agents are endowed with money at the initial node and all agents can finance their purchases of durable goods by either selling the endowments of money or by short-selling money (using the durable bundle as collateral). Here money plays the single role of transferring wealth across time and states of nature. Even in the absence of liquidity frictions money may be essential: a positive price of money may prevail due to its function in completing the markets or in reducing market incompleteness.

Our *first example* of monetary equilibrium illustrates the case where the introduction of money ends up completing the markets. Efficient monetary equilibria always have an asymptotically zero money supply, but are not unambiguously interpretable as price bubbles. Actually, in this example,

[1] As the utility function is assumed to be additively separable but not necessarily recursive.

the economy has a borderline case stationary equilibrium where part of the collateral is always surrendered. For this reason, the monetary bubble can be reinterpreted as a positive fundamental value consisting of marginal utility returns from consumption of the part of the money net supply that is being converted into garnished collateral. Our *second example* illustrates an incomplete markets monetary equilibrium where default or even the borderline situation never occur and, therefore, the bubble is unambiguous since the above reinterpretation is no longer possible. Hence, there are inefficient monetary equilibria that can only be interpreted as price bubbles.

In both examples financial constraints (collateral requirements) are not binding. Contrary to previous examples in the general equilibrium literature (as in the cash-in-advance model by Santos (2006) or in the credit model by Gimenez (2005)), the positive price of money is not due to a positive fundamental value that simply adds up the shadow prices of binding (liquidity or borrowing) constraints.

Our result on the absence of price bubbles, for the Kuhn-Tucker deflators and under uniform lower bounds on additional commodity endowments, applies not only to assets in positive net supply but also to zero net supply assets, and, therefore, contrasts with the results for default-free assets, namely those of Magill and Quinzii (1996). For other deflator processes, we lose the relation between uniformly interior additional endowments and summable deflated commodity prices, but we can still say that asset price bubbles can only occur if deflated commodity prices fail to be summable. Here there seems to be a relationship with the more abstract approach to bubbles proposed (in an Arrow-Debreu setting) by Gilles and Le Roy (1992).

The paper is organized as follows. The second and third sections present the model. The fourth section discusses a crucial property of the default model which says that a consumption and portfolio plan is individually optimal if and only if it satisfies Euler inequalities and a transversality condition on its cost. The sufficiency part is used to establish existence of equilibrium (in Section 5) and to compute some examples of monetary equilibria (in Section 8). The necessity part is used to characterize asset prices (in Section 6). This characterization (which is analogous to the non-arbitrage valuation studied by Araujo, Fajardo and Páscoa (2005)) is the basis for the definitions of fundamental values and for the results on absence of price bubbles (in Section 7).

## 2. The Infinite Horizon Economy with Default and Collateral

*Uncertainty.* We consider a discrete time economy with infinite horizon. The set of dates is $T = \{0, 1, \ldots\}$. We suppose that there is no uncertainty at $t = 0$, and there are finitely many states of nature at the following date, $t = 1$. In general, given a history of realization of the states of nature for the first $t-1$ dates, with $t > 1$, $\overline{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states of nature that may occur at date $t$.

An information set $\xi = (t, \overline{s}_t, s)$, where $t \in T$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. The date associated with node $\xi$ is denoted by $t(\xi)$. There exists a natural order in the information structure: given nodes $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that $\mu$ is a *successor* of the node $\xi$, and write $\mu \geq \xi$, if $t' \geq t$ and $\overline{s}_{t'} = (\overline{s}_t, s, \ldots)$. Given a node $\xi$, we denote by $\xi^+$ the set of immediate successors of $\xi$, that is, the set of nodes $\mu \geq \xi$, where $t(\mu) = t(\xi) + 1$. The (unique) predecessor of node $\xi$ is denoted by $\xi^-$ and the only information set at $t = 0$ is $\xi_0$. We write $\mu > \xi$ to say that $\mu \geq \xi$ but $\mu \neq \xi$.

The set of nodes, called the *event-tree*, is denoted by $D$. Let $D(\xi) = \{\mu \in D : \mu \geq \xi\}$ be the subtree with root $\xi$. The set of nodes with date $T$ in $D(\xi)$ is denoted by $D_T(\xi)$. Finally, let $D^T(\xi) = \bigcup_{k=t(\xi)}^{T} D_k(\xi)$ be the set of successors of $\xi$ with date less or equal than $T$. When $\xi = \xi_0$ notations above will be shorten to $D_T$ and $D^T$.

*Physical markets.* At each node there is a finite ordered set of commodities, $L$, which can be traded and may suffer partial depreciation at the nodes that are immediate successors. At each node $\xi$, the depreciation structure is given by a matrix $Y_\xi = (Y_\xi(l, l'))_{(l,l') \in L \times L}$ with non-negative entries. Thus, if one unit of good $l \in L$ is consumed at node $\xi$, then at each node $\mu \in \xi^+$ we obtain an amount $Y_\mu(l', l)$ of the good $l'$. The structure of depreciation is very general, allowing for instance for goods that are perishable or perfectly durable and also for the transformation of some commodities into others. We denote by $Y_\xi(\cdot, l)$ the l-column of matrix $Y_\xi$ and by $Y_\xi(l, \cdot)$ the l-row. Given a history of nodes $\{\xi_1, \ldots, \xi_n\}$, with $\xi_{j+1} \in \xi_j^+$, the accumulated depreciation factor between $\xi_1$ and $\xi_n$ is denoted by $Y_{\xi_1, \xi_n}$. That is, $Y_{\xi_1, \xi_n}$ is equal to $Y_{\xi_n} Y_{\xi_{n-1}} \cdots Y_{\xi_2}$, when $n > 1$; and equal to the identity matrix when $n = 1$. For simplicity, we are not allowing for storage.

Spot markets for commodity trade are available at each node. We denote by $p_\xi = \big(p_{\xi,l} \ : l \in L\big)$ the row vector of spot prices at the node $\xi \in D$ and by $p = (p_\xi : \xi \in D)$ a process of commodity prices.

*Financial markets.* There is a finite ordered set $J$ of different types of infinite lived securities. Assets may suffer default but are *individually* protected by price-dependent collateral requirements. Collateral may consist of consumption goods (Physical collateral) as well as of assets (Financial collateral). Assets of a given type have the same promises of real deliveries and the same collateral requirements.

We suppose that for each type of securities $j \in J$ there is always an asset that can be issued at every node. In the absence of default assets of the same type can be treated as being the same security. However, when an asset issued at a node $\xi$ defaults at a successor node $\mu > \xi$, it converts into the respective collateral, although agents can constitute at this node $\mu$ new long or short positions on an asset of the same type. Moreover, for the sake of simplicity, whenever there is no possible confusion, we will refer to an asset of type $j$ simply as "asset $j$".

Each type of security $j$ is thus characterized by its net supply at $t = 0$, $e_j \geq 0$; the process $A(\xi, j)$ of real promises, which are defined in the set of nodes $\xi > \xi_0$; and the collateral coefficients, which can vary along the event-tree (see below). We denote by $q_\xi$ the row vector of asset prices at node $\xi \in D$, and by $q = (q_{\xi,j}, (\xi, j) \in D \times J)$ a plan of asset prices in the event-tree.

When assets are short-sold, borrowers have to constitute collateral. At each node $\xi \in D$, collateral requirements per unit of asset $j \in J$ are given by functions $C^P_{\xi,j} : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+$ and $C^F_{\xi,j} : \mathbb{R}^J_+ \to \mathbb{R}^J_+$, where physical collateral requirements $C^P_{\xi,j}(p_\xi, q_{\xi,j})$ may depend on the price of the asset and the prices of the commodities backing it, whereas financial collateral coefficients, $C^F_{\xi,j}(q_\xi)$, may depend on financial prices at $\xi$. Note that collateral is always held by the borrower.

Holders of asset endowments are not required to constitute collateral when selling these endowments. Moreover, when purchasing an unit of an asset it is not possible to distinguish whether this unit was short-sold or is part of someone's endowment. The return from this purchase is the same in both cases (given by the minimum between the promise and the garnishable collateral) and the price is also the same. That is, an holder of an asset endowment holds units of the tradeable asset subject to default and not of the underlying primitive asset free of default with promises $A$, which can not be traded.

Also, we assume, for the sake of simplicity (in the market clearing equations), that assets in positive net supply are only backed by physical collateral,

Assumption A. *For each class $j \in J$ the functions $C^P_{\xi,j}$ and $C^F_{\xi,j}$ are, for every node $\xi \in D$, continuous in its domain, homogenous of degree zero and satisfy $(\,C^P_{\xi,j}(p_\xi, q_{\xi,j}),\, C^F_{\xi,j}(q_\xi)\,) \neq 0$, for all $(p_\xi, q_\xi)$. The commodities and the securities used as collateral for a class $j$ are the same along the event-tree $D$.[2] Moreover, each class in positive net supply, $e_j > 0$, is protected only by physical collateral.*

In addition to price-independent physical collateral requirements (as in Geanakoplos and Zame (2002)), the hypothesis above accommodates the following examples:

*1) Fixed margin.* An asset of type $j$ is backed only by commodity $l$ and, within some bounds, $\underline{c}_{\xi,j}$ and $\overline{c}_{\xi,j}$, the quantity of the commodity serving as collateral, per unit of the asset, varies so that its value is a fixed proportion $m > 1$ of the asset price. That is, $\left(C^P_{\xi,j}(p_\xi, q_{\xi,j})\right)_l = \min\left\{\max\left\{\underline{c}_{\xi,j}, \frac{m q_{\xi,j}}{p_{\xi,l}}\right\}, \overline{c}_{\xi,j}\right\}$.

*2) Mortgage loans.* Asset (of type) $j$ is protected only by physical collateral. Unitary collateral requirements are independent of the price level and, at each node, the depreciated collateral coincides with the current collateral bundle, $C^P_{\xi,j} = Y_{\xi_0,\xi} C^P_{\xi_0,j}$. That is, a given collateral bundle entitles a borrower to a constant short position, throughout a non-default path, by successively using the depreciated collateral as the new collateral.

The next hypothesis rules out *self-collateralization*, that is, the possibility that an asset ends up securing itself, even if this is done through a chain of other assets,

[2]That is, given a commodity $l \in L$, if $\left(C^P_{\xi,j}(p_\xi, q_{\xi,j})\right)_l \neq 0$ then $\left(C^P_{\mu,j}(p'_\xi, q'_{\xi,j})\right)_l \neq 0$ for all node $\mu \in D$ and prices $(p'_\xi, q'_{\xi,j})$. Analogously, given an asset $j' \neq j$ in $J$, if $\left(C^F_{\xi,j}(q_\xi)\right)_{j'} \neq 0$ then $\left(C^F_{\mu,j}(q'_\xi)\right)_{j'} \neq 0$ for all price vectors $q'_\xi$ and nodes $\mu \in D$.

Assumption B. *For each node $\xi \in D$, and any $m$-tuple $(j_1, j_2, \ldots, j_m) \subset J^m$ we have that,* $\left(C^F_{\xi,j_1}(q_\xi)\right)_{j_2} \left(C^F_{\xi,j_2}(q_\xi)\right)_{j_3} \ldots \left(C^F_{\xi,j_m}(q_\xi)\right)_{j_1} = 0.$

Remark 1. A consequence of the assumption above is the existence of a *pyramidal structure* on the set $J$, whose basic layer consists of assets backed only by physical collateral, and the successive layers are collateralized only by assets in the previous layer. More formally, the set $J$ can be decomposed into a disjoint union of sets $(\Lambda_k)_{k\geq 0}$ which are independent of the price level (as a consequence of Assumption A) and are defined by

$$\begin{aligned}\Lambda_0 &= \left\{j \in J : C^P_{\xi_0,j} \neq 0 \ \wedge \ C^F_{\xi_0,j} = 0\right\},\\ \Lambda_k &= \left\{j \in J : \left(C^F_{\xi_0,j}\right)_{j'} \neq 0 \Rightarrow j' \in \bigcup_{r=0}^{k-1} \Lambda_r\right\} - \bigcup_{r=0}^{k-1} \Lambda_r, \quad \forall k > 0.\end{aligned}$$

Moreover, if $\Lambda_k \neq \emptyset$ then $\Lambda_r \neq \emptyset$, for all $r < k$. Thus, the set of assets that are backed only by commodities is non-empty. *Therefore, even without assets in positive net supply, physical collateral exists, as a consequence of the absence of self-collateralization in the economy.*

For convenience of notations, let $\mathrm{CV}_{\xi,j}(p,q) = p_\xi C^P_{\xi,j}(p_\xi, q_{\xi,j}) + q_\xi C^F_{\xi,j}(q_\xi)$ be the value of the unitary collateral requirement of asset $j$, at node $\xi$.

As the only enforcement in case of default is given by the seizure of the collateral, borrowers will pay (and lenders will expect to receive) the minimum between the depreciated value of the collateral and the market value of the original debt. Thus, the (unitary) nominal payment made by asset $j$ at node $\xi$ is given by $D_{\xi,j}(p,q) := \min\{p_\xi A(\xi,j) + q_{\xi,j}, \mathrm{DCV}_{\xi,j}(p,q)\}$, where $\mathrm{DCV}_{\xi,j}(p,q)$ denotes the depreciated value of asset' $j$ collateral coefficients required at the preceding node,

$$\mathrm{DCV}_{\xi,j}(p,q) := p_\xi Y_\xi C^P_{\xi^-,j}(p_{\xi^-}, q_{\xi^-,j}) + \sum_{j'\in J} D_{\xi,j'}(p,q)\,(C^F_{\xi^-,j}(q_{\xi^-}))_{j'},$$

where the recursive rule above is well defined due to Remark 1 above. To shorten notations, we introduce the following row vectors $\mathrm{CV}_\xi(p,q) := (\mathrm{CV}_{\xi,j}(p,q),\, j \in J)$, $D_\xi(p,q) := (D_{\xi,j}(p,q), j \in J)$ and $\mathrm{DCV}_\xi(p,q) := (\mathrm{DCV}_{\xi,j}(p,q),\, j \in J)$.

*Agents.* There exists a finite set, $H$, of infinite-lived agents in the economy.[3] At each node $\xi \in D$, each agent $h \in H$ can choose a *short financial position* $\varphi^h_\xi = (\varphi^h_{\xi,j})_{j\in J}$, an *autonomous long position* $\theta^h_\xi = (\theta^h_{\xi,j})_{j\in J}$ (that is, asset purchases in excess of required financial collateral) and an *autonomous consumption bundle* $x^h_\xi \in \mathbb{R}^L_+$ (that is, consumption in excess of required physical collateral). As short sales are backed by physical or financial collateral, given a plan $z^h_\xi = (x^h_\xi, \varphi^h_\xi, \theta^h_\xi)$, we denote by $\hat{x}^h = \left(\hat{x}^h_\xi\right)_{\xi\in D}$ the agent' $h$ consumption plan and by $\hat{\theta}^h = \left(\hat{\theta}^h_\xi\right)_{\xi\in D}$ the agent' $h$ long-position plan. That is,

$$\hat{x}^h_\xi = x^h_\xi + \sum_{j\in J} C^P_{\xi,j}(p_\xi, q_{\xi,j})\,\varphi^h_{\xi,j}, \qquad \hat{\theta}^h_{\xi,j} = \theta^h_{\xi,j} + \sum_{j'\in J} \left(C^F_{\xi,j'}(q_\xi)\right)_j \varphi^h_{\xi,j'}.$$

[3]Our goal is to use a set up which has been considered to be the most hostile to bubbles, when agents are infinite-lived. For this reason, we did not allow for incomplete participation.

Each agent $h \in H$ is also characterized by financial endowments at the first date $e^h = (e^h_j)_{j\in J} \in \mathbb{R}^J_+$, which satisfy $e^j = \sum_{h\in H} e^h_j$, and by a physical endowment process $w^h \in \mathbb{R}^{D\times L}_+$.

Furthermore, given prices $(p,q) \in \mathbb{P}$, consumer $h$'s objective is to maximize her utility function $U^h : \mathbb{R}^{D\times L} \to \mathbb{R}$, that represents her preferences over the plans $\hat{x}^h$, by choosing a plan $z = (x, \theta, \varphi)$, in the state-space $\mathbb{E} = \mathbb{R}^{D\times L}_+ \times \mathbb{R}^{D\times J}_+ \times \mathbb{R}^{D\times J}_+$ which satisfies the following budgetary constraints

$$p_{\xi_0} x_{\xi_0} + \mathrm{CV}_{\xi_0}(p,q)\varphi_{\xi_0} + q_{\xi_0}\left(\theta_{\xi_0} - \varphi_{\xi_0}\right) \leq p_{\xi_0} w^h_{\xi_0} + q_{\xi_0} e^h, \tag{1}$$

and for all $\xi > \xi_0$,

$$\begin{aligned} p_\xi x_\xi + \mathrm{CV}_\xi(p,q)\varphi_\xi + q_\xi\left(\theta_\xi - \varphi_\xi\right) \\ \leq p_\xi\left(w^h_\xi + Y_\xi x_{\xi^-}\right) + \mathrm{DCV}_\xi(p,q)\varphi_{\xi^-} + D_\xi(p,q)\left(\theta_{\xi^-} - \varphi_{\xi^-}\right). \end{aligned} \tag{2}$$

Thus, given prices $(p,q)$, we define the budget set of agent $h$, $B^h(p,q)$, as the collection of plans $(x,\theta,\varphi) \in \mathbb{E}$ such that equations (1) and (2) hold. Moreover, it follows from Assumption A that we can restrict the price set to $\mathbb{P} := \{(p_\xi, q_\xi)_{\xi\in D} : (p_\xi, q_\xi) \in \Delta^{L+J-1}_+\}$, where $\Delta^{n-1}_+$ denotes the n-dimensional simplex.

Remark 2. The non-negativity condition on the autonomous consumption represents the *physical collateral constraint*. In fact, the later requires $\hat{x}^h_\xi \geq \sum_{j\in J} C^P_{\xi,j}(p_\xi, q_{\xi,j})\, \varphi^h_{\xi,j}$, which is equivalent to $x^h_\xi \geq 0$. Analogously, the non-negativity of the autonomous long position reflects the financial collateral constraint.

*Market clearing.* Real returns from asset endowments have to be taken into account in the market clearing conditions. When an asset does not default, the real returns from asset endowments coincide with the promised real returns. In this first case, the asset will remain with the same positive net supply that it had at the preceding node. However, in the case of default, real returns generated by assets' endowments will be determined by the depreciated physical collateral coefficients. In this second case, the asset can be traded again, as long as the collateral requirements are again satisfied, but the positive net financial supply disappears. In fact, the previous positive net supply has been entirely converted into a supply of garnished collateral.

In the borderline case, when borrowers are indifferent between surrendering the depreciated collateral and honoring the promise, in value terms it does not matter whether the collateral is garnished or the promise is payed, but, for the purposes of market clearing, this choice becomes relevant. This choice will determine also whether the asset's net supply will decrease or not.

For this reason, given prices $(p,q) \in \mathbb{P}$, we introduce, at each node $\xi \neq \xi_0$, delivery rates $\lambda_{\xi,j} \in [0,1]$, which are equal to one when the promise is lower than the value of the garnishable collateral, equal to zero when the opposite strict inequality holds, but may take a value between zero and one in case of equality.

Actually, since the promise and the garnishable collateral coefficients are impersonal, the delivery rates may vary across agents but there is no rationale for such differences. Hence, we can concentrate our attention on outcomes where, in the case of indifference between paying the promise and surrendering the collateral, all agents choose the same combination of these two, that is, the same delivery rates.

Using the delivery rates $\lambda_\xi$, the effective nominal return of an asset $j$ in positive net supply, $D_{\xi,j}(p,q)$, can be seen as the value of a real component plus the value of a financial position,

$$D_{\xi,j}(p,q) = p_\xi \,\left(\lambda_{\xi,j}\, A(\xi,j) + (1-\lambda_{\xi,j})\, Y_\xi C^P_{\xi^-,j}(p_{\xi^-}, q_{\xi^-,j})\right) + q_\xi\, \lambda_{\xi,j},$$

where the real component is either the promised physical delivery or the depreciated physical collateral or a combination of the two.

Definition 1. *An Equilibrium for the Infinite Horizon Economy is a vector of prices* $(p,q) \in \mathbb{P}$ *jointly with individual plans* $\big((x^h,\theta^h,\varphi^h),\, h\in H\big) \in \mathbb{E}^H$, *and delivery rates* $\lambda \in [0,1]^{(D\setminus\{\xi_0\})\times J}$, *such that*

*A. For each agent* $h\in H$, $(x^h,\theta^h,\varphi^h) \in Argmax\,\{U^h(\hat{x}),\,(x,\theta,\varphi)\in B^h(p,q)\}$.

*B. At each node* $\xi \neq \xi_0$, *the asset* $j$ *delivery rate satisfies,*

$$\lambda_{\xi,j} \in Argmin_{\lambda'\in[0,1]}\ [\,\lambda'(p_\xi A(\xi,j) + q_{\xi,j}) + (1-\lambda')\mathrm{DCV}_{\xi,j}(p,q)\,]\,.$$

*C. Asset markets are cleared. That is, for each asset* $j\in J$,

$$\sum_{h\in H}\left(\hat{\theta}^h_{\xi_0,j} - \varphi^h_{\xi_0,j}\right) = e_j\,;$$

$$\sum_{h\in H}\left(\hat{\theta}^h_{\xi,j} - \varphi^h_{\xi,j}\right) = \lambda_{\xi,j}\sum_{h\in H}\left(\hat{\theta}^h_{\xi^-,j} - \varphi^h_{\xi^-,j}\right), \quad \forall \xi\neq\xi_0.$$

*D. Physical markets are cleared.*

$$\sum_{h\in H}\hat{x}^h_{\xi_0} = \sum_{h\in H} w^h_{\xi_0};$$

*and, at each node* $\xi\neq\xi_0$,

$$\sum_{h\in H}\hat{x}^h_\xi = \sum_{h\in H}\left(w^h_\xi + Y_\xi \hat{x}^h_{\xi^-}\right) + \sum_{j\in J}\left(\lambda_{\xi,j}A(\xi,j) + (1-\lambda_{\xi,j})Y_\xi C^P_{\xi^-,j}\right)\sum_{h\in H}\left(\hat{\theta}^h_{\xi^-,j} - \varphi^h_{\xi^-,j}\right).$$

## 3. Assumptions on Agents' Characteristics

We make the following hypotheses on agents' endowments and preferences,

Assumption C. *For each agent, the cumulated resources generated by her own physical endowments up to each node* $\xi$, $W^h_\xi$, *belongs to* $\mathbb{R}^L_{++}$. *That is, for each node* $\xi\in D$, *that occurs as result of an history* $F_\xi := \{\xi_0,\dots,\xi^-,\xi\}$ *we have that* $W^h_\xi := \sum_{\mu\in F_\xi} Y_{\mu,\,\xi}\; w^h_\mu \gg 0$.

We do not need to impose any uniform lower bound in the aggregate cumulated resources, $\sum_{h\in H} W^h_\xi$. Thus we allow for durable commodities whose aggregate resources converge to zero. Also, as commodities can be durable, the traditional assumption that individual endowments of commodities are interior points can be replaced by the weaker assumption that requires only individual accumulated resources to be interior points.

Now, as we allow for assets in positive net supply, the aggregated resources up to a node $\xi$ not only need to take into account the individual contingent physical endowments and the depreciated quantities of past resources, but also the streams of real resources generated by the positive net supply of assets.

As assets in positive net supply are protected only by physical collateral, an *upper bound* for the aggregate physical resources in the economy up to node $\xi$ is given the vector $\mathbb{W}_\xi := \sum_{h\in H} W^h_\xi + \sum_{\mu\in F_\xi} Y_{\mu,\xi} \sum_{j\in J} b^j_\mu e_j$, where $b^j_{\xi_0} = 0$ and $b^j_\xi = (b^j_{\xi,l})_{l\in L}$, with

$$b^j_{\xi,l} = \max_{(p,\,q)\in\Delta_+^{L+J-1}} \left\{ \left(C^P_{\xi,j}(p,q_j)\right)_l \,;\, A(\xi,j)_l \right\}.$$

The following assumption guarantees that agents attain a finite level of utility at feasible consumption plans, even when accumulated endowments are not uniformly bounded along the even-tree.

Assumption D. *The utility function for the agent h, $U^h$, is separable in time and in states of nature, in the sense that $U^h(\hat{x}) := \sum_{\xi\in D} u^h_\xi(\hat{x}(\xi))$, where the functions $u^h_\xi : \mathbb{R}^L \to \mathbb{R}$ are strictly concave, continuous and strictly increasing in their convex effective domain $\mathbb{A}^h_\xi$, which satisfies $\mathbb{R}^L_+ \subset \mathbb{A}^h_\xi$. Moreover, $U^h(\mathbb{W}) < +\infty$.*

The above hypothesis is weaker than Magill and Quinzzi (1996) assumption of uniform impatience, which is satisfied by recursive stationary utility. An even stronger assumption, jointly on preferences and endowments, of uniform impatience, was defined by Santos and Woodford (1997), in the context of exogenous net supplies of assets, as follows: *For each $h \in H$, there exists $\sigma^h \in [0,1)$ such that for any $\xi \in D$,*

$$U^h\left((c^h_\mu)_{\mu\notin D(\xi)},\, c^h_\xi + \tilde{w}_\xi,\, (\sigma\, c^h_\mu)_{\mu>\xi}\right) > U^h((c^h_\mu)_{\mu\in D}),$$

*for all consumption plans satisfying $c^h_\mu \le \tilde{w}_\mu$, with $\mu \in D$, and all $\sigma \ge \sigma^h$, where $\tilde{w}_\mu$ is the aggregate commodity resources up to node $\mu$.*[4]

## 4. Individual Optimality

In this section we present necessary and sufficient conditions for individual optimality. As in positive dynamic programming theory, we will show that the default structure gives to the Lagrangian a sign property under which Euler inequalities jointly with a transversality condition are necessary and sufficient to guarantee the optimality of a consumption-portfolio plan.

Given prices $(p,q) \in \mathbb{P}$, denote by $z_\xi = (x_\xi, \theta_\xi, \varphi_\xi) \in \mathbb{Z} := \mathbb{R}^L \times \mathbb{R}^J \times \mathbb{R}^J$ a vector of consumption-portfolio choices at node $\xi$ and by $g^h_\xi(\,\cdot\,; p,q) : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ the function satisfying,

$$g^h_\xi\left(z_\xi, z_{\xi^-}; p,q\right) \le 0 \quad \Leftrightarrow \quad (p,q,z_\xi,z_{\xi^-}) \text{ verifies the budget constraint of node } \xi,$$

where for convenience of notation we put $z_{\xi_0^-} = 0$. Therefore, at prices $(p,q)$, the objective of the agent $h$ is to find a plan $(z^h_\xi)_{\xi\in D}$ in order to solve

$$(P^h) \qquad \begin{array}{ll} \max & \displaystyle\sum_{\xi\in D} v^h_\xi(z_\xi) \\ \text{s.t.} & \left\{ \begin{array}{lcl} g^h_\xi\left(z_\xi, z_{\xi^-}; p,q\right) & \le & 0, \quad \forall \xi \in D, \\ z_\xi = (x_\xi,\theta_\xi,\varphi_\xi) & \ge & 0, \quad \forall \xi \in D. \end{array} \right. \end{array}$$

[4]Notice that $\tilde{w}_\mu$ includes the real returns from assets' net supply at preceding nodes, which are not exogenous in our model (see Section 3 and footnote 13 in Section 8).

where $v^h_\xi(z_\xi) := u^h_\xi\left(x_\xi + \sum_{j\in J} C^P_{\xi,j}(p_\xi, q_{\xi,j})\,\varphi_{\xi,j}\right)$.

For each real number $\gamma \geq 0$, let $\mathcal{L}^h_\xi(\cdot,\gamma;p,q) : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ be the *Lagrangian function* associated to consumer problem at node $\xi$, which is defined by

(3) $$\mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q) = v^h_\xi(z_\xi) - \gamma\, g^h_\xi(z_\xi, z_{\xi^-}; p, q).$$

As under Assumption D the function $\mathcal{L}^h_\xi(\cdot,\gamma;p,q)$ is concave, we can consider its super-differential set at point $(z_\xi, z_{\xi^-})$, $\partial\mathcal{L}^h_\xi(z_\xi, z_{\xi^-},\gamma;p,q)$, which is defined as the set of vectors $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \mathbb{Z}\times\mathbb{Z}$ such that, for all pair $(z'_\xi, z'_{\xi^-})$,

(4) $$\mathcal{L}^h_\xi(z'_\xi, z'_{\xi^-}, \gamma; p, q) - \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q) \leq (\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \cdot \Big((z'_\xi, z'_{\xi^-}) - (z_\xi, z_{\xi^-})\Big).$$

The above vectors $\mathcal{L}'_{\xi,1}$ and $\mathcal{L}'_{\xi,2}$ are partial super-gradients with respect to the current and past decision variables, respectively.

The following proposition states necessary and sufficient conditions to guarantee that a plan $(z^h_\xi)_{\xi\in D}$ is a solution of the agent' $h$ maximization utility problem. The sufficiency depends crucially on the following *sign property* of the Lagrangian,

(5) $$\mathcal{L}'_{\xi,2} \geq 0, \qquad \forall\, (\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \partial\mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma_\xi; p, q).$$

which is very specific to the default and collateral model. In fact, as for each asset $j \in J$ short sales effective returns $D_{\xi,j}(p,q)$ are not greater than the respective garnishable collateral values, the joint returns from actions taken at immediately preceding nodes are non-negative (see Appendix A).

Proposition 1. *Suppose that Assumptions C and D hold.*
*(i) If a budget feasible plan* $(z^h_\xi)_{\xi\in D} = (x^h_\xi, \theta^h_\xi, \varphi^h_\xi)_{\xi\in D}$ *gives a finite optimum to consumer' h problem, at prices* $(p,q)\in\mathbb{P}$, *then there exist strictly positive multipliers* $(\gamma^h_\xi)_{\xi\in D}$ *for which* $\gamma^h_\xi g^h_\xi(z^h_\xi, z^h_{\xi^-}; p, q) = 0$ *and super-gradients* $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \partial\mathcal{L}^h_\xi(z^h_\xi, z^h_{\xi^-}, \gamma^h_\xi; p, q)$, *satisfying the following transversality condition,*

(TC) $$\lim_{T\to+\infty} \sum_{\xi\in D_T} \mathcal{L}'_{\xi,1}\cdot z^h_\xi = 0.$$

*jointly with the Euler conditions,*

(EE$_1$) $$\mathcal{L}'_{\xi,1} + \sum_{\mu\in\xi^+} \mathcal{L}'_{\mu,2} \leq 0,$$

(EE$_2$) $$\left(\mathcal{L}'_{\xi,1} + \sum_{\mu\in\xi^+} \mathcal{L}'_{\mu,2}\right)\cdot z^h_\xi = 0.$$

*(ii) Reciprocally, the existence of non-negative multipliers* $(\gamma^h_\xi)_{\xi\in D}$ *satisfying conditions above for a budget feasible plan* $z^h = (z^h_\xi)_{\xi\in D}$ *assures the optimality of* $z^h$ *in* $(P^h)$.
*(iii) Using multipliers that satisfies (EE)'s and (TC) as deflators, the present value of endowments is always summable, in the sense that,* $\sum_{\xi\in D}\gamma^h_\xi p_\xi w^h_\xi < +\infty$.

An attainable consumption and portfolio plan satisfying Euler and transversality conditions gives the consumer a finite optimum, as the next corollary points out.

Corollary 1. *If Assumptions C and D hold, in the sufficiency part (ii) of the above theorem, if the plan* $(z^h_\xi)_{\xi\in D} = (x^h_\xi, \theta^h_\xi, \varphi^h_\xi)_{\xi\in D}$ *is attainable with respect to the aggregated resources of the economy (in the sense that* $\hat{x}^h_\xi = x^h_\xi + \sum_{j\in J} C^P_{\xi,j}(p_\xi, q_{\xi,j})\varphi^h_{\xi,j} \leq \mathbb{W}_\xi$*), then the optimum of problem* $(P^h)$ *is finite.*

The above Euler conditions will be used below to characterize the prices of assets and commodities that are compatible will individual optimality, or equivalently, that rule out arbitrage opportunities. As our immediate objective is to guarantee the existence of equilibrium, we postpone this characterization of asset prices to Section 6 (see Proposition 2). It is worth discussing here the interpretation of the above transversality condition and comparing it with related conditions used in the literature.

First, it should be noticed that the above transversality condition is not a constraint that is imposed together with the budget restrictions (as was the case in Hernandez and Santos (1996) or Magill and Quinzii (1996)), it is rather a property that optimal plans should satisfy.

Secondly, as the deflated value of endowments is summable, our transversality condition (CT) can be rewritten as requiring that, as time tends to infinity, the deflated cost of the autonomous consumption goes to zero,

$$\text{(TC}_x\text{)} \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h_\xi p_\xi\, x^h_\xi \;=\; 0;$$

jointly with the cost of the joint operation of constituting collateral and short-selling,

$$\text{(TC}_\varphi\text{)} \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h_\xi \left(\text{CV}_\xi(p,q) - q_\xi\right) \varphi^h_\xi \;=\; 0;$$

and the cost of autonomous asset purchases,

$$\text{(TC}_\theta\text{)} \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h_\xi q_\xi\, \theta^h_\xi \;=\; 0,$$

where $z^h_\xi = (x^h_\xi, \theta^h_\xi, \varphi^h_\xi)$ (see Appendix A).

Corollary 2. *Under the assumptions of the previous proposition, if the process* $(\mathbb{W}_\xi)_{\xi\in D}$*, of upper bounds for the aggregate physical resources, is uniformly bounded by above along the event-tree and the process* $(w^h_\xi)_{\xi\in D}$*, of agent' h new endowments, is uniformly bounded away from zero, then our transversality condition (CT) is equivalent to*

$$\text{(6)} \qquad \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h_\xi q_\xi \left(\hat{\theta}^h_\xi - \varphi^h_\xi\right) = 0.$$

This last transversality condition should not be confused with the usual *exogenous transversality constraint* of models without default, which requires the deflated cost of *any* budget feasible portfolio to go to zero. In fact, it is the sign property of the Lagrangian which allows us to dispense with imposing the usual transversality constraint on all budget feasible portfolios.

## 5. Equilibrium Existence

The literature on equilibrium in default-free economies with infinite-lived agents had to impose transversality or debt constraints in order to obtain existence of equilibrium in the case of models with nominal or numeraire single-period securities, and merely generic existence of equilibrium in more general cases (see Hernandez and Santos (1996) or Magill and Quinzii (1996)).[5] In fact, there were three difficulties, but collateral avoids them and equilibrium always exists.

First, even for finite horizon economies and nominal or numeraire assets, when assets live several periods, the rank of the returns matrix will depend on asset prices and, therefore, unless short-sales are bounded, equilibrium existed, in the default-free model, only for a generic set of economies. Collateral avoids this problem as the physical resources that can serve to collateralize the short-sales (directly or indirectly through other assets) are finite. That is, collateral overcomes the problem, just as it circumvented the price-dependence of the rank of the return matrices of real single-period assets (see Geanakoplos and Zame (2002)).

Second, Ponzi schemes could occur (if debt or transversality restrictions were not imposed), but collateral rules them out, as it did in the case of single-period assets (see Araujo, Páscoa and Torres-Martínez (2002)). Actually, it is the non-negativity of the returns of the joint operation of constituting collateral and short-selling (i.e., the sign property of the Lagrangian discussed in the previous section) that takes care of this problem.[6]

Third, as Hernandez and Santos (1996) pointed out, when asset return matrices are not bounded along the event-tree, equilibria might not exist when infinite-lived real (or numeraire) assets are in zero net supply. In fact, the asset price can be shown to be the series of discounted real returns and would be unbounded, unless marginal rates of substitution tend to zero quickly enough (which would be the case if the asset's net supply were positive, inducing unbounded additional resources). Collateral dispenses with any uniform bounds on assets' promised returns, as the asset price is bounded by the discounted value of the depreciated collateral at the next date, plus perhaps some shadow price of the collateral constraint (see Proposition 2 below).

THEOREM 1. *Under Assumptions A to D there exists an equilibrium.*

SKETCH OF PROOF: The proof follows along the lines of the existence argument for single-period assets in Araujo, Páscoa and Torres-Martinez (2002) (see Appendix D for all the details). Existence for finite horizon economies is established, given that collateral requirements bound feasible short-sales. By a diagonalization argument, the sequence of finite horizon equilibrium variables and associated Lagrange multipliers has a subsequence that converges at every node. The cluster point satisfies individual optimality since it verifies the Euler and transversality conditions of Proposition 1 above.

[5]Hernandez and Santos (1996) were also able to show the existence of equilibrium in the special case where the asset structure consists of a single infinite lived real asset in positive net supply.

[6]In the presence of utility penalties, this sign property is lost and Ponzi schemes may occur (see Páscoa and Seghir (2005), for the case of single-period assets).

## 6. Asset Pricing Properties

In this section we will characterize prices which give a finite optimum to the agents' problems. We refer to those prices as *optimality-compatible*. Of course, it follows from Assumption D that equilibrium prices are always optimality-compatible.

We will treat commodities as if they were assets, because their durability allows us to regard them as Lucas' trees; that is, as rights to benefit from part of it through current consumption, as well as rights to sell its depreciated values in future states of nature.

The following result, which is a consequence of the Euler conditions, asserts that optimality-compatible asset prices can be decomposed in terms of the deflated value of their future deliveries, accrued by the shadow prices of the financial constraints, and non-pecuniary returns, that reflect the utility gains from autonomous consumption or from consumption of physical collateral.

Proposition 2. *Under Assumptions B and D, for each process* $(p,q)\in\mathbb{P}$ *of optimality-compatible prices, there exist, at each node* $\xi$*, strictly positive deflators* $\gamma_\xi$*, non-negative shadow price vectors* $\eta_x(\xi)$*,* $\eta_\theta(\xi)$*,* $\eta_\varphi(\xi)$*, and non-pecuniary returns* $\alpha_\xi \gg 0$*, such that,*

$$\gamma_\xi p_\xi = \sum_{\mu\in\xi^+} \gamma_\mu p_\mu Y_\mu + \eta_x(\xi) + \alpha_\xi; \tag{7}$$

$$\gamma_\xi q_\xi = \sum_{\mu\in\xi^+} \gamma_\mu D_\mu(p,q) + \eta_\theta(\xi); \tag{8}$$

$$\gamma_\xi(\mathrm{CV}_{\xi,j}(p,q) - q_{\xi,j}) = \sum_{\mu\in\xi^+} \gamma_\mu\left(\mathrm{DCV}_{\mu,j}(p,q) - D_{\mu,j}(p,q)\right) + \alpha_\xi\cdot C^P_{\xi,j}(p,q) + \eta_\varphi(\xi,j). \tag{9}$$

*Moreover, the shadow prices of the sign constraints on the long and short positions of asset* $j$*,* $(\eta_\theta(\xi,j),\, \eta_\varphi(\xi,j))$*, are equal to zero when the shadow prices* $\eta_x(\xi,l)$ *are zero, for the commodities that serve as collateral (directly baking asset* $j$ *or indirectly, by baking its financial collateral).*

It follows that, although asset prices can be non-linear functions of their future effective payments, the non-linearities can only arise as a consequence of binding physical collateral constraints (more precisely, when the shadow prices $\eta_x$ of these constraints are strictly positive). In fact, in any equilibrium in which at least one agent made autonomous consumption in those commodities that are used as physical collateral (i.e. this agent has zero sign multipliers for consumption), asset prices satisfy the following (generalized) martingale property: There exists a plan of strictly positive deflators $(\overline{\gamma}_\xi)_{\xi\in D}$ (the Kuhn-Tucker multipliers of that agent) such that,

$$q_{\xi,j} = \sum_{\mu\in\xi^+} \frac{\overline{\gamma}_\mu}{\overline{\gamma}_\xi} D_{\mu,j}(p,q), \qquad \forall\,(\xi,j)\in D\times J. \tag{10}$$

Remark 3. In the context of a two-period economy with physical collateral only, Araujo, Fajardo and Páscoa (2005) had already obtained the same asset pricing conditions as *non-arbitrage conditions*. In fact, just like in Araujo, Fajardo and Páscoa (2005), prices satisfy the conditions (7), (8) and (9) of Proposition 2, if and only if, there do not exist non-negative plans $(x_\xi,\theta_\xi,\varphi_\xi)_{\xi\in D}$ giving with certainty, at each node, either positive returns or an improvement of the utility, without any cost. That is, if and only if, at each node $\xi \in D$, the following conditions can not hold with at least one as a strict inequality,

$$p_\xi x_\xi + \mathrm{CV}_\xi(p,q)\,\varphi_\xi + q_\xi\,(\theta_\xi - \varphi_\xi) \;\leq\; 0; \tag{11}$$

$$p_\mu Y_\mu x_\xi + \mathrm{DCV}_\mu(p,q)\varphi_\xi + D_\mu(p,q)\,(\theta_\xi - \varphi_\xi) \;\geq\; 0, \quad \forall \mu \in \xi^+; \tag{12}$$

$$x_\xi + \sum_{j \in J} C^P_{\xi,j}(p_\xi, q_{\xi,j})\;\varphi_{\xi,j} \;\geq\; 0. \tag{13}$$

Existence of deflators and deviations from linearity could be established, as in Araujo, Fajardo and Páscoa (2005), using a theorem on separation of convex cones.

As financial markets can be incomplete, the processes of deflators and deviations from linearity that satisfy conditions (7), (8) and (9) above will not be necessarily unique. Thus, we denote by $\Gamma$ any arbitrary plan of deflators, shadow prices and non-pecuniary returns that satisfy pricing conditions of Proposition 2, and we refer to it as a process of *valuation coefficients*, compatible with prices $(p,q)$. A process of valuation coefficients in which deflators are given by Kuhn-Tucker multipliers of agent $h$ will be denoted by $\Gamma^h$.

Finally, it follows from the above pricing formulas that, at each node $\xi$, the *financial "haircut"* on asset $j$ (i.e. the difference between the collateral cost and the asset price) must be non-negative. Although this difference may be equal to zero (contrarily to the case where there is only physical collateral, see Araujo, Páscoa and Torres-Martínez (2002)), it follows from equation (9) that the financial haircut is strictly positive when the asset is directly backed by physical collateral. Thus, optimality-compatible prices satisfy,

$$\mathrm{CV}_{\xi,j}(p,q) \geq q_{\xi,j}, \quad \forall j \in J, \tag{14}$$

with a strictly inequality when $C^P_{\xi,j}(p_\xi, q_{\xi,j}) \neq 0$.

These last inequalities, jointly with asset pricing properties of Proposition 2, will be fundamental for obtaining conditions which characterize the existence of speculative bubbles in the prices of assets subject to default.

## 7. Fundamental Values and Speculative Bubbles in Prices

The existence of speculative bubbles in non-arbitrage prices was studied, among others, by Magill and Quinzii (1996) and Santos and Woodford (1997), in the case where agents can not default and where short sales satisfy debt restrictions or exogenous transversality conditions. As in their work, speculation is defined as a deviation of the equilibrium price from the *fundamental value* of the asset, which is the deflated value of future payments and services that an asset yields.

*Speculation in durable goods.* As under incomplete markets, the process of valuation coefficients is not unique, we define the fundamental value of a commodity, at a node $\xi$, as a function of the chosen vector of valuation coefficients.

Note that, although commodities can be durable, their do not generate any payments along the event-tree. Thus, the fundamental value, at $\xi$, of a good $l \in L$ only takes into account the deflated value of the rental services that one unit of this commodity generates in $D(\xi)$. Now, given prices $(p,q) \in \mathbb{P}$ and a process of valuation coefficients $\Gamma$, the rental services, deflated to node $\xi$, that a bundle $y \geq 0$ generates at $\mu \in D(\xi)$ are given by

$$\frac{\gamma_\mu}{\gamma_\xi}\left(p_\mu - \sum_{\nu\in\mu^+}\frac{\gamma_\nu}{\gamma_\mu}p_\nu Y_\nu\right) y.$$

As one unit of commodity $l$ at $\xi$ is transforming itself into a bundle $Y_{\xi,\mu}(\cdot,\, l)$ at $\mu$, the *fundamental value* of $l$ at $\xi$, under the valuation coefficients $\Gamma$, is defined by

$$\begin{aligned} F_l(\xi,p,q,\Gamma) &= \sum_{\mu\in D(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\left(p_\mu - \sum_{\nu\in\mu^+}\frac{\gamma_\nu}{\gamma_\mu}p_\nu Y_\nu\right) Y_{\xi,\mu}(\cdot,\, l) \\ &= \sum_{\mu\in D(\xi)}\frac{1}{\gamma_\xi}\left(\alpha_\mu + \eta_x(\mu)\right) Y_{\xi,\mu}(\cdot,\, l), \end{aligned}$$

where the last equality follows from pricing condition (7) of Proposition 2. Note that, in case of a non-diagonal depreciation matrix, the fundamental value of $l$ will take into account the utility gains $\alpha_{\mu,l'}$, associated with another commodity $l'$. Actually, in this case the depreciation matrix transforms commodities into other goods and can be interpreted as an exogenous production technology.

Definition 2. *Given a process $\Gamma$ of valuation coefficients, we say that a commodity $l \in L$ has a $\Gamma$-bubble at node $\xi$ when the price differs from the fundamental value, i.e. $p_{\xi,l} \neq F_l(\xi,p,q,\Gamma)$.*

Now, applying recursively equation (7) we obtain that, for each $T > t(\xi)$,

$$p_{\xi,l} = \sum_{\mu\in D^T(\xi)}\frac{1}{\gamma_\xi}\left(\alpha_\mu + \eta_x(\mu)\right) Y_{\xi,\mu}(\cdot,\, l) + \sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}p_\mu Y_{\xi,\mu}(\cdot,\, l).$$

As the last term, on the right hand side of the equation above, is non-negative, it follows that the fundamental value of commodity $l$ is well defined, independently of the chosen valuation coefficients $\Gamma$. Moreover, that same term is bounded by the price $p_{\xi,l}$, and, taking the limit as $T$ goes to infinity, we obtain,

$$p_{\xi,l} = F_l(\xi,p,q,\Gamma) + \lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}p_\mu Y_{\xi,\mu}(\cdot,\, l). \tag{15}$$

Hence, the price of a commodity is greater or equal to the fundamental value. Furthermore, a necessary and sufficient condition for the absence of $\Gamma$-bubbles at a node $\xi$ is given by,

$$\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}p_\mu Y_{\xi,\mu}(\cdot,\, l) = 0. \tag{16}$$

Thus, a commodity $l$ that has *finite durability* at a node $\xi$ (i.e. there exists $N > 0$ such that $Y_{\xi,\mu}(\cdot,\, l) = 0$ for all $\mu$ with $t(\mu) > t(\xi) + N$) is free of bubbles. For commodities with infinite durability, the following results give sufficient conditions for the absence of bubbles, Theorem 2. *Given equilibrium prices $(p,q)\in\mathbb{P}$, suppose that Assumptions C and D hold. When markets are complete, commodities are free of bubbles in $D$. With incomplete markets, if there exists an agent $h$ such that the $\Gamma$-deflated value at $\xi$ of date $T$ accumulated physical endowments is asymptotically zero,*

$$\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}p_\mu\,W^h_\mu=0, \tag{17}$$

*then commodities are free of $\Gamma$-bubbles in $D(\xi)$.*

Proof. Given a node $\nu\in D(\xi)$, it follows by Assumption C that, for each $T>t(\nu)$,

$$\sum_{\mu\in D_T(\nu)}\frac{\gamma_\mu}{\gamma_\nu}p_\mu Y_{\nu,\mu}(\cdot,\,l)\le\frac{1}{\min_{g\in L}W^h_{\nu,g}}\sum_{\mu\in D_T(\nu)}\frac{\gamma_\mu}{\gamma_\nu}p_\mu W^h_\mu.$$

Taking the limit as $T$ goes to infinity, we conclude the proof of this case.

In the complete markets case, transversality conditions ($\mathrm{TC}_x$), ($\mathrm{TC}_\varphi$) and ($\mathrm{TC}_\theta$) hold, for all agents and for a *same* deflator $\Gamma$. Adding up these three conditions across all agents, we get condition (17) for the deflator $\Gamma$ and for each agent, at node $\xi_0$. Thus, we assures the absence of $\Gamma$-bubbles in $D$. □

The above sufficient condition, for the absence of bubbles in commodities, holds when the present value at $\xi$ (using $\Gamma$ as valuation coefficient) of an agent's cumulated resources is finite, that is, when

$$\sum_{\mu\in D(\xi)}\frac{\gamma_\mu}{\gamma_\xi}p_\mu\,W^h_\mu<+\infty. \tag{18}$$

Theorem 3. *Given equilibrium prices $(p,q)\in\mathbb{P}$, suppose that Assumptions C-D hold and that agent $h$'s new endowments are uniformly bounded away from zero in $D(\xi)$ (that is there exists an lower bound $\underline{w}\gg 0$ such that $w^h_\mu\ge\underline{w}$, for each $\mu>\xi$). If cumulated depreciation factors $Y_{\xi,\mu}$ are uniformly bounded by above along the event-tree $D(\xi)$, then $\Gamma^h$-bubbles are ruled out in commodities in $D(\xi)$.*

Proof. This follows from equation (16) and item (iii) in Proposition 1. □

Alternatively, to avoid $\Gamma^h$ bubbles in commodity prices, we could have assumed new endowments of agent $h$ in $D(\xi)$ to be at least a fixed proportion $\kappa>0$ of her accumulated resources, i.e. $\kappa\,W^h_\mu\le w^h_\mu$ for all $\mu\in D(\xi)$. That is, we could have required that new resources are not too small relatively to cumulated past resources.

*Bubbles in assets.* The fundamental value of an asset is the present value of its future yields and services. Future yields are the deliveries of perishable goods. Future services include the shadow prices of the financial constraints and the rental values of the delivered durable goods. These goods can be directly delivered, as an original promise or a physical collateral garnishment, or received indirectly, as a physical return associated with the financial collateral garnishment.

These real deliveries are unambiguously anticipated except in the borderline case, when the value of the promise equals the garnishable collateral value. Thus, the fundamental value will depend not just on the deflator compatible with non-arbitrage but also on the believed delivery rates for the borderline nodes.

*For a reason of simplicity, we start with and focus our analysis on the simplest case of assets that are backed only by physical collateral, supposing that, in the borderline case, agents anticipate that assets deliver the original promises. The general case will be studied in Appendix B.*

In this context, given optimality-compatible prices $(p, q)$, let $\Phi(\xi, j) \subset D(\xi)$ be the set of nodes that can be reached from $\xi$ following a path where asset $j$ does not default, that is,

$$\nu \in \Phi(\xi, j) \Leftrightarrow p_\mu A_{\mu,j} + q_{\mu,j} \leq p_\mu Y_\mu C^P_{\mu^-,j}(p_{\mu^-}, q_{\mu^-,j})\,;\ \ \forall \mu \in D:\ \xi < \mu \leq \nu.$$

Moreover, let $\Psi(\xi, j)$ be the set of successors of $\xi$ in which asset $j$ gives default, although promises were honored at preceding successors of $\xi$, that is,

$$\nu \in \Psi(\xi, j) \Leftrightarrow \begin{cases} p_\nu A_{\nu,j} + q_{\nu,j} > p_\nu Y_\nu C^P_{\nu^-,j}(p_{\nu^-}, q_{\nu^-,j})\,; \\ \mu \in \Phi(\xi, j), \ \ , \ \ \forall \mu \in D:\ \xi < \mu < \nu.. \end{cases}$$

By definition, at $\mu \in \Phi(\xi, j)$, one unit of asset $j$'s gives yields and services equal to the fundamental value of the delivery $A(\mu, j)$, that is, $\sum_{l \in L} F_l(\mu, p, q, \Gamma)\, A(\mu, j)_l$. Analogously, at a node $\mu$ in $\Psi(\xi, j)$ asset $j$ generates yields and services equal to $\sum_{l \in L} F_l(\mu, p, q, \Gamma)\, Y_{\mu^-}(l, \cdot)\, C^P_{\mu^-,j}$. Therefore, the fundamental value of asset $j$ at $\xi \in D$, is given by,

$$F_j(\xi, p, q, \Gamma) := \sum_{\mu \in \Phi(\xi,j)} \frac{\gamma_\mu}{\gamma_\xi} \sum_{l \in L} F_l(\mu, p, q, \Gamma)\, A(\mu, j)_l + \sum_{\mu \in \Psi(\xi,j)} \frac{\gamma_\mu}{\gamma_\xi} \sum_{l \in L} F_l(\mu, p, q, \Gamma)\, Y_{\mu^-}(l, \cdot)\, C^P_{\mu^-,j} + \sum_{\mu \in \Phi(\xi,j) \cup \Psi(\xi,j) \cup \{\xi\}} \frac{\eta_\theta(\mu, j)}{\gamma_\xi}, \tag{19}$$

where the last term in the right hand side of equation above is equal to zero, if physical collateral constraints never bind in $D(\xi)$ (see Proposition 2).

It follows that, in the particular case in which asset $j$ does not default in any successor node of $\xi$ and original real promises are composed only of perishable commodities, the fundamental value of asset $j$ becomes,

$$F_j(\xi, p, q, \Gamma) := \sum_{\mu > \xi} \frac{\gamma_\mu}{\gamma_\xi}\, p_\mu A(\mu, j) + \sum_{\mu \geq \xi} \frac{\eta_\theta(\mu, j)}{\gamma_\xi}.$$

Definition 3. *Given prices $(p, q) \in \mathbb{P}$, we say that an asset $j \in \Lambda_0$ is free of $\Gamma$-bubbles at a node $\xi$ when $q_{\xi,j} = F_j(\xi, p, q, \Gamma)$.*

Theorem 4. *Under Assumptions C and D, given equilibrium prices* $(p,q) \in \mathbb{P}$ *and any* $\Gamma$, *the fundamental value of each asset* $j \in \Lambda_0$ *is well defined and satisfies* $q_{\xi,j} \geq F_j(\xi,p,q,\Gamma)$. *Moreover, the following conditions are sufficient for the absence of* $\Gamma$*-bubbles in asset* $j$ *at nodes* $\mu \in D(\xi)$,

$$p_{\xi,l} - F_l(\xi,p,q,\Gamma) = 0, \qquad \forall\, l \in L; \tag{20}$$

$$\lim_{T\to+\infty} \sum_{\mu\in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi}\, q_{\mu,j} = 0. \tag{21}$$

Proof. Using pricing condition (8), we have that, for each $T > t(\xi)$,

$$\gamma_\xi q_{\xi,j} = \sum_{\mu\in D^T(\xi)\cap\Phi(\xi,j)} \gamma_\mu\, p_\mu A(\mu,j) + \sum_{\mu\in D^T(\xi)\cap\Psi(\xi,j)} \gamma_\mu\, p_\mu Y_\mu C^P_{\mu^-,j}(p_{\mu^-},q_{\mu^-,j})$$

$$+ \sum_{\mu\in D_T(\xi)\cap\Phi(\xi,j)} \gamma_\mu\, q_{\mu,j} + \sum_{\mu\in(\Phi(\xi,j)\cup\Psi(\xi,j)\cup\{\xi\})\cap D^T(\xi)} \frac{\eta_\theta(\mu,j)}{\gamma_\xi}\,.$$

Now, at each node $\mu \in D(\xi)$ the price of commodities $p_{\mu,j}$ can be split into a fundamental value, $F_l(\mu,p,q,\Gamma)$, plus a non-negative bubble component, denoted by, $b_l(\mu,p,q,\Gamma)$. Therefore, taking the limit as $T$ goes to infinity, we obtain that,

$$q_{\xi,j} = F_j(\xi,p,q,\Gamma) + \lim_{T\to+\infty} \sum_{\mu\in D_T(\xi)\cap\Phi(\xi,j)} \frac{\gamma_\mu}{\gamma_\xi}\, q_{\mu,j}$$

$$+ \sum_{\mu\in\Phi(\xi,j)} \frac{\gamma_\mu}{\gamma_\xi} \sum_{l\in L} b_l(\mu,p,q,\Gamma) A(\mu,j)_l$$

$$+ \sum_{\mu\in\Psi(\xi,j)} \frac{\gamma_\mu}{\gamma_\xi} \sum_{l\in L} b_l(\mu,p,q,\Gamma)\, Y_\mu(l,\cdot)\, C^P_{\mu^-,j}.$$

Thus, the fundamental value is less than or equal to the asset price. Moreover, when commodities are free of bubbles at node $\xi$, are also free of bubbles in the event-tree $D(\xi)$ and, therefore, when condition (20) holds, bubbles are ruled out provided that condition (21) holds. □

Remark 4. Mortgage Loans. When asset $j$ is a mortgage loan we have that $C^F_{\xi,j} = 0$ and $C^P_{\xi,j} = Y_{\xi_0,\xi}\, C^P_{\xi_0}$. Therefore, if the commodities that asset $j$ promises to deliver or that are used as collateral are free of $\Gamma$-bubbles in $\xi$, then $\Gamma$-bubbles are also ruled out in asset $j$. In fact, absence of commodity bubbles in $\xi$ implies that asset $j$ has a $\Gamma$-bubble only if condition (21) does not hold, but this is impossible, due the non-arbitrage condition (14) and the particular structure of collateral requirements. Hence, a bubble in a mortgage loan is always a consequence of a bubble in a commodity that is used as collateral or is part of the real promises.

The following corollary assures that, if all agents are not lenders at infinity under a same deflator, for which commodities are free of bubbles, then assets with persistent net supply are also free of bubbles for this deflator. Sufficient conditions for absence of bubbles in commodity prices were given by Theorems 2 and 3.

Corollary 4.1 *Consider an economy that satisfies Assumptions C and D. Given equilibrium prices $(p, q) \in \mathbb{P}$, suppose that under a plan $\Gamma$ of valuation coefficients commodities are free of bubbles in $D(\xi)$ and*

$$\lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} q_{\mu,j}\, \hat{\theta}^h_{\mu,j} = 0, \quad \forall h \in H. \tag{22}$$

*Then, asset $j \in \Lambda_0$ is free of $\Gamma$-bubbles in $D(\xi)$ provided that its positive net supply is persistent.*

Proof. By Theorem 4 it suffices to show that equation (21) holds. Now, let $e^j_\mu$ be the total net supply of asset $j$ at node $\mu \in D(\xi)$ (that is, $e^j_\mu = \sum_{h\in H}(\hat{\theta}^h_{\mu,j} - \varphi^h_{\mu,j})$). If $e^j_\mu$ is uniformly bounded away from zero by $\underline{e}^j$, then (22) implies that

$$\lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} q_{\mu,j} \le \lim_{T\to+\infty} \frac{1}{\underline{e}^j} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} q_{\mu,j} \sum_{h\in H} \hat{\theta}^h_{\mu,j} = 0.$$

That is, bubbles are ruled out at $\xi$ and, hence, also throughout $D(\xi)$. □

Condition (22) should not be confused with the transversality condition ($\mathrm{TC}_\theta$), which is necessary for individual optimality and guarantees that, for *personalized Kuhn-Tucker deflators*, the *autonomous lending* converges asymptotically to zero.

However, when markets are complete there is only one non-arbitrage deflator and ,therefore, transversality conditions ($\mathrm{TC}_x$), ($\mathrm{TC}_\varphi$) and ($\mathrm{TC}_\theta$) imply that, even when the obligation of constituting financial collateral makes agents become lenders at infinity, bubbles in prices of assets can arise only if the net supply is asymptotically zero (see Example 1) or actually zero beyond a certain node.

Corollary 4.2 *Suppose that, at equilibrium prices $(p, q) \in \mathbb{P}$, there is only one non-arbitrage process of valuation coefficients in the event-tree $D(\xi)$. Then any asset with persistent positive net supply is free of bubbles at nodes $\mu \ge \xi$.*

Proof. Denote by $e^j_\mu$ the aggregate net supply of an asset $j$ at nodes $\mu \ge \xi$. It follows from ($\mathrm{TC}_x$), ($\mathrm{TC}_\varphi$) and ($\mathrm{TC}_\theta$) that, under the plan $\Gamma$ of valuation coefficient compatible with equilibrium,

$$\lim_{T\to+\infty} \left( \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu \sum_{h\in H} \hat{x}^h_\mu + \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} \sum_{j\in J} q_{\mu,j} e^j_\mu \right) = 0.$$

Therefore, as $e^k_\mu$ is bounded away from zero in the event-tree $D(\xi)$, we have that

$$\lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} q_{\mu,k} = 0.$$

Thus, asset $k$ is free of bubbles, as commodities do not have bubbles in $D(\xi)$ (see Theorem 2). The absence of bubbles at successors of $\xi$ also follows. □

Under incomplete markets, Kuhn-Tucker deflators can vary across agents and, therefore, the transversality conditions ($\mathrm{TC}_x$), ($\mathrm{TC}_\varphi$) and ($\mathrm{TC}_\theta$) may not hold for all agents under a same plan of valuation coefficients. Thus, even without bubbles in commodities, assets with persistent positive net supply can have bubbles (see Example 2). Now, independently of the incompleteness of the markets or the behavior of the positive net supply, when collateral requirements are bounded, we can determine conditions on cumulated resources in order to assure that asset prices are equal to fundamental values,

Corollary 4.3 *Given equilibrium prices $(p, q)$, suppose that Assumptions C and D hold and that asset' $j \in \Lambda_0$ collateral requirements are uniformly bounded in $D(\xi)$. (i) If there exists an agent $h$ which has cumulated physical endowments that are uniformly bounded away from zero in the sub-tree $D(\xi)$, and satisfying also condition (17) under $\Gamma$ at $\xi$, then $\Gamma$-bubbles are ruled out in asset $j$ at the nodes $\mu \geq \xi$. (ii) Alternatively, $\Gamma^h$-bubbles are avoided in $D(\xi)$, when the new endowments of agent $h$ are uniformly bounded away from zero, provided that there exists a matrix $\overline{Y}$ such that $Y_{\xi,\mu} \leq \overline{Y}$, for each node $\mu > \xi$.*

Proof. Suppose that there exists $\overline{C} \in \mathbb{R}^L_+$ such that $C^P_{\mu,j}(p_\mu, q_{\mu,j}) \leq \overline{C}$, for all $\mu > \xi$. Now, by the non-arbitrage condition (14), as $j \in \Lambda_0$, $q_{\mu,j} \leq p_\mu C^P_{\mu,j}(p_\mu, q_{\mu,j})$, and, therefore,

$$\lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} q_{\mu,j} \leq \lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_\mu \overline{C}. \tag{23}$$

(i) Condition (17) guarantees that commodities are free of bubbles, and, when cumulated physical endowment are bounded away from zero, implies that

$$\lim_{T\to+\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma_\mu}{\gamma_\xi} p_{\mu,l} = 0, \qquad \forall\, l \in L,$$

which avoids bubbles in the price of $j$, due equation (23).
(ii) The upper bound in cumulated depreciation structures implies the asset $j$ is free of bubbles, due item (iii) in Proposition 1. □

Let us close this section with some remarks on a few novel aspects of the above concepts of fundamental values and results on price bubbles.

1. Now, *finitely-lived assets may have price bubbles too*. In fact, the price of a finitely-lived asset will have a bubble if the asset pays in durable goods whose prices have bubbles or if the asset defaults and the surrendered physical or financial collateral is subject to price bubbles. In a straightforward extension of our model, we could have allowed for assets with finite life and show that price bubbles would occur if the infinitely-lived commodities or assets serving as collateral are priced at infinity (see Appendix B, for the general case allowing for financial collateral).

2. Coming back to a question raised by Santos and Woodford (1997), there seems to exist now a relation between asset price bubbles and the more abstract concept of price bubble proposed by Gilles and Le Roy (1992), in the Arrow-Debreu contingent claims set up, in terms of the non-summability of the price process, weighted by state-prices. Such a relation seemed to be absent

in models without default, as the summability of the deflated commodity price sequence implied absence of price bubbles only in the case assets in positive net supply (as noticed by Magill and Quinzii (1996)). Now, when assets are allowed to default but are backed by bounded collateral requirements, if the deflated commodity price sequence is summable, then assets are free of price bubbles for this deflator.

## 8. Examples of Monetary Equilibria

In this section we consider an asset, money, that promises zero real returns. As any other asset in the model, money can be short-sold provided that collateral requirements are met. In other words, consumers may use money loans to finance the purchase of a durable bundle that has to be surrendered if they default on these loans. In addition, there may be endowments of money at the initial node. The endowment of money should be seen as a holding of a contract that, due to an institutional arrangement, some privileged agents can sell to finance themselves, or as money that was brought into this economy at the initial node but was generated elsewhere precisely in the same way as collateralized short-sales are done here.[7]

We do not attempt to model the role of money as a medium of exchange. There is an extensive literature, dating back to Clower (1967), that explains why money may be used instead of a credit system that enables agents to purchase goods before selling their own. Such a credit system might require perfect record keeping of past transactions to evaluate the ability of those receiving credit to repay (see Ostroy (1973)). This literature has progressed along two lines: the general equilibrium models with cash-in-advance constraints and the models with randomly matched agents, where cash is a substitute for memorizing bilateral imbalances (as in Kiyotaki and Wright (1989) and Kocherlakota (1998)). Our secured credit approach and the liquidity approach can be seen as opposite ideal representations of economies where credit is used in many, but not all, transactions and not all credit is secured. However, even in secured credit economies, where agents need to keep record of credit given only at immediately preceding dates,[8] money may have a positive price as a result of the incompleteness of credit markets.

Now, collateralizing does not imply convertibility. In fact, the price of money is not pegged to the price of the collateral goods except in borderline equilibria, where $q_\xi = p_\xi Y_\xi C_{\xi^-}$. Even in this case, the apparent convertibility is an endogenous property that may happen to prevail in equilibrium, rather than an institutional arrangement, and the analogy with the gold standard situation does not carry through completely for another reason: short-sales of convertible currencies do not require traders to purchase the equivalent gold bundle, but, in our default model, short-sales of money always require the purchase of the collateral bundle. Alternatively, short-sales could be rethought as being in fact an additional issue by non-privileged agents required to constitute gold reserves.

[7] This cashless, pure credit, environment, where all transactions may be financed by money loans, may portray an ideal world to which modern credit economies seem to be converging, and has also been an appealing abstraction to some monetary theorists, both in the past (see Wicksell (1898)) and recently (see Woodford (2003)).

[8] If default had other penalties besides collateral repossession, such as utility penalties, the price-dependent default rule common to all agents would be replaced by subjective default criteria and cash payments would become a valuable alternative to revealing one's credit history.

However, convertibility would again fail outside of the borderline case. Collateralizing implies only that depreciated collateral values become a ceiling to the price of money and, therefore, the positivity of the price of money is still quite an intriguing issue.[9]

We present two examples of monetary equilibrium in which the fragile features of previous examples in the literature are avoided, as the present value of wealth is finite and financial constraints are non-binding. In the first example, the introduction of money completes the markets and it is a borderline case equilibrium, with an asymptotically zero money supply; in the second example, money reduces but does not eliminate market incompleteness, but there is a persistent money supply.[10]

Example 1. Consider a deterministic economy with two infinite lived agents, $h \in \{1, 2\}$, one durable good and one long lived asset, money. There are asset endowments, $e^h$, only at the original date. Physical endowments are given by $w^h_t$. Let $\hat{x}^h_t$ be the consumption choices of agent $h$. Agents preferences are given by

$$U^h(\hat{x}_0, \hat{x}_1, \ldots) = \sum_{t=0}^{+\infty} \beta^t \sqrt{\hat{x}_t},$$

where $\beta \in (0, 1)$. We take the commodity as the numeraire. The physical collateral coefficient at date $t$ is $C_t = 1$, and the commodity depreciation rate is also constant, given by $\kappa \in (0, 1)$. Denote agent $h$ portfolio by $z^h_t$. At each date $t > 0$, the asset's effective nominal return is $R_t = \min\{q_t, \kappa\}$. We can write $R_t = (1 - \lambda_t)\,\kappa + \lambda_t\, q_t$, where the delivery rate satisfies $\lambda_t = 1$ if $\kappa > q_t$ and $\lambda_t = 0$ if $\kappa < q_t$.

The collateral constraint can be written as $\hat{x}^h_t \geq -z^h_t$ and the budget constraints, for a non-negative plan $(\hat{x}^h_t)_{t\geq 0}$, are given by,

$$\begin{aligned} \hat{x}^h_0 + q_0 z^h_0 &= w^h_0 + q_0 e^h_0; \\ \hat{x}^h_t + q_t z^h_t &= w^h_t + \kappa \hat{x}^h_{t-1} + R_t z^h_{t-1}, \quad \text{for } t > 0. \end{aligned}$$

Market clearing conditions are as follows (see Section 2),

$$\sum_h \hat{x}^h_0 = \sum_h w^h_0; \qquad \sum_h z^h_0 = \sum_h e^h;$$

$$\sum_h \hat{x}^h_t = \sum_h w^h_t + \kappa \sum_h \hat{x}^h_{t-1} + (1 - \lambda_t)\,\kappa \sum_h z^h_{t-1}, \quad \text{for } t > 0;$$

$$\sum_h z^h_t = \lambda_t \sum_h z^h_{t-1}, \quad \text{for } t > 0.$$

We look for an equilibrium without default (that is, where $q_t \leq \kappa$, for all $t > 0$) and with non-binding collateral constraints. By Proposition 1, a budget feasible plan $(\hat{x}^h_t, z^h_t)_{t\geq 0}$ is optimal

[9] Borderline equilibria might be regarded instead as implementing an interest rate property. In the simplest case, where depreciation and collateral coefficients are constant, the nominal interest rate matches the inflation rate computed using the collateral requirements as reference bundle.

[10] Santos and Woodford (1997) had already remarked that an asymptotically zero net supply to an infinitely-lived private sector could accommodate bubbles, as in the "hyperdeflationary" equibrium by Woodford (1994). We show that collateral introduces an endogenous mechanism that allows for the net supply to become asymptotically zero and that, under incomplete markets, robust occurrence of bubbles does not require the net supply to vanish.

for agent $h$ if there exist non-negative multipliers $\gamma_t^h$ such that, the following Euler equations and transversality conditions hold,[11]

$$\gamma_t^h q_t = \gamma_{t+1}^h q_{t+1}, \tag{24}$$

$$\gamma_t^h = \kappa\gamma_{t+1}^h + (u_t^h)'(\hat{x}_t^h), \tag{25}$$

$$0 = \lim_{t\to+\infty} \gamma_t^h(\hat{x}_t^h + q_t z_t^h) \tag{26}$$

where $u_t^h(x) = \beta^t\sqrt{x}$.

Now, let $\delta = \beta^2$ and take $\kappa \in (\frac{\delta}{2}, \delta)$. Suppose that individual endowments are given by,

$$w_0^h = 1 + (-1)^{h+1}\kappa\, s_0,$$
$$w_t^h = \delta^t - \kappa\delta^{t-1} - \kappa\alpha(1-\alpha)^{t-1}(e^h + (-1)^{h+1}s_0), \quad \forall\, t > 0,$$

where, for non-negativity of endowments, we require that

$$\alpha \in \left(\frac{\delta-\kappa}{\kappa}, 1\right), \qquad \delta > 1-\alpha, \qquad s_0 < \frac{\delta-\kappa}{\kappa\alpha} - \max_h e^h, \qquad e^h < \frac{\delta-\kappa}{\kappa\alpha}.$$

Take, for example, $(\beta,\delta,\kappa,\alpha,s_0,e^1,e^2) = (\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{8}, \frac{1}{2}, 0)$.

*Let us compute an equilibrium.* Let, for all $t \geq 0$, $\gamma_t^h = \frac{1}{2(1-\kappa)}$ and $q_t = \kappa$. We argue that the consumption plan $x_t^h = \delta^t$ can be implemented in equilibrium. In fact, we have that $(u_t^h)'(\delta^t) = 0.5$ and, therefore, Euler equations (24) and (25) are satisfied. Moreover, as $R_t = \kappa$, budget restrictions reduce to $z_0^h = (-1)^{h+1}s_0 + e^h$ and $z_t^h = -\alpha(1-\alpha)^{t-1}(e^h + (-1)^{h+1}s_0) + z_{t-1}^h$, for $t \geq 1$.

Hence, budget restrictions hold for $z_t^h = (1-\alpha)^t\,((-1)^{h+1}s_0 + e^h)$. As consumption and portfolio plans tend to zero, but deflators and prices are constant, transversality conditions hold. Moreover, the restrictions on the parameters of the economy imply that collateral constraints are not binding. Therefore, given prices $(p_t, q_t) = (1, \kappa)$, the plan $(x_t^h, z_t^h) = (\delta^t,\, (1-\alpha)^t\,((-1)^{h+1}s_0 + e^h))$ is an optimal choice for agent $h$. Clearly, money net supply is asymptotically zero.

Note that asset is in borderline case for each $t \geq 1$ and, if we take the equilibrium delivery rate $\lambda_t$ to be equal to $1-\alpha$, then all market clearing conditions hold at each date. Now, commodity prices do not have bubbles, since at each date $t \geq 0$, $\lim_{T\to+\infty} \frac{\gamma_T^h}{\gamma_t^h} p_t Y_{t,T} = \lim_{T\to+\infty}\ \kappa^{T-t} = 0$. Moreover, as collateral constraints are not binding, the respective shadow prices are zero and, therefore, if agents believe that in the borderline case the asset pays the original promise, it follows from equation (19) that the fundamental value of money at date $t$ is zero and, as $q_t > 0$, money has a bubble at each date.

As argued in the previous section, the above belief about the deliveries may diverge from the equilibrium delivery rates. When an agent evaluates whether there is speculation or not in an asset, the agent is concerned with values and therefore it is perfectly reasonable to anticipate full delivery in borderline situations, as the computation becomes much simpler. But other beliefs should also be allowed and the above monetary equilibrium may be reinterpreted as a positive fundamental value for some beliefs (or even as a situation where bubbles and positive fundamental values coexist).

[11]Notice that the Euler equations with respect to $\hat{x}_t^h$ and $z_t^h$, conditions (24) and (25), imply the Euler conditions (EE), with respect to $(x_t^h, \theta_t^h, \varphi_t^h)$.

In fact, it follows from equation (B.3) (see Appendix B) that the fundamental value at date $t$, under delivery beliefs $(\tau_t)_{t\geq 1} \in [0,1]^{\mathbb{N}}$, is given by,

$$F(t,p,q,\Gamma,\tau) = \sum_{t'>t}\left(\prod_{t<s<t'}\tau_s\right)(1-\tau_{t'})\kappa = \kappa\left(1-\lim_{T\to+\infty}\prod_{t<s\leq T}\tau_s\right),$$

which implies that the asset has a bubble at date $t$ if and only if $\prod_{t<s\leq T}\tau_s$ converges to a strictly positive limit as $T$ goes to infinity.

Thus, when the limit above is strictly positive and less than one, the asset has a bubble, although the fundamental value is positive. Notice that, in this example, monetary equilibria can be interpreted as a positive fundamental value even though shadow prices of the collateral constraints are zero. □

In the *incomplete markets* case, the diversity of agents' deflators allows for price bubbles, even with persistent positive net supply (as collateral is not surrendered).

Example 2. Consider a stochastic economy where each node $\xi \in D$ has two successors: $\xi^+ = \{\xi^u,\, \xi^d\}$. There are two agents $h \in H = \{1,2\}$, who trade two commodities and a single asset. The first commodity is a perishable good $x$ and the second one is a durable good $y$ subject to a constant rate of depreciation $\kappa \in (0,1)$. The asset, money, is subject to default and backed by constant physical collateral requirements, $C_\xi = (0,2)$. Each agent $h$ has physical endowments $(w^h_{\xi,x},\, w^h_{\xi,y})_{\xi\in D}$, financial endowments $e^h \geq 0$, at the first node, and utility given by,

$$U^h(\hat{x}_\xi,\hat{y}_\xi) = \sum_{\xi\in D}\beta^{t(\xi)}\rho^h(\xi)\left(\hat{x}_\xi + \sqrt{\hat{y}_\xi}\right),$$

where $\beta \in (0,1)$ and $\rho^h(\xi) \in (0,1)$ satisfies $\rho^h(\xi) = \rho^h(\xi^d) + \rho^h(\xi^u)$ and, for each $t \geq 0$, $\sum_{\xi\in D_t}\rho^h(\xi) = 1$. The depreciation factor is such that $\kappa < \delta := \beta^2$ and

$$\begin{aligned}\rho^1(\xi^u) &= \frac{1}{2^{t(\xi)+1}}\rho^1(\xi),\\ \rho^2(\xi^u) &= \left(1-\frac{1}{2^{t(\xi)+1}}\right)\rho^2(\xi).\end{aligned}$$

Let $D^{du}$ be the set of nodes attained after going down followed by up, that is, $D^{du} = \{\kappa \in D :\, \exists\xi,\;\; t(\kappa) = t(\xi)+2\;\wedge \kappa = (\xi^d)^u\;\}$ and let $D^{ud}$ be the set of nodes reached by going up and then down, that is, $D^{ud} = \{\kappa\in D:\, \exists\xi,\;\; t(\kappa) = t(\xi)+2\;\wedge\kappa = (\xi^u)^d\;\}$.

Suppose that agent $h=1$ is the only one endowed with the asset, i.e. $(e^1,e^2) = (e,0)$ and that agent' $h$ physical endowments at initial node are $w^h_{\xi),x} = w^h_{\xi_0,y} = 1$. Moreover, for each $\xi > \xi_0$, define $w^h_{\xi,y} = \delta^{t(\xi)} - \kappa\delta^{t(\xi)-1}$ and $w^h_{\xi,x} = 1 + d^h_\xi$, where

$$d^1_\xi = \begin{cases}\frac{\kappa\, e}{2(1-\kappa)}\beta^{-t(\xi)}, & \text{if}\,\xi\in D^{du};\\ 0, & \text{otherwise.}\end{cases}$$

$$d^2_\xi = \begin{cases}\frac{\kappa\, e}{2(1-\kappa)}\beta^{-t(\xi)}, & \text{if}\,\xi\in \{\xi_0^d\}\cup D^{ud};\\ 0, & \text{otherwise.}\end{cases}$$

Notice that the cumulated endowments of the durable good are tending to zero at the rate $\delta = \beta^2$ and, therefore, given the specific form of the marginal utility, the scarcity of this commodity tends to offset the discount factor and, in equilibrium, each agent should consume the own cumulated endowment. But the shocks on endowments of the perishable good create an opportunity to use money as a way to transfer wealth to states that are more valuable.

In fact, let *prices* be $(p_{\xi,x}, p_{\xi,y}, q_\xi)_{\xi\in D} = (2(1-\kappa)\,\beta^{t(\xi)},\, 1,\, \kappa)_{\xi\in D}$. For these prices, default never occurs (as $q_\xi = \kappa < 2\kappa = p_\xi Y_\xi C_{\xi^-}$). Now, as the price of money is stationary and the price of the perishable good is falling at a rate equal to the discount factor, the decision on whether to consume the endowment shocks or save, will depend only on conditional probabilities. Since agents receive positive shocks in states that are assigned low probabilities, there is an incentive to use these positive shocks to buy money and sell it later in states with higher probabilities. Thus, we will look for an equilibrium where agent 1, the one that starts endowed with money, gets rid of it when going down (to which she attaches a higher conditional probability), but, if afterwards she goes up, she will use the positive perishable endowment shock to buy back money (and be back in a financial position analogous to the one she had at the original node). Clearly, each agent should end up consuming the other agent's positive shock.

Suppose that total consumption of agent $h$ is given by $(\hat{x}^h_\xi,\, \hat{y}^h_\xi) = (1+d^{h'}_\xi,\, \delta^{t(\xi)})$, where $h \neq h'$. As effective payments are $R_\xi = q_\xi$, budget feasibility implies that, for each $\xi \in D$, $z^h_\xi = 2\frac{1-\kappa}{\kappa}\,\beta^{t(\xi)}(d^h_\xi - d^{h'}_\xi) + z^h_{\xi^-}$, where $z^h_{\xi_0^-} := e^h$ and

$$d^h_\xi - d^{h'}_\xi = \begin{cases} \frac{\kappa\, e}{2(1-\kappa)}\beta^{-t(\xi)}\,, & \text{if } \left[\, h=1 \ \wedge\ \xi \in D^{du}\right] \ \vee\ \left[h=2 \ \wedge\ \xi \in \{\xi_0^d\}\cup D^{ud}\right]; \\ -\frac{\kappa\, e}{2(1-\kappa)}\beta^{-t(\xi)}\,, & \text{if } \left[\, h=1 \ \wedge\ \xi \in \{\xi_0^d\}\cup D^{ud}\right] \ \vee\ \left[h=2 \ \wedge\ \xi \in D^{du}\right]; \\ 0\,, & \text{in other case.} \end{cases}$$

Therefore, the *consumption allocations* $(\hat{x}^h_\xi,\, \hat{y}^h_\xi) = (1+d^{h'}_\xi,\, \delta^{t(\xi)})$, where $h \neq h'$, jointly with the *portfolios* $(\theta^1_{\xi_0},\, \theta^1_{\xi^u},\, \theta^1_{\xi^d}, \varphi^1_\xi) = (e,\, e,\, 0,\, 0)$ and $(\theta^2_\xi,\, \varphi^2_\xi) = (e - \theta^1_\xi,\, 0)$ are budget feasible, because there are no short sales, which assures that collateral constraints are never binding. Moreover, market clearing also holds. It remains to guarantee individual optimality.

For this purpose, it suffices (by Proposition 1) to find non-negative multipliers $(\gamma^h_\xi)_{\xi\in D}$ such that the following Euler equations,

$$\begin{aligned} \gamma^h_\xi p_{\xi,x} &= u'_{\xi,x}(1+d^{h'}_\xi,\, \delta^{t(\xi)}) \\ \gamma^h_\xi &= \kappa\left(\gamma^h_{\xi^u} + \gamma^h_{\xi^d}\right) + u'_{\xi,y}(1+d^{h'}_\xi,\, \delta^{t(\xi)}) \\ \gamma^h_\xi q_\xi &= \gamma^h_{\xi^u} q_{\xi^u} + \gamma^h_{\xi^d} q_{\xi^d} \end{aligned}$$

jointly with the transversality condition

$$\sum_{\eta\in D_T} \gamma^h_\eta \left(p_{\eta,x}\,\hat{x}^h_\eta + \hat{y}^h_\eta\right) + \sum_{\eta\in D_T} \gamma^h_\eta q_\eta \theta^h_\eta \quad \longrightarrow \quad 0, \quad \text{as } T \to +\infty$$

hold. However, as $u'_{\xi,x}(1+d^{h'}_\xi,\, \delta^{t(\xi)}) = \beta^{t(\xi)}\rho^h(\xi)$ and $u'_{\xi,y}(1+d^{h'}_\xi,\, \delta^{t(\xi)}) = \frac{\rho^h(\xi)}{2}$, if we chose $\gamma^h_\xi = \rho^h(\xi)\,\frac{1}{2(1-\kappa)}$, we guarantee Euler equations hold. It is easy (see Appendix C) to show that transversality conditions above hold for both agents. Thus, we have found an equilibrium.

In equilibrium, money has an unambiguous bubble. In fact, the price of money is always strictly less than the value of the depreciated collateral (default or the borderline case never occur) and collateral constraints never bind (which implies that shadow prices of financial constraints are zero). Finally, as expected (see Corollary 4.1), agents can not agree in not lending at infinity under a same process of multipliers $(\gamma_\xi^h)_{\xi\in D}$. For instance, agent $h=1$ is a lender at infinity when future is discounted using agent' 2 multipliers,

$$\begin{aligned}\sum_{\eta\in D_T}\gamma_\eta^2 q_\eta \theta_\eta^1 &= \kappa\, e \sum_{\{\eta\in D_T:\, \eta=(\eta^-)^u\}} \gamma_\eta^2 \\ &= \frac{\kappa\, e}{2(1-\kappa)}\left(1-\frac{1}{2^T}\right) \longrightarrow \frac{\kappa\, e}{2(1-\kappa)}>0, \quad \text{as } T\to+\infty. \qquad \square\end{aligned}$$

Notice that both agents perceive finite present values of aggregate wealth. In fact, aggregated endowments up to node $\xi$ are $W_\xi = W_\xi^1 + W_\xi^2 = (2+\sum_h d_\xi^h,\, 2\delta^{t(\xi)})$ and, therefore, given $\xi\in D$, for each $h\in H=\{1,2\}$,

$$\begin{aligned}\sum_{\mu\geq\xi}\frac{\gamma_\mu^h}{\gamma_\xi^h}p_\mu W_\mu &\leq \frac{\gamma_{\xi_0}^h}{\gamma_\xi^h}\sum_{\mu\in D}\frac{\gamma_\mu^h}{\gamma_{\xi_0}^h}p_\mu W_\mu \\ &= \frac{1}{\rho^h(\xi)}\left(4\frac{1-\kappa}{1-\beta}+2\frac{1}{1-\delta}+\kappa e\sum_{\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\}}\rho^h(\mu)\right),\end{aligned}$$

where

$$\begin{aligned}\sum_{\mu\in D^{ud}\cup D^{du}}\rho^h(\mu) &\leq \sum_{t\geq 0}\sum_{\mu\in D_{t+2}\cap(D^{ud}\cup D^{du})}\rho^h(\mu) \\ &= \sum_{t\geq0}\sum_{\mu\in D_t}\left(\frac{1}{2^{t+1}}\left(1-\frac{1}{2^{t+2}}\right)+\frac{1}{2^{t+2}}\left(1-\frac{1}{2^{t+1}}\right)\right)\rho^h(\mu) \\ &\leq \sum_{t\geq0}\frac{1}{2^t}\left(1-\frac{1}{2^{t+2}}\right)<\frac{5}{3}.\end{aligned}$$

We close this section with some comments relating the examples to the results by Santos and Woodford (1997).

1. In both examples, uniform impatience, defined by these authors as a joint requirement on endowments and preferences, fails. If endowments were uniformly bounded from below, then a recursive stationary utility would meet this requirement. This is not the case in the examples.[12] Given any consumption plan and any node, it is not possible to find a uniform coefficient $\sigma\in[0,1)$ such that: adding to current consumption the current aggregate resources can compensate for multiplying future consumption by $\sigma$ (as required in definition of uniformly patience given after

[12] As Levine and Zame (1996) point out, endowments can be made bounded from above or from below by re-scaling the variables, but then utility would no longer be stationary.

Assumption D). The coefficient $\sigma$ can not be uniformly (across all nodes) bounded away from one, as aggregate endowments tend to zero.[13]

It was already known (see Santos and Woodford (1997)) that, in the model without default and under market incompleteness, the lack of uniform impatience could create price bubbles for assets in positive net supply and for deflators with finite present value of aggregate wealth. Let us compare with what happens in both examples. In the first one, markets became complete but bubbles can not be ruled out for such a deflator, since the asset's positive net supply converges endogenously to zero. The second example may look less surprising, as it portrays an incomplete markets equilibrium in the absence of uniform impatience, but it should be noted that non-uniform impatience is allowed by the general conditions for existence of equilibrium in economies with secured assets, whereas this was not the case in the default-free world. It was only in the very special case that borrowing constraints forbade short-sales that examples could be computed, as Ponzi schemes were absent.

In Example 2, collateral constraints are not binding and one might try to infer, from the default-free results, that, in this case, the positive price of money is not robust to adding new assets (assuming the constraints would remain unbinding). However, this can not be infered. Let us recall first the argument for default-free economies. According to Theorem 3.1 in Santos and Woodford (1997), if the supremum, over all state-price processes, of the present values of aggregate wealth is finite, then, for any equilibrium, assets in positive net supply would be free of bubbles for one of these processes. The hypothesis would be satisfied by adding sufficiently productive assets, so that a portfolio could be found at the initial date which, if maintained throughout the event-tree (possibly allowing for exogenous asset conversion), would be capable of super-replicating aggregate endowments at every node (see Santos and Woodford (1997), Lemma 2.4). The thesis would allow, in the case of equilibrium without binding borrowing constraints (and hence a zero fundamental value), to infer that money should have a zero price.

In the collateral model, an analogous theorem could be established, but the hypothesis could not be guaranteed by adding assets, since the effective real returns from a date 0 portfolio are price-dependent (in other words, the conversion of the asset into its depreciated collateral is endogenous) and, once default occurs, future returns fall at the depreciation rate $\kappa$, whereas aggregate endowments of the durable good are given by $2\delta^t$, where $\delta \geq \kappa$. That is, super-replication of aggregate wealth is always default-dependent, depends on equilibrium prices and can not be guaranteed by assumption. Hence, once we allow for default, it is no longer possible to claim the existence of an asset return structure that can always super-replicate aggregate endowments, so that present values of aggregate wealth become finite for every non-arbitrage deflator. This opens more room for asset price bubbles. Moreover, bubbles may also occur in the prices of durable goods, under market incompleteness (as condition (17) fails in Theorem 2), and mortgage loans become subject to speculation, injected by speculation on the collateral (see Remark 4, Section 7).

[13]In Example 2 the aggregate resources at node $\xi$, at equilibrium prices, are just equal to $\sum_h W_\xi^h$, whereas in Example 1, aggregated resources at date $t$ are given by $2\delta^t = \sum_h W_t^h + \sum_{s=1}^t \kappa^{t-s}\,(\alpha\,\kappa)\;(1-\alpha)^{s-1}\sum_h e^h$, where $t = t(\xi) > 0$.

Notice that the robustness of the money bubble might depend instead on the relative spanning roles of money and the additional assets being added to the economy. If the degree of market incompleteness is not too small, money might still be essential.

2. Actually, the second example could be reformulated replacing the collateral constraint by a no short-sales restriction. The equilibrium that we computed is also an equilibrium for that reformulated example: the Euler and transversality conditions are still sufficient for individual optimality (since the Lagrangian function enjoys also the sign property discussed in Section 4 if short-sales are forbidden). This reformulated example illustrates the role of uniform impatience in Theorem 3.3 in Santos and Woodford (1997): without it bubbles occur for deflators with finite present value of aggregate wealth. Notice that short-sale constraints are binding and might remain so if other assets were added.

If this reformulated example were integrated in a traditional model where all the other assets could not default, we could make all deflators yield finite present values of aggregate wealth (by adding assets), but, if the short-sale constraints remained binding at some nodes, it would not be possible to claim that the price of money should become zero.

Suppose instead that the reformulated example were integrated in the collateral model, allowing for an *outside money* (high powered money endowed by some agent), that can not be short-sold, to coexist with collateralized assets (possibly including an inside money, in zero net supply, with collateral or reserve requirements in terms of outside money). Then, binding short-sale constraints and default by other assets would constitute two possible reasons for the non-fragility of the positive price of outside money (and bubbles could pass to assets secured by outside money, if those assets defaulted (see Section 7)).

## 9. Concluding Remarks

This paper shows that collateral allows for robust cases of price bubbles in assets in positive net supply. This class of assets is quite important, as it includes equity contracts and money. Our examples focus precisely on the latter and in a context without any liquidity frictions. Hence, in these examples, bubbles have the intriguing feature of assigning a positive price to an asset having no dividends and also providing no services. In our examples, there are no binding collateral constraints and the monetary equilibria are either (i) bubbles that can be reinterpreted as positive fundamental values due to collateral repossession or (ii) unambiguous money bubbles.

The former occur under an endogenous reduction in assets' net supply, as the collateral takes the place of the promise, and are a new instance for the long standing view on the efficiency properties resulting from a vanishing money supply (see Friedman (1969), Grandmont and Younés (1972, 1973) or Woodford (1994), among others). The latter are compatible with persistent money supply, but can only occur in the incomplete markets case, by taking advantage of the diversity in agents' personalized deflators.

## Appendix A

Proof of Claims in Remark 1. It follows from the definition of the sets $\Lambda_k$ that, $\dot{\bigcup}_{k\geq 0}\Lambda_k \subset J$. Thus, suppose that there exists $j \in J$ such that $j \notin \dot{\bigcup}_{k\geq 0}\Lambda_k$. In this situation, $j \notin \Lambda_0$ and, therefore, it follows from Assumption A, that there exists $j_1 \in J$ such that, for each node $\xi$, $(C^F_{\xi,j})_{j_1} \neq 0$. If $j_1 \in \dot{\bigcup}_{k\geq 0}\Lambda_k$ then asset $j$ is an element of $\dot{\bigcup}_{k\geq 0}\Lambda_k$ too, a contradiction. Hence, $\{j_1, j_2\} \notin \dot{\bigcup}_{k\geq 0}\Lambda_k$.

By analogous arguments we can find a sequence $\{j_1, j_2, \ldots\}$, such that $j_r \notin \dot{\bigcup}_{k\geq 0}\Lambda_k$ and $(C^F_{\xi_0,j_r})_{j_{r+1}} \neq 0$, for each $r \geq 1$. As $\#J < +\infty$, this contradicts Assumption B. Therefore, $J = \dot{\bigcup}_{k\geq 0}\Lambda_k$. Also, by definition, if $\Lambda_k = \emptyset$ then $\Lambda_{k+1} = \emptyset$. So, if $\Lambda_k \neq \emptyset$, then $\Lambda_{k-1} \neq \emptyset$, which implies that $\Lambda_0 \neq \emptyset$. □

Proof of Proposition 1. Let $\partial v^h_\xi(z_\xi)$ be the super-differential set of the function $v^h_\xi$ at the point $z_\xi$,

$$\partial v^h_\xi(z_\xi) := \left\{ v'_\xi \in \mathbb{Z} : \; v^h_\xi(y) - v^h_\xi(z_\xi) \leq v'_\xi \cdot (y - z_\xi), \;\; \forall\, y \in \mathbb{R}^L \right\}. \tag{A.1}$$

It follows that a vector $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \partial \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q)$ if and only there exists $v'_\xi \in \partial v^h_\xi(z_\xi)$ such that

$$\mathcal{L}'_{\xi,1} = v'_\xi - \gamma\, \nabla_1\, g^h_\xi(p,q) \tag{A.2}$$

$$\mathcal{L}'_{\xi,2} = -\gamma\, \nabla_2\, g^h_\xi(p,q), \tag{A.3}$$

where $\nabla g^h(p,q) := (\nabla_1 g^h_\xi(p,q),\, \nabla_2 g^h_\xi(p,q))$ denotes the gradient of the linear function $g^h_\xi$, i.e.

$$\nabla_1\, g^h_\xi(p,q) \;\; := \;\; (p_\xi,\, q_\xi,\, \mathrm{CV}_\xi(p,q) - q_\xi\,) \tag{A.4}$$

$$\nabla_2\, g^h_\xi(p,q) \;\; := \;\; -\,(\, p_\xi Y_\xi,\, D_\xi(p,q),\, \mathrm{DCV}_\xi(p,q) - D_\xi(p,q)) \tag{A.5}$$

As for each asset $j \in J$ short sales effective returns $D_{\xi,j}(p,q)$ are not greater than the respective garnishable collateral values, the joint returns from actions taken at immediately preceding nodes are non-negative and, therefore, the Lagrangian has the following *sign property*: Any vector $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \partial \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma; p, q)$ satisfies $\mathcal{L}'_{\xi,2} \geq 0$.

*(i) Necessity of transversality condition and Euler equations.* Suppose that the plan $(z^h_\xi)_{\xi\in D}$ is optimal for agent $h \in H$ at prices $(p,q)$. For each $T \in \mathbb{N}$, consider the truncated optimization problem ,

$$(P^{h,T}) \qquad \begin{array}{ll} \max & \sum\limits_{\xi\in D^T} u^h_\xi(x_\xi + C^P_\xi(p_\xi, q_{\xi,j})\varphi_\xi) \\ \text{s.t.} & \left\{ \begin{array}{lcl} g^h_\xi\left(z_\xi, z_{\xi^-}; p, q\right) & \leq & 0, \quad \forall \xi \in D^T, \\ z_\xi = (x_\xi, \theta_\xi, \varphi_\xi) & \geq & 0, \quad \forall \xi \in D^T. \end{array} \right. \end{array}$$

It follows from Assumption D and the optimality of $(z^h_\xi)_{\xi\in D}$ that each truncated problem has a solution $(z^{h,T}_\xi)_{\xi\in D^T}$. In fact, if for some $T \in \mathbb{N}$, the truncated problem $(\mathrm{P}^{h,T})$ does not have a solution, then there exists a sequence of plans $\left[(z_\xi(k))_{\xi\in D^T}\right]_{k\geq 1}$ such that

$$\sum_{\xi\in D^T} u^h_\xi(x_\xi(k) + C^P_\xi\, \varphi_\xi(k)) \rightarrow_{k\rightarrow+\infty} +\infty.$$

Thus, there is a $\overline{k}$ such that $U^h(\hat{x}^h) < \sum_{\xi\in D^T} u^h_\xi(x_\xi(\overline{k}) + C^P_\xi\, \varphi_\xi(\overline{k}))$, which contradicts the optimality of $(z^h_\xi)_{\xi\in D}$, because the plan $(\tilde{z}_\xi)_{\xi\in D}$ defined by $\tilde{z}_\xi = z_\xi(\overline{k})$, for each $\xi \in D^T$, and by $\tilde{z}_\xi = 0$ other wise, is budget feasible and improves the utility level of agent $h$.

Moreover, there exist non-negative multipliers $(\gamma^{h,T}_\xi)_{\xi\in D^T}$ such that, the following saddle point property is satisfied, for each nonnegative plan $(z_\xi)_{\xi\in D^T}$ (see Rockafellar (1997), Section 28, Theorem 28.3),

$$\sum_{\xi\in D^T} \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma^{h,T}_\xi; p, q) \leq \sum_{\xi\in D^T} \mathcal{L}^h_\xi(z^{h,T}_\xi, z^{h,T}_{\xi^-}, \gamma^{h,T}_\xi; p, q), \tag{A.6}$$

with $\gamma_\xi^{h,T} g_\xi^h(z_\xi^{h,T}, z_{\xi^-}^{h,T}; p, q) = 0$.

Claim A1. For each $\xi \in D$ and for all $T \geq t(\xi)$,

$$0 \leq \gamma_\xi^{h,T} \leq \frac{U^h(\hat{x}^h)}{\underline{W}_\xi^h \, ||p_\xi||_\Sigma}, \tag{A.7}$$

where, by Assumption C, $\underline{W}_\xi^h := \min_{l \in L} W_{\xi,l}^h > 0$, and $||p_\xi||_\Sigma > 0$, as a consequence of monotonicity of $u_\xi^h$ in the first coordinate. Therefore, for each $\xi \in D$, the sequence $(\gamma_\xi^{h,T})_{T \geq t(\xi)}$ is bounded.

Proof. Given $t \leq T$ and evaluating equation (A.6) in

$$z_\xi = \begin{cases} (W_\xi^h, 0, 0), & \forall \xi \in D^{t-1}, \\ (0, 0, 0), & \forall \xi \in D^T \setminus D^{t-1}, \end{cases}$$

it follows that,

$$\sum_{\xi \in D_t} \gamma_\xi^{h,T} p_\xi W_\xi^h \leq \sum_{\xi \in D^T} v_\xi^h(z_\xi^{h,T}) \leq U^h(\hat{x}^h).$$

As Assumption C guarantees that, for each $\xi \in D$, $\underline{W}_\xi^h := \min_{l \in L} W_{\xi,l}^h > 0$, the result follows. ⊠

Claim A2. For each $0 < t \leq T$,

$$0 \leq \sum_{\xi \in D_t} \gamma_\xi^{h,T} \nabla_2 g_\xi^h(p, q) \cdot z_{\xi^-}^h \leq \sum_{\xi \in D \setminus D^{t-1}} v_\xi^h(z_\xi^h), \tag{A.8}$$

Proof. First, it follows from equation (A.6) and the fact that $\gamma_\xi^{h,T} g_\xi^h(z_\xi^{h,T}, z_{\xi^-}^{h,T}; p, q) = 0$, for all $\xi \in D^T$, that, for each nonnegative allocation $(z_\xi)_{\xi \in D^T}$,

$$\sum_{\xi \in D^T} \mathcal{L}_\xi^h(z_\xi, z_{\xi^-}, \gamma_\xi^{h,T}; p, q) \leq U^h(\hat{x}^h). \tag{A.9}$$

Thus, given a period $t \leq T$, if we evaluate the equation above in

$$z_\xi = \begin{cases} z_\xi^h, & \forall \xi \in D^{t-1}, \\ 0, & \forall \xi \in D^T \setminus D^{t-1}, \end{cases}$$

by budget feasibility of allocation $(z_\xi^h)_{\xi \in D}$, we have

$$\sum_{\xi \in D_t} \gamma_\xi^{h,T} \nabla_2 g_\xi^h(p, q) \cdot z_{\xi^-}^h + \sum_{\xi \in D^T \setminus D^{t-1}} \gamma_\xi^{h,T} p_\xi w_\xi^h \leq \sum_{\xi \in D \setminus D^{t-1}} v_\xi^h(z_\xi^h)$$

which implies,

$$\sum_{\xi \in D_t} \gamma_\xi^{h,T} \nabla_2 g_\xi^h(p, q) \cdot z_{\xi^-}^h \leq \sum_{\xi \in D \setminus D^{t-1}} v_\xi^h(z_\xi^h).$$

This concludes the proof, because the left hand side term, in the inequality above, is non-negative. ⊠

Claim A3. For each $\xi \in D^T \setminus D_T$ and for any plan $y \geq 0$, we have

$$v_\xi^h(y) - v_\xi^h(z_\xi^h) \leq \left( \gamma_\xi^{h,T} \nabla_1 g_\xi^h(p, q) + \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} \nabla_2 g_\mu^h(p, q) \right) \cdot (y - z_\xi^h) + \sum_{\xi \in D \setminus D^T} v_\xi^h(z_\xi^h). \tag{A.10}$$

Proof. It follows from equation (A.9) that, for each $y \geq 0$, we can choose

$$z_\mu = \begin{cases} z_\mu^h, & \forall \mu \neq \xi, \\ y, & \text{for } \mu = \xi\,, \end{cases}$$

in order to guarantee that,

$$v_\xi^h(y) - \gamma_\xi^{h,T} g_\xi^h(y, z_{\xi^-}^h; p, q) - \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} g_\mu^h(z_\mu^h, y; p, q) \le v_\xi^h(z_\xi^h) + \sum_{\xi \in D \setminus D^T} v_\xi^h(z_\xi^h). \tag{A.11}$$

Now, as the function $g_\xi^h(\cdot; p, q)$ is affine and the plan $(z_\xi^h)_{\xi \in D} \in B^h(p, q)$, we have that,

$$\begin{aligned} g_\xi^h(y, z_{\xi^-}^h; p, q) &= \nabla_1 g_\xi^h(p, q) \cdot y - p_\xi w_\xi^h + \nabla_2 g_\xi^h(p, q) \cdot z_{\xi^-}^h \\ &\le \nabla_1 g_\xi^h(p, q) \cdot y - \nabla_1 g_\xi^h(p, q) \cdot z_\xi^h, \end{aligned}$$

and, for each node $\mu \in \xi^+$,

$$\begin{aligned} g_\xi^h(z_\mu^h, y; p, q) &= \nabla_1 g_\xi^h(p, q) \cdot z_\mu^h - p_\xi w_\mu^h + \nabla_2 g_\xi^h(p, q) \cdot y \\ &\le -\nabla_2 g_\xi^h(p, q) \cdot z_\xi^h + \nabla_2 g_\xi^h(p, q) \cdot y. \end{aligned}$$

Substituting the right hand side of equations above in equation (A.11) we conclude the proof. ⊠

The proof of the necessity of ($EE_1$)-($EE_2$) and (TC) is now a direct consequence of the claims above: Claim A1 guarantees that, node by node, the sequence $(\gamma_\xi^{h,T})_{T \ge t(\xi)}$ is bounded. Therefore, as the event-tree is countable, Tychonoff Theorem assures the existence of a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers $(\gamma_\xi^h)_{\xi \in D}$ such that, for each $\xi \in D$, $\gamma_\xi^{h,T_k} \to_{k \to +\infty} \gamma_\xi^h$, and

$$\gamma_\xi^h g_\xi^h(p, q, z_\xi^h, z_{\xi^-}^h) = 0\,; \tag{A.12}$$

$$\lim_{t \to +\infty} \sum_{\xi \in D_t} \gamma_\xi^h \nabla_2 g_\xi^h(p, q) \cdot z_{\xi^-}^h = 0\,, \tag{A.13}$$

where equation (A.12) follows from the strictly monotonicity of $u_\xi^h$ in the first commodity, and equation (A.13) is a consequence of Claim A2 (taking first, the limit as $T$ goes to infinity in equation (A.8) and, afterwards, the limit in $t$). Moreover, taking the limit as $T$ goes to infinity in (A.10) we obtain that, for each $y \ge 0$,

$$v_\xi^h(y) - v_\xi^h(z_\xi^h) \le \left( \gamma_\xi^h \nabla_1 g_\xi^h(p, q) + \sum_{\mu \in \xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p, q) \right) \cdot (y - z_\xi^h). \tag{A.14}$$

Therefore, $\gamma_\xi^h \nabla_1 g_\xi^h(p, q) + \sum_{\mu \in \xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p, q) \in \partial^+ v_\xi^h(z_\xi^h)$, where

$$\partial^+ v_\xi^h(z) := \{v'_\xi \in \mathbb{Z} : \ v_\xi^h(y) - v_\xi^h(z) \le v'_\xi \cdot (y - z), \ \ \forall y \ge 0\}. \tag{A.15}$$

That is, $\partial^+ v_\xi^h(\cdot)$ is the super-differential of the function $v_\xi^h(\cdot) + \delta(\cdot, \mathbb{R}_+^L)$, where $\delta(z, \mathbb{R}_+^L) = 0$, when $z \ge 0$ and $\delta(z, \mathbb{R}_+^L) = -\infty$, in other case. Notice that, for each $z \ge 0$, $\kappa \in \partial\delta(z) \Leftrightarrow 0 \le k(y - z), \ \forall y \ge 0$.

Now, by Theorem 23.8 in Rockafellar (1997), for all $z \ge 0$, if $v'_\xi \in \partial^+ v_\xi^h(z)$ then there exists $\tilde{v}'_\xi \in \partial v_\xi^h(z)$ such that both $v'_\xi \ge \tilde{v}'_\xi$ and $(v'_\xi - \tilde{v}'_\xi) \cdot z = 0$. Thus, it follows from equation (A.14) that there exists, for each $\xi \in D$, a super-gradient $\tilde{v}'_\xi \in \partial v_\xi^h(z_\xi^h)$ such that,

$$\begin{aligned} \gamma_\xi^h \nabla_1 g_\xi^h(p, q) + \sum_{\mu \in \xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p, q) &\ge \tilde{v}'_\xi, \\ \left( \gamma_\xi^h \nabla_1 g_\xi^h(p, q) + \sum_{\mu \in \xi^+} \gamma_\mu^h \nabla_2 g_\mu^h(p, q) \right) \cdot z_\xi^h &= \tilde{v}'_\xi \cdot z_\xi^h. \end{aligned}$$

Now, by definition, $\big(\tilde{v}'_\xi - \gamma_\xi^h \nabla_1 g_\xi^h(p, q), \ -\gamma_\xi^h \nabla_2 g_\xi^h(p, q)\big) \in \partial \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q)$. Therefore, there exists, for each $\xi \in D$, a vector $(\mathcal{L}'_{\xi,1}, \mathcal{L}'_{\xi,2}) \in \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q)$ of super-gradients, such that Euler conditions hold.

Furthermore, the transversality condition is a direct consequence of equation (A.13) jointly with Euler equations. In fact,

$$\sum_{\xi \in D_t} \gamma_\xi^h \nabla_2 g_\xi^h(p,q) \cdot z_{\xi^-}^h = -\sum_{\xi \in D_t} \mathcal{L}'_{\xi,2} \cdot z_{\xi^-}^h = \sum_{\xi \in D_{t-1}} \mathcal{L}'_{\xi,1} \cdot z_\xi^h,$$

and therefore (A.13) assures that

$$\lim_{t \to +\infty} \sum_{\xi \in D_t} \mathcal{L}'_{\xi,1} \cdot z_\xi^h = 0.$$

Finally, it follows from Euler equations, using the sign property of the Lagrangian, that $\tilde{v}'_\xi - \gamma_\xi^h \nabla_1 g_\xi^h(p,q) \leq 0$. As utility functions $u_\xi^h$ are strictly increasing in the first argument, we know that $\tilde{v}'_\xi$ has a strictly positive first coordinate. Thus, we have that $p_{\xi,1}\gamma_\xi^h > 0$, which implies that the multipliers $\gamma_\xi^h$ are strictly positive, for each $\xi \in D$.

*(ii) Sufficiency of transversality condition and Euler equations.* It follows from equations ($\text{EE}_1$) and ($\text{EE}_2$) that, for each $T \geq 0$,

(A.16) $$\sum_{\xi \in D^T} \mathcal{L}_\xi^h(z_\xi, z_{\xi^-}, \gamma_\xi^h; p, q) - \sum_{\xi \in D^T} \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q) \leq \sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot (z_\xi - z_\xi^h).$$

As, at each node $\xi \in D$ we have that $\gamma_\xi^h g_\xi(z_\xi^h, z_{\xi^-}^h; p, q) = 0$, each budget feasible allocation $(z_\xi)_{\xi \in D}$ must satisfy

$$\sum_{\xi \in D^T} u_\xi^h(\hat{x}_\xi) - \sum_{\xi \in D^T} u_\xi^h(\hat{x}_\xi^h) \leq \sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot (z_\xi - z_\xi^h).$$

Using the transversality condition (TC) we obtain that,

$$\limsup_{T \to +\infty} \sum_{\xi \in D^T} u_\xi^h(\hat{x}_\xi) - U^h(\hat{x}^h) \leq \limsup_{T \to +\infty} \sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot z_\xi.$$

Now, Euler conditions imply that

$$\sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot z_\xi = -\sum_{\mu \in D_{T+1}} \mathcal{L}'_{\mu,2} \cdot z_{\mu^-} \leq 0,$$

where the last inequality follows from the *sign property* $\mathcal{L}'_{\mu,2} \geq 0$, satisfied at each node of the event-tree. Finally, we have that

$$\limsup_{T \to +\infty} \sum_{\xi \in D^T} u_\xi^h(\hat{x}_\xi) = U^h(\hat{x}) \leq U^h(\hat{x}^h),$$

which guarantees that the allocation $(z_\xi^h)_{\xi \in D}$ is optimal.

*(iii) Summability of individual endowments.* As we pointed out in inequality (A.16), the existence of multipliers $(\gamma_\xi^h)_{\xi \in D}$ that satisfy Euler and transversality condition implies that,

$$\sum_{\xi \in D^T} \mathcal{L}_\xi^h(0, 0, \gamma_\xi^h; p, q) - \sum_{\xi \in D^T} \mathcal{L}_\xi^h(z_\xi^h, z_{\xi^-}^h, \gamma_\xi^h; p, q) \leq -\sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot z_\xi^h,$$

and, therefore,

$$\sum_{\xi \in D^T} \gamma_\xi^h p_\xi w_\xi^h \leq U^h(\hat{x}^h) - \sum_{\xi \in D_T} \mathcal{L}'_{\xi,1} \cdot z_\xi^h.$$

Using the transversality condition (TC), we have $\sum_{\xi \in D} \gamma_\xi^h p_\xi w_\xi^h < +\infty$. □

Proof of Claims after Proposition 1. Budget feasibility and Assumption D implies that

$$\lim_{T \to +\infty} \sum_{\mu \in D_T} \gamma_\mu^h \nabla_2 g_\mu^h(p,q) \cdot z_{\mu^-}^h = \lim_{T \to +\infty} \sum_{\mu \in D_T} \gamma_\mu^h \nabla_1 g_\mu^h(p,q) \cdot z_\mu^h - \lim_{T \to +\infty} \sum_{\mu \in D_T} \gamma_\mu^h p_\mu w_\mu^h.$$

Therefore, as deflated endowments are summable, using Euler conditions we assure that our transversality condition is equivalent to

$$\lim_{T\to+\infty}\sum_{\mu\in D_T}\gamma_\mu^h\nabla_1 g_\mu^h(p,q)\cdot z_\mu^h=0.$$

This concludes the proof. □

Proof of Corollary 2. It follows that there is a $\kappa\in(0,1)$ such that, for each node $\xi\in D$, $\kappa\mathbb{W}_\xi\le w_\xi^h$. Therefore,

$$0\le\sum_{\xi\in D_T}\gamma_\xi^h p_\xi\hat{x}_\xi^h\le\sum_{\xi\in D_T}\gamma_\xi^h p_\xi\mathbb{W}_\xi\le\frac{1}{\kappa}\sum_{\xi\in D_T}\gamma_\xi^h p_\xi w_\xi^h\to 0,\quad \text{as}\, T\to+\infty.$$

Since $\hat{x}_\xi^h=x_\xi^h+C_\xi^P\varphi_\xi^h$ and $\hat{\theta}_\xi^h=\theta_\xi^h+C_\xi^F\varphi_\xi^h$, it follows that the transversality conditions ($\text{TC}_x$), ($\text{TC}_\theta$) and ($\text{TC}_\varphi$) are equivalent to

$$\lim_{T\to+\infty}\sum_{\xi\in D_T}\gamma_\xi^h q_\xi\left(\hat{\theta}_\xi^h-\varphi_\xi^h\right)=0.$$

□

Proof of Proposition 2. As prices $(p,q)\in\mathbb{P}$ give a finite optimum to agents problems, if we denote by $(z_\xi^h)_{\xi\in D}$ the optimal plan of agent $h$ at prices $(p,q)$, it follows from Proposition 1 that there exist non-negative multipliers $(\gamma_\xi^h)_{\xi\in D}$ and super-gradients $(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})\in\partial\mathcal{L}_\xi^h(z_\xi^h,z_{\xi^-}^h,\gamma_\xi^h;p,q)$ which satisfy Euler conditions.

Now, as $(\mathcal{L}'_{\xi,1},\mathcal{L}'_{\xi,2})\in\partial\mathcal{L}_\xi^h(z_\xi^h,z_{\xi^-}^h,\gamma_\xi^h;p,q)$, there exists $v'_\xi\in\partial v_\xi^h(z_\xi^h)$ such that

$$\mathcal{L}'_{\xi,1}=v'_\xi-\gamma\,\nabla_1\,g_\xi^h(p,q) \tag{A.17}$$

$$\mathcal{L}'_{\xi,2}=-\gamma\,\nabla_2\,g_\xi^h(p,q). \tag{A.18}$$

Therefore, if we define the *non-negative* vector

$$(\eta_x(\xi),\eta_\theta(\xi),\eta_\varphi(\xi))=-\mathcal{L}'_{\xi,1}-\sum_{\mu\in\xi^+}\mathcal{L}'_{\mu,2}\in\mathbb{Z}, \tag{A.19}$$

using equations (A.17) and (A.18) and the fact that

$$v'_\xi\in\partial v_\xi^h(z_\xi^h)\Leftrightarrow\exists\alpha_\xi\in\partial u_\xi^h(\hat{x}_\xi^h):\, v'_\xi=\Big(\alpha_\xi,\,0,\,(\alpha_\xi C_{\xi,j}^P(p_\xi,q_{\xi,j}))_{j\in J}\Big),$$

we obtain pricing equations (7), (8) and (9), as the super gradients of $u_\xi^h$ are vectors with strictly positive entries.

Moreover, multiplying equation (7) by the physical collateral requirement of asset $j$ and adding equation (8) multiplied by the financial collateral requirement of asset $j$, we obtain that

$$\gamma_\xi\text{CV}_{\xi,j}(p,q)=\sum_{\mu\in\xi^+}\gamma_\mu\text{DCV}_{\mu,j}(p,q)+\alpha_\xi\cdot C_{\xi,j}^P(p,q)+\eta_x(\xi)C_{\xi,j}^P(p_\xi,q_{\xi,j})+\eta_\theta(\xi)C_{\xi,j}^F(q_\xi).$$

Thus, using equations (8) and (9) we have that

$$\eta_\varphi(\xi,j)=\eta_x(\xi)C_{\xi,j}^P(p_\xi,q_{\xi,j})+\eta_\theta(\xi)C_{\xi,j}^F(q_\xi)-\eta_\theta(\xi,j).$$

Therefore, if $j\in\Lambda_0$ and the shadow prices $\eta_x(\xi,l)$ are zero, for the commodities that serve as collateral for asset $j$, then $\eta_x(\xi)C_{\xi,j}^P(p_\xi,q_{\xi,j})=0$. Thus, the non-negativity of shadow prices implies that $(\eta_\theta(\xi,j),\eta_\varphi(\xi,j))=0$. Analogously, if $j\in\Lambda_k$, with $k>0$, and the shadow prices $\eta_x(\xi,l)$ of those commodities that serve as physical collateral (directly or indirectly, via other assets in previous layers of the pyramiding structure) are zero, then a recursive argument assures that $\eta_x(\xi)C_{\xi,j}^P(p_\xi,q_{\xi,j})=0$ and $\eta_\theta(\xi)C_{\xi,j}^F(q_\xi)=0$, which implies that $(\eta_\theta(\xi,j),\eta_\varphi(\xi,j))=0$. □

## Appendix B. Bubbles in Assets. The general case.

As mentioned in Section 7, the definition of the fundamental value of an asset depends not only on the process of valuation coefficients, but also on believed delivery rates if the borderline case occurs. Just as there is no reason to pick as a process of valuation coefficients the Kuhn-Tucker deflators of a particular agent, instead of any other compatible with non-arbitrage conditions, there is also no reason to pick the equilibrium delivery rates instead of any other rates that may treat differently the deliveries only in the borderline case. For purposes of valuation, anticipating that agents pay the promise or anticipating that they surrender the collateral (or a convex combination of the two) are equally perfectly acceptable when agents are indifferent between these two actions. Recall that each agent does not care about this choice and does not know what are the other agents' choices. Thus, given equilibrium prices $(p, q)$ we define fundamental values in terms of any believed delivery rates compatible with individual rationality, that is, any process $\tau = (\tau_{\xi,j}) \in [0,1]^{(D\setminus\{\xi_0\})\times J}$, such that,

$$\tau_{\xi,j} = \begin{cases} 1 & \text{if } p_\xi A(\xi,j) + q_{\xi,j} < \mathrm{DCV}_{\xi,j}(p,q)\,, \\ 0 & \text{if } p_\xi A(\xi,j) + q_{\xi,j} > \mathrm{DCV}_{\xi,j}(p,q). \end{cases}$$

Therefore, given delivery rates, the physical bundle, $PD_{\mu,j}(p,q,\tau)$, that one unit of asset $j$, that was negotiated at node $\xi$, delivers at a node $\mu \in \xi^+$ consists of the part of the promises $A_{\mu,j}$ that are effectively delivered and also of the physical deliveries made by the garnished collateral. More precisely,

$$\text{(B.1)} \qquad PD_{\mu,j}(p,q,\tau) = \tau_{\mu,j}A_{\mu,j} + (1-\tau_{\mu,j})\left( Y_\mu C^P_{\xi,j}(p_\xi, q_{\xi,j}) + \sum_{j'\neq j} PD_{\mu,j'}(p,q,\tau)\,(C^F_{\xi,j}(q_\xi))_{j'} \right),$$

Analogously, the financial (unitary) deliveries are,

$$\text{(B.2)} \qquad FD_{\mu,j}(p,q,\tau) = (1-\tau_{\mu,j}) \sum_{j'\neq j} \left(\tau_{\mu,j'}1_{j'} + FD_{\mu,j'}(p,q,\tau)\right)\,(C^F_{\xi,j}(q_\xi))_{j'},$$

where $1_{j'}$ denotes the portfolio composed only by one unit of asset $j'$. Thus, effective deliveries can be rewritten as, $D_{\mu,j}(p,q) = p_\mu PD_{\mu,j}(p,q,\tau) + q_\mu FD_{\mu,j}(p,q,\tau) + \tau_{\mu,j}q_{\mu,j}$. Therefore, using pricing equation (8) we obtain that,

$$\begin{aligned} q_{\xi,j} &= \sum_{\mu\in\xi^+} \frac{\gamma_\mu}{\gamma_\xi}\left(p_\mu PD_{\mu,j}(p,q,\tau) + q_\mu FD_{\mu,j}(p,q,\tau) + \tau_{\mu,j}q_{\mu,j}\right) + \frac{\eta_\theta(\xi,j)}{\gamma_\xi} \\ &= \sum_{\mu>\xi}\left(\prod_{\xi<\eta<\mu}\tau_{\eta,j}\right)\left[\frac{\gamma_\mu}{\gamma_\xi}\left(p_\mu PD_{\mu,j}(p,q,\tau) + q_\mu FD_{\mu,j}(p,q,\tau)\right) + \frac{\eta_\theta(\mu^-,j)}{\gamma_\xi}\right] \\ &\qquad + \lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)} q_{\mu,j}\prod_{\xi<\eta\leq\mu}\tau_{\eta,j}\,. \end{aligned}$$

As financial deliveries of an asset $j \in \Lambda_0$ are always zero, its fundamental value at node $\xi$ under $(\Gamma, \tau)$, which is the deflated value of their future yields and services, is given by

$$\text{(B.3)} \qquad F_j(\xi,p,q,\Gamma,\tau) = \sum_{\mu>\xi}\left(\prod_{\xi<\eta<\mu}\tau_{\eta,j}\right)\left[\frac{\gamma_\mu}{\gamma_\xi}\sum_{l\in L} F_l(\mu,p,q,\Gamma)\,PD_{\mu,j}(p,q,\tau)_l + \frac{\eta_\theta(\mu^-,j)}{\gamma_\xi}\right].$$

Note that, when agents anticipate that, in the borderline case, asset $j$ pays the original promises, equation above coincides with the definition given by (19). Moreover, as we can order assets in a pyramiding structure, we can define recursively the fundamental value at $\xi$, under $(\Gamma,\tau)$, of an asset $j\in\Lambda_k$ via

$$\begin{aligned}
&F_j(\xi,p,q,\Gamma,\tau)\\
&\quad=\sum_{\mu>\xi}\left(\prod_{\xi<\eta<\mu}\tau_{\eta,j}\right)\left[\frac{\gamma_\mu}{\gamma_\xi}\sum_{l\in L}F_l(\mu,p,q,\Gamma)\,PD_{\mu,j}(p,q,\tau)_l+\frac{\eta_\theta(\mu^-,j)}{\gamma_\xi}\right]\\
&\qquad+\sum_{\mu>\xi}\left(\prod_{\xi<\eta<\mu}\tau_{\eta,j}\right)\left[\frac{\gamma_\mu}{\gamma_\xi}\sum_{j'\in J}F_{j'}(\mu,p,q,\Gamma,\tau)\,FD_{\mu,j}(p,q,\tau)_{j'}\right].
\end{aligned}$$

It follows that the fundamental value at $\xi$ is always well defined and less than or equal to the asset price, $q_{\xi,j}$.

Definition 4. Given equilibrium prices $(p,q)\in\mathbb{P}$, we say that an asset $j\in\Lambda_k$ has a $(\Gamma,\tau)$-bubble at a node $\xi$ when $q_{\xi,j}>F_j(\xi,p,q,\Gamma,\tau)$.

Analogously to Theorem 4, given an asset $j\in\Lambda_k$, when commodities are free of $\Gamma$-bubbles and assets in previous layers do not have bubbles under $(\Gamma,\tau)$, the asset $j$' price coincides with the fundamental value if the future price is asymptotically zero,

Theorem 4-A. *Under Assumptions C and D, given equilibrium prices $(p,q)$, the following conditions are sufficient for the absence of a $(\Gamma,\tau)$-bubble in asset $j\in\Lambda_k$ at nodes in $D(\xi)$,*

$$p_{\xi,l}-F_l(\xi,p,q,\Gamma)=0,\quad \forall\, l\in L; \tag{B.4}$$

$$q_{\mu,j'}-F_{j'}(\mu,p,q,\Gamma,\tau)=0,\quad \forall\,\mu>\xi;\ \forall j'\in\Lambda_r,\ r<k\,; \tag{B.5}$$

$$\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\,q_{\mu,j}=0. \tag{B.6}$$

Proof. Given an asset $j\in\Lambda_k$, it follows from the definition of the $(\Gamma,\tau)$-fundamental-value and equation (B.3) that, when commodities, and assets in $\bigcup_{r<k}\Lambda_r$, are free of bubbles, we have $q_{\xi,j}=F_j(\xi,p,q,\Gamma,\tau)$ if and only if

$$\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\frac{\gamma_\mu}{\gamma_\xi}\,q_{\mu,j}\prod_{\xi<\eta\le\mu}\tau_{\eta,j}=0.$$

As condition (B.6) implies equation above, we conclude the proof. □

Thus, under bounded collateral requirements, bubbles can ruled out for assets in $\Lambda_k$, provided that the individual endowments of some agent are well behaved, in the following sense,

Corollary 4.3-A *Given equilibrium prices $(p,q)$, suppose that Assumptions C and D hold and that each asset in $\bigcup_{r\le k}\Lambda_r$ has uniformly bounded collateral requirements. Then, independently of the delivery rates $\tau$ that are chosen, each asset $j\in\Lambda_k$ is free of $(\Gamma,\tau)$-bubbles in $D(\xi)$ provided that, either*

a. *There exists an agent $h$ which has cumulated physical endowments that are uniformly bounded away from zero in the sub-tree $D(\xi)$, and satisfying also condition (17) under $\Gamma$ at $\xi$; or,*
b. *There is an agent $h$ that has new endowments, $(w^h_\mu)_{\mu\ge\xi}$, uniformly bounded away from zero, and there exists a matrix $\overline{Y}$ such that $Y_{\xi,\mu}\le\overline{Y}$, for each node $\mu>\xi$.*

Proof. The proof is analogous to the proof of Corollary 4.3, uses Theorem 4-A instead of Theorem 4, but requires a recursive application of the arguments, layer by layer in the pyramid structure of assets. □

## Appendix C. Transversality conditions of Example 2.

Transversality condition in autonomous long positions,

$$\begin{aligned}
\sum_{\eta\in D_T} \gamma^1_\eta q_\eta \theta^1_\eta &= \kappa\, e \,\frac{1}{2(1-\kappa)} \sum_{\{\eta\in D_T:\, \eta=(\eta^-)^u\}} \rho^1(\eta)\\
&= \kappa\, e\,\frac{1}{2(1-\kappa)}\frac{1}{2^T} \longrightarrow 0;\\
\sum_{\eta\in D_T} \gamma^2_\eta q_\eta \theta^2_\eta &= \kappa\, e \,\frac{1}{2(1-\kappa)} \sum_{\{\eta\in D_T:\, \eta=(\eta^-)^d\}} \rho^2(\eta)\\
&= \kappa\, e\,\frac{1}{2(1-\kappa)}\frac{1}{2^T} \longrightarrow 0;
\end{aligned}$$

Tranversality condition in consumption,

$$\begin{aligned}
\sum_{\eta\in D_T} \gamma^h_\eta\, p_{\eta,x}\, \hat{x}^h_\eta &= \beta^T \sum_{\eta\in D_T} \rho^h(\eta)(1+d^{h'}_\eta)\\
&= \beta^T + \frac{\kappa\, e}{2(1-\kappa)} \sum_{\{\eta\in D_T:\, d^{h'}_\eta\neq 0\}} \rho^h(\eta)\\
&\leq \beta^T + \frac{\kappa\, e}{2(1-\kappa)}\frac{1}{2^{T-1}}\left(1-\frac{1}{2^T}\right)\sum_{\eta\in D_{T-2}} \rho^h(\eta) \longrightarrow 0\\
\sum_{\eta\in D_T} \gamma^h_\eta\, p_{\eta,y} \hat{y}^h_\eta &= \delta^T \frac{1}{2(1-\kappa)} \longrightarrow 0
\end{aligned}$$

## Appendix D. Proof of Equilibrium Existence Theorem

As in Araujo, Páscoa and Torres-Martinez (2002), we prove first that there exists equilibria in finite horizon economies. Then, an equilibrium for the infinite horizon economy will be found as a limit of equilibria of truncated economies, when the time horizon goes to infinity.

**Equilibria in truncated economies.** Define, for each $T\in\mathbb{N}$, a truncated economy, $\mathcal{E}^T$, with $T+1$ dates, in which agents are restricted to consume and trade assets in the event-tree $D^T$. Thus, using the notation of Section 3 and given prices $(p,q)\in\mathbb{P}^T$, where

$$\mathbb{P}^T := \{(p,q)=(p_\xi,q_\xi)_{\xi\in D^T}\in(\mathbb{R}^L_+\times\mathbb{R}^J_+)^{D^T} \,:\, ||p_\xi||_\Sigma + ||q_\xi||_\Sigma = 1, \quad \forall \xi\in D^T\},$$

each agent $h\in H$ has the objective to chose, at each node $\xi\in D^T$, a vector $z^{h,T}_\xi=(x^{h,T}_\xi,\theta^{h,T}_\xi,\varphi^{h,T}_\xi)\in\mathbb{Z}$ in order to solve the (truncated) individual problem,

$$(P^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi\in D^T} v^h_\xi(z_\xi)\\ \text{s.t.} & \left\{\begin{array}{rcl} g^h_\xi\left(z_\xi, z_{\xi^-}; p,q\right) &\leq& 0, \quad \forall \xi\in D^T,\\ z_\xi=(x_\xi,\theta_\xi,\varphi_\xi) &\geq& 0, \quad \forall\xi\in D^T.\end{array}\right.\end{array}$$

where, as defined early, given $z_\xi=(x_\xi,\theta_\xi,\varphi_\xi)$, the function $v^h_\xi(z_\xi)=u^h_\xi(x_\xi+\sum_{j\in J} C^P_{\xi,j}(p_\xi,q_{\xi,j})\,\varphi_{\xi,j})$.

Now, let $B^{h,T}(p,q)$ be the truncated budget set of agent $h$ in $\mathcal{E}^T$. That is, the set of plans $(z_\xi)_{\xi\in D^T}$ that satisfy the restrictions of problem $P^{h,T}$ above.

An *equilibrium* for the truncated economy $\mathcal{E}^T$ is given by a vector of prices $(p^T,q^T)\in\mathbb{P}^T$ jointly with delivery rates $\lambda^T_\xi=(\lambda^T_{\xi,j})$, for each node $\xi\in D^T\setminus\{\xi_0\}$, and individual plans $z^{h,T}_\xi=(x^{h,T}_\xi,\theta^{h,T}_\xi,\varphi^{h,T}_\xi)_{\xi\in D^T}$ such that: (1) $z^{h,T}$ is an optimal solution for $P^{h,T}$, at prices $(p^T,q^T)$; (2) Physical and financial market clear, in the sense of Definition 1, at each node $\xi\in D^T$ (i.e. condition C and D hold); and (3) for each node

$\xi \in D^T \setminus \{\xi_0\}$, delivery rates $\lambda^T_{\xi,j}$ satisfy condition B of Definition 1.

As a first step, let us restrict the space of consumption and portfolios allocations in $\mathcal{E}^T$ to a compact set. Assumption A assures that collateral requirements are different from zero (as vectors), which implies that, for each pair $(\xi, j) \in D^T \times J$,

$$\left(\underline{c}^P_{\xi,j},\, \underline{c}^F_{\xi,j}\right) := \left(\min_{(p,q)\in\mathbb{P}} ||C^P_{\xi,j}(p_\xi, q_{\xi,j})||_\Sigma,\ \min_{(p,q)\in\mathbb{P}} ||C^P_{\xi,j}(q_\xi)||_\Sigma\right) \in \mathbb{R}^2_+ \setminus \{0\}.$$

Moreover, Assumptions A and B guarantee that assets can be ordered in a pyramiding structure, whose basic layer (composed by assets backed only by physical collateral) is non-empty (see Remark 1). Thus, the *market feasible allocations*, that is, the non-negative allocations $(x^h_\xi, \theta^h_\xi, \varphi^h_\xi)_{(h,\xi)\in H\times D^T}$ that satisfy market clearing conditions $C$ and $D$ of Definition 1, are bounded in $D^T$.

In fact, autonomous consumption allocations, $(x^h_\xi)_{(h,\xi)\in H\times D^T}$ are bounded by above, node by node, by the aggregated physical endowments. The short-sales, $(\varphi^h_{\xi,j})_{(h,\xi)\in H\times D^T}$, of an asset $j$ that is backed by physical collateral are bounded, at each node $\xi \in D^T$, by $\sum_{l\in L} \mathbb{W}_{\xi,l}$ divided by the minimum quantity of (non-zero) collateral, $\underline{c}^P_{\xi,j}$. Thus, the autonomous long positions $(\theta^h_{\xi,j})_{(h,\xi)\in H\times D^T}$, of an asset $j$ with non-zero physical collateral, are also bounded, because are less than or equal to the aggregate short sales plus the initial positive net supply. Finally, the short and (autonomous) long positions of assets that are backed only by financial collateral are bounded by above too. It is sufficient to apply the previous arguments recursively along the different layers in which we divided the set $J$.

Therefore, we can restrict, without loss of generality, the space of plans of each agent $h \in H$ to the convex and compact set $K^T := \{z = (x, \theta, \varphi) \in \mathbb{R}^{L\times D^T}_+ \times \mathbb{R}^{J\times D^T}_+ \times \mathbb{R}^{J\times D^T}_+ : ||z||_\Sigma \leq 2\Upsilon^T\}$, which has in its interior the vector $\Upsilon^T$ of upper bounds for the feasible allocations in $D^T$.

Note that, if we assure the existence of equilibrium for the compact (and truncated) economy, denoted by $\mathcal{E}^T(K^T)$, the finite horizon economy $\mathcal{E}^T$ has also an equilibrium, given that optimal allocation of $\mathcal{E}^T(K^T)$ will be interior points of set $K^T$, budget sets are convex and utility functions are concave (see Claim B3 below).

It follows that, in order to find an equilibrium for the truncated economy $\mathcal{E}^T$ it is sufficient to find an equilibrium for $\mathcal{E}(K^T)$: prices $(p^T, q^T) \in \mathbb{P}^T$, delivery rates $\lambda^T_\xi = (\lambda^T_{\xi,j})$, for each node $\xi \in D^T \setminus \{\xi_0\}$, and allocations $(z^{h,T}_\xi)_{\xi\in D^T} = (x^{h,T}_\xi, \theta^{h,T}_\xi, \varphi^{h,T}_\xi)_{\xi\in D^T}$, compatible with conditions B, C and D of Definition 1, such that, for each agent $h$, the plan $(z^{h,T}_\xi)_{\xi\in D^T}$ solves,

$$(P^{h,T}(K^T)) \qquad \begin{array}{ll} \max & \sum_{\xi\in D^T} v^h_\xi(z_\xi) \\ \text{s.t.} & \begin{cases} g^h_\xi\left(z_\xi, z_{\xi^-}; p^T, q^T\right) \leq 0, & \forall \xi \in D^T, \\ (z_\xi)_{\xi\in D^T} = (x_\xi, \theta_\xi, \varphi_\xi)_{\xi\in D^T} \in K^T. \end{cases} \end{array}$$

*Generalized Games.* In order to prove the existence of equilibrium in the *compact economy*, we use a generalized game. Before describing the game, we define, for each pair $(\xi, j)$ in $D^T \times J$ the following quantities,

$$\text{(D.1)} \qquad \overline{c}^P_{\xi,j} = \max_{(p,q)\in\mathbb{P}} ||C^P_{\xi,j}(p_\xi, q_{\xi,j})||_\Sigma$$

$$\text{(D.2)} \qquad \overline{c}^F_{\xi,j} = \max_{(p,q)\in\mathbb{P}} ||C^P_{\xi,j}(q_\xi)||_\Sigma.$$

Now, in the generalized game, that will be denoted by $\mathcal{G}^T$, each consumer $h \in H$ takes prices $(p, q) \in \mathbb{P}^T$ as given and solves the compact truncated problem above. Moreover, associated with each pair $(\xi, j) \in D^T \times J$, there are two fictitious players. Given prices, one of these players has the objective to find collateral bundles

$$(M^P_{\xi,j}, M^F_{\xi,j})_{j\in J} \in K(\xi,j) := \{(M_1,M_2)\in \mathbb{R}^L_+ \times \mathbb{R}^J_+ : (||M_1||_\Sigma, ||M_2||_\Sigma) \leq (\overline{c}^P_{\xi,j}, \overline{c}^F_{\xi,j})\},$$

in order to solve,

$$\text{(D.3)} \qquad \min_{(M^P,M^F)\in K(\xi,j)} \left( || M^P_j - C^P_{\xi,j}(p_\xi, q_{\xi,j})||^2 + || M^F_j - C^F_{\xi,j}(q_\xi)||^2 \right),$$

The other player, given prices $(p,q)\in\mathbb{P}^T$, will be choose a real number $\lambda_{\xi,j}\in[0,1]$ in order to solve the problem,

$$\text{(D.4)} \qquad \min_{\lambda\in[0,1]} \left[\lambda(p_\xi A(\xi,j) + q_{\xi,j}) + (1-\lambda)\mathrm{DCV}_{\xi,j}(p,q)\right],$$

where, for each $j\in J$ and $(p,q)\in\mathbb{P}^T$, the vector $(A(\xi_0,j),\mathrm{DCV}_{\xi_0,j}(p,q)) := (0,2)$, which implies that $\lambda_{\xi_0,j} = 1$, for each $j\in J$. Finally, associated to each node in $D^T$ there is an auctioneer who, given plans $(z^h_\xi)_{(h,\xi)\in H\times D^T} \in \prod_{h\in H} K^T$, collateral bundles $(M^P_{\xi,j}, M^F_{\xi,j})_{j\in J} \in \prod_{j\in J} K(\xi,j)$ and delivery rates $\lambda_{\xi,j}\in[0,1]$ has the objective to find prices $(p_\xi,q_\xi)\in\Delta^{L+J-1}_+$ in order to maximize the function,

$$\text{(D.5)} \quad p_\xi \sum_{h\in H}\left( x^h_\xi + \sum_{j\in J} M^P_{\xi,j}\varphi^h_{\xi,j} - w^h_\xi - Y_\xi x^h_{\xi^-} - Y_\xi \sum_{j\in J} M^P_{\xi^-,j}\varphi^h_{\xi^-,j}\right)$$

$$+\sum_{j\in J} q_{\xi,j}\sum_{h\in H}\left(\theta^h_{\xi,j} + \sum_{j'\in J}(M^F_{\xi,j'})_j\,\varphi^h_{\xi,j'} - \varphi^h_{\xi,j} - \lambda_{\xi,j}\left(\theta^h_{\xi^-,j} + \sum_{j'\in J}(M^F_{\xi^-,j'})_j\,\varphi^h_{\xi^-,j'} - \varphi^h_{\xi^-,j}\right)\right)$$

$$-p_\xi \sum_{(h,j)\in H\times J}\left(\lambda_{\xi,j}A(\xi,j) + (1-\lambda_{\xi,j})Y_\xi M^P_{\xi^-,j}\right)\left(\theta^h_{\xi^-,j} + \sum_{j'\in J}(M^F_{\xi^-,j'})_j\,\varphi^h_{\xi^-,j'} - \varphi^h_{\xi^-,j}\right),$$

where, $z^h_\xi = (x^h_\xi,\theta^h_\xi,\varphi^h_\xi)$ and, for convenience of notations, for each $(h,j)\in H\times J$ we put $(x^h_{\xi_0^-},\theta^h_{\xi_0^-,j},\varphi^h_{\xi_0^-,j}) = (0,e^h_j,0)$ and $(M^P_{\xi_0^-}, M^F_{\xi_0^-}, Y_{\xi_0}) = (0,0,0)$.

An equilibrium for $\mathcal{G}^T$ is given by a vector $\left[(p^T,q^T);(z^{h,T}_\xi)_{h\in H};\lambda^T_\xi,(M^{P,T}_{\xi,j},M^{F,T}_{\xi,j})_{j\in J}\right]_{\xi\in D^T}$ that solves simultaneously the problems above. That is, (1) given the prices $(p^T,q^T)\in\mathbb{P}^T$, the plan $(z^{h,T}_\xi)_{\xi\in D^T}$ is a solution of $P^{h,T}(K^T)$, (2) given the individual plans $(z^{h,T})_{h\in H}$, and the collateral bundles $(M^{P,T},M^{F,T})$, for each node $\xi\in D^T$, the auctioneer which chooses prices solves (D.5), and (3) given prices, each player whose task is to choose collateral requirements or payment rates is also optimizing.

Lemma D1. For each $T\in\mathbb{N}$ there is an equilibrium for the generalized game $\mathcal{G}^T$.

Proof. The objective function of each participant in the game is continuous and quasi-concave in the own strategy. For fictitious players and auctioneers, the correspondences of admissible strategies are continuous, with non-empty, convex and compact values. Also, the budget restriction correspondence of each agent, $(p,q) \twoheadrightarrow B^{h,T}(p,q)\cap K^T$, has non-empty, convex and compact values. Therefore, in order to find an equilibrium of the generalized game (as a fixed point of the set function given by the product of optimal strategies correspondences), it is sufficient to prove that budget set correspondences are continuous.

The upper hemi-continuity follows from compact values and closed graph properties, that are a direct consequence of continuity of functions $g^h_\xi$. Thus, as is usual in general equilibrium, the main difficulty resides in showing the lower hemi-continuity property. Now, as for each price $(p,q)\in\mathbb{P}^T$ the set $B^{h,T}(p,q)\cap K^T$ is convex and compact, it is sufficient to assure that the (relative) interior correspondence $(p,q)\twoheadrightarrow int(B^{h,T}(p,q))\cap K^T$ has non-empty values. But this last property follows from Assumption C.

In fact, cumulated endowments are such that $W_\xi^h \gg 0$, for each agent $h \in H$, and, therefore, given any plan of prices $(p,q) \in \mathbb{P}^T$, the plan

$$(x_\xi;\theta_\xi;\varphi_\xi) = \left( \frac{W_\xi^h}{2^{t(\xi)+1}} - \sum_{j\in J} C_{\xi,j}^P(p_\xi,q_{\xi,j})\epsilon_\xi;\ 0;\ \epsilon_\xi(1,1,\ldots,1) \right),$$

where

$$\epsilon_\xi = \frac{1}{2}\min_{l\in L} \left\{ \frac{W_{\xi,l}^h}{2^{t(\xi)+1}\left(1+\sum_{j\in J} C_{\xi,j}^P(p_\xi,q_{\xi,j})_l\right)} ;\ \frac{w_{\xi,l}^h}{1+\mathrm{DCV}_{\xi,j}(p,q)} \right\}$$

is budget feasible and belongs to the relative interior of the set $B^{h,T}(p,q)\cap K$. □

Lemma D2. For each $T \in \mathbb{N}$ there is an equilibrium for $\mathcal{E}^T(K^T)$.

Proof. The previous claim guarantees that there exists an equilibrium for the generalized game $\mathcal{G}^T$, which will be denoted by $\left[(p^T,q^T);(z_\xi^{h,T})_{h\in H};\lambda_\xi^T,(M_{\xi,j}^{P,T},M_{\xi,j}^{F,T})_{j\in J}\right]_{\xi\in D^T}$. By definition, the payment rates $\lambda_{\xi,j}^{h,T}$ satisfy condition B of Definition 1 and each agent $h\in H$ solves problem $P^{h,T}(K^T)$ by choosing the plan $(z_\xi^{h,T})_{\xi\in D^T}$. Thus, it is sufficient to verify, for each node $\xi\in D^T$, the validity of conditions C and D of Definition 1.

Now, it follows from players' objective functions that, for each $(\xi,j)\in D^T\times J$, the collateral bundles satisfy $M_{\xi,j}^{P,T} = C_{\xi,j}^P(p_\xi^T,q_{\xi,j}^T)$, $M_{\xi,j}^{F,T} = C_{\xi,j}^F(q_\xi^T)$ and, for each node $\xi>\xi_0$, the effective payments satisfy,

$$D_{\xi,j}(p^T,q^T) = \lambda_{\xi,j}^T(p_\xi^T A(\xi,j)+q_{\xi,j}^T) + (1-\lambda_{\xi,j}^T)\mathrm{DCV}_{\xi,j}(p^T,q^T).$$

Therefore, as

$$\sum_{h\in H} g_\xi^h(z_\xi^{h,T},z_{\xi^-}^{h,T},p^T,q^T) \le 0, \tag{D.6}$$

the optimal value of auctioneers objective functions is less than or equal to zero. This implies that conditions C and D of Definition 1 are satisfied as inequalities. That is, there does not exist excess demand in physical and financial markets.

Thus, as the individual demands for commodities or assets are bounded by the aggregate supply of resources, the optimal bundles that were chosen by the agents are interior points of $K^T$. Therefore, monotonicity of utility function implies that inequality (D.6) holds as equality, which implies that Walras' law is satisfied, at each node in $D^T$.

The existence of an optimal solution for problem $P^{h,T}(K^T)$ in the interior of the set $K^T$ implies that $p_\xi^T \gg 0$ and, therefore, condition C of Definition 1 holds, as a direct consequence of Walras' law, strictly positive commodity prices and the absence of excess demand in physical markets. By analogous arguments, condition D of Definition 1 holds, at a node $\xi\in D^T$, for those assets $j\in J$ which have a strictly positive price $q_{\xi,j}^T>0$.

Finally, given a node $\xi\in D^T$, denote by $\tilde{J}_\xi \subset J$ the set of assets that have zero price at $\xi$, i.e. $q_{\xi,j}^T=0$ and let $\Delta(\theta_\xi^T,\theta_{\xi^-}^T,p^T,q^T,\lambda^T)_{\xi,j}$ be the excess demand, associated with long positions $(\theta_\xi^T,\theta_{\xi^-}^T) = (\theta_\xi^{h,T},\theta_{\xi^-}^{h,T})_{h\in H}$, of asset $j$ at node $\xi$ (it follows from previous arguments that $\Delta(\theta_\xi^T,\theta_{\xi^-}^T,p^T,q^T,\lambda^T)_{\xi,j}\le 0$).

Now, if $j\in\tilde{J}_\xi$, then optimality of agents' allocations assures that the asset does not deliver any payment at the successor nodes $\mu\in\xi^+$ (if this nodes are in $D^T$). Therefore, if we change the portfolio allocation $(\theta_\xi^{h,T})_{h\in H}$ to

$$\tilde{\theta}_\xi^{h,T} = \theta_\xi^{h,T} - \frac{1}{\#H}\Delta(\theta_\xi^T,\theta_{\xi^-}^T,p^T,q^T,\lambda^T)_{\xi,j},$$

we assure that, at node $\xi$, and for asset $j$, condition D holds. Moreover, the new allocation is budget feasible, optimal and we do not lose the market clearing condition in physical markets at node $\mu \in \xi^+$, as asset $j$ does not deliver any payment at these nodes.

However, the total supply of asset $j$ at nodes $\mu \in \xi^+$ can change and, therefore, in order to apply the trick above, node by node, asset by asset, to obtain an optimal allocation that satisfies Condition D for each asset, it is sufficient to prove that, after changing portfolios at a node $\xi$, the new excess demand, at nodes $\mu \in \xi^+$, $\Delta(\theta^T_\mu, \tilde{\theta}^T_\xi, p^T, q^T, \lambda^T)_{\mu,j}$ is still less than or equal to zero and $\Delta(\theta^T_\mu, \tilde{\theta}^T_\xi, p^T, q^T, \lambda^T)_{\mu,j} < 0$ only for assets in $\tilde{J}_\mu$.

In general, it follows by the definition of $\tilde{\theta}^{h,T}_\xi$ that $\Delta(\theta^T_\mu, \tilde{\theta}^T_\xi, p^T, q^T, \lambda^T)_{\mu,j} \leq \Delta(\theta^T_\mu, \theta^T_\xi, p^T, q^T, \lambda^T)_{\mu,j}$. Now, as at each $\mu \in \xi^+$, $D_{\xi,j}(p^T, q^T) = 0$ if $q^T_{\mu,j} > 0$ then asset $j$ defaults at node $\mu$, which implies that $\lambda^T_{\mu,j} = 0$. Thus, $\Delta(\theta^T_\mu, \tilde{\theta}^T_\xi, p^T, q^T, \lambda^T)_{\mu,j} = \Delta(\theta^T_\mu, \theta^T_\xi, p^T, q^T, \lambda^T)_{\mu,j}$, which concludes the proof. □

Lemma D3. For each $T \in \mathbb{N}$, there is an equilibrium for $\mathcal{E}^T$.

Proof. In the previous claim we found an equilibrium for the compact economy $\mathcal{E}^T(K^T)$. We affirm that this equilibrium constitutes also an equilibrium for $\mathcal{E}^T$. As feasibility conditions are satisfied, it is sufficient to prove that, for each agent $h \in H$, the plan $\tilde{z}^{h,T}$, which is a solution of $P^{h,T}(K^T)$, solves problem $P^{h,T}$.

Now, if the plan $\tilde{z}^{h,T}$ is not optimal in $P^{h,T}$ then there is another plan $z \in B^{h,T}(p^T, q^T)$ such that $\sum_{\xi \in D^T} v^h_\xi(z_\xi) > \sum_{\xi \in D^T} v^h_\xi(\tilde{z}^{h,T}_\xi)$. Thus, as $B^{h,T}(p^T, q^T)$ is a convex set and $\tilde{z}^{h,T} \in B^{h,T}(p^T, q^T)$, for each $\pi \in [0,1)$ the plan $z^\pi = (z^\pi_\xi)_{\xi \in D^T}$ defined by $z^\pi_\xi = \pi \tilde{z}^{h,T}_\xi + (1-\pi) z_\xi$ is also budget feasible and satisfies

$$\sum_{\xi \in D^T} v^h_\xi(z^\pi_\xi) > \sum_{\xi \in D^T} v^h_\xi(\tilde{z}^{h,T}_\xi),$$

which is a consequence of concavity of functions $v^h_\xi$. Therefore, independently of the value of $\pi \in [0,1)$, the plan $z^\pi \notin K^T$, which contradicts the fact that $\tilde{z}^{h,T} \in int\,(K^T)$. □

**Asymptotic equilibria.** For each $T \in \mathbb{N}$, fix an equilibrium $\left[(p^T, q^T); (z^{h,T}_\xi)_{h \in H}; \lambda^T_\xi\right]_{\xi \in D^T}$ of $\mathcal{E}^T$. We know that there exist non-negative multipliers $(\gamma^{h,T}_\xi)_{\xi \in D^T}$ such that, $\gamma^{h,T}_\xi g^h_\xi(z^{h,T}_\xi, z^{h,T}_{\xi^-}; p, q) = 0$, and the following saddle point property is satisfied, for each nonnegative plan $(z_\xi)_{\xi \in D^T}$ (see Rockafellar (1997), Section 28, Theorem 28.3),

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(z_\xi, z_{\xi^-}, \gamma^{h,T}_\xi; p^T, q^T) \leq \sum_{\xi \in D^T} \mathcal{L}^h_\xi(z^{h,T}_\xi, z^{h,T}_{\xi^-}, \gamma^{h,T}_\xi; p^T, q^T). \tag{D.7}$$

Thus, analogously to Claim A1 in Appendix A, it follows that, for each $\xi \in D$ and for all $T \geq t(\xi)$,

$$0 \leq \gamma^{h,T}_\xi \leq \frac{\sum_{\xi \in D^T} v^h_\xi(z^{h,T}_\xi)}{\underline{W}^h_\xi \, ||p^T_\xi||_\Sigma}, \tag{D.8}$$

where $\underline{W}^h_\xi = \min_{l \in L} W^h_{\xi,l} > 0$. As $v^h_\xi(z^{h,T}_\xi) \leq u^h_\xi(\mathbb{W}_\xi)$ it follows that,

$$0 \leq \gamma^{h,T}_\xi < \frac{U^h(\mathbb{W})}{\underline{W}^h_\xi \, ||p^T_\xi||_\Sigma}. \tag{D.9}$$

Lemma D4. For each node $\xi \in D$, there is a strictly positive lower bound for the sequence $(||p^T_\xi||_\Sigma)_{T > t(\xi)}$.

Proof. Given $\xi \in D$ and $T > t(\xi)$, optimality of $z^{h,T}$ in $P^{h,T}$ implies that $\mathrm{CV}_{\xi,j}(p^T, q^T) \geq q^T_{\xi,j}$, for each $j \in J$ (by the same argument as in the proof of Proposition 2, in Appendix A). Thus, given an asset $j \in \Lambda_0$, we have that $q^T_{\xi,j} \leq p^T_\xi C^P_{\xi,j}(p^T, q^T_{\xi,j}) \leq \overline{c}^P_{\xi,j}\,||p^T_\xi||_\Sigma$. Analogously, for an asset $j \in \Lambda_k$, with $k > 0$, we have

$$q^T_{\xi,j} \leq \overline{c}^P_{\xi,j}\,||p^T_\xi||_\Sigma + \overline{c}^F_{\xi,j} \sum_{j' \in \Lambda_r,\, r<k} q^T_{\xi,j'}.$$

Thus, it follows from our pyramiding structure of asset that, for each $j \in J$, there is $\overline{m}_{\xi,j} > 0$ such that, $q^T_{\xi,j} \leq \overline{m}_{\xi,j}||p^T_\xi||_\Sigma$. Thus, adding in $j$, we obtain that $||q^T_\xi||_\Sigma \leq ||p^T_\xi||_\Sigma \sum_{\mathrm{J}\in J} \overline{m}_{\xi,j}$. Finally, as $||q^T_\xi||_\Sigma = 1 - ||p^T_\xi||_\Sigma$, at each node $\xi \in D$, and independently of $T$,

$$||p^T_\xi||_\Sigma \geq \frac{1}{1 + \sum_{\mathrm{J}\in J} \overline{m}_{\xi,j}} > 0.$$

□

Therefore, it follows from equation (D.9) that, for each node $\xi \in D$, the sequence of equilibrium allocations and Kuhn-Tucker multipliers, $\left[(p^T_\xi, q^T_\xi); (z^{h,T}_\xi, \gamma^{h,T}_\xi)_{h\in H}; \lambda^T_\xi\right]_{T>t(\xi)}$, is bounded. Therefore, applying Tychonoff Theorem we find, as in the proof of Proposition 1, a subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that, for each $\xi \in D$,

$$\left[(p^{T_k}_\xi, q^{T_k}_\xi); (z^{h,T_k}_\xi, \gamma^{h,T_k}_\xi)_{h\in H}; \lambda^{T_k}_\xi\right]_{T_k>t(\xi)} \to \left[(\overline{p}_\xi, \overline{q}_\xi); (\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{h\in H}; \overline{\lambda}_\xi\right], \quad \text{as}\, k \to +\infty.$$

Moreover, the limit allocations $\left[(\overline{z}^h_\xi)_{\xi\in D}\right]_{h\in H}$ are budget feasible, at prices $(\overline{p}, \overline{q}) \in \mathbb{P}$, and satisfy market feasibility conditions at each node in the event-tree. Thus, in order to assure that $\left[(\overline{p}_\xi, \overline{q}_\xi); (\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{h\in H}; \overline{\lambda}_\xi\right]_{\xi\in D}$ is an equilibrium we just need, by the results of Section 4, to verify that, for each aent $h \in H$, $(\overline{z}^h_\xi, \overline{\gamma}^h_\xi)_{\xi\in D}$ satisfies the Euler and the transversality conditions.

Lemma D5. For each $t > 0$ we have that,

$$\text{(D.10)} \qquad 0 \leq \sum_{\xi\in D_t} \overline{\gamma}^h_\xi \nabla_2 g^h_\xi(\overline{p}, \overline{q}) \cdot \overline{z}^h_{\xi^-} \leq \sum_{\xi \in D\setminus D^{t-1}} v^h_\xi(\overline{z}^h_\xi),$$

Moreover, for each node $\xi D$ and for all plan $y \geq 0$, we have

$$\text{(D.11)} \qquad v^h_\xi(y) - v^h_\xi(\overline{z}^h_\xi) \leq \left(\overline{\gamma}^h_\xi \nabla_1 g^h_\xi(\overline{p}, \overline{q}) + \sum_{\mu\in\xi^+} \overline{\gamma}^h_\mu \nabla_2 g^h_\mu(\overline{p}, \overline{q})\right) \cdot (y - \overline{z}^h_\xi).$$

Proof. The proof is analogous to those made in Claims A2 and A3 (Appendix A), changing prices $(p, q)$ by $(p^T, q^T)$, and taking the limit as $t$ goes to infinity. □

Thus, as

$$\sum_{\xi \in D\setminus D^{t-1}} v^h_\xi(\overline{z}^h_\xi) \leq \sum_{\xi \in D\setminus D^{t-1}} u^h_\xi(\mathbb{W}_\xi),$$

we have that

$$\overline{\gamma}^h_\xi \nabla_1 g^h_\xi(\overline{p}, \overline{q}) + \sum_{\mu\in\xi^+} \overline{\gamma}^h_\mu \nabla_2 g^h_\mu(\overline{p}, \overline{q}) \in \partial^+ v^h(\overline{z}^h_\xi),$$

$$\lim_{t\to+\infty} \sum_{\xi\in D_t} \overline{\gamma}^h_\xi \nabla_2 g^h_\xi(\overline{p}, \overline{q}) \cdot \overline{z}^h_{\xi^-} = 0,$$

which implies, by the same arguments made in the proof of Proposition 1 (see Appendix A) that the Euler equations and the transversality condition hold. Therefore, it follows from Proposition 1 that the allocation $(\overline{z}^h_\xi)_{\xi\in D}$ is optimal for agent $h \in H$, which concludes the proof of the theorem.

## References


[1] Araujo, A., J. Fajardo, and M.R. Páscoa (2005): "Endogenous Collateral ," *Journal of Mathematical Economics*, 41, 439-462.

[2] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2002): "Collateral Avoids Ponzi Schemes in Incomplete Markets," *Econometrica* 70, 1613-1638.

[3] Bewley, T. (1980): "The Optimum Quantiy of Money," in *Models of Monetary Economics*, ed. by J. Kareken and N. Wallace. Minneapolis: Federal Reserve Bank.

[4] Clower, R.W. (1967): "A reconsideration of the Microfundations of Monetary Theory," *Western Economic Journal*, 6, 1-8.

[5] Friedman, M. (1969): "The Optimum Quantity of Money," Aldine Publishing Company, Chicago.

[6] Geanakoplos, J. and W.R. Zame (2002): "Collateral and the Enforcement of Intertemporal Contracts," Yale University Working Paper.

[7] Gale, D. (1973): "Pure Exchange Equilibrium of Dynamic Economic Models," *Journal of Economic Theory*, 6, 12-36.

[8] Gimenez, E. (2005): "On the Positive Fundamental Value of Money with Short-Sale Constraints: A Comment on Two Examples," Vigo University Working paper, Spain.

[9] Grandmont, J.-M., and Y. Younès (1972): "On the Role of Money and the Existence of a Monetary Equilibrium," *Review of Economic Studies*, 39, 355-372.

[10] Grandmont, J.-M., and Y. Younès (1973): "On the Efficiency of a Monetary Equilibrium," *Review of Economic Studies*, 40, 149-165.

[11] Hernandez, A., and M. Santos (1996): "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102-130.

[12] Kiyotaki, N. and R. Wright (1989): "On Money as a Medium of Exchange", *Journal of Political Economy*, 97, 927-54.

[13] Kocherlakota, N. (1998): "Money is Memory", *Journal of Economic Theory*, 81, 232-51.

[14] Magill, M., and M. Quinzii (1996): "Incomplete Markets over an Infinite Horizon: Long-lived Securities and Speculative Bubbles," *Journal of Mathematical Economics*, 26, 133-170.

[15] Ostroy, J. (1973): "The Informational Efficiency of Monetary Exchange", *American Economic Review*, 63, 597-60.

[16] Páscoa, M.R. and A. Seghir (2005): "Harsh Default Penalties Lead to Ponzi Schemes ", American University of Beirut Working paper.

[17] Rockafellar, R.T. (1997): "Convex Analysis," *Princeton University Press*, Princeton, New Yersey.

[18] Samuelson, P. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Poltical Economy*, 66, 467-482.

[19] Santos, M. (2006): "The Value of Money in a Dynamic Equilibrium Model" *Economic Theory*, 27, 39-58.

[20] Santos, M., and M. Woodford (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19-57.

[21] Wicksell, K. (1898): "Interest and Prices". English translation London: Macmillan, for the Royal Economic Society, 1936.

[22] Woodford M. (1994): "Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy," *Economic Theory*, 4, 345-380.

[23] Woodford M. (2003): "Interest and Prices," *Princeton University Press*, Princeton, New Yersey.



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