# Collateral, default penalties and almost finite-time solvency

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**Abstract**

We argue that it is possible to adapt the approach of imposing restrictions on available plans through finitely effective debt constraints, introduced by Levine and Zame (1996), to encompass models with default and collateral. Along this line, we introduce in the setting of Araujo, Páscoa and Torres-Martínez (2002) and Páscoa and Seghir (2008) the concept of almost finite-time solvency. We show that the conditions imposed in these two papers to rule out Ponzi schemes implicitly restrict actions to be almost finite-time solvent. We define the notion of *equilibrium with almost finite-time solvency* and look on sufficient conditions for its existence. Assuming a mild assumption on default penalties, namely that agents are myopic with respect to default penalties, we prove that existence is guaranteed (and Ponzi schemes are ruled out) when actions are restricted to be almost finite-time solvent. The proof is very simple and intuitive. In particular, the main existence results in Araujo et al. (2002) and Páscoa and Seghir (2008) are simple corollaries of our existence result.

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## 1 Introduction

A central issue that arises in sequential markets models with an infinite horizon is the nature of the borrowing constraints imposed on the participants of the economy. This problem has no counterpart in finite horizon economies, since the requirement that agents must balance their debts at the terminal date implies limits on debt at earlier dates. In the absence of a terminal date agents will seek to renew their credit by successively postponing the repayment

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of their debts until infinite. The existence of such schemes (so-called Ponzi schemes) causes agents' decision problem to have no solution even in cases where the system of prices does not offer (local) arbitrage opportunities. Therefore, for an equilibrium to exist when time extends to infinite, one must impose a mechanism (i.e., specify borrowing constraints) that limits the rate at which agents accumulate debt, namely that avoids the existence of Ponzi schemes.

Roughly speaking, three approaches have been proposed in the literature to deal with the specification of debt constraints in infinite horizon sequential markets models. The crucial difference among these lines of research hinges on the specific assumptions made about the enforcement of payments (the possibility of default) as well as the proposed default punishment.

The first approach, due to Magill and Quinzii (1994), Levine and Zame (1996) and Hernández and Santos (1996), assumes full enforcement of payments (i.e., default is not allowed). Magill and Quinzii (1994) argue for *self-imposed debt constraints* that prevent agents from considering trading strategies that lead to unlimited debt. In Magill and Quinzii (1994) the budget constraint is defined according to a particular set of subjective current value price processes. The problem with this characterization of budget sets is that this set of personalized prices is somehow related to marginal utilities which are not typically observable objects and therefore cannot be monitored by an agency. The specification of budget sets proposed by Hernández and Santos (1996) does not suffer from this weakness since the valuation operator takes into account the whole set of non-arbitrage price systems. Levine and Zame (1996) (See also Levine and Zame (2002)) offer an alternative formulation, based on the idea that at each node, all the debt can be repaid in finite time, that is, they require the debt constrains to be *finitely effective*. The formulation makes perfect sense in an incomplete markets setting, and it has the nice characteristic that a broad class of debt constraints are equivalent or reduced to the finitely effective debt constraints.[1]

The second approach builds on the work of Kehoe and Levine (1993), Zhang (1997) and Alvarez and Jermann (2000). In this framework default is permitted but there is a tough punishment for it: if agents do not honor their debts, they are excluded from participating in the credit markets in future periods. In such a framework the authors focus on constraints (so-called *participating constraints*) that are tight enough to prevent default at equilibrium but simultaneously to allow as much risk sharing as possible.

The final approach argues for debt constraints that do not necessarily imply full enforcement of payment at equilibrium, namely it treats default as an equilibrium phenomenon. In Araujo et al. (2002) and Kubler and Schmedders (2003) borrowers are required to constitute collateral either in terms of durable goods or in terms of physical financial assets which are in positive net supply and cannot be sold short (e.g., Lucas' trees). When the repossession of collateral is the only enforcement mechanism, then an equilibrium always exists. Combining short-sales with the purchase of collateral constitutes a joint operation that yields

[1]See Levine and Zame (1996) Section 5 and Hernández and Santos (1996) Section 3.

non-negative returns. By non-arbitrage, the price of the collateral exceeds the price of the asset. Therefore, agents cannot transfer wealth from tomorrow to the current period and in that way Ponzi schemes are ruled out.

In a recent paper, Páscoa and Seghir (2008) have shown that Ponzi schemes may reappear in collateralized economies when there is an additional enforcement mechanism besides collateral guarantees. The additional enforcement mechanism in Páscoa and Seghir (2008) takes the form of the linear utility penalties introduced by Shubik and Wilson (1977) and used, among others by Dubey and Shubik (1979), Dubey, Geanakoplos and Shubik (1990), Zame (1993), Araujo, Monteiro and Páscoa (1996), Araujo, Monteiro and Páscoa (1998) and Dubey, Geanakoplos and Shubik (2005). Default penalties might be interpreted as the consequence in terms of utility of *extra-economic* punishment such as prison terms or pangs of conscience. Páscoa and Seghir (2008) proved that existence of equilibria is compatible only with moderate default penalties. Harsh default penalties may induce payments besides the value of the collateral leading to Ponzi schemes. In the same spirit Revil and Torres-Martínez (2007) show that the non-existence result established in Páscoa and Seghir (2008) goes beyond the specific enforcement mechanism these authors consider. Existence of Ponzi-schemes is consistent with any other enforcement mechanism that is *effective*, i.e., it enforces payments besides the value of the collateral.

The purpose of this paper is twofold. First, our aim is to show that there is a close relation between the budget sets defined by finitely effective debt constraints (Levine and Zame (1996)), and the budget sets defined through collateral obligations (Araujo et al. (2002) and Páscoa and Seghir (2008)). In that respect, we link two approaches that have been considered to be distinct to each other. Finitely effective debt constraints are relevant in models where payments can be fully enforced. However, when full enforcement is not possible, requiring finite-time solvency does not make sense since agents can default at any period. We appropriately modify the definition of finitely effective debt constraints to encompass economies with default. When payments are fully enforced, our concept of finite effective debt coincides with the concept introduced by Levine and Zame (1996). We subsequently show that the conditions imposed in Araujo et al. (2002) and Páscoa and Seghir (2008) implicitly restrict actions to be almost finite-time solvent.

Equipped with the appropriate definition of debt constraints our second objective is to show the existence of what we term *equilibrium with almost finite-time solvency*. Assuming a mild assumption on default penalties, namely that agents are myopic with respect to default penalties, we prove that existence is guaranteed (i.e., Ponzi schemes are ruled out) when actions are restricted to be almost finite-time solvent. The proof is very simple and intuitive. Moreover, it turns out that the existence results in Araujo et al. (2002) and Páscoa and Seghir (2008) are straightforward corollaries of our existence result.

The paper is structured as follows. In Section 2 we set out the model, notation and standard equilibrium concept. Section 3 contains the assumptions imposed on the characteristics of the economy. In Section 4 we present and discuss the new constraint we imposed on budget feasible plans. We define the concept of competitive equilibrium with almost finite-time solvency and highlight its relation with the other equilibrium concepts introduced in Araujo et al. (2002) and Páscoa and Seghir (2008). Section 5 is devoted to the main condition we impose on default penalties: myopia of agents and we prove in Section 6 that this condition is sufficient for existence of a competitive equilibrium with almost finite-time solvency.

## 2 The Model

The model is essentially the one developed in Araujo et al. (2002). We consider a stochastic economy $\mathscr{E}$ with an infinite horizon.

### 2.1 Uncertainty and time

Let $\mathscr{T} = \{0, 1, \ldots, t, \ldots\}$ denote the set of time periods and let $S$ be a (infinite) set of states of nature. The available information at period $t$ in $\mathscr{T}$ is the same for each agent and is described by a finite partition $\mathscr{F}_t$ of $S$. Information is revealed along time, i.e., the sequence $(\mathscr{F}_t)_{t\in\mathscr{T}}$ is increasing. Every pair $(t, \sigma)$ where $\sigma$ is a set in $\mathscr{F}_t$ is called a node. The set of all nodes is denoted by $D$ and is called the event tree. We assume that there is no information at $t = 0$ and we denote by $\xi_0 = (0, S)$ the initial node. If $\xi = (t, \sigma)$ belongs to the event tree, then $t$ is denoted by $t(\xi)$. We say that $\xi' = (t', \sigma')$ is a successor of $\xi = (t, \sigma)$ if $t' \geqslant t$ and $\sigma' \subset \sigma$; we use the notation $\xi' \geqslant \xi$. We denote by $\xi^+$ the set of immediate successors defined by

$$\xi^+ = \{\xi' \in D : \ t(\xi') = t(\xi) + 1\}.$$

Because $\mathscr{F}_t$ is finer than $\mathscr{F}_{t-1}$ for every $t > 0$, there is a unique node $\xi^-$ in $D$ such that $\xi$ is an immediate successor of $\xi^-$. Given a period $t \in \mathscr{T}$ we denote by $D_t$ the set of nodes at period $t$, i.e., $D_t = \{\xi \in D : \ t(\xi) = t\}$. The set of nodes up to period $t$ is denoted by $D^t$, i.e., $D^t = \{\xi \in D : \ t(\xi) \leqslant t\}$.

### 2.2 Agents and commodities

There exists a finite set $L$ of durable commodities available for trade at every node $\xi \in D$. Depreciation of goods is represented by a family $(Y(\xi))_{\xi\in D}$ of linear functionals $Y(\xi)$ from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. The bundle $Y(\xi)z$ is obtained at node $\xi$ if the bundle $z \in \mathbb{R}^L_+$ is consumed at node $\xi^-$. At each node there are spot markets for trading every good. We let $p = (p(\xi))_{\xi\in D}$ be the spot price process where $p(\xi) = (p(\xi, \ell))_{\ell\in L} \in \mathbb{R}^L_+$ is the price vector at node $\xi$.

There is a finite set $I$ of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment process $\omega^i = (\omega^i(\xi))_{\xi\in D}$ where $\omega^i(\xi) = (\omega^i(\xi, \ell))_{\ell\in L} \in \mathbb{R}^L_+$ is the endowment available at node $\xi$. Each agent chooses a consumption process $x = (x(\xi))_{\xi\in D}$ where $x(\xi) \in$

$\mathbb{R}^L_+$. We denote by $X$ the set of consumption processes. The utility function $U^i : X \longrightarrow [0, +\infty]$ is assumed to be additively separable, i.e.,

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi))$$

where $u^i : \mathbb{R}_+ \longrightarrow [0, \infty)$.

*Remark* 2.1. As in Araujo et al. (2002), we allow $U^i(x)$ to be infinite for some consumption process $x$ in $X$. In Levine and Zame (1996) and Levine and Zame (2002), the consumption set is restricted to uniformly bounded from above consumption processes and the function $U^i$ is assumed to have finite values.

## 2.3 Assets and collateral

There is a finite set $J$ of short-lived real assets available for trade at each node. For each asset $j$, the bundle yielded at node $\xi$ is denoted by $A(\xi, j) \in \mathbb{R}^L_+$. We let $q = (q(\xi))_{\xi \in D}$ be the asset price process where $q(\xi) = (q(\xi, j))_{j \in J} \in \mathbb{R}^J_+$ represents the asset price vector at node $\xi$. At each node $\xi$, denote by $\theta^i(\xi) \in \mathbb{R}^J_+$ the vector of purchases and denote by $\varphi^i(\xi)$ the vector of short-sales at node $\xi$.

Following Araujo et al. (2002) (see also Geanakoplos (1997) and Geanakoplos and Zame (2002)), assets are collateralized in the sense that for every unit of asset $j$ sold at a node $\xi$, agents should buy a collateral $C(\xi, j) \in \mathbb{R}^L_+$ that protects lenders in case of default. Implicitly we assume that payments can be enforced only through the seizure of the collateral. At a node $\xi$, agent $i$ should deliver the promise $V(p, \xi)\theta^i(\xi^-)$ where

$$V(p, \xi) = (V(p, \xi, j))_{j \in J} \quad \text{and} \quad V(p, \xi, j) = p(\xi)A(\xi, j).$$

However, agent $i$ may decide to default and chooses a delivery $d^i(\xi, j)$ in units of account. Since the collateral can be seized, this delivery must satisfy

$$d^i(\xi, j) \geqslant D(p, \xi, j)\varphi^i(\xi^-, j)$$

where

$$D(p, \xi, j) = \min\{p(\xi)A(\xi, j), p(\xi)Y(\xi)C(\xi^-, j)\}.$$

Following Dubey et al. (2005), we assume that agent $i$ feels a disutility $\lambda^i_j(s) \in [0, +\infty]$ from defaulting. If an agent defaults at node $\xi$, then he suffers at $t = 0$, the disutility

$$\sum_{j \in J} \lambda^i(\xi, j) \frac{\left[V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)\right]^+}{p(\xi) \cdot v(\xi)}.$$

where $v(\xi) \in \mathbb{R}^L_{++}$ is exogenously specified.

In that case, agent $i$ may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have

$$d^i(\xi,j) > D(p,\xi,j)\varphi^i(\xi^-,j).$$

The possibility of default forces us to add delivery rates $\kappa(\xi) = (\kappa(\xi,j))_{j\in J}$. Each asset $j$ delivers to lenders the fraction $V(\kappa,p,\xi,j)$ per unit of asset purchased defined by

$$V(\kappa,p,\xi,j) = \kappa(\xi,j)V(p,\xi,j) + (1-\kappa(\xi,j))D(p,\xi,j).$$

## 2.4 Solvency constraints

We let $A$ be the space of adapted processes $a = (a(\xi))_{\xi\in D}$ with

$$a(\xi) = (x(\xi),\theta(\xi),\varphi(\xi),d(\xi))$$

where

$$x(\xi) \in \mathbb{R}^L_+, \quad \theta(\xi) \in \mathbb{R}^J_+, \quad \varphi(\xi) \in \mathbb{R}^J_+, \quad d(\xi) \in \mathbb{R}^J_+$$

and by convention

$$a(\xi_0^-) = (x(\xi_0^-),\theta(\xi_0^-),\varphi(\xi_0^-),d(\xi_0^-)) = (0,0,0,0).$$

In each decision node $\xi \in D$, agent $i$'s choice $a^i = (x^i,\theta^i,\varphi^i,d^i)$ in $A$ must satisfy the following constraints:

**(a)** solvency constraint:

$$\begin{aligned} & p(\xi)x^i(\xi) + \sum_{j\in J} d^i(\xi,j) + q(\xi)\theta^i(\xi) \\ & \qquad \leqslant p(\xi)[\omega^i(\xi) + Y(\xi)x^i(\xi^-)] + V(\kappa,p,\xi)\theta^i(\xi^-) + q(\xi)\varphi^i(\xi), \end{aligned} \tag{2.1}$$

**(b)** collateral requirement:

$$C(\xi)\varphi^i(\xi) \leqslant x^i(\xi), \tag{2.2}$$

**(c)** minimum delivery

$$\forall j \in J, \quad d^i(\xi,j) \geqslant D(p,\xi,j)\varphi^i(\xi^-,j). \tag{2.3}$$

### 2.5 The payoff function

Assume that $\pi = (p, q, \kappa)$ is a process of prices and delivery rates. Consider that agent $i$ has chosen the plan $a = (x, \theta, \varphi, d) \in A$. He gets the utility $U^i(x) \in [0, \infty]$ defined by

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi))$$

but he suffers the disutility $W^i(p, a) \in [0, \infty]$ defined by

$$W^i(p, a) = \sum_{\xi > \xi_0} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

We would like to define the payoff $\Pi^i(p, a)$ of the plan $a$ as the following difference

$$\Pi^i(p, a) = U^i(x) - W^i(p, a).$$

Unfortunately, $\Pi^i(p, a)$ may not be well-defined if both $U^i(x)$ and $W^i(p, a)$ are infinite. We propose to consider the binary relation $\succ_{i,p}$ defined on $A$ by

$$\widetilde{a} \succ_{p,i} a \Longleftrightarrow \exists \varepsilon > 0, \quad \exists T \in \mathbb{N}, \quad \forall t \geqslant T, \quad \Pi^{i,t}(p, \widetilde{a}) \geqslant \Pi^{i,t}(p, a) + \varepsilon$$

where

$$\Pi^{i,t}(p, a) = U^{i,t}(x) - W^{i,t}(p, a), \quad U^{i,t}(x) = \sum_{\xi \in D^t} u^i(\xi, x(\xi))$$

and

$$W^{i,t}(p, a) = \sum_{\xi \in D^t \setminus \{\xi_0\}} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

Observe that if $\Pi^i(p, \widetilde{a})$ and $\Pi^i(p, a)$ exist in $\mathbb{R}$ then

$$\widetilde{a} \succ_{p,i} a \Longleftrightarrow \Pi^i(p, \widetilde{a}) > \Pi^i(p, a).$$

The set $\mathrm{Pref}^i(p, a)$ of plans strictly preferred to plan $a$ by agent $i$ is defined by

$$\mathrm{Pref}^i(p, a) = \{\widetilde{a} \in A : \ \widetilde{a} \succ_{i,p} a\}.$$

### 2.6 The equilibrium concept

We denote by $\Pi$ the set of prices and delivery rates $(p, q, \kappa)$ satisfying

$$\forall \xi \in D, \quad p(\xi) \in \mathbb{R}^L_{++}, \quad q(\xi) \in \mathbb{R}^J_{+}, \quad \kappa(\xi) \in [0, 1]^J \tag{2.4}$$

and

$$\sum_{\ell \in L} p(\xi, \ell) + \sum_{j \in J} q(\xi, j) = 1.$$

We denote by $\mathrm{cl}\,\Pi$ the closure of $\Pi$ under the weak topology.[2]

Given a process $(p, q, \kappa)$ of commodity prices, asset prices and delivery rates, we denote by $B^i(p, q, \kappa)$ the set of plans $a = (x, \theta, \varphi, d) \in A$ satisfying constraints (2.1), (2.2) and (2.3). The demand $d^i(p, q, \kappa)$ is defined by

$$d^i(p, q, \kappa) = \{a \in B^i(p, q, \kappa) : \ \mathrm{Pref}^i(p, a) \cap B^i(p, q, \kappa) = \emptyset\}.$$

**Definition 2.1.** *A competitive equilibrium for the economy $\mathscr{E}$ is a family of prices and delivery rates $(p, q, \kappa) \in \Pi$ and an allocation $\boldsymbol{a} = (a^i)_{i \in I}$ with $a^i \in A$ such that*

**(a)** *for every agent $i$, the plan $a^i$ is optimal, i.e.,*

$$a^i \in d^i(p, q, \kappa),$$

**(b)** *commodity markets clear at every node, i.e.,*

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \tag{2.5}$$

*and for all $\xi \neq \xi_0$,*

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} \left[ \omega^i(\xi) + Y(\xi) x^i(\xi^-) \right], \tag{2.6}$$

**(c)** *asset markets clear at every node, i.e., for all $\xi \in D$,*

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi), \tag{2.7}$$

**(d)** *deliveries match at every node, i.e., for all $\xi \neq \xi_0$ and all $j \in J$,*

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \tag{2.8}$$

The set of allocations $\boldsymbol{a} = (a^i)_{i \in I}$ in $A$ satisfying the market clearing conditions (b) and (c) is denoted by F. Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation $\boldsymbol{b}$ such that $a^i = b^i$. The set of physically feasible plans is denoted by $\mathrm{F}^i$. We denote by $\mathrm{Eq}(\mathscr{E})$ the set of competitive equilibria for the economy $\mathscr{E}$.

[2]The process $(p, q, \kappa)$ belongs to $\mathrm{cl}\,\Pi$ if the condition $p(\xi) \in \mathbb{R}^L_{++}$ in (2.4) is replaced by $p(\xi) \in \mathbb{R}^L_{+}$.

## 3 Assumptions

For each agent $i$, we denote by $\Omega^i = (\Omega^i(\xi))_{\xi \in D}$ the process of accumulated endowments, defined recursively by

$$\Omega^i(\xi_0) = \omega^i(\xi_0) \quad \text{and} \quad \forall \xi > \xi_0, \quad \Omega^i(\xi) = Y(\xi)\Omega^i(\xi^-) + \omega^i(\xi).$$

The process $\sum_{i \in I} \Omega^i$ of accumulated aggregate endowments is denoted by $\Omega$. This section describes the assumptions imposed on the characteristics of the economy. It should be clear that these assumptions always hold throughout the paper.

**Assumption 3.1** (Agents)**.** For every agent $i$,

**(A.1)** the process of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \overline{\Omega}^i \in \mathbb{R}^L_+, \quad \forall \xi \in D, \quad \Omega^i(\xi) \in \mathbb{R}^L_{++} \quad \text{and} \quad \Omega^i(\xi) \leqslant \overline{\Omega}^i,$$

**(A.2)** for every node $\xi$, the utility function $u^i(\xi, \cdot)$ is concave, continuous and strictly increasing with $u^i(\xi, 0) = 0$,

**(A.3)** the infinite sum $U^i(\Omega)$ is finite.

**Assumption 3.2** (Commodities)**.** For every node $\xi$ the depreciation function $Y(\xi)$ is not zero.

**Assumption 3.3** (Financial assets)**.** For every asset $j$ and node $\xi$, the collateral $C(\xi, j)$ is not zero.

*Remark* 3.1. Assumptions (3.1), (3.2) and (3.3) are classical in the literature of infinite horizon models with collateral requirements (see e.g., Araujo et al. (2002) and Páscoa and Seghir (2008)).

*Remark* 3.2. Observe that Assumptions (A.2) and (A.3) imply that when restricted to the order interval $[0, \Omega]$, the function $U^i$ is weakly continuous. For the sake of completeness, we give the straightforward proof in Appendix A.1.

We recall a particular case of our framework that is widely used in the literature (see e.g. Araujo and Sandroni (1999)).

**Definition 3.1.** *The economy $\mathscr{E}$ is said standard if Assumptions (3.1), (3.2) and (3.3) are satisfied and if for each agent i, there exists*

**(S.1)** *a discount factor $\beta_i \in (0, 1)$,*

**(S.2)** *a sequence $(P^i_t)_{t \geqslant 1}$ of beliefs about nodes at period $t$ represented by a probability $P^i_t \in \text{Prob}(D_t)$,*

**(S.3)** *an instantaneous felicity function* $v^i : D \times \mathbb{R}^L_+ \to [0,\infty)$,

**(S.4)** *an instantaneous default penalty* $\mu^i(\xi, j) \in (0,\infty)$ *for each node* $\xi > \xi_0$,

*such that for each node* $\xi \in D$,

$$u^i(\xi,\cdot) = [\beta_i]^{t(\xi)} P^i_{t(\xi)}(\xi) v^i(\xi,\cdot)$$

*for each* $j \in J$,

$$\lambda^i(\xi, j) = [\beta_i]^{t(\xi)} P^i_{t(\xi)}(\xi) \mu^i(\xi, j)$$

*and the processes* $(A(\xi, j))_{\xi>\xi_0}$, $(\mu^i(\xi, j))_{\xi>\xi_0}$ *and* $(G(\xi, j))_{\xi\in D}$ *are uniformly bounded from above, where*

$$G(\xi, j) = \frac{1}{\max_{\ell\in L} C(\xi, j, \ell)}.$$

## 4 Almost finite-time solvent plans

Observe that if $\lambda^i(\xi, j)$ is zero for every asset $j$ at every node $\xi$, then our model reduces to the one in Araujo et al. (2002). In this setting equilibrium always exists. Combining short-sales with the purchase of collateral constitutes a joint operation that yields non-negative returns. By non-arbitrage, the price of the collateral exceeds the price of the asset. Therefore, agents cannot transfer wealth from tomorrow to the current period and Ponzi schemes are ruled out. In a recent paper, Páscoa and Seghir (2008) proved that harsh default penalties may induce effective payments over collateral requirements and lead to Ponzi schemes.

When the default penalty is infinite and the collateral requirement is zero, our model reduces to the standard one as in Magill and Quinzii (1994) and Levine and Zame (1996). If no additional (possibly endogenous) debt constraints were imposed, then an equilibrium could not possibly exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing. Levine and Zame (1996) (see also Levine and Zame (2002)) formalize the so-called *finitely effective* debt constraints by requiring that agents should be capable of repaying almost all the debt in finite time.

We propose to adapt in our setting these endogenous debt constraints. Fix a process $\pi = (p, q, \kappa)$ of prices and delivery rates. At the initial node $\xi_0$, agent $i$ makes plans for infinite consumption and investment. Consider the case where agent $i$ anticipates (or fears) that, at every possible node $\xi$, his demand for credits at this node may be questioned by an authority. In order to convince this authority that he is reliable, he must prove that he can pay back his debt in at most a finite number of periods after $t(\xi)$, i.e., he must prove that there is a possible plan of consumption and investment from period $t(\xi)+1$ to $T \geqslant t(\xi)+1$ such that, at the virtually terminal node $T$, he does not need to ask for new loans in order to pay his debt. More formally, we may consider the following definition.

**Definition 4.1.** *A plan $a \in B^i(p,q,\kappa)$ is said to have finitely effective debt, if for each period $t \geqslant 0$, there exists a period $T > t$ and a plan $\widehat{a}$ also in the budget set $B^i(p,q,\kappa)$ such that*

**(i)** *up to period $t$ both plans coincide, i.e.,*

$$\forall \xi \in D^t, \quad \widehat{a}(\xi) = a(\xi),$$

**(ii)** *at every node in period $T$, there is solvency without new loans, i.e.,*

$$\forall \xi \in D_T, \quad \widehat{\varphi}(\xi) = 0,$$

**(iii)** *the plan $\widehat{a}$ is a $T$-horizon plan, i.e.,*

$$\forall \xi \in D, \quad t(\xi) > T \Longrightarrow \widehat{a}(\xi) = 0.$$

Consider the following notation. If $a$ is a plan in $A$ and $t$ is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in $A^t$ which coincides with $a$ for every node $\xi \in D^t$. In other words, a plan $a$ has a finitely effective debt if for each period $t \geqslant 0$, there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$$

where $B^T$ is the set of plans $a$ in $A$ satisfying

$$\forall \xi \in D_T, \quad \varphi(\xi) = 0 \quad \text{and} \quad \forall \xi \in D, \quad t(\xi) \geqslant T+1 \Longrightarrow a(\xi) = (0,0,0,0).$$

This concept was introduced by Levine and Zame (1996) for models without default, i.e., models for which the financial authority can enforce payments: it may force agents to sell their current and future endowments (by short-selling assets). However, when the authority is not capable of enforcing payments, imposing finitely effective debt constraints does not make sense since agents can default at any period. Indeed, let $a = (x, \theta, \varphi, d)$ be a plan in $B^i(p,q,\kappa)$ and $t$ be any period. Consider the plan $\widehat{a}$ defined by

$$\forall \xi \in D, \quad \widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) \leqslant t \\ (\omega^i(\xi), 0, 0, D(p,\xi)\varphi(\xi^-)) & \text{if} \quad t(\xi) = t+1 \\ (0,0,0,0) & \text{if} \quad t(\xi) > t+1. \end{cases}$$

This plan belongs to the budget set $B^i(p,q,\kappa)$ and coincides with $a$ on every node up to period $t$.

In our framework, enforcement mechanisms are limited to the seizure of collateral. Agents can always default up to the minimum value between their debt and the depreciated value of their collateral. Therefore, there is no room for an authority to control debt along time. We propose another interpretation of endogenous debt constraints. Assume that the authority has the legal ability, when the debt carried out by an agent becomes larger and larger, to impose at any period $t$ that agents can participate in the financial market only for a finite number $\tau$ of periods after $t$. Assume that a negotiation is possible between agents and the financial authority such that the number $\tau$ of periods can be chosen by agents. Each agent anticipates this possibility and behaves accordingly in the following sense. When making a plan $a^i$, agent $i$ takes in consideration that the financial authority may force him, at any period $t$, to stay in the financial market no more than a finite number $\tau$ of periods, i.e., at date $T = t + \tau$, agent $i$ must leave the market. Therefore, agent $i$ also plans that, for every period $t$, he can find another plan $\widehat{a}$ and a *terminal* date $T > t$, such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$$

but also that the payoff he gets at the terminal node $T$ with the plan $\widehat{a}$ is not too far from the payoff he would get with the initial plan $a$.

The formal definition is as follows.

**Definition 4.2.** *A plan $a$ in the budget set $B^i(p,q,\kappa)$ is said to be almost finite-time solvent if for every period $t \geqslant 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that*

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T, \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]} \quad \text{and} \quad \Pi^{i,T}(p,\widehat{a}) \geqslant \Pi^{i,T}(p,a) - \varepsilon.$$

When the default penalty is infinite, our concept of almost finite-time solvent plans coincides with the concept introduced by Levine and Zame (1996) of plans with finitely effective debt.

**Proposition 4.1.** Assume that the default penalty is infinite and consider a budget feasible plan $a \in B^i(p,q,\kappa)$ with a finite utility $U^i(x) < \infty$. The plan $a$ is almost finite-time solvent if and only if it has a finitely effective debt.

*Proof of Proposition 4.1.* Let $a$ be a budget feasible plan, i.e., $a \in B^i(p,q,\kappa)$ with a finite utility $U^i(x) < \infty$. Since the default penalty is infinite, agent $i$ never plans to default and we get

$$\Pi^i(p,a) = U^i(x).$$

It is obvious that if $a$ is almost finite-time solvent then it has a finitely effective debt. The converse deserves more details. Assume that the plan $a$ has a finitely effective debt. Fix a period $t$ and $\varepsilon > 0$. If we apply the definition to the period $t$, we get the existence of a period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}.$$

Unfortunately, we don't know if $U^{i,T}(\widehat{x}) \geqslant U^{i,T}(x) - \varepsilon$. However, we know that the utility $U^i(x)$ is finite. Therefore, there exists $t' > t$ such that

$$\sum_{s>t'} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \leqslant \varepsilon. \tag{4.1}$$

Now, applying the definition of *finitely effective debt* for the period $t'$, there exist a period $T > t'$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T \quad \text{and} \quad a\mathbf{1}_{[0,t']} = \widehat{a}\mathbf{1}_{[0,t']}.$$

Since $T > t'$ we can use (4.1) to get

$$\begin{aligned} U^{i,T}(\widehat{x}) &\geqslant U^{i,t'}(\widehat{x}) \\ &\geqslant U^{i,t'}(x) \\ &\geqslant U^{i,T}(x) - \sum_{t'<s\leqslant T} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\ &\geqslant U^{i,T}(x) - \varepsilon. \end{aligned}$$

□

We denote by $B^i_\star(p,q,\kappa)$ the set of all plans in $B^i(p,q,\kappa)$ which are almost finite-time solvent.

**Definition 4.3.** *A competitive equilibrium with almost finite-time solvency for the economy $\mathscr{E}$, is a family of prices and delivery rates $(p,q,\kappa) \in \Pi$ and an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in A$ such that conditions (b), (c) and (d) are satisfied together with*

**(a')** *for every agent $i$, the plan $a^i$ is almost finite-time solvent and optimal among all almost finite-time budget feasible plans, i.e.,*

$$a^i \in d^i_\star(p,q,\kappa) := \{a \in B^i_\star(p,q,\kappa) : \ \mathrm{Pref}^i(p,a) \cap B^i_\star(p,q,\kappa) = \emptyset\}.$$

We denote by $\mathrm{Eq}_\star(\mathscr{E})$ the set of competitive equilibria with almost finite-time solvency for the economy $\mathscr{E}$. We propose to compare our equilibrium concept with those proposed in Araujo et al. (2002) and Páscoa and Seghir (2008).

### 4.1 No default penalty

Observe that if default is not allowed or if there are default penalties for it, then $B^i_\star(p,q,\kappa)$ may be a strict subset of $B^i(p,q,\kappa)$. However, in the model proposed in Araujo et al. (2002), any budget feasible allocation with a finite utility is finite-time solvent. This is a consequence of the absence of default penalties or explicit economic punishments.

**Proposition 4.2.** Assume that there is no default penalty and let $a=(x,\theta,\varphi,d)$ be a plan in the budget set $B^i(p,q,\kappa)$. If $U^i(x)$ is finite then $a$ is almost finite-time solvent, i.e., $a$ belongs to $B^i_\star(p,q,\kappa)$.

*Proof of Proposition 4.2.* Fix an agent $i$ and consider a plan $a$ that is budget feasible, i.e., $a\in B^i(p,q,\kappa)$ with a finite utility, i.e.,

$$\sum_{\xi\in D} u^i(\xi,x(\xi))<\infty.$$

Fix a period $t\geqslant 1$ and $\varepsilon>0$. Since $U^i(x)$ is finite, there exists $T\geqslant t+1$ such that

$$\sum_{\xi\in D_T} u^i(\xi,x(\xi))\leqslant \varepsilon.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi)=\begin{cases} a(\xi) & \text{if} \quad t(\xi)<T \\ (\omega^i(\xi),0,0,\widehat{d}(\xi)) & \text{if} \quad t(\xi)=T \\ (0,0,0,0) & \text{if} \quad t(\xi)>T \end{cases}$$

where

$$\forall \xi\in D_T, \quad \forall j\in J, \quad \widehat{d}(\xi,j)=D(p,\xi,j)\varphi(\xi^-,j).$$

Observe that the plan $\widehat{a}$ is budget feasible, belongs to $B^T$ and satisfies

$$\widehat{a}\mathbf{1}_{[0,T-1]}=a\mathbf{1}_{[0,T-1]}.$$

In order to prove that the plan $a$ is almost finite-time solvent, we need to compare $U^{i,T}(\widehat{x})$

and $U^{i,T}(x)$. Observe that

$$\begin{aligned}
U^{i,T}(\widehat{x}) &= U^{i,T-1}(x) + \sum_{\xi \in D_T} u^i(\xi, \omega^i(\xi)) \\
&\geqslant U^{i,T-1}(x) \\
&\geqslant U^{i,T}(x) - \sum_{\xi \in D_T} u^i(\xi, x(\xi)) \\
&\geqslant U^{i,T}(x) - \varepsilon.
\end{aligned}$$

Since $T - 1 \geqslant t$, this implies that the plan $a$ is almost finite-time solvent. □

In general the two sets $\mathrm{Eq}(\mathscr{E})$ and $\mathrm{Eq}_\star(\mathscr{E})$ are not comparable. Actually, when there is no loss of utility in case of default, both sets coincide.

**Proposition 4.3.** If there is no default penalty then $(\pi, \boldsymbol{a})$ is a competitive equilibrium if and only if it is a competitive equilibrium with almost finite-time solvency, i.e., the sets $\mathrm{Eq}(\mathscr{E})$ and $\mathrm{Eq}_\star(\mathscr{E})$ coincide.

*Proof of Proposition 4.3.* Let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}(\mathscr{E})$ be a competitive equilibrium. Fix an agent $i \in I$. In order to prove that $a^i$ belongs to the demand $d^i_\star(\pi)$, it is sufficient to prove that $a^i$ is an almost finite-time solvent plan. Since $\boldsymbol{a}$ is feasible we have $x^i(\xi) \leqslant \Omega(\xi)$. From (A.3), we get that $U^i(x^i)$ is finite. The desired result follows from Proposition 4.2.

Now let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}_\star(\mathscr{E})$ be a competitive equilibrium with almost finite-time solvency. We only have to prove that $a^i$ belongs to $d^i(\pi)$ for each agent $i$. Fix an agent $i$ and assume by contradiction that there exists a plan $a$ in $B^i(\pi)$ such that $U^i(x) > U^i(x^i)$. If $U^i(x)$ is finite then, applying Proposition 4.2, we get that $a \in B^i_\star(\pi)$: contradiction. Therefore, we must have $U^i(x) = \infty$, implying that there exists $T \geqslant 1$ such that

$$U^{i,T}(x) > U^i(x^i).$$

Consider the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) \leqslant T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T+1 \\ (0,0,0,0) & \text{if} \quad t(\xi) > T+1 \end{cases}$$

where

$$\forall \xi \in D_{T+1}, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Since the plan $\widehat{a}$ is budget feasible and has a finite horizon, it is almost finite-time solvent and belongs to $B^i_\star(p,q,\kappa)$. Moreover we have

$$U^i(\widehat{x}) = U^{i,T}(x) + \sum_{\xi \in D_{T+1}} u^i(\xi, \omega^i(\xi)) > U^i(x^i).$$

This contradicts the optimality of $x^i$ in $B^i_\star(p,q,\kappa)$. □

### 4.2 $\alpha$-moderate default penalties

Before introducing the main condition imposed on default penalties by Páscoa and Seghir (2008), we need to introduce some notations. For each asset $j$ and node $\xi$, we denote by $M(\xi, j)$ the real number

$$\min_{\ell \in L} \frac{\Omega(\xi,\ell)}{C(\xi,j,\ell)}.$$

Observe that under Assumption 3.2, we have $M(\xi, j) < \infty$. Finally, for every node $\xi \neq \xi_0$ we let

$$H(\xi, j) = M(\xi^-, j) \sup_{p \in \Delta(L)} \frac{\left[pA(\xi,j) - pY(\xi)C(\xi^-, j)\right]^+}{p\nu(\xi)}.$$

The quantity $H(\xi, j)$ is the maximum amount in real terms that an agent may default on asset $j$ if his plan is feasible. The proof of the following proposition is straightforward and omitted.

**Proposition 4.4.** If $a$ in $A$ is a plan physically feasible and $(p,q,\kappa)$ in $\Pi$ is a process of prices and delivery rates, then for each node $\xi$ and each asset $j$, we have

$$\varphi(\xi, j) \leqslant M(\xi, j) \quad \text{and} \quad \left[V(p,\xi,j)\varphi(\xi^-, j) - d(\xi, j)\right]^+ \leqslant H(\xi, j).$$

Páscoa and Seghir (2008) introduced the concept of *$\alpha$-moderate* default penalties. Fix a process $\alpha = (\alpha(\xi))_{\xi \in D}$ with $\alpha(\xi) \in (1,\infty)^J$.

**Definition 4.4.** *Default penalties are said $\alpha$-moderate with respect to utility functions, if for each agent $i$, for each period $t$, there exists $T > t$ such that*

$$\sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j)\alpha(\xi, j) H(\xi, j) \leqslant \sum_{\xi \in D_T} u^i(\xi, \omega^i(\xi)). \tag{4.2}$$

In other words, when default penalties are $\alpha$-moderate then, sometime in the future, the penalty associated with a maximal default for a feasible plan, is less than the utility from consuming the current endowment.

*Remark* 4.1. Actually Páscoa and Seghir (2008) replace condition (4.2) by the following more restrictive condition:

$$\forall \xi \in D_T, \quad \sum_{j \in J} \lambda^i(\xi, j)\alpha(\xi, j)H(\xi, j) \leqslant u^i(\xi, \omega^i(\xi)).$$

We let $A_\alpha$ be the set of all of processes $a$ in $A$ satisfying

$$\exists \lambda \geqslant 0, \quad \forall \xi \in D, \quad \forall j \in J, \quad \varphi(\xi, j) \leqslant \lambda \alpha(\xi, j)M(\xi, j).$$

A plan $a$ belonging to $A_\alpha$ is said to be $\alpha$-constrained. We denote by $B^i_\alpha(p, q, \kappa)$ the set of all plans in $B^i(p, q, \kappa)$ which are $\alpha$-constrained.

*Remark* 4.2. Observe that the constraints imposed in the definition of $A_\alpha$ are not binding at equilibrium since $\alpha(\xi, j) > 1$. Actually, if $\boldsymbol{a}$ in $A^I$ is a physically feasible allocation then each plan $a^i$ is automatically $\alpha$-constrained for each $i$, more precisely, we have

$$\forall \xi \in D, \quad \forall j \in J, \quad \varphi(\xi, j) < \alpha(\xi, j)M(\xi, j).$$

**Definition 4.5.** *An $\alpha$-constrained competitive equilibrium for the economy $\mathscr{E}$, is a family of prices and delivery rates $(p, q, \kappa) \in \Pi$ and an allocation $\boldsymbol{a} = (a^i)_{i \in I}$ with $a^i \in A$ such that conditions (b), (c) and (d) are satisfied together with*

**($\mathbf{a}^\alpha$)** *for every agent $i$, the plan $a^i$ is $\alpha$-constrained budget feasible and optimal among all $\alpha$-constrained budget feasible plans, i.e.,*

$$a^i \in d^i_\alpha(p, q, \kappa) = \{a \in B^i_\alpha(p, q, \kappa) : \ \mathrm{Pref}^i(p, a) \cap B^i_\alpha(p, q, \kappa) \neq \emptyset\}.$$

We denote by $\mathrm{Eq}_\alpha(\mathscr{E})$ the set of $\alpha$-constrained competitive equilibria for the economy $\mathscr{E}$. In general the two sets $\mathrm{Eq}_\alpha(\mathscr{E})$ and $\mathrm{Eq}_\star(\mathscr{E})$ are not comparable. Actually, when default penalties are $\alpha$-moderate, the set $\mathrm{Eq}_\star(\mathscr{E})$ is a subset of $\mathrm{Eq}_\alpha(\mathscr{E})$.

**Proposition 4.5.** If default penalties are $\alpha$-moderate, then every competitive equilibrium with almost finite-time solvency is actually an $\alpha$-constrained competitive equilibrium, i.e.,

$$\mathrm{Eq}_\star(\mathscr{E}) \subset \mathrm{Eq}_\alpha(\mathscr{E}).$$

*Proof of Proposition 4.5.* Let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}_\star(\mathscr{E})$ be a competitive equilibrium with almost finite-time solvency. Fix an agent $i \in I$. Since $a^i$ is physically feasible, we already know that it is $\alpha$-constrained, i.e., $a^i \in B^i_\alpha(\pi)$. Let us prove that $a^i$ belongs to the demand $d^i_\alpha(\pi)$. Assume by way of contradiction that there exists an $\alpha$-constrained plan $\overline{a}$ in $B^i_\alpha(\pi)$, $\varepsilon > 0$ and $T^1 \in \mathbb{N}$ satisfying

$$\forall T \geqslant T^1, \quad \Pi^{i,T}(p, \overline{a}) > \Pi^{i,T}(p, a^i) + \varepsilon. \tag{4.3}$$

Since $a^i$ is physically feasible, we have

$$\forall \xi \in D, \quad x^i(\xi) \leqslant \Omega(\xi).$$

It follows from Assumption (A.2) and (A.3) that

$$U^i(x^i) \leqslant U^i(\Omega) < \infty.$$

Hence

$$\lim_{T \to \infty} \Pi^{i,T}(p, a^i) = \Pi^i(p, a^i).$$

Therefore, there exists $T^2 \geqslant T^1$ such that

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p, a^i) + \varepsilon > \Pi^i(p, a^i). \tag{4.4}$$

Combining (4.3) and (4.4) we get

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p, \overline{a}) > \Pi^i(p, a^i).$$

Let $\beta \in (0,1)$ and pose

$$\widetilde{a} = \beta \overline{a} + (1 - \beta) a^i.$$

Observe that we still have

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p, \widetilde{a}) > \Pi^i(p, a^i).$$

Recall that there exists $\lambda \geqslant 0$ such that

$$\forall \xi \in D, \quad \forall j \in J, \quad \overline{\varphi}(\xi, j) \leqslant \lambda \alpha(\xi, j) M(\xi, j).$$

Recall that $a^i$ is physically feasible, implying that $\varphi^i(\xi, j) \leqslant M(\xi, j)$ for each node $\xi$ and each asset $j$. It then follows that

$$\forall \xi \in D, \quad \forall j \in J, \quad \widetilde{\varphi}(\xi, j) \leqslant \left[\beta \lambda \alpha(\xi, j) + (1 - \beta)\right] M(\xi, j).$$

Since $\alpha(\xi, j) > 1$, we can choose $\beta$ close enough to 0 such that

$$\forall \xi \in D, \quad \widetilde{\varphi}(\xi, j) \leqslant \alpha(\xi, j) M(\xi, j).$$

Since default penalties are $\alpha$-moderate, there exists $T^3 > T^2$ such that

$$\sum_{\xi \in D_{T^3}} \left[ u^i(\xi, \omega^i(\xi)) - \sum_{j \in J} \lambda^i(\xi, j) \alpha(\xi, j) H(\xi, j) \right] \geqslant 0.$$

Let now $\widehat{a}$ be the plan defined by

$$\widehat{a}(\xi) = \begin{cases} \widetilde{a}(\xi) & \text{if} \quad t(\xi) \leqslant T^3 - 1 \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T^3 \\ (0,0,0,0) & \text{if} \quad t(\xi) > T^3 \end{cases}$$

where

$$\forall \xi \in D_{T^3}, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\widetilde{\varphi}(\xi^-, j).$$

By construction, the plan $\widehat{a}$ is finite-time solvent and budget feasible. In particular, it is almost finite-time solvent and belongs to $B^i_\star(\pi)$. We propose to compare the payoffs of $\widehat{a}$ and $a^i$.

$$\begin{aligned} \Pi^i(p, \widehat{a}) &\geqslant \Pi^{i,T^3-1}(p, \widetilde{a}) + \sum_{\xi \in D_{T^3}} \left[ u^i(\xi, \omega^i(\xi)) - \sum_{j \in J} \lambda^i(\xi, j)\alpha(\xi, j)H(\xi, j) \right] \\ &\geqslant \Pi^{i,T^3-1}(p, \widetilde{a}) > \Pi^i(p, a^i). \end{aligned}$$

This contradicts the optimality of $a^i$ in $B^i_\star(p, q, \kappa)$. □

## 5 Myopic agents and equilibrium existence

It was proved in Levine and Zame (1996) that finitely effective debt constraints are compatible with equilibrium when the default penalty is infinite and no collateral is required. A natural question concerns the possible extension of this existence result to our framework when default penalties are not infinite and collateral requirements are not zero. The answer is yes, provided that agents are myopic with respect to default penalties as defined hereafter.

**Definition 5.1.** *Agent $i$ is said to be myopic with respect to default penalties if for each agent $i$, we have*

$$\liminf_{T \to \infty} \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j)H(\xi, j) = 0.$$

*Remark* 5.1. Assuming myopic agents with respect to default penalties is a very mild assumption since it is automatically satisfied for every standard economy.

Observe that if default penalties are moderate with respect to utility functions then, for each $i$ we have

$$\begin{aligned} \liminf_{T \to \infty} \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j)H(\xi, j) &\leqslant \liminf_{T \to \infty} \sum_{\xi \in D_T} u^i(\xi, \omega^i(\xi)) \\ &\leqslant \liminf_{T \to \infty} \sum_{\xi \in D_T} u^i(\xi, \Omega^i(\xi)). \end{aligned} \tag{5.1}$$

It then follows from Assumption (A.3) that every agent is myopic with respect to default penalties.

**Proposition 5.1.** If default penalties are moderate then every agent is myopic with respect to them.

When agents are myopic with respect to default penalties, any budget and physically feasible plan $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$ is actually almost finite-time solvent. This result will turn out to be crucial when proving existence of equilibrium.

**Proposition 5.2.** If agent $i$ is myopic with respect to default penalties, then every budget and physically feasible plan is actually almost finite-time solvent. In other words, we have

$$B^i(p,q,\kappa) \bigcap \mathrm{F}^i \subset B^i_\star(p,q,\kappa).$$

*Proof of Proposition 5.2.* Fix an agent $i$ and consider a plan $a$ that is budget and physically feasible, i.e., $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$. Fix a period $t \geqslant 1$ and $\varepsilon > 0$. Since the allocation $a$ is physically feasible, we have $x(\xi) \leqslant \Omega(\xi)$, implying that

$$\sum_{\xi \in D} u^i(\xi, x(\xi)) < \infty.$$

Therefore there exists $T^0 \geqslant 1$ such that

$$\forall T \geqslant T^0, \quad \sum_{\xi \in D_T} u^i(\xi, x(\xi)) \leqslant \frac{\varepsilon}{2}.$$

Since agent $i$ is myopic with respect to default penalties, there exists $T > \max\{t, T^0\}$ such that

$$\sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) \leqslant \frac{\varepsilon}{2}.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) < T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T \\ (0,0,0,0) & \text{if} \quad t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan $\widehat{a}$ satisfies

$$\widehat{a} \in B^i(p,q,\kappa) \cap B^T \quad \text{and} \quad \widehat{a}\mathbf{1}_{[0,T-1]} = a\mathbf{1}_{[0,T-1]}.$$

Moreover,

$$\begin{aligned}
\Pi^{i,T}(p,\widehat{a}) &= \Pi^{i,T-1}(p,\widehat{a}) \\
&+ \sum_{\xi \in D_T} \left[ u^i(\xi,\omega^i(\xi)) - \sum_{j \in J} \lambda^i(\xi,j) \frac{\left[V(p,\xi,j) - D(p,\xi,j)\right] \varphi(\xi^-,j)}{p(\xi)v(\xi)} \right] \\
&\geqslant \Pi^{i,T-1}(p,a) - \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi,j) H(\xi,j) \\
&\geqslant \Pi^{i,T-1}(p,a) - \frac{\varepsilon}{2} \\
&\geqslant \Pi^{i,T}(p,a) - \frac{\varepsilon}{2} - \sum_{\xi \in D_T} u^i(\xi,x(\xi)) \\
&\geqslant \Pi^{i,T}(p,a) - \varepsilon.
\end{aligned}$$

Since $T-1 \geqslant t$, this implies that the plan $a$ is almost finite-time solvent. □

The main result of this paper is the following generalization of Theorem 2 in Araujo et al. (2002) and Theorem 4.1 in Páscoa and Seghir (2008). We prove that, in order to rule out Ponzi schemes, it is not necessary to assume that default penalties are moderate with respect to utility functions. It is sufficient to assume that every agent is myopic with respect to default penalties.

**Theorem 5.1.** *If every agent is myopic with respect to default penalties then a competitive equilibrium with almost finite-time solvency exists.*

As a direct consequence of Proposition 4.3, we obtain the main existence result in Araujo et al. (2002, Theorem 2) as a corollary of Theorem 5.1.

**Corollary 5.1** (Araujo et al. (2002))**.** *If there is no default penalty then there exists a competitive equilibrium, i.e.,* $\mathrm{Eq}(\mathscr{E}) \neq \emptyset$.

As a direct consequence of Proposition 4.5, we obtain the existence result in Páscoa and Seghir (2008, Theorem 4.1) as a corollary of Theorem 5.1.

**Corollary 5.2** (Páscoa and Seghir (2008))**.** *If default penalties are moderate with respect to utility functions then there exists a constrained competitive equilibrium, i.e., the set* $\mathrm{Eq}_\alpha(\mathscr{E}) \neq \emptyset$.

*Remark* 5.2. Páscoa and Seghir (2008) claim to prove that not only the set $\mathrm{Eq}_\alpha(\mathscr{E})$ is non-empty when default penalties are $\alpha$-moderate, but also that the set $\mathrm{Eq}(\mathscr{E})$ is non-empty. However, in order to get existence of $\widetilde{T}$ in the arguments of the proof of their main result (Páscoa and Seghir (2008, Theorem 4.1, p. 15)), they implicitly consider $\alpha$-constrained plans.

## 6 Proof of Theorem 5.1

Fix $\tau \in \mathscr{T}$ with $\tau > 0$. We denote by $A^\tau$ the set

$$\forall \xi \in D, \quad t(\xi) > \tau \Longrightarrow a(\xi) = 0.$$

Recall that $B^\tau$ denotes the set of plans $a \in A^\tau$ satisfying the additional condition

$$\forall \xi \in D, \quad t(\xi) = \tau \Longrightarrow \varphi(\xi) = 0.$$

Given a process $(p,q,\kappa) \in \Pi$, we denote by $B^{i,\tau}(p,q,\kappa)$ the set defined by

$$B^{i,\tau}(p,q,\kappa) = B^i(p,q,\kappa) \cap B^\tau.$$

**Definition 6.1.** *A competitive equilibrium for the truncated economy $\mathscr{E}^\tau$ is a family of prices and delivery rates $\pi = (p,q,\kappa) \in \Pi$ and an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in B^\tau$ such that*

**(a)** *for every agent $i$, the plan $a^i$ is optimal, i.e.,*

$$a^i \in d^{i,\tau}(p,q,\kappa) = \operatorname{argmax}\{\Pi^{i,\tau}(p,a) : a \in B^{i,\tau}(p,q,\kappa)\}, \tag{6.1}$$

**(b)** *commodity markets clear at every node up to period $\tau$, i.e.,*

$$\sum_{i\in I} x^i(\xi_0) = \sum_{i\in I} \omega^i(\xi_0) \tag{6.2}$$

*and for all $\xi \in D^\tau \setminus \{\xi_0\}$,*

$$\sum_{i\in I} x^i(\xi) = \sum_{i\in I} \left[\omega^i(\xi) + Y(\xi)x^i(\xi^-)\right], \tag{6.3}$$

**(c)** *asset markets clear at every node up to period $\tau - 1$, i.e., for all $\xi \in D^{\tau-1}$,*

$$\sum_{i\in I} \theta^i(\xi) = \sum_{i\in I} \varphi^i(\xi), \tag{6.4}$$

**(d)** *deliveries match up to period $\tau$, i.e., for all $\xi \in D^\tau \setminus \{\xi_0\}$ and all $j \in J$,*

$$\sum_{i\in I} V(\kappa,p,\xi,j)\theta^i(\xi^-,j) = \sum_{i\in I} d^i(\xi,j). \tag{6.5}$$

*Remark* 6.1. Observe that if a plan $a$ belongs to $B^\tau$, then $\Pi^{i,\tau}(p,a)$ and $\Pi^i(p,a)$ coincide for every price process $p$.

*Remark* 6.2. Observe that if $(\pi,\boldsymbol{a})$ is a competitive equilibrium for the truncated economy $\mathscr{E}^\tau$, then without any loss of generality, we can assume that $q(\xi)=0$ and $\theta(\xi)=0$ for every terminal node $\xi\in D_\tau$.

It is claimed in Páscoa and Seghir (2008) that a competitive equilibrium for every truncated economy $\mathscr{E}^\tau$ exists, and that commodity prices are uniformly bounded away from 0. For the sake of completeness, we postpone to Appendix A.2 a simple proof of this result.

**Proposition 6.1.** There exists a process $m=(m(\xi))_{\xi\in D}$ of strictly positive numbers $m(\xi)>0$ such that for every period $\tau$, there exists a competitive equilibrium $(\pi^\tau,\boldsymbol{a}^\tau)$ of the truncated economy $\mathscr{E}^\tau$ satisfying $\left\|p(\xi)\right\|\geqslant m(\xi)$ at every node $\xi\in D^{\tau-1}$.

For each $\tau\in\mathscr{T}$ with $\tau\geqslant 1$, we let $(\pi^\tau,\boldsymbol{a}^\tau)$ be a competitive equilibrium for the economy $\mathscr{E}^\tau$ where $\pi^\tau=(p^\tau,q^\tau,\kappa^\tau)$ and $\boldsymbol{a}^\tau=(a^{i,\tau})_{i\in I}$. Each process $\pi^\tau$ belongs to $\operatorname{cl}\Pi$ which is weakly compact as a product of compact sets. Passing to a subsequence if necessary, we can assume that the sequence $(\pi^\tau)_{\tau\in\mathscr{T}}$ converges to a process $\pi=(p,q,\kappa)$ in $\operatorname{cl}\Pi$. Following Proposition 6.1, for each node $\xi\in D$, we have $\left\|p(\xi)\right\|\geqslant m(\xi)>0$. In particular, for each period $t$ and every plan $a\in A$, the payoff $\Pi^{i,t}(p,a)$ is well-defined.

By feasibility at each node $\xi$, we get for each $j$

$$x^{i,\tau}(\xi)\leqslant\Omega(\xi),\quad \varphi^{i,\tau}(\xi,j)\leqslant M(\xi,j)\quad\text{and}\quad \theta^{i,\tau}(\xi,j)\leqslant M(\xi,j).$$

This implies that the sequence $(x^{i,\tau}(\xi),\varphi^{i,\tau}(\xi),\theta^{i,\tau}(\xi))_{\tau\in\mathscr{T}}$ is uniformly bounded. By optimality, the delivery $d^{i,\tau}(\xi,j)$ is always lower than $V(p^\tau,\xi,j)\varphi^{i,\tau}(\xi^-,j)$ and therefore the sequence $(d^{i,\tau}(\xi))_{\tau\in\mathscr{T}}$ is uniformly bounded. Passing to a subsequence if necessary, we can assume that for each $i$, the sequence $(a^{i,\tau})_{\tau\in\mathscr{T}}$ converges to a process $a^i\in A$.

We claim that $(\pi,\boldsymbol{a})$ is a competitive equilibrium with almost finite-time solvency for the economy $\mathscr{E}$. It is straightforward to check that each plan $a^i$ belongs to the budget set $B^i(p,q,\kappa)$ and that the feasibility conditions (2.5), (2.6), (2.7) and (2.8) are satisfied. Applying Proposition 5.2, we get that the plan $a^i$ is almost finite-time solvent. We propose now to prove that $a^i$ is optimal among almost finite-time solvent plans, i.e., $\operatorname{Pref}^i(p,a^i)\cap B^i_\star(p,q,\kappa)$ is empty. Assume by way of contradiction that there exists a plan $\overline{a}$ in the budget set $B^i_\star(p,q,\kappa)$, $\varepsilon>0$ and $T^1\in\mathbb{N}$ satisfying

$$\forall T\geqslant T^1,\quad \Pi^{i,T}(p,\overline{a})>\Pi^{i,T}(p,a^i)+\varepsilon. \tag{6.6}$$

Since $a^i$ is physically feasible, we have

$$\forall\xi\in D,\quad x^i(\xi)\leqslant\Omega(\xi).$$

It follows from Assumptions (A.2) and (A.3) that

$$U^i(x^i) \leqslant U^i(\Omega) < +\infty.$$

This implies that

$$\lim_{T\to\infty} \Pi^{i,T}(p,a^i) = \Pi^i(p,a^i).$$

It follows that there exists $T^2 \geqslant T^1$ such that

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p,a^i) + \frac{\varepsilon}{2} > \Pi^i(p,a^i). \tag{6.7}$$

Since the plan $\overline{a}$ is almost finite-time solvent, there exists $T > T^2$ and $\widehat{a}$ in the truncated budget set $B^i(p,q,\kappa) \cap B^T$ such that

$$\widehat{a}\mathbf{1}_{[0,T^2]} = \overline{a}\mathbf{1}_{[0,T^2]} \quad \text{and} \quad \Pi^{i,T}(p,\widehat{a}) \geqslant \Pi^{i,T}(p,\overline{a}) - \frac{\varepsilon}{4}. \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) we get

$$\Pi^{i,T}(p,\widehat{a}) > \Pi^i(p,a^i) + \frac{\varepsilon}{4}.$$

We let $\psi^i$ be the correspondence from $A$ to $A^T$ defined by

$$\forall a \in A, \quad \psi^i(a) = \left\{ b \in B^T \ : \ \Pi^{i,T}(p,b) > \frac{\varepsilon}{4} + \Pi^i(p,a) \right\}.$$

Let $F^i$ be the correspondence from $\Pi \times A$ to $A^T$ defined by

$$\forall (\pi,a) \in \Pi \times A, \quad F^i(\pi,a) = B^{i,T}(\pi) \cap \psi^i(a).$$

Following the arguments in Páscoa and Seghir (2008), we have the following continuity result.

**Lemma 6.1.** *The correspondence $F^i$ is lower semi-continuous for product topologies on $\Pi \times A$.*

Observe that

$$\widehat{a} \in F^i((p,q,\kappa),a^i).$$

We proved that there exists a strictly increasing sequence $(T_n)_{n\in\mathbb{N}}$ with $T_n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}((p_n,q_n,\kappa_n),a^i_n) = ((p,q,\kappa),a^i)$$

where

$$(p_n,q_n,\kappa_n) = (p^{T_n},q^{T_n},\kappa^{T_n}), \quad a^i_n = a^{i,T_n}.$$

Since $F^i$ is lower semi-continuous, there exists $\nu$ large enough such that

$$\widehat{a}_\nu \in F^i((p_\nu, q_\nu, \kappa_\nu), a^i_\nu).$$

In particular we have

$$\widehat{a}_\nu \in B^{i,T_\nu}(p_\nu, q_\nu, \kappa_\nu) \cap B^{T_\nu} \quad \text{and} \quad \Pi^{i,T_\nu}(p_\nu, \widehat{a}_\nu) \geqslant \Pi^i(p_\nu, a^i_\nu) + \frac{\varepsilon}{4}.$$

This contradicts the optimality of $a^i_\nu$.[3]

We have thus proved that for each $i$, the plan $a^i$ is almost-finite solvent and satisfies

$$\mathrm{Pref}^i(p, a^i) \cap B^i_\star(p, q, \kappa) = \emptyset.$$

This means that $a^i$ belongs to the demand set $d^i_\star(\pi)$. We already proved that all markets clear. This means that $(\pi, \boldsymbol{a})$ is a competitive equilibrium with almost finite-time solvency.

## 7 Conclusion

This paper shows that it is possible to adapt the approach of restricting action plans to have finite effective debt, introduced in the work of Levine and Zame (1996), to models with default and collateralized promises. Working in this direction we introduce in the framework developed by Araujo et al. (2002) and Páscoa and Seghir (2008) the concept of almost finite-time solvency and show that the restrictions imposed in these two papers to rule out Ponzi schemes imply that plans are almost finite-time solvent. We also define the notion of what we term equilibrium with almost finite-time solvency and provide sufficient conditions for its existence. It turns out that the existence results in Araujo et al. (2002) and Páscoa and Seghir (2008) can be derived as straightforward corollaries of our existence result.

## A Appendix

We collect in this appendix the proofs of some technical results.

### A.1 Continuity on order intervals

Assumptions (A.2) and (A.3) imply that when restricted to the order interval $[0, \Omega]$, the function $U^i$ is weakly continuous.

**Proposition A.1.** The function $U^i$ is weakly continuous on $[0, \Omega^i]$.

[3] Recall that $((p_\nu, q_\nu, \kappa_\nu), \boldsymbol{a}_\nu)$ is a competitive equilibrium of the truncated economy $\mathscr{E}^{T_\nu}$.

For the sake of completeness, we give the straightforward proof of this result.

*Proof of Proposition A.1.* For every period $\tau \in \mathbb{N}$, we let $U^{i,\tau}$ be the function defined on $X$ by

$$U^{i,\tau}(x) = \sum_{\xi \in D^\tau} u^i(\xi, x(\xi)).$$

Let $(x^n)_{n\in\mathbb{N}}$ be a sequence of consumption processes in $[0,\Omega]$, weakly converging to $x$.[4] Fix $\varepsilon > 0$. From Assumption (A.3), the utility $U^i(\Omega)$ is finite. Then there exists $\tau \in \mathbb{N}$ such that

$$U^i(\Omega) - U^{i,\tau}(\Omega) = \sum_{\xi \in D\setminus D^\tau} u^i(\xi, \Omega^i(\xi)) \leqslant \frac{\varepsilon}{4}.$$

Observe that for each $n$

$$\begin{aligned} \left|U^i(x) - U^i(x^n)\right| &\leqslant \left|U^{i,\tau}(x) - U^{i,\tau}(x^n)\right| + 2 \sum_{\xi \in D\setminus D^\tau} u^i(\xi, \Omega^i(\xi) \\ &\leqslant \left|U^{i,\tau}(x) - U^{i,\tau}(x^n)\right| + \frac{\varepsilon}{2}. \end{aligned} \tag{A.1}$$

From Assumption (A.2), each utility function $u^i(\xi,\cdot)$ is continuous. Since $(x^n)_{n\in\mathbb{N}}$ converges weakly to $x$, there exists $n_\varepsilon > 0$ large enough such that

$$\forall n \geqslant n_\varepsilon, \quad \left|U^{i,\tau}(x) - U^{i,\tau}(x^n)\right| \leqslant \frac{\varepsilon}{2}. \tag{A.2}$$

Combining (A.1) and (A.2), we get the desired result. □

## A.2 Proof of Proposition 6.1

We consider the following modification of the normalization of the default penalty. For every $\varepsilon > 0$ and every period $\tau$, we let

$$W^{i,\tau}_\varepsilon(\pi, a) = \sum_{\xi \in D^\tau \setminus \{\xi_0\}} \sum_{j\in J} \lambda^i(\xi, j) \frac{\left[V(p,\xi,j)\varphi(\xi^-, j) - d(\xi,j)\right]^+}{p(\xi)v(\xi) + \varepsilon \left\|q(\xi)\right\|}$$

and

$$\Pi^{i,\tau}_\varepsilon(\pi, a) = U^{i,\tau}(x) - W^{i,\tau}_\varepsilon(\pi, a).$$

When the process $\pi$ belongs to $\operatorname{cl}\Pi$, the functions $(W^{i,t}_\varepsilon)_{t\geqslant 1}$ are well-defined for every $\varepsilon > 0$. A pair $(\pi, \boldsymbol{a})$ where $\pi \in \Pi$ and $\boldsymbol{a} = (a^i)_{i\in I}$ is an allocation with $a^i \in B^\tau$, is said to be a competitive equilibrium of the truncated economy $\mathcal{E}^\tau_\varepsilon$ if market clearing conditions (6.2), (6.3), (6.4) and (6.5) are satisfied and the optimality condition (6.1) is replaced by

[4] Remember that the weak topology on $X$ is metrizable.

**($\mathbf{a}_\varepsilon$)** for every agent $i$, the plan $a^i$ is optimal with respect to $\Pi^{i,\tau}_\varepsilon$, i.e.,

$$a^i \in d^{i,\tau}_\varepsilon(p,q,\kappa) = \operatorname{argmax}\{\Pi^{i,\tau}_\varepsilon(\pi,a) \,:\, a \in B^{i,\tau}(\pi)\}.$$

Observe that for every process $\pi$ of prices and delivery rates in $\operatorname{cl}\Pi$, the quantity $p(\xi)v(\xi) + \varepsilon \left\|q(\xi)\right\|$ is never 0. It is now very easy to adapt the arguments in Araujo et al. (2002) and prove that a competitive equilibrium $(\pi, \boldsymbol{a})$ for the truncated economy $\mathscr{E}^\tau_\varepsilon$ exists for any $\varepsilon > 0$ where $\pi \in \operatorname{cl}\Pi$. Since utility functions are strictly increasing, we must have $p(\xi) \in \mathbb{R}^L_{++}$ for each node $\xi \in D^\tau$. We propose to exhibit an exogenous lower bound $m(\xi)$ for every node $\xi$ with $t(\xi) < \tau$. Fix a node $\xi \in D^{\tau-1}$, $\alpha > 0$ and an agent $i \in I$. Let $\widetilde{a}^i_\alpha$ be the plan in $B^\tau$ defined for every node $\zeta \in D^\tau$ by

$$\widetilde{a}^i_\alpha(\zeta) = \begin{cases} a^i(\zeta) & \text{if} \quad \zeta \notin \{\xi\} \cup \xi^+ \\ (x^i(\xi) + f(\pi,\xi)\alpha \mathbf{1}_L, \theta^i(\xi), \varphi(\xi) + \alpha \mathbf{1}_J, d^i(\xi)) & \text{if} \quad \zeta = \xi \\ (x^i(\zeta), \theta^i(\zeta), \varphi^i(\zeta), \widetilde{d}^i_\alpha(\zeta)) & \text{if} \quad \zeta \in \xi^+ \end{cases}$$

where

$$f(\pi,\xi) = \frac{\left\|q(\xi)\right\| - p(\xi)C(\xi)}{\left\|p(\xi)\right\|} \quad \text{with} \quad C(\xi) = \sum_{j \in J} C(\xi, j)$$

and for every $j$,

$$\widetilde{d}^i_\alpha(\zeta) = d^i(\zeta, j) + \alpha D(p, \zeta, j).$$

In other words, we propose to short-sell at node $\xi$ an additional quantity $\alpha > 0$ of each asset $j$ and to increase consumption of each good by $f(\pi,\xi)\alpha$ units. At each successor node $\zeta \in \xi^+$, we propose to “fully" default on additional short-sales. By doing so, at node $\xi$ we get an additional amount of $\alpha \left\|q(\xi)\right\|$ units of accounts from short-selling. In order to satisfy the constraint imposed by the collateral requirements, we should purchase the bundle $\alpha C(\xi)$ at node $\xi$. This is possible if $\left\|q(\xi)\right\| \geqslant p(\xi)C(\xi)$. In other words, if $f(\pi,\xi) \geqslant 0$ then the plan $\widetilde{a}^i_\alpha$ belongs to the budget set $B^{i,\tau}(\pi)$ for every $\alpha > 0$. We propose to compare the payoffs of the two plans $a^i$ and $\widetilde{a}^i_\alpha$.

First observe that

$$U^{i,\tau}(\widetilde{x}^i_\alpha) - U^{i,\tau}(x^i) = u^i(\xi, x^i(\xi) + f(\pi,\xi)\alpha \mathbf{1}_L) - u^i(\xi, x^i(\xi)).$$

Moreover, since for each $\zeta \in \xi^+$

$$\left[V(p,\zeta,j)\{\varphi(\xi,j) + \alpha\} - \{d(\zeta,j) + \alpha D(p,\zeta,j)\}\right]^+$$

is lower than

$$\left[V(p,\zeta,j)\varphi(\xi,j) - d(\zeta,j)\}\right]^+ + \left[V(p,\zeta,j)\alpha - \alpha D(p,\zeta,j)\right]^+$$

we get

$$\begin{aligned}\Pi_\varepsilon^{i,\tau}(\pi,\widetilde{a}_\alpha^i)-\Pi_\varepsilon^{i,\tau}(\pi,a^i) &\geqslant u^i(\xi,x^i(\xi)+f(\pi,\xi)\alpha\mathbf{1}_L)-u^i(\xi,x^i(\xi))\\ &- \alpha\sum_{\zeta\in\xi^+}\sum_{j\in J}\lambda^i(\zeta,j)\frac{\left[V(p,\zeta,j)-D(p,\zeta,j)\right]^+}{p(\zeta)v(\zeta)+\varepsilon\left\|q(\zeta)\right\|}.\end{aligned}$$

Let us denote by $\delta_\varepsilon^{i,\tau}$ the real number defined by

$$\delta_\varepsilon^{i,\tau}=\lim_{\alpha\to 0^+}\frac{\Pi_\varepsilon^{i,\tau}(\pi,\widetilde{a}_\alpha^i)-\Pi_\varepsilon^{i,\tau}(\pi,a^i)}{\alpha},$$

and let us denote by $\nabla^+u^i(\xi,x^i(\xi))$ the vector in $\mathbb{R}^L_{++}$ which $\ell$-th coordinate $\nabla^+_\ell u^i(\xi,x^i(\xi))$ is defined by[5]

$$\nabla^+_\ell u^i(\xi,x^i(\xi))=\lim_{\beta\to 0^+}\frac{u^i(\xi,x^i(\xi)+\beta\mathbf{1}_{\{\ell\}})-u^i(\xi,x^i(\xi))}{\beta}.$$

Then

$$\begin{aligned}\delta_\varepsilon^{i,\tau} &\geqslant \left\|\nabla^+u^i(\xi,x^i(\xi))\right\|f(\pi,\xi)-\sum_{\zeta\in\xi^+}\sum_{j\in J}\lambda^i(\zeta,j)\frac{\left[V(p,\zeta,j)-D(p,\zeta,j)\right]^+}{p(\zeta)v(\zeta)+\varepsilon\left\|q(\zeta)\right\|}\\ &\geqslant \left\|\nabla^+u^i(\xi,\Omega^i(\xi))\right\|f(\pi,\xi)-\sum_{\zeta\in\xi^+}\sum_{j\in J}\lambda^i(\zeta,j)\frac{\left[V(p,\zeta,j)-D(p,\zeta,j)\right]^+}{p(\zeta)v(\zeta)}\\ &\geqslant \left\|\nabla^+u^i(\xi,\Omega^i(\xi))\right\|f(\pi,\xi)-\sum_{\zeta\in\xi^+}\sum_{j\in J}\lambda^i(\zeta,j)\frac{H(\zeta,j)}{M(\zeta,j)}.\end{aligned}$$

Therefore, if

$$f(\pi,\xi)>g(\xi):=\frac{\sum_{\zeta\in\xi^+}\sum_{j\in J}\lambda^i(\zeta,j)\frac{H(\zeta,j)}{M(\zeta,j)}}{\left\|\nabla^+u^i(\xi,\Omega^i(\xi))\right\|}$$

then $\Pi_\varepsilon^{i,\tau}(\pi,\widetilde{a}_\alpha^i)>\Pi_\varepsilon^{i,\tau}(\pi,a^i)$ for $\alpha>0$ small enough. It follows that we must have

$$\frac{1-\left\|p(\xi)\right\|-p(\xi)C(\xi)}{\left\|p(\xi)\right\|}=f(\pi,\xi)\leqslant g(\xi).$$

Hence there exists $m(\xi)>0$ depending only on the primitives of the economy $\mathscr{E}$ such that $\left\|p(\xi)\right\|\geqslant m(\xi)$.

[5] The existence of $\nabla^+_\ell u^i(\xi,x^i(\xi))$ is a consequence of the concavity of $u^i(\xi,\cdot)$. The strict monotonicity of $u^i(\xi,\cdot)$ implies that $\nabla^+_\ell u^i(\xi,x^i(\xi))$ is strictly positive.

Consider now the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ defined by

$$\forall n \in \mathbb{N}, \quad \varepsilon_n = \frac{1}{n+1}.$$

For each $n \in \mathbb{N}$, there exists an equilibrium $(\pi_n, \boldsymbol{a}_n)$ of the truncated economy $\mathscr{E}^\tau_{\varepsilon_n}$. Following standard arguments, there exists a process $\pi \in \mathrm{cl}\,\Pi$ of prices and delivery rates and a process $\boldsymbol{a}$ of plans $a^i \in B^\tau$ such that, passing to a subsequence if necessary, the sequence $(\pi_n, \boldsymbol{a}_n)_{n\in\mathbb{N}}$ converges to $(\pi, \boldsymbol{a})$. Since for each $n$, we have $\left\|p_n(\xi)\right\| \geqslant m(\xi)$ for every non-terminal node $\xi \in D^{\tau-1}$, passing to the limit, we get that $\left\|p(\xi)\right\| \geqslant m(\xi)$, in particular $p(\xi) > 0$ for each $\xi \in D^\tau$.[6] Therefore the payoff $\Pi^{i,\tau}(p,a)$ is well-defined for every plan $a \in B^\tau$. It is now standard to prove that the limit $(\pi, \boldsymbol{a})$ is actually a competitive equilibrium of the truncated economy $\mathscr{E}^\tau$.

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[6] Recall that for every terminal node $\xi \in D_\tau$, we have the normalization $\left\|p_n(\xi)\right\| = 1$, implying that $p(\xi) > 0$.

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