Asymmetric Contests with Conditional Investments*

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Abstract

This paper investigates equilibrium behavior in a class of games that models asymmetric competitions with unconditional and conditional investments. Such competitions include lobbying settings, labor-market tournaments, and R&D races, among others. I provide an algorithm that constructs the unique equilibrium in these games, and apply it to study contests in which a fraction of each competitor's investment is sunk and the rest is paid only by the winners. Complete-information all-pay auctions are a special case.

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1 Introduction

Many competitions are characterized by asymmetries among competitors. One example is the competition for rents in a regulated market. Since firms often compete by engaging in lobbying activities, firm-specific attributes that affect the cost and effectiveness of lobbying may lead to asymmetries in competition. Such firm-specific attributes include cost of capital and geographic location, which affect lobbying costs, and quality of political connections, which affect lobbying efficacy. When these attributes vary across firms, different firms face different costs of achieving any desired level of influence through lobbying. Similar cost asymmetries result from heterogeneity across competitors in competitions for promotions and in research and development (R&D) races.

These competitions typically include sunk investments: competitors exert effort, make outlays, or bear other non-refundable costs in the course of the competition. These investments are *unconditional* of the competition's outcome. In addition, competitors sometimes commit to investments *conditional* on winning. For example, in a lobbying setting, a firm might commit to building factories and creating new jobs if it is granted a monopoly position. In some rent-seeking scenarios, "a decision-maker accepts contingent rewards or bribes offered by interested parties."[1] In an R&D race, a better prototype commits the firm to higher expenditures conditional on winning.

This paper investigates equilibrium behavior in a multiprize *contest* model that is rich enough to accommodate general asymmetries and both unconditional and conditional investments. In a contest, players compete for one of several identical prizes. Each player chooses a "score", and the players with the highest scores obtain one prize each. In a lobbying setting, for example, a player's score represents the influence he achieves. Conditional on winning or losing, a player's payoff decreases continuously with his chosen score. This formulation allows for a wide degree of heterogeneity among players, including differing production technologies, costs of capital, and abilities. In addition, the difference in a player's payoff between winning and losing may depend on his chosen score. This accommodates a combination of unconditional and conditional investments.[2] The primitives of the model are commonly known, capturing players' knowledge of the asymmetries among them. This helps interpret players' payoffs as "economic rents", in contrast to "information rents" that arise in models of competition with private information.[3] Contests are defined

[1] Bernheim and Whinston (1986).

[2] It also accommodates player-specific risk attitudes, and player- and score-dependent valuations for a prize.

[3] Examples of such models include Moldovanu & Sela (2001, 2006), Kaplan, Luski, Sela, & Wettstein

in Section 2. The payoff result of Siegel (2009), who characterizes equilibrium payoffs in a more general model of contests, provides a closed-form formula for players' equilibrium payoffs.[4]

The main result of this paper is a constructive characterization of the unique equilibrium for a large class of contests. The equilibrium is constructed by an algorithm whose input is a set of parameters that describe a contest, and whose output is players' equilibrium strategies. Once a contest is specified, each player's equilibrium performance, expenditures, and probability of winning a prize can be deduced. The degree of rent dissipation and level of efficiency, relevant for deriving policy implications and contest design, can be calculated as well.[5] None of these can be derived from equilibrium payoffs alone. The framework can also be used to examine how specific features of competition affect the competition's outcome. I consider a class of contests parameterized by the size of the unconditional component of the investment, and investigate the effects of varying the size of this component.

A special case of contests is the single-prize and multiprize all-pay auction with complete-information (henceforth: all-pay auction). The all-pay auction, along with its variants, has been used extensively to model rent-seeking and lobbying activities (Hillman & Samet (1987), Hillman & Riley (1989), Baye, Kovenock & de Vries (1993), González-Díaz (2007)), competitions for a monopoly position (Ellingsen (1991)), waiting in line (Clark & Riis (1998)), sales (Varian (1980)), and R&D races (Dasgupta (1986)), competitions for multiple prizes (Clark & Riis (1998) and Barut & Kovenock (1998)), the effect of lobbying caps (Che & Gale (1998, 2006) and Kaplan & Wettstein (2006)), and R&D races with endogenous prizes (Che & Gale (2003)). In an all-pay auction, all investments are unconditional and players' costs are linear. Contests generalize all-pay auctions by allowing for non-linear, asymmetric costs, and accommodating both unconditional and conditional investments.[6]

Section 3 begins the analysis by considering contests with two players and one prize.

(2002)), and Parreiras & Rubinchik (2006).

[4]Siegel (2009) allows for players' payoffs to weakly decrease in score, but does not solve for equilibrium.

[5]Rent dissipation and efficiency are context-specific and depend, for example, on whether expenditures are productive or wasteful.

[6]Other models of competition postulate a probabilistic relation between competitors' efforts and prize allocation. See the papers by Edward Lazear and Sherwin Rosen (1981) and Gordon Tullock (1980). For a comprehensive treatment of the literature on competitions with sunk investments see Nitzan (1994) and Konrad (2007).

In such contests, players compete "head to head" by choosing scores from (essentially) the same interval. Using this property, I show that there is a unique equilibrium and provide a closed-form formula for players' equilibrium strategies.[7]

The analysis of multiprize contests is much more challenging, because players do not necessarily compete head to head. That is, different players may compete by choosing scores from different intervals. Moreover, when players are sufficiently heterogeneous, a player may choose scores from multiple intervals. This is demonstrated in the example of Figures 2 and 3.

To overcome these difficulties, I identify key properties of any "well-behaved" equilibrium of an $(m+1)$-player contest for m prizes. These properties, detailed in Section 3.1, imply that under a mild technical condition players' equilibrium strategies can be uniquely constructed by considering an appropriate "supply function" for the hazard rates of players' strategies as a function of score.[8] Section 3.2 provides an algorithm that does this. Using the fact that a well-behaved equilibrium exists, Section 3.3 rules out the existence of additional equilibria. This step is not straightforward, since little can be assumed about the structure of equilibria that are not well behaved, if they exist.

The equilibrium uniqueness result for $(m+1)$-player contests implies that contests with $n > m$ players have at *most* one equilibrium in which precisely $m+1$ players participate. The participation result of Siegel (2009) provides a sufficient condition for precisely $m+1$ players to participate in any equilibrium. Therefore, the algorithm constructs the unique equilibrium for a large class of multiplayer, multiprize contests.

Section 4 applies these results to investigate the effects of conditional investments when competitors compete by using the same underlying production technology. In this class of *simple* contests, a positive fraction $\alpha \leq 1$ of each competitor's costs is sunk, and the remaining $1-\alpha$ is paid only by the winners of the $m \geq 1$ prizes. Competitors may differ in how efficiently they employ the common underlying technology. This difference in efficiency is captured by player-specific cost coefficients, which are multiplied by a common cost function representing the common production technology. Competitors may also differ in their valuations for a prize.

I show that simple contests have a unique equilibrium, in which every player chooses a score from an interval. Moreover, after normalizing each player's efficiency by dividing his

[7] This result generalizes those of Kaplan & Wettstein (2006) and Che & Gale (2006), who considered two-player contests with ordered cost functions and no conditional investments.

[8] The construction also uses knowledge of players' equilibrium payoffs, which are characterized in Siegel (2009).

cost coefficient by his valuation for a prize, I show that the equilibrium strategies of more efficient players first-order stochastically dominate those of less efficient players, and that more efficient players win prizes more often than less efficient players. As α approaches 0, the most efficient players obtain a prize with near certainty. When players differ only in their valuations for a prize, as α approaches 0 the allocation of prizes becomes efficient and expenditures are maximized.

The limit of the equilibria as α approaches 0 is an equilibrium of the corresponding "discriminatory-price auction," in which all investments are conditional on winning. In this equilibrium, no player chooses weakly dominated strategies with positive probability, and the equilibrium is robust to the tie-breaking rule. This provides a selection criterion among the continuum of equilibria of the discriminatory-price auction, which are not payoff equivalent. A special case is the complete-information first-price auction.

When $\alpha = 1$, all investments are unconditional. I provide a closed-form formula for players' equilibrium strategies in this case. If, in addition, the common production technology is linear, we have a multiprize all-pay auction.[9]

Appendix A contains an example of an equilibrium that is not well-behaved. Appendix B contains proofs of results from Section 3. Appendix C contains proofs of results from Section 4.

2 The Model

In a *contest*, n players compete for m homogeneous prizes, $0 < m < n$. The set of players $\{1, \ldots, n\}$ is denoted by N. Players compete by each choosing a score, simultaneously and independently. Player i chooses a score $s_i \in S_i = [0, \infty)$. Each of the m players with the highest scores wins one prize. In case of a relevant tie, any procedure may be used to allocate the tie-related prizes among the tied players.

Given scores $s = (s_1, \ldots, s_n)$, one for each player, *player i's payoff* is

$$u_i(s) = P_i(s) v_i(s_i) - (1 - P_i(s)) c_i(s_i)$$

where $v_i : S_i \to \mathbb{R}$ is *player i's valuation for winning*, $c_i : S_i \to \mathbb{R}$ is *player i's cost of*

[9] This result generalizes and corrects that of Clark & Riis (1998). They constructed an equilibrium for the multiprize all-pay auction and claimed it was unique. Their proof of uniqueness relied on showing that in any equilibrium the best response set of each player is an interval. Their proof of this latter claim was incorrect.

losing, and $P_i : \times_{j \in N} S_j \to [0,1]$ is *player i's probability of winning*, which satisfies

$$P_i(s) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i \\ 1 & \text{if } s_j < s_i \text{ for } N - m \text{ or more players } j \neq i \\ \text{any value in } [0,1] & \text{otherwise} \end{cases}$$

such that $\sum_{j=1}^{n} P_j(s) = m$.

Note that a player's probability of winning depends on all players' scores, but his valuation for winning and cost of losing depend only on his chosen score. The primitives of the contest are commonly known.

I consider contests that meet the following Assumptions B1-B3.

B1 v_i *and* $-c_i$ *are continuous and strictly decreasing.*

B2 $v_i(0) > 0$ *and* $\lim_{s_i \to \infty} v_i(s_i) < c_i(0) = 0$.

Assumption B2 says that prizes are valuable, and that sufficiently high investments are prohibitively costly. Assumption B1 captures the all-pay component of contests. It is not satisfied by complete-information first-price auctions, for example, since a player pays nothing if he loses, so $c_i \equiv 0$. But assumption B1 *is* satisfied when an all-pay element is introduced, e.g., when every bidder pays some positive fraction of his bid whether he wins or not, and only the winner pays the balance of his bid. Assumptions B1 and B2 are depicted in Figure 1.

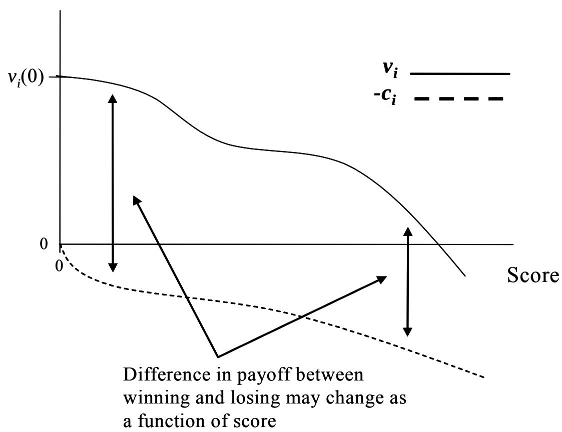

Figure 1: Assumptions B1 and B2

As Figure 1 shows, the payoff difference between winning and losing may depend on the player's chosen score. This accommodates a combination of unconditional and conditional investments, as shown in Section 4. A non-constant payoff difference can also be used to model different risk attitudes and prizes of varying values.

In a *separable contest*, this payoff difference is constant, so $v_i(s_i) = V_i - c_i(s_i)$ and $u_i(s) = P_i(s) V_i - c_i(s_i)$, where $V_i = v_i(0) > 0$. The value V_i can be thought of as player i's valuation for a prize, which does not depend on his chosen score, and $c_i(s_i)$ can be thought of as player i's cost of choosing score s_i, which does not depend on whether he wins or loses. If a given score can be achieved in different ways, $c_i(s_i)$ corresponds to the least costly way of achieving it. All investments are unconditional, and players are risk neutral. Figure 2 below depicts a separable contest with non-linear costs. Separable contests with linear costs are single- and multiprize complete-information all-pay auctions (Hillman & Samet (1987), Hillman & Riley (1989), Clark & Riis (1998)).[10]

Assumption B3, which completes the description of contests, uses the following definition.

Definition 1 (i) *Player i's* reach r_i *is the score at which his valuation for winning is* 0*. That is,* $r_i = v_i^{-1}(0)$*. Re-index players in (any) decreasing order of their reach, so that* $r_1 \geq r_2 \geq \ldots \geq r_n$*.*

(ii) *Player $m+1$ is the* marginal player*.*

(iii) *The* threshold T *of the contest is the reach of the marginal player:* $T = r_{m+1}$*.*

Assumption B3 states that the reach of the marginal player is different from that of the other players.

B3 Only the marginal player's reach is equal to the threshold, so $r_i \neq T$ for every player $i \neq m+1$.

In a separable contest, a player's reach is the score whose cost is the player's valuation for a prize. For example, in an all-pay auction a player's reach is his valuation for a prize, so Assumption B3 is met if the marginal player's valuation for a prize is different from those of the other players.

[10] Formally, $v_i(s_i) = V_i - s_i, c_i(s_i) = s_i$, and ties are resolved by randomizing uniformly, where V_i is bidder i's valuation for a prize.

The model of contests described here is a special case of Siegel's (2009) all-pay contest model. The main difference is that he allows for weakly decreasing v_i and $-c_i$.[11]

2.1 Existing Results

This subsection lists four results, Theorem 1, Theorem 2, Lemma 1, and Lemma 2 which I use in solving for equilibrium. They are immediate corollaries of results in Siegel (2009).[12] The first result characterizes players' equilibrium payoffs (without solving for equilibrium), and uses the following definition.

Definition 2 *Players i's* power w_i *is his valuation for winning at the threshold. That is,* $w_i = v_i(T)$. *In particular, the marginal player's power is* 0.

Assumptions B1 and B3 imply that only the marginal player has power 0. In an all-pay auction, for example, a player's power is equal to his valuation for a prize less that of the marginal player.

Theorem 1 *In any equilibrium of a contest, the expected payoff of every player equals the maximum of his power and* 0.

In addition to giving a closed-form formula for players' equilibrium payoffs, Theorem 1 shows that players $1, \ldots, m$ have strictly positive expected payoffs, and players $m+1, \ldots, n$ have expected payoffs of 0. It also shows that a player's expected payoff does not depend on his cost of losing.

A player *participates* in an equilibrium of a contest if with strictly positive probability he chooses scores associated with strictly positive costs of losing. The second result provides a sufficient condition for players $m+2, \ldots, n$ not to participate in any equilibrium.

Theorem 2 *If the normalized costs of losing and valuations for winning for the marginal player are, respectively, strictly lower and weakly higher than those of player* $i > m+1$, *that is*

$$\frac{c_{m+1}(x)}{v_{m+1}(0)} < \frac{c_i(x)}{v_i(0)} \text{ for all } x \geq 0 \text{ such that } c_i(x) > 0$$

[11] Instead of Assumptions B1-B3, Siegel (2009) makes three assumptions (A1-A3), and specifies two Generic Conditions (Cost Condition and Power Condition) on players' valuations for winning and costs of losing. Assumptions B1 and B2 imply his Assumptions A1-A3 and Cost Condition, and Assumptions B1 and B3 imply his Power Condition. In particular, a contest here is a generic contest of Siegel (2009).

[12] The results follow, respectively, from Theorem 1, Theorem 2, Corollary 1, and Theorem 1, Lemma 1, and footnote 19 of Siegel (2009).

and

$$\frac{v_{m+1}(x)}{v_{m+1}(0)} \geq \frac{v_i(x)}{v_i(0)} \text{ for all } x \geq 0$$

then player i does not participate in any equilibrium. In particular, if these conditions hold for all players $m+2,\ldots,n$, then in any equilibrium only the $m+1$ players $1,\ldots,m+1$ may participate.

The third result states that an equilibrium always exists.

Lemma 1 *Every contest has a Nash equilibrium.*

The fourth result enumerates four properties of any equilibrium.

Lemma 2 *In any equilibrium of a contest, (1) no score in $(0,T)$ is chosen with strictly positive probability by any player, (2) every score in $(0,T)$ is a best response for at least two players, (3) no score higher than T is a best response for any player, and (4) players $1,\ldots,m+1$ participate.*

3 Solving for Equilibrium

A player's (mixed) strategy is a probability distribution over $[0,\infty)$, and an equilibrium is a profile of strategies, one for each player, such that each player's strategy assigns probability 1 to the player's best-responses given the other players' strategies. I describe a strategy of player i by a cumulative distribution function (CDF) G_i, which for every $x \geq 0$ specifies the probability $G_i(x)$ that player i chooses a score lower or equal to x. A key step in solving for equilibrium is identifying players' best-response sets, or strategy supports.

Consider first the relatively simple case of two players and one prize. Part (2) of Lemma 2 shows that the players must behave in a way that makes every positive score up to the threshold a best response for both of them. That both players have interval best-response sets pins down the unique equilibrium even without knowing players' payoffs.

Theorem 3 *In a two-player, single-prize contest, the unique equilibrium is given by* $(G_1,G_2)(x) = \left(\frac{c_2(x)}{v_2(x)+c_2(x)}, \frac{w_1+c_1(x)}{v_1(x)+c_1(x)}\right)$ *on* $[0,\ T]$.[13]

[13] When there are more than two players, since players 1 and 2 participate in any equilibrium (part (4) of Lemma 2), a single-prize contest has at most one equilibrium in which two players participate. Example 3 in Siegel (2009) shows that a single-prize contest may have multiple equilibria when more than two players participate.

Proof. It is straightforward to verify that (G_1, G_2) is an equilibrium. For uniqueness, consider some equilibrium of the contest. By part (2) of Lemma 2, both players are indifferent among all scores in $(0, T)$. Moreover, no player can have an atom at T, otherwise the other player would not have best responses slightly below T, contradicting part (2) of Lemma 2. Therefore, the equilibrium has the form $\left(\frac{d_2+c_2(x)}{v_2(x)+c_2(x)}, \frac{d_1+c_1(x)}{v_1(x)+c_1(x)}\right)$ on $(0, T]$ for some d_1 and d_2, and neither CDF can reach 1 before T. By part (3) of Lemma 2, the CDFs of both players must reach 1 at exactly T. Consequently, $d_1 = v_1(T) = w_1$ and $d_2 = v_2(T) = 0$, so the equilibrium coincides with (G_1, G_2). ■

Section 4.1 below provides an application of this result to certain non-separable contests. Applied to separable contests, Theorem 3 extends the results of Kaplan & Wettstein (2006) and Che & Gale (2006), who considered two-player separable contests with ordered cost functions.

I now turn to multiprize contests. In this case, part (4) of Lemma 2 shows that more than two players participate in any equilibrium. This raises the possibility that different players may compete on different sets of scores. Moreover, players may have non-interval best-response sets. This is indeed the case when players have relative cost advantages at different scores. In fact, players' behavior can be quite "pathological".[14]

The challenge is therefore to (1) guarantee the existence of and solve for a "well-behaved" equilibrium and (2) rule out "pathological" equilibria. I do both for a large class of contests, which nests single- and multiprize all-pay auctions. To this end, I consider *regular contests*, which are contests that meet the following regularity condition.

R *The valuations for winning and costs of losing of players* $1, \ldots, m+1$ *are piecewise analytical on* $[0, T]$.[15].

Consider the three-player, two-prize, separable regular contest depicted in Figure 2.[16] Its unique equilibrium is depicted in Figure 3, drawn as cumulative probability distributions. In the equilibrium, the best-response set of player 2 is $(0, x_1] \cup [x_2, 1]$, and that of player 3 is $[0, x_3] \cup [x_4, 1]$. Each player is defined as being "active" on his best-response

[14] Appendix A depicts an equilibrium in which a player's best-response set is the Cantor set.

[15] A function f is piecewise analytical on $[0, T]$ if $[0, T]$ can be divided into a finite number of closed intervals such that the restriction of f to each interval is analytical. Analytical functions include polynomials, the exponent function, trigonometric functions, and power functions. Sums, products, compositions, reciprocals, and derivatives of analytical functions are analytical (see, for example, Chapman (2002)).

[16] Players' cost functions are given in Appendix B.7.

set, with the possible inclusion of 0.[17] As Figure 3 shows, different players compete on different sets of scores.

The algorithm described in Section 3.2 constructs the equilibrium by identifying the players active on each interval (denoted in curly brackets), and the "switching points" above which the set of active players changes ($x_k, 1 \leq k \leq 4$, and 1).

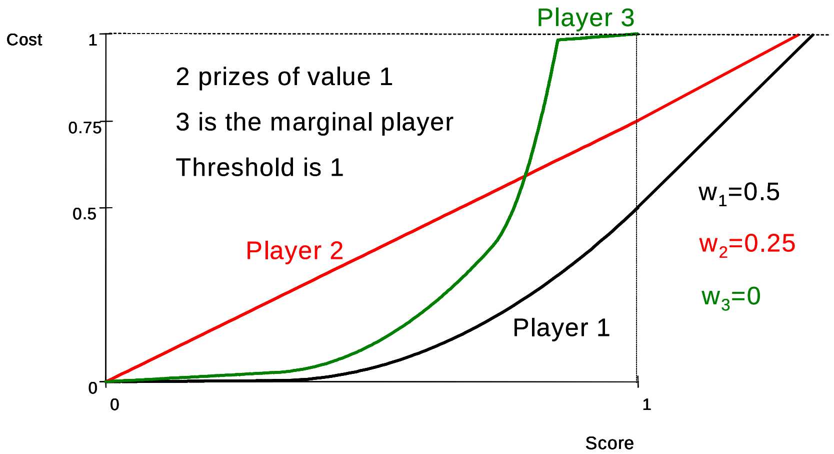

Figure 2: Player's costs, reaches, and powers

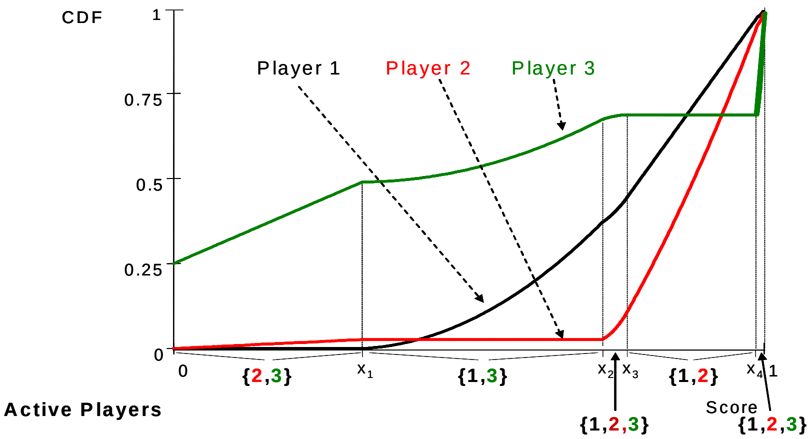

Figure 3: The unique equilibrium

[17]In the equilibria of Baye et al. (1993), who considered a single-prize all-pay auction that violates Assumption B3, a player's best-response set may be the union of 0 and a single interval whose lower endpoint is strictly positive. All such equilibria disappear when players' valuations are perturbed slightly to produce unique players with the first- and second-highest valuations (so that Assumption B3 holds). This leaves a single equilibrium, in which the best-response set of each player is an interval (or the singleton 0). A similar perturbation produces a single equilibrium, in which the best-response set of each player is an interval, in González-Díaz (2007). In contrast, the non-interval property that arises here is "fundamental" in nature: it is robust to perturbations in the contest's specification, and, moreover, a player's best-response set may consist of several disjoint intervals of positive length.

For such a construction to be possible, the equilibrium must be "well-behaved", in that for every $x < T$ the set of players active immediately to the right of x must remain constant. This is formalized by the following definition.

Definition 3 *An equilibrium is* constructible *if for every score $x < T$ there exists some $\bar{x} > x$ such that for each player either every score in $(x, \bar{x})$ is a best response, or no score in $(x, \bar{x})$ is a best response. I refer to equilibria that are not constructible as* non-constructible.

The algorithm solves for a constructible equilibrium $G = (G_1, \ldots, G_{m+1})$ of an $(m+1)$-player regular contest. This is done by using three properties of constructible equilibria to generate a profile of CDFs, and then showing that these CDFs form a constructible equilibrium. The three properties are derived in Section 3.1. I begin with an informal overview of these properties and the algorithm.

The first property is that only player $m+1$ has an atom at 0, and the size of the atom is determined by players' payoffs (which are given by Theorem 1). This determines $G(0)$. The second property is that for every $x < T$ the value of G on some right-neighborhood of x can be uniquely determined from $G(x)$ and the set of players active immediately to the right of x. This value coincides with the solution to a set of simultaneous equations derived from the condition that active players obtain their equilibrium payoff and inactive players' CDFs do not increase. The third property is that the set of players active to the right of $x < T$ is uniquely determined by $G(x)$, and the first switching point above x can be uniquely determined as well. This is seen by showing that $G(x)$, players' valuations for winning and costs of losing in a right-neighborhood of x, and players' payoffs jointly define a "supply function" for hazard rates whose unique positive fixed point identifies the set of players active to the right of x. This uses Condition R. The first switching point above x is the first point that violates one of the following two equilibrium conditions when G is defined to the right of x as described in the second property above. First, no player should be able to obtain more than his equilibrium payoff. Thus, an inactive player becomes active when other players' CDFs, and therefore his probability of winning, become sufficiently high. Second, players' CDFs must be non-decreasing. Thus, an active player becomes inactive if his CDF would otherwise decrease.

Combining these three properties, the algorithm constructs G by proceeding from 0 to T. Beginning at 0, the set of active players to the right of 0 is determined from $G(0)$, and G is defined up to and including the first switching point above 0. The process is repeated until T is reached. I show that the number of switching points that result is finite, and the resulting G is indeed a constructible equilibrium. In the course of the construction, a player may become active and inactive several times, leading to non-interval supports.

This is what happens to players 2 and 3 in the equilibrium of Figure 3. The construction of this equilibrium is described at the end of Section 3.2 as an application of the algorithm.[18]

Because the local conditions combined with players' payoffs uniquely determine G at every point, the algorithm constructs the unique constructible equilibrium. Theorem 5 in Section 3.3 rules out the existence of non-constructible equilibria using the fact that a constructible equilibrium exists. The remainder of Section 3.3 considers implications for n-player contests.

3.1 Properties of a Constructible Equilibrium

Suppose that G is a constructible equilibrium of an $(m+1)$-player regular contest. The value of G at 0 is given by the following lemma, which doesn't rely on constructibility.

Lemma 3 $G_i(0) = 0$ *for* $i < m+1$*, and* $G_{m+1}(0) = \min_{i \leq m} \frac{w_i}{v_i(0)} < 1$*.*

The proofs of this and other results in this section are in Appendix B.

Now, choose y in $(0, T)$ and suppose y is a best response for player i. Since there are $m+1$ players and the powers of players $1, \ldots m+1$ are non-negative, payoffs equal powers (Theorem 1). Thus, y is a best response for player i if and only if

$$P_i(y)\, v_i(y) - (1 - P_i(y))\, c_i(y) = w_i,$$

where $P_i(y)$ is the probability that player i wins a prize when other players choose scores according to G. Equivalently,

$$1 - P_i(y) = 1 - \frac{w_i + c_i(y)}{v_i(y) + c_i(y)}. \tag{1}$$

Since there are m prizes and G is continuous at $y > 0$ (Part (1) of Lemma 2), the expression on on left-hand side of Equation (1) equals $\Pi_{j \in N \setminus \{i\}} (1 - G_j(y))$, i.e., the probability that all other players choose scores higher than y. Since $v_i(y) + c_i(y)$ is the gain from winning relative to losing, I refer to the right-hand side of Equation (1) as *player i's normalized excess payoff at y* and denote it by $q_i(y)$,

$$q_i(y) = 1 - \frac{w_i + c_i(y)}{v_i(y) + c_i(y)} = \frac{v_i(y) - v_i(T)}{v_i(y) + c_i(y)} > 0.$$

[18] A contest can be thought of as a silent game of timing, in which every player i chooses a quitting time s_i, and each of the m players who are the last to quit wins a prize. A region on which a player is not active is an interval of times at which the player never quits.

Thus,

$$\Pi_{j\in N\setminus\{i\}}\left(1-G_j\left(y\right)\right)=q_i\left(y\right) \tag{2}$$

if and only if y in $(0,T)$ is a best response for player i.

Let x be a score in $[0,T)$. Considering scores y slightly higher than x, I denote the set of players for whom all such scores are best responses by $A^+(x)$, and refer to it as the set of *players active to the right of* x:

$$A^+(x)=\{i\in N:\text{Equation (2) holds for all } y\in(x,z) \text{ for some } z>x\}.$$

That $A^+(x)$ is well defined follows from constructibility of G. Let

$$\bar{x}=\sup\left\{z\in(x,T):A^+(y)=A^+(x) \text{ for all } y\in[x,z)\right\}.$$

I refer to $\bar{x}$ as the first *switching point* above x, i.e., the first score higher than x at which the set of players active to the right of x changes.

I now show that $G(x)$ and $A^+(x)$ determine the value of G on $[x,\bar{x}]$. By constructibility and continuity of G, if $j\notin A^+(x)$, then G_j does not increase on $[x,\bar{x}]$. Thus, given $G(x)$ and $A^+(x)$, Equation (2) for players $j\in A^+(x)$ and score y in $(x,\bar{x}]\setminus T$ leads to a system of $|A^+(x)|$ equations (where $|A|$ denotes the cardinality of a set A) in $|A^+(x)|$ unknowns $(1-G_j(y))$. The unique solution is given by the following lemma.[19]

Lemma 4 *Let* $D=\Pi_{j\notin A^+(x)}\left(1-G_j(x)\right)$ *(if* $A^+(x)=N$*, then* $D=1$*). For every* y *in* $[x,\bar{x}]\cap[x,T)$*,*

$$G_i(y)=\begin{cases}1-\dfrac{\Pi_{j\in A^+(x)}q_j(y)^{\frac{1}{|A^+(x)|-1}}}{q_i(y)D^{\frac{1}{|A^+(x)|-1}}} & \text{if } i\in A^+(x)\\ G_i(x) & \text{if } i\notin A^+(x)\end{cases} \tag{3}$$

and $G_i(T)=1$.

I now show that $G(x)$ uniquely determines $A^+(x)$, and then show that $\bar{x}$ is uniquely determined as well. For every x in $[0,T)$, let

$$A(x)=\{i\in N:\text{Equation (2) holds with } x \text{ in place of } y\}. \tag{4}$$

I refer to $A(x)$ as the set of *players active at* x. For any x in $(0,T)$, these are the players for whom x is a best response. By right-continuity of q_i and G at every x in $[0,T)$, $A^+(x)\subseteq A(x)$. This inclusion provides an upper bound on $A^+(x)$: *players who are active*

[19] $|A^+(x)|\geq 2$ is guaranteed by part (2) of Lemma 2.

to the right of x must be active at x. But this bound is not tight: $A^+(x)$ may be a strict subset of $A(x)$.[20] Nevertheless, $A(x)$ uniquely determines $A^+(x)$. To see this, rewrite Equation (2) in terms of marginal percentage changes as follows.

Denote by $\varepsilon_i(y) = -\frac{q_i'(y)}{q_i(y)} > 0$ player i's *semi-elasticity at $y < T$*, and by $h_j(y) = -\frac{(1-G_j(y))'}{1-G_j(y)}$ player j's *hazard rate at y*, where all derivatives denote right-derivatives.[21] For i in $A^+(x)$, by Equation (2), player i's normalized excess payoff at $y > x$ equals the *product* of the other players' probabilities of choosing scores higher than y, for y sufficiently close to x. Thus, taking natural logs and differentiating Equation (2), $\varepsilon_i(y)$ equals the *sum* $\sum_{j\in N\setminus\{i\}} h_j(y)$ of the other players' hazard rates at y. Since players who are not in $A^+(x)$ have hazard rates of 0 at y, we have

$$\text{for every } i \text{ in } A^+(x)\,,\ \varepsilon_i(y) = \sum_{j\in A^+(x)\setminus\{i\}} h_j(y)\,. \tag{5}$$

By right-continuity, Equation (5) holds at $y = x$. In addition, since no player can obtain more than his power on a right-neighborhood of x, we have

$$\text{for every } i \text{ in } A(x)\,,\ \varepsilon_i(x) \geq \sum_{j\in A^+(x)\setminus\{i\}} h_j(x) \ \text{ with equality for } i\in A^+(x)\,. \tag{6}$$

Letting $H(x) = \sum_{j\in A^+(x)} h_j(x)$ and noting that $h_i(x) > 0$ implies that player i is in $A^+(x)$, Equation (5) and Inequality (6) can be combined as

$$\forall i \in A(x) : h_i(x) = \max\{H(x) - \varepsilon_i(x)\,, 0\}\,. \tag{7}$$

Equation (7) pins down players' hazard rates at x. To see this, think of the right-hand side of Equation (7) with H in place of $H(x)$ as player i's "supply curve" of "hazard rate" as a function of "price" H. Then $S_x(H) = \sum_{i\in A(x)} \max\{H - \varepsilon_i(x)\,, 0\}$ is the aggregate supply of hazard rates at x given H. In equilibrium, by adding up Equation (7) for $i \in A(x)$ and noting that $h_j(x) = 0$ for $j \in A(x)\setminus A^+(x)$, the aggregate hazard rates supplied by players in $A(x)$ must equal the actual aggregate hazard rate $H(x)$. Thus, $H(x)$ must satisfy $S_x(H(x)) = H(x)$. To determine $H(x)$ from S_x, note that S_x is a piecewise linear function, whose slope increases by 1 every time H exceeds the semi-elasticity of a player in

[20] This is the case at x_1 in Figure 3, since $A(x_1) = \{1, 2, 3\}$ and $A^+(x_1) = \{1, 3\}$. The correspondence $x \rightrightarrows A(x)$ can be thought of as "right upper hemi-continuous". In general, however, it is not "right lower hemi-continuous".

[21] By Condition R, and since G is given by Equation (3), q_i and G are right-continuously differentiable. $q_i'(y)$ is strictly negative since $q_i'(y) = \left(\frac{v_i(y)-v_i(T)}{v_i(y)+c_i(y)}\right)' = \frac{\overbrace{v_i'(y)}^{\text{Negative}}\overbrace{(c_i(y)+v_i(T))}^{\text{Positive}}+\overbrace{c_i'(y)}^{\text{Positive}}\overbrace{(v_i(T)-v_i(y))}^{\text{Negative}}}{(v_i(y)+c_i(y))^2}$

$A(x)$. Since all semi-elasticities are positive and $|A(x)| \geq 2$,[22] $S'_x(0) = 0$ and $H(x) > 0$. So, S_x is a convex function that starts below the diagonal and reaches a slope of at least 2. Therefore, it intersects the diagonal precisely once above 0, at $H(x)$ (see Figure 4 below).

Since players with positive hazard rates at x are in $A^+(x)$, if $\varepsilon_i(x) < H(x)$ for a player $i \in A(x)$, then $i \in A^+(x)$.[23] Since a player $l \in A(x)$ must obtain his power immediately to the right of x to be in $A^+(x)$, if $\varepsilon_l(x) > H(x)$, then $l \notin A^+(x)$. This is depicted in Figure 4: $A(x) = \{i, j, l\}$, and $A^+(x) = \{i, j\}$, since $\varepsilon_i(x) < H(x), \varepsilon_j(x) < H(x)$, and $\varepsilon_l(x) > H(x)$. Also, S'_x does not increase at $\varepsilon_k(x)$, since player k is not active at x ($k \notin A(x)$).

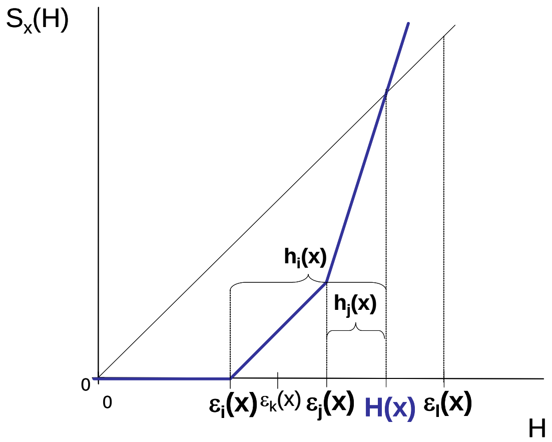

Figure 4: The function S_x and its fixed point $H(x)$

A complication arises when $\varepsilon_i(x) = H(x)$ for a player i in $A(x)$. The correct assignment of such a player is important, since his semi-elasticity slightly above x may differ from the aggregate hazard rate, so he may or may not be active to the right of x, which in turn influences the aggregate hazard rate. That this assignment can be determined unambiguously follows from the assumption of piecewise analytical costs (Condition R). This is shown in Lemma 7 in Appendix B. The lemma gives a closed-form formula for $H(x)$ and provides the following procedure for deciding whether a player i in $A(x)$ is in $A^+(x)$. Compare $\varepsilon_i(x)$ and $H(x)$; if they are equal, compare their first right-derivatives, etc. (This will "generically" stop at the first derivatives.) The lowest order derivatives of $\varepsilon_i(x)$ and $H(x)$ that differ determine whether player i is in $A^+(x)$ ($<$ means the player is in $A^+(x)$, $>$ means the player is not in $A^+(x)$). If all derivatives are equal, then player i is in $A^+(x)$.

[22] Part (2) of Lemma 2.

[23] By definition of $H(x)$ as the fixed point of $S_x(H)$, there are at least two such players, so $|A^+(x)| \geq 2$.

Now consider the first switching point $\bar{x} < T$ above x. This is the first score for which $A^{+}(x) \neq A^{+}(\bar{x})$. If $j \in A^{+}(\bar{x}) \backslash A^{+}(x)$, then $j \in A(\bar{x}) \backslash A^{+}(x)$, so j obtains his power at $\bar{x}$. If, on the other hand, $j \in A^{+}(x) \backslash A^{+}(\bar{x})$, then $h_j(\bar{x}) = 0$. Thus, to identify $\bar{x}$ consider the first point $y > x$ such that Equation (2) holds for a player $j \notin A^{+}(x)$, or $h_j(y) = 0$ for a player $j \in A^{+}(x)$, or y is a concatenation point of the cost function of a player in $A^{+}(x)$ (recall that costs are piecewise-defined functions), or $y = T$. If $y \neq T$, using Equation (4) determine $A(y)$ from $G(y)$, and use $H(y)$ to determine $A^{+}(y)$ from $A(y)$ as described above. If $A^{+}(y) \neq A^{+}(x)$, then y is the switching point $\bar{x}$. If $A^{+}(y) = A^{+}(x)$, then y is not a true switching point, and the search continues above y for the next candidate switching point. This can only repeat a finite number of times before $\bar{x}$ is identified.[24]

3.2 The Algorithm

The properties derived in the Section 3.1 suggest the following algorithm for constructing a candidate constructible equilibrium G on $[0, T]$. Define $G(0)$ as in Lemma 3 and set $x = 0$. Define $A(x)$ from $G(x)$ using Equation (4). Determine $A^{+}(x)$ from $A(x)$ via S_x and its unique fixed point $H(x)$ as described above. Identify $\bar{x}$, the first switching point higher than x, as described above. Define G on $[x, \bar{x}]$ using Equation (3). If $\bar{x} \neq T$, set $x = \bar{x}$ and continue.

For every score x in $(0, T)$ which has been reached in this process, the following points are true.

1. G is continuous and non-decreasing on $(0, x)$ by construction.

2. $\left(1 - \Pi_{j \in N \backslash \{i\}} (1 - G_j(x))\right) v_i(x) - \left(\Pi_{j \in N \backslash \{i\}} (1 - G_j(x))\right) c_i(x) \leq w_i$, with equality if $h_i(x) > 0$, by construction.

3. $G(x) \in (0, 1)$. This follows from the continuity and monotonicity of G up to y, since $G(0) < 1$ (Lemma 3), and if $G_i(x) \geq 1$, then every player $j \neq i$ would obtain strictly more than his power by choosing a score slightly lower than x, violating point 2.

4. $|A(x)| \geq 2$. This can be seen by induction on the number of switching points up to x, since (i) $|A(0)| \geq 2$ $\left(A(0) = \left\{i \in N : w_i \leq \min_{j < m+1} \frac{w_j}{v_j(0)}\right\}\right)$, (ii) if $|A(y)| \geq 2$ then $|A^{+}(y)| \geq 2$ for any $y < x$ (see footnote 23), and (iii) $A^{+}(y) \subseteq A(\bar{y})$ for any $y < x$, by construction.

[24] Finiteness can be shown using analyticity, similarly to the proof of Lemma 7.

Points 3 and 4 show that the algorithm can proceed from any score $y < T$ that has been reached. To show that the algorithm terminates, it suffices to show that the number of switching points is finite.

Lemma 5 *The number of switching points in* $[0, T]$ *identified by the algorithm is finite. In addition,* $A(x) = N$ *for all* x *sufficiently close to* T.

The construction will therefore reach T by applying the steps above a finite number of times. Thus, the output G is characterized by a partition into a finite number of intervals of positive length, on the interior of which the set of active players remains constant. The value of G on each interval is given by Equation (3). To show that G is an equilibrium, it remains to show that $G_i(T) = 1$.

Lemma 6 *For every player* i, $\lim_{x \to T} G_i(x) = G_i(T) = 1$.

Theorem 4 G *is a constructible equilibrium, which is continuous above* 0.

Proof. G is a profile of probability distribution functions, since it is right-continuous on $[0, T]$, weakly increasing, and $G(T) = 1$ (point 1 above and Lemma 6). It is continuous above 0 (point 1 above and Lemma 6). No player can obtain more than his power, and G_i is strictly increasing only where player i obtains precisely his power (point 2 above). Thus, best responses are chosen with probability 1, so G is an equilibrium. By the construction procedure, G is constructible (for every $x < T$, every score in $(x, \bar{x})$ is a best response for players in $A^+(x)$ and no score in $(x, \bar{x})$ is a best response for players in $N \backslash A^+(x)$). ■

For an illustration, consider the supply function S_x and its positive fixed point $H(x)$ in the context of Figure 3 above. $A(0) = A^+(0) = \{2, 3\}$. As x increases from 0 to T, the set of active players changes. At the switching point x_1, player 1 becomes active since he obtains his power. This changes S_x and $H(x)$ discontinuously. As a result, $H(x_1)$ falls below player 2's hazard rate, and he becomes inactive immediately above x_1. At x_2, player 2 rejoins the set of active players, and all three players are active up to x_3. Thus, the addition of an active player may or may not cause another to become inactive. At x_3, player 3's hazard rate reaches 0, and he becomes inactive immediately above x_3. Player 3 rejoins the set of active players at x_4, and all three players remain active up to the threshold.[25]

[25] Bulow & Levin (2006) constructed the equilibrium of a game in which players have linear costs and compete for heterogeneous prizes. Their construction proceeds from the top, without first identifying players' equilibrium payoffs. This is possible because each player's best-response set is an interval and players' marginal costs are identical. Such a procedure does not work here, since the set of players active to the *left* of x cannot be uniquely determined from $G(x)$ and players' payoffs using local conditions.

3.3 Equilibrium Uniqueness and Implications

Since the value of G at 0, the switching points, and the corresponding sets of active players are uniquely determined, the algorithm constructs the unique *constructible* equilibrium of an $(m+1)$-player regular contest. Theorem 5 rules out the existence of non-constructible equilibria, using the fact that a constructible equilibrium exists.[26]

Theorem 5 *If an $(m+1)$-player contest, regular or not, has a constructible equilibrium, then that is the unique equilibrium of the contest.*

The following theorem summarizes the results for $(m+1)$-player regular contests.

Theorem 6 *An $(m+1)$-player regular contest has a unique equilibrium, which is constructible. It is characterized by a partition of $[0, T]$ into a finite number of closed intervals with disjoint interiors of positive length, such that the set of active players is constant on the interior of each interval. Thus, each player's best-response set is a finite union of intervals. All players are active on the last interval.*

Proof. Immediate from Lemma 5, Theorem 4, and Theorem 5. ■

Appendix A shows that when players' costs are not piecewise analytical, so the contest is not regular, non-constructible equilibria may exist.

I now turn to regular contests with any number of players.

Theorem 7 *Regular contests have* at most *one equilibrium in which precisely $m+1$ players participate. The candidate for this equilibrium is the unique equilibrium of the reduced contest with players $1, \ldots, m+1$. It is an equilibrium of the original contest if and only if players $m+2, \ldots, n$ cannot obtain strictly positive payoffs by participating. If they can, then in every equilibrium at least $m+2$ players participate.*

Proof. Immediate from Theorem 6 and part (4) of Lemma 2. ■

In some regular contests, players $m+2, \ldots, n$ do not participate in any equilibrium. A sufficient condition is given by Theorem 2. For such contests, a much stronger result can be stated.

[26] Clark & Riis (1998) and, to the best of my reading, Bulow & Levin (2006), who constructed equilibria of similar games with a continuum of pure strategies in which more than two players participate, did not rule out the existence of equilibria that are not constructible. Such equilibria do not arise in Baye et al. (1996), but are not ruled out in the setting of González-Díaz (2007), who extends the analysis of Baye et al. (1996) to more general costs.

Theorem 8 *Regular contests in which players $m+2,\ldots,n$ do not participate in any equilibrium have a unique equilibrium. In this equilibrium, players $m+2,\ldots,n$ choose 0 with certainty, and players $1,\ldots,m+1$ behave as in the unique equilibrium of the reduced contest with players $1,\ldots,m+1$. A sufficient condition for this is that the conditions of Theorem 2 hold for every player $m+2,\ldots,n$.*

Proof. By Theorem 6, the only candidate equilibrium is the one described in the statement of the theorem. This is an equilibrium of the contest, because every contest has an equilibrium (Lemma 1). ■

Note that the conditions of Theorem 2 do not place any restrictions on how the valuations for winning and costs of losing relate among players $N\backslash\{m+1\}$. Thus, Theorem 8 applies to a wide class of contests.

4 Simple Contests

This section studies a class of contests that allows for a combination of unconditional and conditional investments, while accommodating a limited degree of asymmetry among players. All-pay auctions are a special case. I show that contests in this class have a unique equilibrium, in which every player's best response set is an interval. I also explore the connection between the unconditional component of the investment and how efficiently prizes are allocated.

Suppose that all players share a common underlying technology, captured by a strictly increasing function $c(\cdot)$ with $c(0)=0$, but may differ in how well they employ this technology. This difference is captured by every player i's idiosyncratic *cost coefficient* $\gamma_i>0$.[27] A fraction α in $(0,1]$ of the cost is sunk. The remainder of the cost, $1-\alpha$, is borne only if the player wins a prize. For every player i, we therefore have $v_i(s_i)=V_i-\gamma_i c(s_i)$ and $c_i=\alpha\gamma_i c(s_i)$, where $V_i>0$ is player i's valuation for a prize. A contest in this family is called a *simple contest* (see Figure 5 below).[28] When $\alpha=1$, all investments are sunk.[29] If,

[27] For example, firms competing in an R&D race may have access to a similar technology, but may differ in the skill and innovativeness of their workers.

[28] Other contests can also accommodate player- and score-specific fractions $\alpha_i(s_i)$ and valuations $V_i(s_i)$. Of course, the more general is the class of contests considered, the weaker are the conclusions regarding players' behavior in the unique equilibrium specified by the algorithm of Section 3.2.

[29] The contest is then similar to the one in Moldovanu & Sela (2001). The informational assumptions, however, are different. In their model, all players are ex-ante symmetric. The individual coefficients γ_i

in addition, $c(x) = x$ and $\gamma_i = 1$ for every player i, we have an all-pay auction.

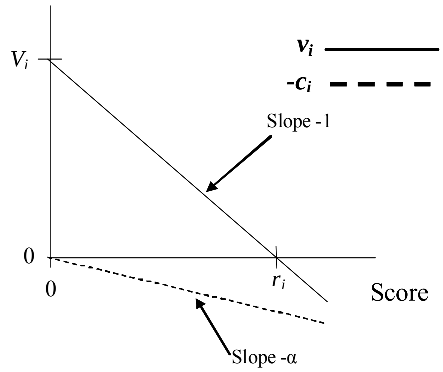

Figure 5: The valuation for winning and cost of losing for player i with $\gamma_i = 1$ in a simple contest with $c(x) = x$

The reach r_i of player i satisfies $v_i(r_i) = 0$, so $r_i = c^{-1}\left(\frac{V_i}{\gamma_i}\right)$. Since c is strictly increasing and players are ordered in decreasing order of reach, $\frac{V_1}{\gamma_1} \geq ... \geq \frac{V_N}{\gamma_N}$. The contest's threshold is $T = r_{m+1} = c^{-1}\left(\frac{V_{m+1}}{\gamma_{m+1}}\right)$, so the range of scores over which players compete, $[0, T]$, is independent of α. For Assumption B3 to hold, assume that $\frac{V_{m+1}}{\gamma_{m+1}}$ is different from $\frac{V_i}{\gamma_i}$ for all $i \neq m+1$. Theorem 1 then shows that the equilibrium payoff of player $i < m+1$ is

$$w_i = v_i(T) = V_i - \gamma_i c\left(c^{-1}\left(\frac{V_{m+1}}{\gamma_{m+1}}\right)\right) = V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}.$$

We therefore have the following corollary of Theorem 1.

Corollary 1 *In a simple contest, the payoff of every player $i < m+1$ is $V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}$. The payoffs of players $m+1, \ldots, N$ are 0. Payoffs are independent of α and c.*

Corollary 1 shows that the payoff of every player $i < m+1$ increases in his valuation for a prize and in the marginal player's cost coefficient, and decreases in the player's cost coefficient and in the marginal player's valuation for a prize. In particular, the payoff of player i is not affected by the characteristics of any player in $N \setminus \{i, m+1\}$.

Aggregate expenditures equal the allocation value of the prizes less players' utilities. We therefore obtain the following corollary of Theorem 1.

are privately known and drawn iid from a commonly known distribution. Moldovanu & Sela (2001) solve for the symmetric equilibrium, and do not characterize all equilibria. In contrast, the model here is of complete information and has a unique equilibrium.

Corollary 2 *In a simple contest in which* $V_1 = \ldots = V_N = V$*, aggregate expenditures are* $mV - \sum_{i=1}^{m}\left(V - \gamma_i \frac{V}{\gamma_{m+1}}\right) = V\sum_{i=1}^{m}\left(\frac{\gamma_i}{\gamma_{m+1}}\right)$*, and are independent of* α *and* c*.*

Corollary 2 shows that when all valuations are equal, as is the case when prizes are monetary, aggregate expenditures increase in valuations and in the cost coefficients of players $1, \ldots, m$, and decrease in the marginal player's cost coefficient.

4.1 Simple Contests with a Single Prize

Theorems 2 and 3 show that a simple contest with a single prize has a unique equilibrium, described in Theorem 3. We therefore have the following corollary.

Corollary 3 *A simple contest with a single prize has a unique equilibrium. In this equilibrium, players* $3, \ldots, n$ *choose* 0 *with certainty, and the CDFs of players* 1 *and* 2 *are* $\left(G_1^{\alpha}, G_2^{\alpha}\right)(x) = \left(\frac{\alpha\gamma_2 c(x)}{V_2-(1-\alpha)\gamma_2 c(x)}, \frac{V_1-\gamma_1\frac{V_2}{\gamma_2}+\alpha\gamma_1 c(x)}{V_1-(1-\alpha)\gamma_1 c(x)}\right)$ *on* $\left[0, c^{-1}\left(\frac{V_2}{\gamma_2}\right)\right]$.

The corollary shows that the unique equilibrium is not independent of α and c. It is straightforward to verify that player 1's CDF first-order stochastically dominates (FOSD) that of player 2. Therefore, player 1 wins the prize with higher probability than player 2 (see Corollary 5 below). Moreover, $\frac{\partial G_1^{\alpha}(x)}{\partial\alpha}, \frac{\partial G_2^{\alpha}(x)}{\partial\alpha} > 0$ for all x in $(0, T)$, so the equilibria for lower values of α FOSD those of higher values of α: players tend to choose higher scores as α decreases. It can also be shown that player 1's probability of winning increases (and that of player 2 decreases) as α decreases.

In the limit, as α approaches 0, player 1 chooses T with probability 1 and wins the prize with probability 1, and player 2 chooses scores lower or equal to x with probability $\frac{V_1-\gamma_1\frac{V_2}{\gamma_2}}{V_1-\gamma_1 c(x)}$. This is an equilibrium of the limit game in which every player i's payoff when choosing s_i is 0 if he loses and $V_i - \gamma_i c(s_i)$ if he wins (see Section 4.2.1 below). A special case is the complete-information first-price auction.

4.2 Simple Contests with Multiple Prizes

From now on assume that $c(\cdot)$ is piecewise analytical. The following result is an immediate corollary Theorems 8, since players $m+2, \ldots, n$ do not participate in any equilibrium (the conditions of Theorem 2 hold).

Corollary 4 *A simple contest with multiple prizes has a unique equilibrium. In the unique equilibrium, the strategies of players* $1, \ldots, m+1$ *are given by the algorithm, and players* $m+2, \ldots, N$ *choose* 0 *with certainty.*

Let $a_i = \frac{\gamma_i}{V_i}$, and note that a_i is increasing in i. The following result shows that in the unique equilibrium, the best response set of every player $i \leq m+1$ is an interval whose upper bound is T and whose lower bound increases in the player's reach (or, equivalently, decreases in a_i).

Theorem 9 *In the unique equilibrium of a simple contest, every player $i \leq m+1$ is active on the interval $\left[s_i^l, T\right]$ for some $s_i^l \geq 0$, with $s_m^l = s_{m+1}^l = 0$. For $i, j \leq m$, $s_i^l \leq s_j^l$ if and only if $a_j \leq a_i$. Players $m+2, \ldots, N$ choose 0 with certainty.*

Appendix C contains the proof of Theorem 9 and of other results in this section. The outline of the proof is as follows. First, players with higher reach become active for the first time at higher scores. Second, semi-elasticities are increasing in reach at every score, so when a player becomes active his semi-elasticity is higher than those of the other active players. Thus, no active players become inactive as a result of a new player becoming active.[30] Third, the semi-elasticity of an active player never increases above the aggregate hazard rate.[31] This shows that once a player becomes active he stays active until the threshold. Figure 6 depicts players' atoms and the regions in which players are active in the unique equilibrium.

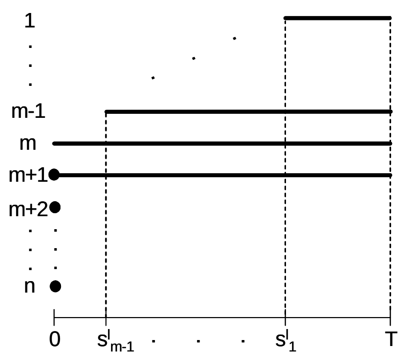

Figure 6: The unique equilibrium of a simple contest

A corollary of Theorem 9 and the fact that players' semi-elasticities are increasing in reach is that players' equilibrium CDFs can be ranked in terms of FOSD.

[30] Contrast this with what happens in Figure 3 at the switching point x_1: player 1 becomes active, and because his semi-elasticity is sufficiently lower than that of player 2, player 2 becomes inactive immediately above x_1.

[31] Contrast this with what happens in Figure 3 at the switching point x_3: the hazard rate of player 3 drops to 0, so he becomes inactive immediately above x_3.

Corollary 5 *For any α in $(0,1]$ and $i<j$, the CDF of player i FOSD that of player j. This implies that player i chooses higher scores than player j, on average, and also that player i wins a prize with higher probability than player j.*[32]

Using Theorem 9, I now derive an expression for players' equilibrium strategies. Recall that for any $y<T$ the formula for player i's CDF at y is given by Equation (3). Since each player is active on an interval, $D=1$, and the switching points are the points s_i^l at which players become active. Because players with higher reaches become active at higher scores, for every $y<T$ there is a unique $j=1,\ldots,m$ such that y is in $\left[s_j^l, s_{j-1}^l\right)$ (where $s_0^l=T$) and the set of players active to the right of s_j^l is $j,\ldots,m+1$. Therefore, for y in this $\left[s_j^l, s_{j-1}^l\right)$ we have

$$G_i(y)=\begin{cases} 1-\frac{\Pi_{k=j}^{m+1} q_k(y)^{\frac{1}{m+1-j}}}{q_i(y)} & \text{if } \ i\geq j \\ 0 & \text{if } \ i<j \end{cases}.$$

Substituting

$$q_i(y)=\frac{v_i(y)-v_i(T)}{v_i(y)+c_i(y)}=\frac{a_i\left(\frac{1}{a_{m+1}}-c(x)\right)}{1-(1-\alpha)a_i c(x)}$$

for $i\geq j$ and simplifying, we obtain

$$G_i(y)=1-\left(\frac{1}{a_{m+1}}-c(y)\right)^{\frac{1}{m+1-j}}\frac{\Pi_{k=j}^{m+1}\left[\frac{a_k}{1-(1-\alpha)a_k c(y)}\right]^{\frac{1}{m+1-j}}}{\frac{a_i}{1-(1-\alpha)a_i c(y)}}. \tag{8}$$

We still have to identify the scores s_i^l at which players become active. For simplicity assume that the a_is are distinct.[33] Recall that $s_m=s_{m+1}=0$. The score s_i^l at which player $i<m$ becomes active is the lowest score x at which he obtains his power. That is, s_i^l is the lowest score x that satisfies

$$\left(1-\prod_{d=i+1}^{m+1}(1-G_d(x))\right)(1-a_i c(x))-\left(\prod_{d=i+1}^{m+1}(1-G_d(x))\right)\alpha a_i c(x)=w_i=1-\frac{a_i}{a_{m+1}},$$

or

$$\left(\prod_{d=i+1}^{m+1}(1-G_d(x))\right)(a_i c(x)(1-\alpha)-1)-a_i c(x)=-\frac{a_i}{a_{m+1}}.$$

[32] In contrast to the two-player case, a player's CDF for low values of α does not always FOSD his CDF for high values of α.

[33] A similar analysis works when they are not distinct, but the notation becomes more cumbersome.

After substituting G_d with the right-hand side of Equation (8) (with active players $i+1,\ldots,m+1$) and some algebraic manipulation, it can be shown that s_i^l is the lowest score x that satisfies

$$\frac{\Pi_{k=i+1}^{m}a_k}{a_i^{m-i}}=\frac{1}{(1-a_{m+1}c(x))}\frac{\Pi_{k=i+1}^{m+1}(1-a_kc(x)(1-\alpha))}{(1-a_ic(x)(1-\alpha))^{m-i}}. \tag{9}$$

Equation (9) characterizes s_i^l implicitly, and provides a simple closed-form expression for s_i^l when $\alpha=1$ (see Section 4.2.2 below). It can also be used to show the following result.

Theorem 10 *For $\alpha<1$ and $i<m$, s_i^l decreases in α. As α approaches 0, s_i^l approaches T.*

Theorem 10 has the following implication regarding the allocation of prizes. Call a simple contest *β-efficient*, for some β in $(0,1)$, if each of the players $1,\ldots,m$ (players with positive power) obtains a prize with probability at least β in the unique equilibrium of the contest.

Corollary 6 *Choose a family of simple contests parameterized by α. For any $\beta<1$, every simple contest in the family with a small enough $\alpha>0$ is β-efficient.*

In particular, the corollary shows that when players differ only in their valuations for a prize, so that players $1,\ldots,m$ are those with the highest valuations for a prize, allocative efficiency of the prizes can be approached arbitrarily closely by reducing the unconditional investment component α. The limiting equilibrium corresponding to $\alpha=0$ is efficient (see Section 4.2.1 below). Since players' payoffs remain the same for all $\alpha>0$ (Corollary 1), this immediately implies that as α approaches 0, expenditures approach their maximal value,

$$\sum_{i=1}^{m}V_i-\sum_{i=1}^{m}\left(V_i-\gamma_i\frac{V_{m+1}}{\gamma_{m+1}}\right)=V_{m+1}\sum_{i=1}^{m}\frac{\gamma_i}{\gamma_{m+1}}.$$

Although players' individual expenditures and probabilities of winning, as well as aggregate expenditures, change with α, a simple change-of-variable argument can be used to show that they are independent of c.

4.2.1 The Case $\alpha=0$

The game with $\alpha=0$ is not a contest, since Assumption B3 is violated. Instead, it is a complete-information, discriminatory-price multiprize auction in which player i's cost of bidding x is $\gamma_i c(x)$ if he wins and 0 if he loses. This game has many equilibria, some of which involve players playing weakly-dominated strategies, and some of which rely on specific tie-breaking rules. Different equilibria lead to different payoffs.[34] Considering the limit of the equilibria of simple contests as α approaches 0, we obtain the following equilibrium of the game with $\alpha = 0$. Players $1, \ldots, m-1$ bid T with probability 1. Players $m+2, \ldots, n$ bid 0 with probability 1. Players m and $m+1$ bid the limit of the equilibria of two-player simple contests as α approaches 0. As shown in Section 4.1, this means that player m bids T with probability 1, and player $m+1$ bids according to the CDF $G_{m+1}(x) = \frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}}}{V_m - \gamma_m c(x)}$. In particular, only player $m+1$ employs a mixed strategy. Players $1, \ldots, m$ win with certainty and players $m+1, \ldots, n$ lose with certainty. Players' payoffs are given by Theorem 1. Thus, taking the fraction of the all-pay component to 0 can serve as a selection criterion that delivers a unique equilibrium of the limit game. This equilibrium is robust to the tie-breaking rule, and is "close" to the equilibria of "nearby" contests with a small all-pay component. A special case is the complete-information, pay-your-bid multiprize auction.

4.2.2 The Case $\alpha = 1$

When $\alpha = 1$ the contest is separable and all investments are unconditional. Equation (8) then simplifies to

$$G_i(y) = 1 - \left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{1}{m+1-j}} \frac{\Pi_{k=j}^{m+1} a_k^{\frac{1}{m+1-j}}}{a_i} \tag{10}$$

Equation (9) provides the following closed-form expression for s_i^l, $i < m$ (recall that $s_m^l = s_{m+1}^l = 0$):

$$\frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} = \frac{1}{\left(1 - a_{m+1} c\left(s_i^l\right)\right)} \Rightarrow s_i^l = c^{-1}\left(\frac{1}{a_{m+1}} - \frac{a_i^{m-i}}{\Pi_{k=i+1}^{m+1} a_k}\right). \tag{11}$$

The special case of $c(x) = x$ and $\gamma_i = 1$ is the multiprize all-pay auction, first analyzed by Clark & Riis (1998). Setting $c(x) = x$ and $\gamma_i = 1$ in Equations (10) and (11) delivers the equilibrium described in their Proposition 1.[35] The analysis of Clark & Riis (1998)

[34] For example, a weak player could submit a bid whose cost exceeds his valuation for a prize, thereby forcing a stronger player to submit a high bid. As long as the weak player loses for sure, he does not incur any costs. This cannot happen under Assumption B3.

[35] Theorem 9 applied to multiprize all-pay auctions corrects two imprecisions in Clark & Riis (1998). The first is that they claimed uniqueness of equilibrium but provided an incorrect proof of this claim, as

(page 279) provides a recursive closed-form formula for each player's expenditures and probability of winning, which are independent of c.

5 Conclusion

This paper has investigated equilibrium behavior in a multiprize contest model featuring asymmetric contestants and a combination of unconditional and conditional investments. These features are common to many real-world competitions, and may lead to complicated equilibrium behavior. I have solved for the unique equilibrium in two-player, single-prize contests, and provided an algorithm that constructs the unique equilibrium within a large class of multiprize contests. What matters for equilibrium uniqueness in a contest for m prizes is that only the strongest $m+1$ players participate, which is implied when weak players are everywhere disadvantaged relative to the marginal player. Many existing models of competition satisfy this condition.

As an application of the algorithm and the equilibrium uniqueness result, I investigated the class of simple contests. Simple contests accommodate a limited degree of asymmetry among players and both unconditional and conditional investments. Equilibrium behavior in simple contests is relatively simple, since every player chooses a score from an interval. This explains players' behavior in the multiprize all-pay auction, which is a simple contest in which all investments are unconditional and all costs are linear. When players differ only in their valuations for a prize, higher allocative efficiency can be achieved by reducing the unconditional component of the investment. Doing so, however, increases aggregate expenditures, since players' payoffs are independent of the unconditional component. When investments are wasteful, this implies a trade-off between increasing allocative efficiency, which suggests decreasing the unconditional component, and decreasing aggregate expenditures, which suggests increasing the unconditional component.

Classes of contests corresponding to real-world competitions can be studied using the tools developed in this paper. Such studies can address various policy questions and issues of contest design. The richness of the contest framework enhances the potential empirical validity of such studies.

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discussed in footnote 9 above. The second is that their footnote 6 claims that multiple equilibria arise when two or more players have the same valuation. Theorem 9 shows that the equilibrium is unique even if several players have the same valuation for a prize, as long as the valuation of the marginal player is different from those of the other players.

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A A Non-Constructible Equilibrium[36]

The example depicts a separable contest with three players and one prize of common value 1 (so $v_i(\cdot) = 1 - c_i(\cdot)$). I construct an equilibrium (C, G, G) of the contest, in which player 1's best-response set is the Cantor set. To define player 1's cost function, let $c_1(x) = F(x) = x$ and modify $F(x)$ by mimicking the construction of the Cantor set on $[0,1]$ in the following way. At every removed (a,b) and for every $x \in (a,b)$, let $F(x) = a + \frac{(x-a)^2}{b-a}$. Denote the resulting function by $\tilde{F}$. Then $\tilde{F}(0) = 0, \tilde{F}(1) = 1$, and $\tilde{F}$ is continuous, strictly increasing, equals x precisely on the Cantor set, and is strictly lower than x on its complement. In particular, if player 1's probability of winning when playing x is $\tilde{F}(x)$, then his best-response set is the Cantor set. To achieve this, let C be the Cantor function, and recall that it is continuous and weakly increasing, with $C(0) = 0$ and $C(1) = 1$. Let $G(x)$ satisfy $\tilde{F}(x) = 1 - (1 - G(x))(1 - G(x))$ for all $x \in [0,1]$. That is,

$$G(x) = 1 - \sqrt{1 - \tilde{F}(x)}$$

Then $G(x)$ is continuous and strictly increasing, with $G(0) = 0$ and $G(1) = 1$. Now, define player 2 and 3's cost functions as

$$c_2(x) = c_3(x) = 1 - (1 - G(x))(1 - C(x)) = 1 - \left(\sqrt{1 - \tilde{F}(x)}\right)(1 - C(x))$$

Then c_2 and c_3 are continuous and strictly increasing, with $c_2(0) = c_3(0) = 0$ and $c_2(1) = c_3(1) = 1$. It is straightforward to verify that (C, G, G) is an equilibrium of the contest, in which player 1's best-response set is the Cantor set.

B Proofs of the Results of Section 3

B.1 Proof of Lemma 3

Since positive payoffs imply winning with positive probability at every best response, the Tie Lemma in Siegel (2009) shows that players in $1, \ldots, m$ do not have an atom at 0. Since there are no atoms above 0 and every $x > 0$ is a best response of at least two players (Lemma 2), $G_{m+1}(0) \geq \min_{i \leq m} \frac{w_i}{v_i(0)}$. Since no player should be able to obtain more than his power by choosing a score slightly above 0, $G_{m+1}(0) \leq \min_{i \leq m} \frac{w_i}{v_i(0)}$. Strictly decreasing valuations for winning also imply that $w_i = v_i(T) < v_i(0)$, so $G_{m+1}(0) < 1$.

B.2 Proof of Lemma 4

Choose $y \in (x, \bar{x})$, and let $p_i(y) = 1 - G_i(y)$. Since $q_i(y) > 0$ and $p_i(y), D > 0$ (all players choose scores up to the threshold by the Threshold Lemma of Siegel (2009) and

[36] I thank George Mailath for encouraging me to provide this example.

strictly decreasing valuations for winning), Equation (2) for $i \in A^{+}(x)$ can be rewritten as $\Pi_{j\in A^{+}(x)\setminus\{i\}} p_j(y) = \frac{q_i(y)}{D} > 0$. Taking natural logs,

$$\sum_{j\in A^{+}(x)\setminus\{i\}} \ln p_j(y) = \ln q_i(y) - \ln D.$$

This is a system of $|A^{+}(x)|$ linear equations in $|A^{+}(x)|$ unknowns $p_j(y)$. Denote by $I_{M\times M}$ and $1_{M\times M}$ the identity matrix and a matrix of ones, respectively, of dimensions $M \times M$. Then, in vector notation,

$$\left(1_{|A^{+}(x)|\times|A^{+}(x)|} - I_{|A^{+}(x)|\times|A^{+}(x)|}\right) \ln p(y) = \ln q(y) - \ln D$$

Since $(1_{M\times M} - I_{M\times M})^{-1} = \left(\frac{1}{M-1} \cdot 1_{M\times M} - I_{M\times M}\right)$, we have

$$\ln p_i(y) = \frac{1}{|A^{+}(x)|-1} \sum_{j\in A^{+}(x)} \ln q_j(y) - \ln q_i(y) - \frac{1}{|A^{+}(x)|-1} \ln D$$

which gives the result for $y \in (x, \bar{x})$. For $y \in \{x, \bar{x}\}, \bar{x} \neq T$, the result follows from left- and right-continuity on $[0, T)$. And $G(T) = 1$ because by part (3) of Lemma 2 no player has best responses above T.

B.3 Statement and Proof of Lemma 7

For a set $B \subseteq A(x)$ of at least two players and $y > x, H \geq 0$, let

$$S_y^B(H) = \sum_{j\in B} \max\{H - \varepsilon_j(y), 0\},$$

and denote by $H^B(y)$ the unique positive fixed point of S_y^B. Note that $S_x(\cdot) = S_x^{A(x)}(\cdot)$ and $H(x) = H^{A(x)}(x)$. The following lemma characterizes $A^{+}(x)$.

Lemma 7 *$A^{+}(x)$ is exactly all players $i \in A(x)$ with $\varepsilon_i(y) \leq H^{A(x)}(y)$ on some right-neighborhood of x.*[37]

I first establish that the set of players described in the lemma is well defined. For this, I show that $H^{A(x)}(y)$ on some right-neighborhood of x can be computed as follows.

[37] $A^{+}(x)$ can also be constructed as follows. Order the players in $A(x)$ in any non-decreasing order of semi-elasticity on some right-neighborhood of x. $A^{+}(x)$ is the subset $\{1, \ldots, L(x)\} \subseteq A(x)$, where $L(x)$ is the highest $l \geq 2$ (in this ordering) such that

$$\frac{1}{l-1} \sum_{j\in A(y), j\leq l} \varepsilon_j(y) - \varepsilon_l(y) \geq 0$$

on this right-neighborhood of x. This follows from solving the system of Equations (5) and using the arguments in the proof of Lemma 7.

Order the players in $A(x)$ in any non-decreasing order of semi-elasticity on some right-neighborhood of x (this can be done since semi-elasticities are analytical on some right-neighborhood of x, and an analytical function with an accumulation point of roots is identically 0 in the connected component of the accumulation point). Let $L(x)$ be the highest $l \geq 2$ (in this ordering) such that

$$\frac{1}{l-1} \sum_{j \in A(x), j \leq l} \varepsilon_j(y) - \varepsilon_l(y) \geq 0$$

on some right-neighborhood of x (that $L(x)$ is well defined follows from analyticity as well). Then

$$H^{A(x)}(y) = \frac{1}{L(x)-1} \sum_{j \in A(x), j \leq L(x)} \varepsilon_j(y)$$

on this right-neighborhood of x. This follows from the uniqueness of the fixed point of $S_y^{A(x)}(y)$. Consequently, $H^{A(x)}$ is analytical on this right-neighborhood of x.

Therefore, for every player $i \in A(x)$ and all scores y slightly above x, either (i) $\varepsilon_i(y) < H^{A(x)}(y)$, (ii) $\varepsilon_i(y) > H^{A(x)}(y)$, or (iii) $\varepsilon_i(y) = H^{A(x)}(y)$. To determine which of these obtains for player $i \in A(x)$, consider the lowest $k \geq 0$ such that the k^{th} derivative of $\varepsilon_i(x) - H^{A(x)}(x)$ is different from 0 (by analyticity this expression is infinitely differentiable). If this k^{th} derivative is negative, then (i) obtains; if the derivative is positive, then (ii) obtains; if all derivatives are 0, then (iii) obtains (by analyticity). This is the method described in Section 3, since $H(x) = H^{A(x)}(x)$.

It remains to show that $A^+(x)$ is exactly the set of players described by the lemma. For this, note that the aggregate hazard rate $H(y) = H^{A^+(x)}(y)$ at scores y slightly above x is bounded from below by $H^{A(x)}(y)$, since $A^+(y) \subseteq A(x)$ and increasing the set of players with respect to which the supply function is calculated can only lower the fixed point. Therefore, if (i) or (iii) obtain for player $i \in A(x)$ then $\varepsilon_i(y) \leq H^{A(x)}(y) \leq H(y)$ so $i \in A^+(x)$. Suppose that (ii) obtains but $i \in A^+(x)$. This means that $H^{A(x)}(y) < \varepsilon_i(y) \leq H(y)$ for all y slightly above x. Therefore, there is a player $j \in A(x) \backslash A^+(x)$ with $\varepsilon_j(y) < \varepsilon_i(y)$ for all y slightly above x.[38] But player j is getting more than his power slightly above x, since $\varepsilon_j(y) < \varepsilon_i(y) \leq H(y)$ for all y slightly above x.

B.4 Proof of Lemma 5

Lemma 8 *$\forall \tilde{x} < T$, the number of switching points in $[0, \tilde{x}]$ is finite.*

Proof. I assume analytical valuations for winning and costs of losing (the obvious extension applies to piecewise analyticity). Choose $\tilde{x} < T$ and rank players' semi-elasticities at every

[38] Players j with $\varepsilon_j(y) \geq \varepsilon_i(y)$ for all y slightly above x do not affect whether $H^{A(x)}(y)$ is higher or lower than $\varepsilon_i(y)$, so if all players $j \in A(x)$ with $\varepsilon_j(y) < \varepsilon_i(y)$ for all y slightly above x were in $A^+(x)$, then $\varepsilon_i(y) \leq H^{A^+(x)}(y)$ would mean that $\varepsilon_i(y) \leq H^{A(x)}(y)$.

score in $[0, \tilde{x}]$. Since semi-elasticities are analytical, this ranking can change only finitely many times. Thus, $[0, \tilde{x}]$ can be divided into a finite number of intervals such that the ranking of players' semi-elasticities is constant on each interval. Fix one such interval J. For every subset $B \subseteq N$ of at least two players and every $x \in J$, denote by $t_B(x)$ the highest semi-elasticity of a player who can join the set of active players B and maintain a non-negative hazard rate: $t_B(x) = \frac{1}{|B|-1} \sum_{j \in B} \varepsilon_j(x)$ (the aggregate hazard rates of players in B). Since semi-elasticities are analytical, so is $t_B(\cdot)$. Thus, the interval J can be divided into a finite number of subintervals such on every subinterval each function in $\{\varepsilon_i - t_B : i \in N, B \subseteq N, |B| \geq 2\}$ is either positive, negative, or 0. Clearly, on any such subinterval $L \subseteq J$ an active player can become inactive only if a player with a strictly lower semi-elasticity becomes active (recall that the order of players' semi-elasticities doesn't change on J). Now, suppose in contradiction that the number of switching points in L is infinite. This implies that some player i becomes inactive and active an infinite number of times, which, by the above, implies that some player j with semi-elasticity strictly lower than that of i becomes inactive and active an infinite number of times. Continuing in this way, we obtain a contradiction since the number of players is finite. ■

The following two lemmas show that there are no switching points in some left-neighborhood of T.

Lemma 9 *$\exists \tilde{x} < T$ such that $\forall i \in N : \varepsilon_i(x) < H(x)$ for every $x \in (\tilde{x}, T)$.*

Proof. First, $\forall i, j$:

$$\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_j(x)} = \lim_{x \to T} \frac{q_i'(x)}{q_i(x)} \frac{q_j(x)}{q_j'(x)} = \frac{q_i'(T)}{q_j'(T)} \lim_{x \to T} \frac{q_j(x)}{q_i(x)} = \frac{q_i'(T)}{q_j'(T)} \frac{q_j'(x)}{q_i'(x)} = 1$$

where the penultimate equality follows from l'Hopital's rule.
Let $\varepsilon_{\min}(x) = \min_{i \in N} \varepsilon_i(x)$ for $x < T$. Then, by the above, $\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_{\min}(x)} = 1$, so $\frac{\varepsilon_i(x)}{\varepsilon_{\min}(x)} < \frac{n}{n-1}$ for all $x > \tilde{x}$ for some $\tilde{x}$ sufficiently close to T. To conclude, it suffices to show that $\forall x > \tilde{x} : H(x) \geq \frac{n}{n-1} \varepsilon_{\min}(x)$. Let $S_x^{\min}(H) = n \max\{H - \varepsilon_{\min}(x), 0\}$. Then $\forall H : S_x(H) \leq S_x^{\min}(H)$ and since $\frac{n}{n-1} \varepsilon_{\min}(x)$ is the unique positive fixed point of $S_x^{\min}$, $H(x) \geq \frac{n}{n-1} \varepsilon_{\min}(x)$. ■

Since active players with semi-elasticities strictly lower than the aggregate hazard rate remain active, in order to complete the proof it suffices to show the following.

Lemma 10 *Every player i has scores x arbitrarily close to T such that $q_i(x) = \Pi_{j \neq i}(1 - G_j(x))$.*

Proof. Suppose, in contradiction, that $\forall x \in (\tilde{x}_i, T) : q_i(x) < \Pi_{j \neq i}(1 - G_j(x))$ for some player i and some $\tilde{x}_i > \tilde{x}$ of the previous lemma. Then, $f(x) = \sum_{j \neq i} \ln(1 - G_j(x)) - \ln q_i(x) > 0$. Since $i \notin A(x)$,

$$\forall x \in (\tilde{x}_i, T) : H(x) = \frac{|A^+(x)|}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \varepsilon_j(x) = \sum_{j \in N} \frac{G_i'(x)}{(1 - G_i(x))} = \sum_{j \neq i} \frac{G_i'(x)}{(1 - G_i(x))}$$

Thus,

$$f'(x) = \varepsilon_i(x) - \frac{|A^+(x)|}{|A^+(x)|-1} \sum_{j \in A^+(x)} \varepsilon_j(x) \leq \varepsilon_i(x) - \frac{n-1}{n-2} \sum_{j \in A^+(x)} \varepsilon_j(x) =$$

$$-\frac{1}{n-2} \varepsilon_{\min}(x) + o(\varepsilon_{\min}(x))$$

as $x \to T$, by the proof of the previous lemma. Since

$$-\frac{1}{n-2} \int_x^T \varepsilon_{\min}(y)\, dy = \lim_{z \to T} \frac{1}{n-2} (\ln q_{\min}(z) - \ln q_{\min}(x)) = -\infty$$

f crosses 0 at a score in $(\tilde{x}_i T)$, a contradiction. ■

B.5 Proof of Lemma 6

By Lemma 5, $\exists \tilde{x} < T$ such that $\forall x \in (\tilde{x}, T), A(x) = N$. Equation (3) now implies that

$$\forall x \in (\tilde{x}, T), \forall i \in N : \ln(1 - G_i(x)) = \frac{1}{n-1} \sum_{j \in N} \ln q_j(x) - \ln q_i(x)$$

To show that $G_i(x) \underset{x \to T}{\to} 1$, it therefore suffices to show that

$$\frac{1}{n-1} \sum_{j \in N} \ln q_j(x) - \ln q_i(x) \underset{x \to T}{\to} -\infty$$

Since $\ln q_i(x) \underset{x \to T}{\to} -\infty$, it suffices to show that

$$\frac{1}{n-1} \frac{\sum_{j \in N} \ln q_j(x)}{\ln q_i(x)} > 1 + \delta \text{ for some } \delta > 0$$

for large enough x. The inequality follows from l'Hopital's rule and the fact that $\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_j(x)} = 1$, shown in the proof of Lemma 5.

B.6 Proof of Theorem 5

Suppose that the contest has a constructible equilibrium G. For expositional simplicity, I assume that the number of switching points in G is finite. It is straightforward to accommodate a countably infinite number of switching points by defining the limit of a sequence of switching points to be a switching point and modifying the proof appropriately.

In the following propositions, x_k denotes switching point k in G. The last switching point is T. $A(x)$ and $A^+(x)$ are defined as in Section 3. Choose any equilibrium $\tilde{G}$ of the contest, and recall that $\tilde{G}$ is continuous above 0 (part (1) of Lemma 2). $\tilde{A}(x)$ denotes the set of players active at x under $\tilde{G}$, i.e., the set of players defined by Equation (4) with $\tilde{G}$ instead of G. Using $A^+(x)$, I will show that $A(x) = \tilde{A}(x)$ for all $x \in [0, T]$. The following lemma shows that doing so is sufficient.

Lemma 11 *Let $\tilde{x} \in [0, T]$. If $\forall x \in [0, \tilde{x}] : \tilde{A}(x) = A(x)$, then $\forall x \in [0, \tilde{x}] : \tilde{G}(x) = G(x)$.*

Proof. Denote by $\bar{x}$ the infimum of the scores on which $\tilde{G}$ differs from G. Since $\tilde{G}(0) = G(0)$ (Lemma 3 does not rely on analyticity), G and $\tilde{G}$ are continuous on $(0, T)$ (part (1) Lemma 2), and $\tilde{G}(T) = G(T) = 1$ (part (3) of Lemma 2), we have $\tilde{G}(\bar{x}) = G(\bar{x})$. If $\bar{x} < \tilde{x}$, then by constructibility of G and because $\forall x \in [0, \tilde{x}] : \tilde{A}(x) = A(x)$, the algorithm shows that $\tilde{G}(x) = G(x)$ on a right-neighborhood of $\bar{x}$, a contradiction. Thus, $\bar{x} = \tilde{x}$. ■

Now, let x_k be the highest positive switching point such that $\tilde{A}(x) = A(x)$ on $[0, x_k]$, and suppose in contradiction that $x_k < T$. Choose $x \in (x_k, x_{k+1})$ such that $\tilde{A}(x) \neq A(x)$. Since $x_k < T$, such an x exists otherwise Lemma 11 and continuity would imply that $\tilde{A}(x_{k+1}) = A(x_{k+1})$. The following lemmas show that $\tilde{A}(x) \subseteq A(x)$ and $A(x) \subseteq \tilde{A}(x)$, which completes the proof.

Lemma 12 *$\tilde{A}(x) \subseteq A(x)$.*

Proof. Suppose $\tilde{A}(x) \nsubseteq A(x)$, and let $i_0 \in \tilde{A}(x) \backslash A(x)$. Since $i_0 \notin A(x)$, we have

$$\frac{w_{i_0} + c_{i_0}(y)}{v_{i_0}(y) + c_{i_0}(y)} > P_{i_0}(x) = 1 - \Pi_{j \neq i_0}(1 - G_j(x))$$

or

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} < \Pi_{j \neq i_0}(1 - G_j(x))$$

and since $i_0 \in \tilde{A}(x)$, we have

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} = \Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right)$$

so

$$\Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \neq i_0}(1 - G_j(x))$$

Let $J_1 = N \backslash \{i_0\}$. Then

$$\Pi_{j \in J_1}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1}(1 - G_j(x)) \tag{12}$$

By the Threshold Lemma of Siegel (2009), the expression on each side of Inequality (12) is a product of $n - 1$ strictly positive numbers. Therefore, there exists a player $i_1 \in J_1$ such that

$$\Pi_{J_1 \backslash \{i_1\}}\left(1 - \tilde{G}_j(x)\right) < \Pi_{J_1 \backslash \{i_1\}}(1 - G_j(x)) \tag{13}$$

(otherwise multiplying the products of all subsets of size $n - 2$ for G and for $\tilde{G}$ would lead to a contradiction).

Now, note that $\forall i \in N : \tilde{G}_i(x_k) = G_i(x_k)$, by Lemma 11, and since $\tilde{G}_i$ is non-decreasing

$$\forall i \notin A^+(x) : \left(1 - \tilde{G}_i(x)\right) \leq (1 - G_i(x)) \tag{14}$$

Let $K_1 = N \backslash J_1 = \{i_0\}$. Since $A^+(x) \subseteq A(x)$ and $i_0 \notin A(x)$, by Inequality (14)

$$\left(1 - \tilde{G}_{j \in K_1}(x)\right) \leq \left(1 - G_{j \in K_1}(x)\right) \tag{15}$$

By Inequalities (13) and (15),

$$\Pi_{j \in J_1 \cup K_1 \backslash \{i_1\}} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1 \cup K_1 \backslash \{i_1\}} \left(1 - G_j(x)\right) \tag{16}$$

Since $N = J_1 \cup K_1$, Inequality (16) shows that $i_1 \notin A^+(x_k)$, otherwise i_1 would obtain under $\tilde{G}$ more than his power by choosing x.
Now repeat the process above, letting $J_{r+1} = J_r \backslash \{i_r\}, K_{r+1} = K_r \cup \{i_r\}$. By induction on r, Inequalities (12),(13),(15), and (16) hold with J_r instead of J_1, K_r instead of K_1, and i_k instead of i_1, so $K_r \cap A^+(x_k) = \phi$. A contradiction is reached at stage $n-1$, since $|K_{n-1}| = N - 1$ but $|A^+(x_k)| \geq 2$. ■

Corollary 7 $\forall j \notin A(x), \forall y \in (x_k, x_{k+1}) : \tilde{G}_j(y) = G_j(y) = G(x_k)$.

Proof. Immediate from $\tilde{A}(y) \subseteq A(y)$ applied to all points $y \in (x_k, x_{k+1})$. ■

The next two lemmas establish that $A(x) \subseteq \tilde{A}(x)$.

Lemma 13 *If $A(x) \nsubseteq \tilde{A}(x)$, then $\tilde{G}_i(x) > G_i(x)$ for some $i \in A(x) \backslash \tilde{A}(x)$.*

Proof. Let $B = A(x) \backslash \tilde{A}(x)$, and suppose that $\forall j \in B : \tilde{G}_j(x) \leq G_j(x)$. This implies that $\exists j \in B : \tilde{G}_j(x) < G_j(x)$. Indeed, by Corollary 7 and Equation (3) with $\tilde{A}(x)$ instead of $A^+(x)$ and x instead of y, once G and $\tilde{G}$ agree on $(N/A(x)) \cup B = N/\tilde{A}(x)$ we obtain $\tilde{G}(x) = G(x)$ and therefore $A(x) = \tilde{A}(x)$.
To show that $\exists i \in B$ such that $\tilde{G}_i(x) > G_i(x)$, the following observation is required. Fix some values $\bar{G}_j(x)$ for $j \notin A(x)$ and use Equation (3) to solve for the values $\bar{G}_l(x), l \in A(x)$. Maintaining the value $\bar{G}_l(x)$ for player $l \in A(x)$ and solving for $A(x) \backslash \{l\}$ using Equation (3) gives the same solutions. If we now lower $\bar{G}_l(x)$ and solve for $A(x) \backslash \{l\}$, then the values $\bar{G}_j(x)$ of all players $j \in A(x) \backslash \{l\}$ strictly increase (this is easily seen from Equation (3), since D increases). Observe also that setting $\bar{G}_j(x) = \tilde{G}_j(x)$ for $j \notin \tilde{A}(x)$ and solving for the values $\bar{G}_i(x), i \in \tilde{A}(x)$ using Equation (3) with $\tilde{A}(x)$ instead of $A(x)$ gives $\bar{G}(x) = \tilde{G}(x)$.
Now, consider the following process by which $\tilde{G}(x)$ is "reached" from $G(x)$. Set $\bar{G}_l(x)$ equal to $G_l(x) = \tilde{G}_l(x)$ for $l \notin A(x)$. Take a player $j \in B$ for whom $\tilde{G}_j(x) < G_j(x)$. Then, solving for $A(x) \backslash \{j\}$ using $\bar{G}_j(x) = G_j(x)$ as described above and then lowering $\bar{G}_j(x)$ to $\tilde{G}_j(x)$ and solving again for $A(x) \backslash \{j\}$, raises the solutions above $G_l(x)$ for all $l \in A(x) \backslash \{j\}$. Thus, if $B = \{j\}$ then $A(x) \backslash \{j\} = \tilde{A}(x)$ and $\bar{G}_l(x) = \tilde{G}_l(x), l \notin \tilde{A}(x)$, so the solutions for $\bar{G}_i, i \in \tilde{A}(x)$ coincide with $\tilde{G}$ and player j obtains under $\tilde{G}$ at x more than his power (since under G at x player $j \in A(x)$ obtains precisely his power). If $B \neq \{j\}$, continue this process: take a player $l \in B \backslash \{j\}$ and lower $\bar{G}_l(x)$ obtained

in the previous step to $\tilde{G}_l(x)$;[39] solve for $A(x)\setminus\{j,l\}$, and remember that after lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ all players in $A(x)\setminus\{j\}$ obtained precisely their power, and $l\in A(x)\setminus\{j\}$. Since $\bar{G}_l(x)$ decreases to $\tilde{G}_l(x)$, the solutions for all players in $A(x)\setminus\{j,l\}$ strictly increase and $\tilde{G}_j(x)$ does not change, so player l now obtains more than his payoff. Continuing in this way and recalling that $\bar{G}=\tilde{G}$ once $\bar{G}$ and $\tilde{G}$ agree on $N/\tilde{A}(x)$, we see that the last player in B obtains more than his power under $\tilde{G}$ at x, a contradiction. Therefore, $\tilde{G}_i(x)>G_i(x)$ for some player $i\in B$. ∎

Lemma 14 $A(x)\subseteq\tilde{A}(x)$.

Proof. Suppose $A(x)\not\subseteq\tilde{A}(x)$. By the previous lemma, $\tilde{G}_i(x)>G_i(x)$ for some $i\in A(x)\setminus\tilde{A}(x)$. Since $\tilde{G}_i(x_k)=G_i(x_k)$, $\tilde{G}_i(y)>G_i(y)$ for some $y\in(x_k,x)$ such that $i\in\tilde{A}(y)$.[40] This means that $\tilde{A}(y)\neq A(y)$ (otherwise Corollary 7 and Equation (3) would imply $\tilde{G}_i(y)=G_i(y)$). Let $\hat{B}=A(y)\setminus\tilde{A}(y)$.
Now perform a procedure similar to the one described in the previous lemma, reaching $\tilde{G}(y)$ from $G(y)$. Begin with players $l\in\hat{B}$ for whom $\tilde{G}_l(y)>G_l(y)$. Raising $\bar{G}_l(y)$ from $G_l(y)$ to $\tilde{G}_l(y)$ decreases the solutions for all other players, so the order of raising does not matter - the solutions must be raised for all players $l\in\hat{B}$ for whom $\tilde{G}_l(y)>G_l(y)$. If the solutions of any remaining players in $\hat{B}$ now need to be raised to reach their level in $\tilde{G}$, continue the raising process until no more players in $\hat{B}$ need their solutions raised. It cannot be that $\hat{B}$ is exhausted, since $\tilde{G}_i(y)>G_i(y)$ and so far the solutions of all players in $\tilde{A}(y)$ have been repeatedly decreased, starting from their level in G. Thus, there remains a non-empty set $\bar{B}\subseteq\hat{B}$ of players whose solutions must now be decreased to reach their level in $\tilde{G}$. Decreasing these solutions increases the solutions for all other players. By the argument used in the previous lemma, the last player whose solution is decreased receives too high a payoff under $\tilde{G}$ at y. ∎

B.7 The Example of Section 3

Cost functions are $c_2(x)=\frac{3x}{4}$,

$$c_1(x)=\begin{cases}\frac{x}{100} & \text{if } 0\leq x\leq 0.31948\\ \frac{x}{100}+1.0581\left(x-(0.31948)\right)^2 & \text{if } 0.31948<x\leq 1\\ 0.5+1.45\left(x-1\right) & \text{if } 1<x\end{cases}$$

[39] Since $\tilde{G}_l(x)\leq G_l(x)$ and lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ raised the solutions $\bar{G}_l(x)$ for all $l\in A(x)\setminus\{j\}$, we have $\tilde{G}_l(x)<\bar{G}_l(x)$.

[40] Let $\bar{z}=\sup_{z\in[x_k,x)}\left\{\tilde{G}_i(z)=G_i(z)\right\}$. By continuity of $\tilde{G}_i$ and G_i, $\bar{z}<x$. If for all $y\in(\bar{z},x)$ we had $i\notin\tilde{A}(y)$ then $\tilde{G}_i$ would not increase on $(\bar{z},x)$ so we would have $\tilde{G}_i(x)\leq G_i(x)$.

and

$$c_3(x)=\begin{cases}\frac{x}{12} & \text{if} \quad 0\le x\le 0.31948\\ \frac{x}{12}+1.9794\left(x-(0.31948)\right)^2 & \text{if} \quad 0.31948<x\le 0.7259\\ 0.387\,44+1.6923\left(x-0.7259\right)+25\left(x-0.7259\right)^2 & \text{if} \quad 0.7259<x\le 0.85\\ 0.98247+\frac{(1-0.98247)}{0.15}\left(x-0.85\right) & \text{if} \quad 0.85<x\end{cases}$$

These cost functions give powers of 0, $\frac{1}{4}$ and $\frac{1}{2}$. Perturbing the cost functions slightly does not change the qualitative aspects of the equilibrium.

C Proofs of the Results of Section 4

C.1 Proof of Theorem 9

That players $m+2,\dots,N$ do not participate was shown in Corollary 4. Denote by s_i^l the lowest score at which player i is active. Because $m,m+1\in A(0)$ (see point 4 in Section 3.2), we have $s_m^l=s_{m+1}^l=0$. Now, suppose that for $i,j\le m$, $s_i^l\le s_j^l$. Since i and j have positive powers and are not active below s_i^l, $G_i\left(s_i^l\right)=G_j\left(s_i^l\right)=0$. Thus, $P_i\left(s_i^l\right)=P_j\left(s_i^l\right)$.[41] Since

$$\frac{u_i\left(s_i^l\right)}{V_i}=P_i\left(s_i^l\right)\left(1-a_ic\left(s_i^l\right)\right)-\left(1-P_i\left(s_i^l\right)\right)\alpha a_ic\left(s_i^l\right)=P_i\left(s_i^l\right)\left(1-\left(1-\alpha\right)a_ic\left(s_i^l\right)\right)-\alpha a_ic\left(s_i^l\right)$$

and

$$\frac{u_j\left(s_i^l\right)}{V_j}=P_j\left(s_i^l\right)\left(1-a_jc\left(s_i^l\right)\right)-\left(1-P_j\left(s_i^l\right)\right)\alpha a_jc\left(s_i^l\right)=P_j\left(s_i^l\right)\left(1-\left(1-\alpha\right)a_jc\left(s_i^l\right)\right)-\alpha a_jc\left(s_i^l\right),$$

we have

$$\begin{aligned}\frac{u_i\left(s_i^l\right)}{V_i}-\frac{u_j\left(s_i^l\right)}{V_j}&=P_i\left(s_i^l\right)\left(1-\alpha\right)c\left(s_i^l\right)\left(a_j-a_i\right)+\alpha c\left(s_i^l\right)\left(a_j-a_i\right)\\&=\left(a_j-a_i\right)c\left(s_i^l\right)\left(\alpha+P_i\left(s_i^l\right)\left(1-\alpha\right)\right).\end{aligned}$$

Also, $\frac{w_i}{V_i}=1-a_ic(T)$ and $\frac{w_j}{V_j}=1-a_jc(T)$ so $\frac{w_i}{V_i}-\frac{w_j}{V_j}=\left(a_j-a_i\right)c(T)$. Since $\frac{w_i}{V_i}=\frac{u_i\left(s_i^l\right)}{V_i}$ and $\frac{w_j}{V_j}\ge\frac{u_j\left(s_i^l\right)}{V_j}$, we have

$$0\ge\left(\frac{w_i}{V_i}-\frac{w_j}{V_j}\right)-\left(\frac{u_i\left(s_i^l\right)}{V_i}-\frac{u_j\left(s_i^l\right)}{V_j}\right)=\left(a_j-a_i\right)\left(c(T)-c\left(s_i^l\right)\left(\alpha+P_i\left(s_i^l\right)\left(1-\alpha\right)\right)\right)$$

[41] If $s_i^l=0$, we consider the limit of the probabilities of winning as the score approaches 0 from above, and similarly for $u_i\left(s_i^l\right)$ and $u_j\left(s_i^l\right)$.

and since $\alpha > 0$ and $P_i\left(s_i^l\right) \leq 1$, $\left(\alpha + P_i\left(s_i^l\right)(1-\alpha)\right) > 0$ so the above inequality holds if and only if

$$(a_j - a_i)\left(c(T) - c\left(s_i^l\right)\right) \leq 0.$$

Since $c\left(s_i^l\right) < c(T)$, this implies that $a_j \leq a_i$.

It remains to show that if a player becomes active at some score, then he remains active until the threshold. To do this, let us derive some properties of player's semi-elasticities. We must first normalize players' payoffs so that the prize value is 1 for all players. To this end, note that the contest is strategically equivalent to a contest in which all valuations equal 1 and in which player i's cost is $a_i c$ instead of $\gamma_i c$. We then have

$$q_i(x) = \frac{v_i(x) - v_i(T)}{v_i(x) + c_i(x)} = \frac{a_i\left(\frac{1}{a_{m+1}} - c(x)\right)}{1 - (1-\alpha)\, a_i c(x)}$$

and

$$\varepsilon_i(x) = -\frac{q_i'(x)}{q_i(x)} = \frac{c'(x)\left(a_i(\alpha-1) + a_{m+1}\right)}{\left(1 - c(x)\, a_{m+1}\right)\left(a_i c(x)(\alpha-1) + 1\right)}.$$

Viewed as a function of a_i, we obtain

$$\frac{\partial \varepsilon_i(x)}{\partial a_i} = -\frac{c'(x)(1-\alpha)}{\left(a_i c(x)(\alpha-1)+1\right)^2} \leq 0$$

so at every score players with a higher reach have higher semi-elasticities. This means that when a new player becomes active all existing active players remain active, and that whether an active player remains active depends only on his semi-elasticity and those of players with lower reaches. In particular, players m and $m+1$ are always active, since their semi-elasticities are always the lowest. To show that players $1,\ldots,m-1$ are active on an interval, observe that the ratio of semi-elasticities of players $j > i$ is non-decreasing in score:

$$\left(\frac{\varepsilon_j(x)}{\varepsilon_i(x)}\right)' = \left(\frac{\left(a_j(\alpha-1) + a_{m+1}\right)\left(a_i c(x)(\alpha-1)+1\right)}{\left(a_i(\alpha-1) + a_{m+1}\right)\left(a_j c(x)(\alpha-1)+1\right)}\right)'$$
$$= \frac{a_{m+1} - (1-\alpha)\, a_j}{a_{m+1} - (1-\alpha)\, a_i} \frac{(1-\alpha)(a_j - a_i)\, c'(x)}{\left(a_j c(x)(\alpha-1)+1\right)^2} \geq 0$$

and also that this ratio is at most 1 (since players with a higher reach have higher semi-elasticities). Suppose in contradiction that there is a player who is not active on an interval, and let $i \leq m-1$ be the player with the highest index among such players. Suppose that player i is active at s_i. Denote by $H^{i+1,\ldots,m+1}(\cdot)$ the fixed point of the "supply function" defined using the semi-elasticities of players $i+1,\ldots,m+1$, and consider a score $s_i' \in [s_i, T]$. By definition of player i and because players with higher reaches become active at higher scores, players $m+1,\ldots,i+1$ are active at s_i'. So, because the semi-elasticities of all players $1,\ldots,i-1$ are weakly higher than that of player i, $\varepsilon_i\left(s_i'\right) \leq H\left(s_i'\right)$ if and only if $\varepsilon_i(s_i') \leq H^{i+1,\dots,m+1}(s_i')$. Let $b = \frac{\varepsilon_i(s_i')}{\varepsilon_i(s_i)}$. For all $j > i$, since $\frac{\varepsilon_j(s_i')}{\varepsilon_i(s_i')} \geq \frac{\varepsilon_j(s_i)}{\varepsilon_i(s_i)}$, we have $\frac{\varepsilon_j(s_i')}{\varepsilon_j(s_i)} \geq \frac{\varepsilon_i(s_i')}{\varepsilon_i(s_i)} = b$. Therefore,

$$H^{i+1,\dots,m+1}(s_i') = \frac{1}{m-i}\sum_{j=i+1}^{m+1}\varepsilon_j(s_i') \geq \frac{1}{m-i}\sum_{j=i+1}^{m+1} b\varepsilon_j(s_i) = bH^{i+1,\dots,m+1}(s_i).$$

Because player i is active at s_i, $\varepsilon_i(s_i) \leq H(s_i)$ so $\varepsilon_i(s_i) \leq H^{i+1,\dots,m+1}(s_i)$. Therefore, $\varepsilon_i(s_i') = b\varepsilon_i(s_i) \leq bH^{i+1,\dots,m+1}(s_i) \leq H^{i+1,\dots,m+1}(s_i')$ so $\varepsilon_i(s_i') \leq H(s_i')$. This shows that once a player becomes active he remains active until the threshold.

C.2 Proof of Corollary 5

First, $s_j^l \leq s_i^l$, so $G_i(x) \leq G_j(x)$ for any $x \leq s_i^l$. By Theorem 9, both players are active on on $\left[s_i^l, T\right]$. Third, $\varepsilon_i \geq \varepsilon_j$, so because both players are active on $\left[s_i^l, T\right]$, the equilibrium construction algorithm shows us that $h_i \leq h_j$ on $\left[s_i^l, T\right]$. So i starts dropping out later and drops out more slowly than j, which means that G_i FOSD G_j. To see this, recall that $h_i(x) = -\frac{(1-G_i(x))'}{1-G_i(x)}$, so $h_i \leq h_j$ implies that $\frac{(1-G_i)'}{1-G_i} \geq \frac{(1-G_j)'}{1-G_j}$. This implies that for $y \in \left[s_i^l, T\right]$,

$$\begin{aligned} 0 \leq \int_{s_i^l}^{y}\left(\frac{(1-G_i(x))'}{1-G_i(x)} - \frac{(1-G_j(x))'}{1-G_j(x)}\right)dx &= \ln\left(\frac{1-G_i(x)}{1-G_j(x)}\right)\Big|_{s_i^l}^{y} \\ &= \ln\left(\frac{1-G_i(y)}{1-G_j(y)}\right) - \ln\left(\frac{1-G_i\left(s_i^l\right)}{1-G_j\left(s_i^l\right)}\right). \end{aligned}$$

Because $G_i\left(s_i^l\right) \leq G_j\left(s_i^l\right)$, we have $\frac{1-G_i\left(s_i^l\right)}{1-G_j\left(s_i^l\right)} > 1$, so by taking exponents the previous inequality implies $\frac{1-G_i(y)}{1-G_j(y)} \geq 1$, or $G_j(y) \geq G_i(y)$, as required.

This FOSD implies that probability of winning is higher than that of player j for any given score, and hence also in expectation. To see this, note that by choosing $x > 0$ i beats j with probability $G_j(x)$, whereas by choosing x j beats i with probability $G_i(x)$. Therefore, because $G_j(x) \geq G_i(x)$, for any given score i wins with at least as high a probability as j does, i.e., $P_i(x) \geq P_j(x)$. Therefore, $P_i = \int P_i(x)\,dG_i \geq \int P_j(x)\,dG_i$, and because $P_j(\cdot)$ is non-decreasing, by FOSD $\int P_j(x)\,dG_i \geq \int P_j(x)\,dG_j = P_j$.

C.3 Proof of Theorem 10

Let

$$f(\alpha, x) = \frac{1}{(1-a_{m+1}c(x))}\frac{\Pi_{k=i+1}^{m+1}(1-a_k c(x)(1-\alpha))}{(1-a_i c(x)(1-\alpha))^{m-i}},$$

and note that f is differentiable in $\alpha < 1$ and $x < T$. Denote by $s_i^l(\alpha)$ the lowest x such that $f(\alpha, x) = \frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}}$. Note that $\frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} > 1$ (because a_k increases in k) and $f(\alpha, 0) = 1$

(since $c(0)=0$). Suppose that when α increases to α' the value of f at $s_i^l(\alpha)$ increases. Then, because $f\left(\alpha', s_i^l(\alpha)\right) > \frac{\Pi_{k=i+1}^m a_k}{a_i^{m-i}}$ and $f(\alpha', 0) < \frac{\Pi_{k=i+1}^m a_k}{a_i^{m-i}}$, the intermediate value theorem shows that $s_i^l(\alpha') < s_i^l(\alpha)$. Therefore, to show that s_i^l decreases in α it suffices to show that $\frac{\partial f(a,x)}{\partial \alpha} > 0$ for $x \in (0,T)$. Since

$$\frac{\partial f}{\partial \alpha} = \frac{\left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k\in\{i+1,\ldots,m+1\}\setminus j} (1-a_k c(x)(1-\alpha))\right)(1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}}{\left((1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}\right)^2}$$

$$-\frac{\Pi_{k=i+1}^{m+1}(1-a_k c(x)(1-\alpha))\left((1-a_{m+1}c(x))(m-i)(1-a_i c(x)(1-\alpha))^{m-i-1} a_i c(x)\right)}{\left((1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}\right)^2},$$

it suffices to show that

$$\left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k\in\{i+1,\ldots,m+1\}\setminus j} (1-a_k c(x)(1-\alpha))\right)(1-a_i c(x)(1-\alpha)) >$$

$$\Pi_{k=i+1}^{m+1}(1-a_k c(x)(1-\alpha))\left((m-i)a_i c(x)\right).$$

For this inequality to hold, it suffices that for every $j = i+1, \ldots m+1$,

$$a_j c(x) > a_i c(x) \text{ and } 1-a_i c(x)(1-\alpha) > 1-a_j c(x)(1-\alpha),$$

and this holds since a_k increases in k. Therefore s_i^l decrease in $\alpha < 1$ for every player $i = 1,\ldots,m-1$.

Now consider what happens to s_i^l as α approaches 0. For $x < T$,

$$f(0,x) = \frac{1}{(1-a_{m+1}c(x))} \frac{\Pi_{k=i+1}^{m+1}(1-a_k c(x))}{(1-a_i c(x))^{m-i}} = \frac{\Pi_{k=i+1}^{m}(1-a_k c(x))}{(1-a_i c(x))^{m-i}} \le 1.$$

Therefore, by uniform continuity of $f(\alpha,x)$ on $[0,\tilde{\alpha}]\times[0,x]$ for any $\tilde{\alpha}\in(0,1)$, s_i^l must approach T as α approaches 0.

C.4 Proof of Corollary 6

Choose $\beta < 1$. By Theorem 10 there exist $\tilde{x} < T$ and $\tilde{\alpha} > 0$ such that for all $\alpha < \tilde{\alpha}$ and $i < m$, $s_i^l > \tilde{x}$. Choose such $\tilde{x}$ and $\tilde{\alpha}$ that also satisfy (1) $\frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}} + \alpha\gamma_m c(\tilde{x})}{V_m - (1-\alpha)\gamma_m c(\tilde{x})} > \beta$ and (2) $\frac{\alpha\gamma_{m+1}c(\tilde{x})}{V_{m+1}-(1-\alpha)\gamma_{m+1}c(\tilde{x})} < 1-\beta$. Consider the unique equilibrium G of a such a simple contest with $\alpha < \tilde{\alpha}$. Since $G_i(\tilde{x}) = 0$ for $i = 1,\ldots,m-1$ and $G_i(0) = 1$ for $i = m+2,\ldots,n$, Corollary 3 shows that the CDFs of players $m+1$ and m on $[0,\tilde{x}]$ are given by (1) and (2). Since $s_i^l > \tilde{x}$ for $i = 1,\ldots,m-1$, each of these $m-1$ players beats player $m+1$, and therefore wins a prize, with probability of at least β. Player m chooses scores higher than $\tilde{x}$ with probability of at least β, and therefore wins a prize with probability of at least β^2.