Quantum Games Have No News for Economists[1]

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This version: September 1, 2005

First version: December 3, 2005

Quantum computing offers the possibility of massively parallel computing that scales to large problems in a way not possible for classical computers. It also may make possible rapid and secure forms of communication that are not possible with classical devices. As an offshoot of the quantum computing literature, a small set of papers has started to examine quantum games. The question naturally arises: what if anything does quantum game theory have for economists? This brief note attempt to summarize the sometimes impenetrable notation used in quantum physics. I argue that quantum games fall within the existing framework of correlated equilibrium, cheap-talk equilibrium and mechanism design theory, where the correlation and/or communication devices are limited in a way not terribly relevant to economic theory. The notation is taken from Cleve et al [2004], and interpreted using the Wikipedia.

A quantum mechanical system has a state $\langle\psi|$, which is simply a k-dimensional complex valued row vector of unit length. The notation $|\psi\rangle$ refers to the conjugate of $\langle\psi|$, that is, the transposed k-dimensional column vector consisting of complex conjugates of $\langle\psi|$. Since the length of a complex vector is the square root of the inner product of that vector with its conjugate, the condition that the state has unit length is simply $\langle\psi|\psi\rangle = 1$. A complex $k \times k$ matrix X is positive semi-definite if $\langle\psi|X|\psi\rangle \geq 0$ for all complex k-vectors $\langle\psi|$. A measurement system consists of a finite collection $\{X^a\}$ of $k \times k$ complex positive semi-definite matrices, where $a \in A$, a finite set, and

$$\sum_{a \in A} X^a = I,$$

[1] Based on conversations with Eddie Dekel and Larry Samuelson. I also benefited from discussion with Pierfrancesco La Mura.

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where I is the identity matrix. When the measurement system is applied to the state $\langle\psi|$, the probability that the measurement takes on the value a is imply $\langle\psi| X^a |\psi\rangle$.

Quantum physicists also have their own notation for canonical bases of complex k-space. Suppose the underlying classical state has two components each of which can take on two values $\{0,1\}$. Then the quantum state has four dimensions, corresponding to each of the four classical states $\{00,01,10,11\}$ and reflecting the value of the first and second component. Then the canonical basis of complex 4-space, which we would usually write as $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ would be written by a physicist as $\{\langle 00|,\langle 01|,\langle 10|,\langle 11|\}$. Complex vectors are then written as linear combinations of these basis vectors, so for example $(\langle 00| + \langle 11|)/\sqrt{2} = (1/\sqrt{2},0,0,1/\sqrt{2})$, which of course has unit length.

Next we suppose that there are two players. These players have access to a k_1k_2 dimensional state $\langle\psi|$. A measurement system for player i consists of a finite collection $\{X_i^a\}$, $a \in A_i$ of $k_i \times k_i$ complex positive semi-definite matrices. The Kronecker product of two measurement matrices is the $k_1k_2 \times k_1k_2$ matrix

$$X_1^a \otimes X_2^b \equiv \begin{bmatrix} X_{1,11}^a X_2^b & \dots & X_{1,1k_1}^a X_2^b \\ \vdots & \ddots & \vdots \\ X_{1,k_1 1}^a X_2^b & \dots & X_{1,k_1k_1}^a X_2^b \end{bmatrix},$$

which may be familiar to economists from seemingly unrelated regression theory where Kronecker products frequently appear in covariance matrices. When both players apply their measurements, and the state is $\langle\psi|$, the probability of the pair of measurements $(a,b) \in A_1 \times A_2$ is given by $\langle\psi| X_1^a \otimes X_2^b |\psi\rangle$. Since the probability of player 1's measurement depends on the measurement taken by player 2, this is referred to in quantum mechanics as *quantum entanglement*. It should be emphasized that although this is rarely explicitly stated in the quantum game literature, it is assumed that the underlying state $\langle\psi|$ is common knowledge among the players, although of course, the realized value of measurements based on that state is not known.

We now consider a game in which the pure strategy spaces are A_i for player i – the outcome of quantum mechanical measurement, in other words, will correspond to a choice of strategy for that player. Payoffs are real valued $\pi_i(a_i, a_{-i})$, where as usual a_{-i} refers to the strategy of the player other than player i. However, in a quantum game, players play by choosing a measurement system from a feasible set of systems Σ_i. The system

$$X_i^b = \begin{cases} I & b = a \\ 0 & b \neq a \end{cases}$$

corresponds to the pure strategy a; this is ordinarily assumed to be in the feasible set. More generally

$$X_i^b = \alpha_i(b) I$$

corresponds to the mixed strategy that plays b with probability $\alpha_i(b)$. These also should be feasible. But quantum games may allow other non-classical measurement systems as well – and indeed, even if a player plays a "classical" pure or mixed strategy, an opponent through quantum entanglement may be able to correlate play with that player.

At this point the literature on quantum games faces a modeling decision, although they do not recognize it as such. In one model, players make their measurements, then decide what to do based on the measurement. In this case, we may think of the result of the measurement as a recommendation on how to play. We then define an equilibrium to be a pair of measurement systems such that each player knowing her opponent's measurement system finds it optimal to follow the recommendation made by her own system. In this case, it should be apparent that an equilibrium is a special case of a correlated equilibrium – all that matters to player is the joint distribution of recommendations over strategy profiles – that this is generated by a quantum mechanical system is not relevant to incentives. This point was first made in Meyer [2004].

However, the quantum game literature has taken two different turns. One possibility that is considered is that players have private information prior to submitting their measurements. Consider, for example, the pure coordination game examined by Cleve et al. In the variant described by Dahl and Landsburg [2005] Alice and Bob each independently with 50-50 probability are asked either "Do you like cats?" or "Do you like dogs?" If they agree they both get one, unless both are asked "Do you like cats?" in which case they get one if they disagree. Otherwise they get 0. After receiving the question, they then submit their measurements to the quantum device, get their recommendation on whether to say "yes" or "no" and submit either answer they prefer to receive their payoff. In any Nash, and moreover, in any correlated equilibrium, of the game they can win at most ¾ of the time. If they use a particular quantum device to coordinate their actions, Cleve et al show that they win $\cos^2(\pi/8) > 3/4$ of the time. A similar example was previously described by La Mura [2003]. The key point here is the players are implicitly allowed to send messages (measurements) to a machine that then gives them advice. This is not an example of a correlated equilibrium: it is an example of a cheap-talk equilibrium. If we can design an arbitrary "cheap talk" device to which players can submit messages and get advice, they can win all the time. Each simply announces the question they were asked, and if they were both asked about cats, Alice says "no" and Bob "yes" otherwise they both say "yes." This is of course the basic point of the cheap talk literature going back to the early work of Crawford and Sobel [1982] and Farrell [1987]. This particular branch of the quantum tree has simply rediscovered cheap talk.

These examples do, however, give an indication of just how complicated quantum mechanics can be even in the simplest problem – the Dahl and Landsburg [2005] version of the example asserts on page 1 that quantum probabilities violate the ordinary laws of probability – something which the fact that $\langle\psi|X^a|\psi\rangle \geq 0$ and $\Sigma_a \langle\psi|X^a|\psi\rangle = 1$ should persuade you is not true. They then continue on to analyze on p. 4 the cats and dogs example – mixing the usual laws of probability with the supposed quantum laws.

The La Mura [2003] example is instructive.[3] In that example each player is one of three types A,B and C with equal probability of 1/3rd. Each chooses one of three measurements, x, y, z. Because of the quantum entanglement, if they choose the same measuring device, then the probabilities they are told to say "yes", "no" are given by

	"yes"	**"no"**
"yes"	0	1/2
"no"	1/2	0

while if they choose different measuring devices the probabilities are

[3] I am grateful to Pierfrancesco La Mura for explaining this to me.

	"yes"	**"no"**
"yes"	3/8	1/8
"no"	1/8	3/8

Of course, we can implement this scheme with a classical communications device that receives the messages x, y, z from the players and replies by advising them according to these probabilities. Note that there is no issue of "classical" versus "quantum" laws of probability here.

The point that deserves some emphasis, however, is that not all communications devices can be implemented by quantum correlating devices. In the La Mura example suppose that type A chooses x, type B chooses y and type C chooses z. Then a player 1 with type A faces the following probabilities

	A		B		C	
	"yes"	"no"	"yes"	"no"	"yes"	"no"
"yes"	0	4/8	3/8	1/8	3/8	1/8
"no"	4/8	0	1/8	3/8	1/8	3/8

where opponents of each type have probability of 1/3rd. The goal of this game is to avoid agreeing with the same type, and to agree with different types. This scheme accomplishes that – if the advice is followed and the opponent is also type A the probability of agreement is 0. On the other hand, if the opponent is type B or C the probability of agreement is 3/8. But notice the following fact: given the message ("yes," for example) received by player 1, the conditional probability of each type of opponent remains 1/3rd.[4] In this sense no communication takes place. This is fundamental to quantum mechanics – by simply entangling states, no communication in this sense can ever take place. That is, based on quantum entanglement, a player's measurement may not reveal anything about

[4] It is also the case that the conditional probability his opponent says "yes" remain 1/2, but this is less fundamental in that a player's signal may contain information about what signal his opponent received. For example, it they are both restricted to use the same device, then each players signal reveals exactly what signal his opponent received.

what measurement device the other player used.[5] It may, however, reveal information about the signal received by the other player introducing a correlation, or may reveal information about the joint distribution of the measurement device and the signal, as it does in this example: physicists sometimes refer to this as "pseudo-communication." Note that quantum pseudo-communication may have advantages of security; or may be available when other "true" communication devices are not, in which case the quantum constraints become relevant.

There is second branch of quantum games, which does not consider private information, but is focused instead on solving games with dominant strategies such as the Prisoner's Dilemma. Benjamin and Hayden [2001] is an example of such a model. Since it is obvious that after you get your recommendation, regardless of what quantum principles may be involved in making it, it is still best to follow your dominant strategy. So they assume (implicitly) that following the recommendation of the device is not optional, but is rather a binding commitment. The only way to make sense of this is to assume that the results of player measurements do not go back to the players, but rather go to a machine that then implements the decisions. But if we are going to build a machine that takes input from players and makes choices on their behalf, we are by no means limited to the simplistic machines considered in the quantum games literature. Indeed, the problem of building machines to make choices based on player submissions is exactly the problem considered in the mechanism design literature. So it should not be surprising, for example, that it might be possible to get cooperation in a "quantum" prisoner's dilemma game. If we are allowed to build machines, a simple machine consistent with players' original strategy spaces is to add a single strategy "Z." If a player plays a strategy in the original space, then the machine implements that action for her. If she chooses Z and her opponent chooses Z, then the machine assigns both to cooperate. If she chooses Z and her opponents does not, then the machine assigns her to defect. It is obvious that Z weakly dominates all other strategies in this mechanism.

While at the moment economists may have little to learn from quantum games, there are legitimate issues that the literature may address in the future: for example, quantum correlation devices may impose limitations on the feasible set of correlated

[5] This is known as the impossibility of Bell's telephone – if it failed, then an entanglement could be used for communication.

equilibria while offering levels of security not attainable by classical devices. Or it may be that in modeling evolution at the molecular level, quantum devices play an important correlating role.

For further reading on quantum games oriented towards economists and game theorists, Campos [2005] has a nice exposition.

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