Noisy talk

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We investigate strategic information transmission with communication error, or *noise*. Our main finding is that adding noise can improve welfare. With quadratic preferences and a uniform type distribution, welfare can be raised for almost every bias level by introducing a sufficiently small amount of noise. Furthermore, there exists a level of noise that makes it possible to achieve the best payoff that can be obtained by means of any communication device. As in the model without noise, equilibria are interval partitional; with noise, however, coding (the measure of the message space used by each interval of the equilibrium partition of the type space) becomes critically important.

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1. Introduction

In many situations of economic interest, decision makers seek advice from better-informed experts. Examples include lobbying, management consulting, and financial advice. In these situations frequently the interests of experts and decision makers do

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not coincide. This creates an incentive for the expert to use her information strategically, and for the decision maker to interpret advice in light of the expert's bias. The seminal analysis of strategic information transmission between a biased expert and an uninformed decision maker was provided by Crawford and Sobel (1982) (henceforth CS). In their model a privately informed *sender* sends a costless message to a *receiver* who takes an action that affects the payoff of both parties.

In this paper we investigate strategic information transmission when there is communication error, which we refer to as *noise*. With some probability, independent of the message sent, observed messages are drawn from a fixed error distribution; otherwise messages go through as sent. Our main finding is that adding noise can improve welfare. In the uniform–quadratic model, i.e. with quadratic preferences and a uniform type distribution, welfare can be raised for almost every bias level by introducing a sufficiently small amount of noise. In addition there exists a level of noise that makes it possible to achieve the best possible payoff that can be obtained by means of any communication device.

As is the case in the CS model, all equilibria of the noise model are interval partitional: the sender's message reveals only in which of a number of intervals of the state space the true state lies. But unlike in the CS model, in the noise model there may be infinitely many actions (countable or uncountable) induced in equilibrium. Even restricting to equilibria with a fixed number of intervals, once noise is introduced there is generally a continuum of equilibria, each of which induces a different outcome. This multiplicity is a consequence of the fact that, unlike in the CS model, *coding*, the measure of the message space used by each interval of the equilibrium partition of the type space, matters when there is noise. Our welfare results are achieved with equilibria that induce a finite partition and with a *front-loading* coding scheme: types in the lowest interval of the equilibrium partition randomize over almost all of the available messages; each other interval is identified with a single distinct message, sent by all types in that interval. This front-loading construction and consequently our welfare results extend to an environment in which noise levels are correlated with messages, provided the function that maps messages into noise levels has a sufficiently large range. Interestingly, in this environment the best payoff that can be obtained by means of any communication device can be approximated to any desired degree without manipulating the noise level, which is determined endogenously in equilibrium.

Communication errors have been studied in information theory, pioneered by Shannon (1948). There the problem is faithful transmission of messages in the presence of some fundamental source of noise, abstracting from strategic considerations. In Shannon's original model, the only source of noise lies in the *channel* through which signals (translated messages) are passed from sender to receiver, but in a broader context we may think of errors as arising also in the translation process. Suppose, for example, that Alice is trying to pass on some information to Bob by means of verbal communication. Three sources of error are possible: she may fail to choose appropriate words to express her thoughts; he may not hear correctly what she says; or he may misunderstand the meaning of her words.[1]

[1] This final source of potential error is particularly important when the words used are vague. Vagueness

The paper is structured as follows. In Section 2, we describe the formal details of the model and provide a partial characterization of the equilibrium set. Section 3 provides a closer examination of the set of equilibria in the uniform–quadratic case. In Section 4 we consider the welfare properties of noise equilibria, again in the uniform–quadratic case. Section 5 examines some extensions of the model, and Section 6 concludes.

1.1 *Related literature*

To our knowledge, the idea that noisy communication channels can improve information transmission is first discussed by Myerson (1991, pages 285–288). He considers a two-state, three-action cheap talk game; if player 1 is able to send a message to player 2 by means of a carrier pigeon that arrives only half the time, then communication is possible when it would have been impossible with direct, reliable messages. In the communicative equilibrium, player 1 sends a message in only one of the two states. If the pigeon arrives, player 2 knows that he is in that state; if not, he cannot determine whether the pigeon got lost or was never sent. In this way, an outcome is achieved that is better for both players than would have been possible in the absence of noise.

Also related to the current project is the extensive literature on general communication devices (see e.g. Forges 1986 and Myerson 1986). Such a device (often thought of as an impartial mediator) receives inputs (messages) from each player and transmits outputs according to a matrix of transition probabilities. Forges and Myerson show that allowing the players to use these devices can expand the set of equilibrium outcomes in games. Clearly, a communication device could be used to replicate the noise mechanism considered here, or to reproduce the effects of Myerson's unreliable carrier pigeon. But communications devices are much more general than noise mechanisms.

More recently, Goltsman et al. (2007) investigate optimal communication devices (which they call *mediators*) in the context of the uniform–quadratic version of the CS model. They derive an upper bound on the receiver's payoff in any equilibrium; we show in Section 4.2.1 that if the level of noise can be chosen appropriately, our front-loading equilibrium construction achieves this upper bound. They also show that if the receiver is able to commit to using an *arbitrator* to make decisions for him on the basis of messages received from the sender, he can obtain a strictly higher expected payoff. (In a related paper, Kováč and Mylovanov (2006) study arbitration in a more general framework.) Ganguly and Ray (2007) also analyze communication devices in the uniform–quadratic version of the CS model. Their main result concerns devices that are N-*simple*: they receive N messages and submit N recommendations. Such devices cannot improve on the N-step CS equilibrium if the bias lies below some bound which depends on N.

A paper by Krishna and Morgan (2004) shows that allowing multiple rounds of (two-way) communication in the CS framework can also result in equilibria that Pareto dominate those of the original model.[2] They consider a first round of communication, a

is a pervasive phenomenon in natural language—consider the use of terms such as "tall," "red," and "good."

[2]Aumann and Hart (2003) also examine games with multiple rounds of pre-play communication, and provide a complete characterization of the set of equilibrium outcomes. Since they consider games with a finite set of states, their results do not apply to the CS model.

meeting, in which the sender and receiver exchange messages simultaneously, followed by a single transmission from sender to receiver. During the meeting, the sender reveals in which of two elements of the state space the true state lies, and the two agents also send random messages to determine whether the meeting should be deemed a "success" or a "failure." These random messages effectively induce a lottery over outcomes such that neither agent can affect the probability of success or failure. If the meeting was a success, then the sender reveals more information about the true state during the second round of communication; otherwise no more information is revealed. (Clearly this kind of communication could also be replicated using a communication device: Ganguly and Ray show this formally.) Krishna and Morgan establish the remarkable result that it is almost always possible to construct equilibria in which, relative to the best CS equilibrium, the information gain when the meeting is a success outweighs the information loss when it is a failure, leading to a Pareto improvement. This kind of equilibrium is able to improve on the CS equilibria by leveraging the risk aversion of the sender; in the face of risk about whether or not additional information will be conveyed in the second round, she is willing to give up more information in the first round. In the uniform–quadratic case most commonly used in applications of the CS model, the welfare results of Krishna and Morgan are similar to our own, although we show that the probability of error can lead to welfare improvements for more extreme values of the sender's bias. But the underlying source of the welfare gain is very different.

Three recent papers introduce different kinds of perturbations into the CS model. First, Kartik et al. (2007) (henceforth KOS) study strategic information transmission when messages directly affect payoffs, either because the sender faces a cost of lying or receivers are credulous. They show that if the state space is unbounded there are fully revealing equilibria. Unlike in their environment, in the noise model analyzed here messages do not have an intrinsic meaning and therefore the notions of deception and language inflation that play an important role in KOS have no content. On the other hand, the issue of coding, i.e. how the message space is used by the various sender types, that plays a crucial role in our analysis does not arise for KOS. Further, in our model there are no fully revealing equilibria, regardless of whether we choose the state space to be bounded or not. One parallel between the two papers is that in both there is a sense in which sender types who separate themselves achieve their ideal points, on average in the KOS model with heterogeneously sophisticated receivers and in the no-noise event in the noise model.

Kartik (2007) looks at a perturbation of the CS model in which the sender has an explicit convex cost of misreporting. He finds that only the most informative CS equilibria can arise as limits of monotonic equilibria as the cost of misreporting converges to zero. Finally, in a closely related paper, Chen (2006) modifies the CS model by including a small proportion of behavioral types, *honest* senders and *naive* receivers. Using an additional monotonicity restriction, she shows that there is a unique equilibrium. This equilibrium approaches the maximally-informative CS equilibrium in the limit as the proportions of honest senders and naive receivers converge to zero. In contrast, in the noise model messages have no exogenous meaning and so it is hard to make sense of the notion of honesty. From a technical standpoint, in the noise model we obtain welfare results for a range of strictly positive noise levels, not only in the limit as noise tends to zero; monotonicity of equilibrium is not a significant constraint on the equilibrium set; and monotonicity in conjunction with the fact that the message space is a continuum does not pose existence problems (unlike in Chen's model).

We conclude this review section by mentioning two more variants of the CS model. Olszewski (2004) examines a model in which the receiver has private information, there is positive probability that the sender is a behavioral type who always tells the truth rather than being strategic, and the strategic sender prefers to be perceived as the honest type. If the latter concern is sufficiently strong, there is a unique equilibrium which is fully revealing. In Olszewski's model the receiver can ask for more or less information; when the sender cares about the receiver's action, in addition to being perceived as honest, asking for more information may create an incentive for lying. Both of these results are predicated on having behavioral types and messages with intrinsic meanings. In our model, as already discussed, messages acquire meaning only endogenously from the sender's equilibrium strategy, and there are no behavioral types. In Morgan and Stocken's (2003) variation of the CS model the receiver is uncertain about the sender's bias. They find equilibria in which there is full separation on a portion of the type space: sender types whose preferences are perfectly aligned with those of the receiver are able to perfectly reveal sufficiently low states of the world. In contrast, in the noise model the possibility of an error in information transmission ensures that the receiver never learns the sender's type for certain, even in equilibria that involve separation on a portion of the type space.

2. The model

2.1 *Setup*

We investigate communication between a privately informed sender, S, and a receiver, R. The agents' payoffs depend on the sender's information or *type*, $\theta \in T = [0,1]$, and the receiver's action, $a \in \mathbb{R}$. We assume that θ is drawn from a common-knowledge distribution F with an everywhere positive density f on the support T. The payoff of a sender of type θ when the receiver takes action a is $U^S(a,\theta,b)$, where b is a parameter measuring her *bias* relative to the receiver. The payoff of a receiver who takes action a is $U^R(a,\theta)$ when the sender's type is θ. The functions U^S and U^R are assumed to be twice continuously differentiable. We assume that $U^R(a,\theta) \equiv U^S(a,\theta,0)$ for all (a,θ).[3] We use subscripts to denote partial derivatives; e.g., $U^S_{12}(a,\theta,b)$ stands for the cross-partial derivative of U^S with respect to its first and second argument, evaluated at (a,θ,b). We assume that for each realization of θ and each value of b there exists an action a such that $U^S_1(a,\theta,b) = 0$ and for each θ there exists an action a' such that $U^R_1(a',\theta) = 0$; $U^S_{11}(a,\theta,b) < 0 < U^S_{12}(a,\theta,b)$ for all a, θ, and b; and $U^R_{11}(a,\theta) < 0 < U^R_{12}(a,\theta)$ for all a and θ. Thus, given the sender's private information θ and her bias b, a unique action,

[3]We make this assumption to maintain consistency with the Crawford and Sobel framework. Since we are interested only in values of $b \neq 0$, there is no loss of generality.

called her "ideal action" and denoted $a^S(\theta, b)$, maximizes her payoff; similarly, given θ, the receiver has a unique ideal action, denoted $a^R(\theta)$.[4] Note that each player's ideal action is increasing in θ. Finally, we assume that $U^S_{13}(a, \theta, b) > 0$ everywhere, so that an increase in b shifts the sender's preferences further away from the receiver's. Henceforth we disregard the case where the sender and receiver have identical preferences, assuming without loss of generality that $b > 0$, so that $a^R(\theta) < a^S(\theta, b)$ for all θ.

The timing of the game is as follows. The sender observes the value of θ and then sends a message $m \in M = [0,1]$. The sender's message is subject to *error*: with probability ϵ, the receiver observes a message m' that is a draw from the error distribution G on the message space M; otherwise, the receiver observes the message m sent by the sender. We assume that he cannot distinguish between received messages that are the result of an error and messages that were sent intentionally. The error distribution G is independent of the sender's type and of the message sent, and has a density g that is everywhere positive on M. Finally, the receiver takes some action $a \in \mathbb{R}$. We consider values of $\epsilon \in (0,1)$, and refer to this game as the *noise model*. In the degenerate case when $\epsilon = 0$, the game collapses to that of Crawford and Sobel—the *CS model*.

2.2 *Equilibrium*

A behavior strategy for the sender $\sigma : T \to \Delta(M)$ specifies the distribution of messages she sends for each value of θ; for the receiver, given the strict concavity of U^R in a, it is without loss of generality to restrict attention to pure strategies $\rho : M \to \mathbb{R}$ that describe the action he chooses for each message he might receive.

In a perfect Bayesian equilibrium (henceforth equilibrium) strategies are optimal given players' beliefs and beliefs are derived from Bayes' rule whenever possible. For a sender of type θ, this means that every message m that she sends must maximize the weighted average of her expected payoff if the message is received as intended and her expected payoff if there is an error, i.e.

$$\begin{aligned} m &\in \arg\max_{m'} \left((1-\epsilon) U^S(\rho(m'), \theta, b) + \epsilon \int_0^1 U^S(\rho(m''), \theta, b) g(m'')\, dm'' \right) \\ &= \arg\max_{m'} U^S(\rho(m'), \theta, b). \end{aligned}$$

(The simplification is possible because the probability of an error, ϵ, and the error distribution, g, are independent of the message actually sent.[5]) Now consider the receiver. Let $\mu(\theta \mid m)$ denote his beliefs about θ conditional on receiving message m. Since $\epsilon > 0$ and g is everywhere positive, Bayes' rule is always well-defined and gives us

$$\mu(\theta \mid m) = \frac{((1-\epsilon)\sigma(m \mid \theta) + \epsilon g(m)) f(\theta)}{\int_0^1 ((1-\epsilon)\sigma(m \mid \theta') + \epsilon g(m)) f(\theta')\, d\theta'}.$$

[4]That is, $a^S(\theta, b) = \arg\max_a U^S(a, \theta, b)$ and $a^R(\theta) = \arg\max_a U^R(a, \theta)$.

[5]We relax the first assumption in Section 5.1.

On receiving message m, the receiver chooses the (unique) action that maximizes his expected payoff given these beliefs:

$$\rho(m) = \arg\max_{a'} \int_0^1 U^R(a', \theta)\, d\mu(\theta \mid m).$$

Definition 1. A *perfect Bayesian equilibrium of the noise model* is a strategy for the sender, $\sigma : T \to \Delta(M)$, a strategy for the receiver, $\rho : M \to \mathbb{R}$, and a set of beliefs for the receiver, $\mu : M \to \Delta(T)$, such that

1. for all $\theta \in T$: $m \in \arg\max_{m'} U^S(\rho(m'), \theta, b)$, for all $m \in \text{supp}(\sigma(\cdot \mid \theta))$,
2. for all $m \in M$: $\rho(m) = \arg\max_{a'} \int_0^1 U^R(a', \theta)\, d\mu(\theta \mid m)$, and
3. $\mu(\theta \mid m) = \dfrac{((1-\epsilon)\sigma(m \mid \theta) + \epsilon g(m)) f(\theta)}{\int_0^1 ((1-\epsilon)\sigma(m \mid \theta') + \epsilon g(m)) f(\theta')\, d\theta'}$.

For given parameter values, the set of equilibria is very large, and it is difficult to provide a complete characterization. In this section we derive a number of results about the nature of the equilibrium set; in the next section, we provide more results in the context of a specific example (an extension of the well-known uniform–quadratic case of Crawford and Sobel).

We start by introducing some new notation and terminology. Since the sender can influence the receiver's actions only in the no-noise event, it is useful to define $\omega(\sigma, \rho, \theta)$ as the distribution of actions that is induced by type θ in the no-noise event when the sender uses strategy σ and the receiver uses strategy ρ. We call two equilibria with corresponding strategy pairs (σ, ρ) and (σ', ρ') *outcome equivalent* if for every sender type θ, $\omega(\sigma, \rho, \theta) = \omega(\sigma', \rho', \theta)$, and *essentially outcome equivalent* if $\omega(\sigma, \rho, \theta) = \omega(\sigma', \rho', \theta)$ for all but a set of types θ that is at most countable. We can now state our first result.

Proposition 1. *In every equilibrium (σ, ρ, μ), the set of types θ for whom any given action a is in the support of $\omega(\sigma, \rho, \theta)$ is a (possibly empty) interval. If the interior of the interval corresponding to the action a is nonempty, then for every θ in the interior of the interval $\omega(\sigma, \rho, \theta)$ is the degenerate distribution that assigns probability 1 to a. Furthermore, every equilibrium is essentially outcome equivalent to an equilibrium in which each type θ induces a single action.*

The proof of all results in this section can be found in the Appendix. According to Proposition 1 almost every type induces precisely one action, and the set of types that induce any given action is an interval of the type space. The types in this interval may, however, use different strategies. If we are concerned only with outcomes, Proposition 2 shows that it is without loss of generality to confine attention to equilibria in which these types behave identically.

Proposition 2. *Consider an equilibrium in which the type space is partitioned into intervals, with types in any given interval inducing the same action and types in distinct intervals inducing distinct actions. There is an outcome-equivalent equilibrium in which for any non-degenerate intervals I and I' ($I \neq I'$) that are elements of this partition*

(i) all types in I use the same *distribution over messages, and*

(ii) this distribution is equal to the error distribution, G, restricted to a subset M_I of the message space, with $M_I \cap M_{I'} = \varnothing$.

We say that a message is *unused* if it is not in the support of any type's distribution over messages. Our next result shows that we can, without loss of generality, assume that the entire message space is used.

Proposition 3. *Every equilibrium is outcome equivalent to an equilibrium in which there are no unused messages.*

There is a close connection between Propositions 1–3 and Theorem 1 of Crawford and Sobel (1982), but there are also several differences. Their Theorem 1 states that every equilibrium of the CS model is outcome equivalent to an equilibrium in which types in the interior of a given element I of the equilibrium partition, (θ_{i-1}, θ_i), randomize uniformly over messages in $[\theta_{i-1}, \theta_i]$; but the mixing distribution used is not important, nor is the set of messages used, as long as each partition element uses a distinct set of messages. More precisely, one could construct an outcome-equivalent equilibrium in which types in (θ_{i-1}, θ_i) randomize over messages in some arbitrary set $M_I \subseteq M$, according to some arbitrary distribution h_I, as long as the message sets used by each interval are disjoint. On the other hand, Proposition 2 describes an equilibrium in which types in a given partition element I randomize according to the error distribution restricted to message set M_I; in this case (as is evident from the proof) it is crucial that this particular distribution is used. Intuitively, if a different distribution is used, then the receiver's posterior probability of an error, and hence his action, will depend on which message in M_I is observed. Additionally, the size of the set M_I is important, in a sense that is made precise in Proposition 4 below.

Let $\mathscr{M}$ denote a finite ordered N-tuple $(M_1, \dots, M_N)$ of measurable sets that partition the message space and let λ_G be the measure that the error distribution G induces on M. Define $\Lambda_G(\mathscr{M}) \equiv (\lambda_G(M_1), \dots, \lambda_G(M_N))$ as the ordered N-tuple of probabilities of the components of $\mathscr{M}$. We refer to $\mathscr{M}$ as a *message-set vector* and say that two message-set vectors $\mathscr{M}$ and $\mathscr{M}'$ are *G-distinguished* if $\Lambda_G(\mathscr{M}) \neq \Lambda_G(\mathscr{M}')$. An equilibrium is *adapted to $\mathscr{M}$* if there is a partition of the type space into N intervals $T_1, \dots, T_N$ such that for $i = 1, \dots, N$ the mixed strategy of each type in T_i is G restricted to M_i. Denote by $O(\mathscr{M})$ the set of equilibrium outcomes (joint distributions over types and actions) of equilibria that are adapted to $\mathscr{M}$. The following result highlights the importance of coding, which determines the measures of the sets of messages M_i used by each element T_i of the equilibrium partition of the type space.

Proposition 4. *If $\mathscr{M}$ and $\mathscr{M}'$ are G–distinguished, then $O(\mathscr{M}) \cap O(\mathscr{M}') = \varnothing$. Otherwise, $O(\mathscr{M}) = O(\mathscr{M}')$.*

Our next result concerns the relationship between the set of equilibria of the CS model and the set of equilibria of the noise model when the level of noise is low. This proposition requires an additional assumption, Crawford and Sobel's monotonicity condition (M) (see page 1444 of their paper).[6] This condition is satisfied by all standard versions of their model used in applications, such as the uniform–quadratic case; a precise definition can be found in the Appendix. We call an equilibrium in which N actions are induced an *N-step equilibrium*; if each action is induced by a set of types with positive measure, we refer to a *non-degenerate N-step equilibrium*. For any non-degenerate N-step equilibrium with sender strategy σ, $\mathbb{P}(\sigma)$ denotes the boundary points between the intervals of the corresponding partition viewed as a point in $\mathbb{R}^{N-1}$.

Proposition 5. *Assume, in the CS model, that condition (M) holds and that there exists a non-degenerate N-step equilibrium with sender strategy σ. Then for all $\delta > 0$ there exists $\tilde{\epsilon} > 0$ such that for all noise levels $\epsilon \in (0, \tilde{\epsilon})$ and for any N-element partition $\mathscr{M}$ of the message space, there exists an equilibrium of the noise model with sender strategy σ_ϵ that is adapted to $\mathscr{M}$ and satisfies $|\mathbb{P}(\sigma_\epsilon) - \mathbb{P}(\sigma)| < \delta$.*

Proposition 4 tells us that, for fixed ϵ, if $\mathscr{M}$ and $\mathscr{M}'$ are distinct, they cannot produce the same equilibrium outcome. Together with Proposition 5, this implies that near any non-degenerate N-step CS equilibrium there is an $N-1$-dimensional set of equilibria of the noise model all of which induce different equilibrium outcomes.

The final result of this section says that full separation of types is not possible in a noise equilibrium. More precisely, consider a given noise equilibrium in which each type induces precisely one action (by Proposition 1, every noise equilibrium is essentially outcome equivalent to an equilibrium of this kind). Slightly abusing notation, let $\omega : T \to \mathbb{R}$ be the *outcome function*, where $\omega(\theta)$ is the action induced by type θ. Then we say that this equilibrium is *separating* if ω is one-to-one.

Proposition 6. *The noise model has no separating equilibrium.*

It is worth noting that this result holds even if the state space is not bounded.

3. Equilibria in the uniform–quadratic case

As noted earlier, it is difficult to give a complete characterization of the equilibrium set. In this section we take a small step in that direction, concentrating on the well-known *uniform–quadratic* case introduced by Crawford and Sobel (1982). The remainder of the

[6]In the CS model, the boundary types that separate elements of the equilibrium partition solve a difference equation with appropriate initial conditions. Condition (M) ensures that the solutions of this difference equation vary monotonically with initial conditions. The proof strategy is to use condition (M) to find one solution to the CS-difference equation in which the length of the first interval $[0, \theta_1)$ is too small and another where it is too large for an equilibrium. By continuity, the same will be true for the corresponding difference equation in the noise model. The intermediate-value theorem then ensures that the noise model has a nearby equilibrium.

paper focuses on this case, except where explicitly indicated otherwise. In the uniform–quadratic case, the sender's type θ is drawn from the uniform distribution on the unit interval; the sender's and receiver's payoff functions are given by

$$U^S(a,\theta,b) = -(\theta + b - a)^2$$
$$U^R(a,\theta) = -(\theta - a)^2.$$

Notice that the ideal actions of the sender and the receiver are $\theta + b$ and θ respectively. We also assume that the message space $M = [0,1]$, and the error distribution G is uniform on $[0,1]$.

In this section, we restrict attention to values of bias $0 < b < \frac{1}{2}$, since for larger values no communication is possible and every equilibrium is therefore outcome equivalent to pooling. There are two key differences between the equilibria of the models with and without noise. First, in the noisy case there can be a continuum of equilibrium outcomes of a given number of steps (see Section 3.1 below), while in the CS case, Crawford and Sobel show that every N-step equilibrium (if any exist) yields the same outcome. Second, we show in Sections 3.2 and 3.3 that, as long as the level of noise is high enough, there are equilibria with an infinite and even an uncountable number of steps; in the CS case, on the other hand, every equilibrium has a finite number of steps.

3.1 *Two-step equilibria*

As a starting point, we provide a characterization of the set of two-step equilibrium outcomes. In a two-step equilibrium, two distinct actions a_1 and a_2 are induced; assume without loss of generality that $a_1 < a_2$. By Proposition 1, the set of types inducing action a_i is an interval of the state space, I_i $(i = 1,2)$. Let $\theta_1 \in (0,1)$ denote the boundary type between the two intervals,[7] so $I_1 = [0,\theta_1)$ and $I_2 = [\theta_1,1]$.[8] Since we are interested only in outcomes, it follows from Proposition 2 that we can restrict attention to equilibria in which types in I_1 randomize uniformly over messages in M_1 and types in I_2 randomize uniformly over messages in M_2, for some $M_1, M_2 \subseteq [0,1]$ with $M_1 \cap M_2 = \varnothing$; further, by Proposition 3 we can assume that $M_1 \cup M_2 = [0,1]$. Finally, Proposition 4 tells us that the outcome is affected only by the measure of M_1 and M_2 (with respect to the error distribution), and not the exact composition of these sets. Let λ_1 denote the measure of M_1, so that $1 - \lambda_1$ is the measure of M_2. Then the actions chosen by the receiver on receiving messages in M_1 and M_2 are respectively

$$a_1 = \frac{(1-\epsilon)\theta_1 \frac{\theta_1}{2} + \epsilon\lambda_1 \frac{1}{2}}{(1-\epsilon)\theta_1 + \epsilon\lambda_1}$$
$$a_2 = \frac{(1-\epsilon)(1-\theta_1)\frac{\theta_1+1}{2} + \epsilon(1-\lambda_1)\frac{1}{2}}{(1-\epsilon)(1-\theta_1) + \epsilon(1-\lambda_1)}.$$

[7] If $\theta_1 = 0$ or $\theta_1 = 1$, then each interval induces the same action (see expressions for a_1 and a_2 below), and we have a one-step equilibrium.

[8] Or $I_1 = [0,\theta_1]$ and $I_2 = (\theta_1,1]$; or $I_1 = [0,\theta_1]$ and $I_2 = [\theta_1,1]$. The boundary type could belong to either or both intervals, though by Proposition 1 all types in the interior of each interval induce only one action. Henceforth we assume that the boundary type belongs only to the second interval; this assumption does not affect the outcome for any other type.

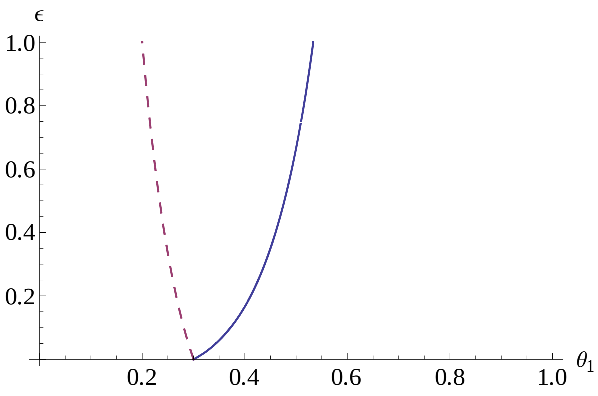

FIGURE 1. Two-step equilibrium partitions, $b = \frac{1}{10}$.

Since $a_1 < a_2$, a necessary and sufficient condition for equilibrium is that the sender of type θ_1 is indifferent between a_1 and a_2, or

$$\theta_1 + b = \tfrac{1}{2}(a_1 + a_2). \tag{1}$$

Let $\theta_1^*(b, \epsilon, \lambda_1)$ denote the relevant solution[9] to this equation (and therefore the equilibrium boundary type), when it exists.

Case 1: $b < \frac{1}{4}$. A two-step equilibrium exists for all $\lambda_1 \in [0, 1]$. The lower and upper bounds on $\theta_1^*(b, \epsilon, \lambda_1)$ are realized when $\lambda_1 = 0$ and $\lambda_1 = 1$ respectively, and are given by[10]

$$\underline{\theta}_1 = \frac{3 - 4b + 4b\epsilon - \sqrt{(3 - 4b + 4b\epsilon)^2 - 8(1 - 4b)(1 - \epsilon)}}{4(1 - \epsilon)} = \theta_1^*(b, \epsilon, 0)$$
$$\overline{\theta}_1 = \frac{1 - 4b(1 - \epsilon) - 4\epsilon + \sqrt{1 + 8b(2b(1 - \epsilon) - 1)(1 - \epsilon) + 8\epsilon}}{4(1 - \epsilon)} = \theta_1^*(b, \epsilon, 1).$$

(Notice that when $\epsilon = 0$, both of these expressions are equal to $\frac{1}{2}(1 - 4b)$, the unique boundary value for a two-step equilibrium in the CS model.) It can be shown that $\theta_1^*(b, \epsilon, 0) < \theta_1^*(b, \epsilon, 1)$, and furthermore, $\theta_1^*(b, \epsilon, \lambda_1)$ is continuous and strictly increasing in λ_1. Thus any value between these two bounds is attainable as an equilibrium boundary type for appropriate choice of λ_1, and allowing the first step of the equilibrium partition to use a larger proportion of the message space shifts the boundary between the two steps to the right.

Figure 1 illustrates the lower bound (dotted line) and upper bound (solid line) as a function of the noise level when $b = \frac{1}{10}$.

[9]That is, $\theta_1^*(b, \epsilon, \lambda_1)$ denotes the solution to equation (1) that lies strictly between 0 and 1.
[10]The derivation of this and all other results in this section can be found in the Appendix.

Case 2: $b \geq \frac{1}{4}$. A two-step equilibrium exists as long as $\lambda_1 \neq 0$. In this case the lower bound on the set of equilibrium boundary values is 0, and $\theta_1^*(b,\epsilon,\lambda_1)$ approaches this value as λ_1 tends to 0. The upper bound on $\theta_1^*(b,\epsilon,\lambda_1)$ is given by the same expression as before, and is again attained when $\lambda_1 = 1$. As before, $\theta_1^*(b,\epsilon,\lambda_1)$ (when defined) is continuous and strictly increasing in λ_1.

3.2 *An infinite partition*

We now show that the noise model, unlike the CS model, has equilibria with infinitely many steps.[11] Such equilibria exist as long as the level of noise is high enough.

Proposition 7. *If* $\epsilon \geq 2b/(1+\sqrt{2b})^2$, *then the noise model has an equilibrium with infinitely many steps.*

Note that the level of noise required is increasing in b, and tends to 0 as b tends to 0. The proof of Proposition 7 (in the Appendix) is constructive. To give some flavor of the construction, here we describe the sender's strategy. Consider the following (infinite) partition of the type space:

$$\{\{0\},\ldots,[\theta_{-3},\theta_{-2}),[\theta_{-2},\theta_{-1}),[\theta_{-1},1]\},$$

where boundary types $\theta_{-1},\theta_{-2},\ldots$ form a (descending) geometric progression; since $\theta_i \to 0$ as $i \to -\infty$, the set does indeed partition $[0,1]$. Types in each partition element, except the final one $[\theta_{-1},1]$, randomize uniformly over a set of messages that is proportional to the size of the element, while types in $[\theta_{-1},1]$ randomize uniformly over the leftover messages.

3.3 *Uncountable partitions*

Proposition 7 states that, as long as the level of noise is at least some threshold value, the noise model has an equilibrium with a countably infinite number of steps. If ϵ is strictly larger than this value, we can find an equilibrium of the noise model with uncountably many steps. This can be shown using a construction similar to that used to prove Proposition 7, except that at the left-hand end of the type space (i.e. for low values of θ) the sender adopts a fully-revealing strategy, with every type sending a distinct message. For the sake of exposition, however, we here present a weaker result (with a tighter constraint on the value ϵ) that can be proved by means of a simpler construction.

[11]A recent paper by Gordon (2007) also demonstrates the existence of equilibria with infinitely many steps in a framework that is based on the CS model. Gordon adopts a reduced-form approach in which the receiver's preferences are represented indirectly by a mapping from sets of types to actions; intuitively, this mapping gives the receiver's ideal action if the sender's message indicates that her type is in a given set. (Note that in the noise model (unlike the CS model), this mapping would depend on the coding of messages, and thus could not be treated as exogenous.) Gordon's Theorem 4 states that equilibria with infinitely many steps exist as long as preferences satisfy a *moderate audience* condition, which says that the lowest sender type has a negative bias while the highest sender type has a positive bias. To establish their finiteness result, Crawford and Sobel impose restrictions on preferences that rule out the moderate audience condition.

Proposition 8. *If $\epsilon > 2b$, then the noise model has an equilibrium with uncountably many steps.*

Again we relegate the details of the equilibrium construction to the Appendix, describing only the sender's strategy here. Consider the following (uncountably infinite) partition of the type space:

$$\{\{\{\theta\}\}_{\theta\in[0,\theta^*]},\ldots,(\theta_{-3},\theta_{-2}],(\theta_{-2},\theta_{-1}],(\theta_{-1},1]\},$$

where boundary types $\theta_{-1},\theta_{-2},\ldots$ form a descending sequence that tends to θ^*. The sender strategy is given by:

- if $\theta\in[0,\theta^*]$, send message $m=s(\theta)$ where s is a strictly increasing differentiable function with $s(0)=0$;
- if $\theta\in(\theta_{i-1},\theta_i]$ $(i\leq 0)$, randomize uniformly over messages in $(\zeta(\theta_{i-1}-\theta^*)+s(\theta^*),\zeta(\theta_i-\theta^*)+s(\theta^*)]$, where $\zeta(1-\theta^*)+s(\theta^*)=1$.

Each of the singleton elements of the partition, then, sends a single message, while each nondegenerate-interval element randomizes over some nondegenerate interval of the message space. It is worth noting that each sender type in the fully revealing region, $[0,\theta^*]$, induces her ideal action.

4. Welfare results in the uniform–quadratic case

The results of the previous section suggest a sense in which, if the information transmission process is noisy, more communication is possible—we found noise equilibria in which the sender's messages partition the state space more evenly and into more elements than is possible in any equilibrium of the CS model; furthermore, introducing noise allows us to construct communicative equilibria for values of b that are so high that the only equilibrium of the CS model is totally uninformative (specifically, $b\in\left[\frac{1}{4},\frac{1}{2}\right)$ in the uniform–quadratic case).

We call any changes in the agents' payoff resulting from changes in the equilibrium partition the *strategic effect* of noise. What is the source of this effect? Recall that in a equilibrium of the CS model communication is imperfect because the sender and the receiver do not agree on the action that should be chosen for any type. In the presence of noise, the receiver has to take into account the possibility that a given message was received in error; his expectation of the sender's type is a weighted average of the expectation given that the message was transmitted faithfully and the expectation given that there was an error. Compared with the noiseless case, the receiver's expectations are distorted towards the *ex ante* mean. In particular, the meaning of a message that signals a low type is distorted upwards; this implies that the receiver's action will also be distorted upwards, and hence closer to the ideal action of the sender (given positive bias). For low types, then, noise brings the effective preferences of the sender and receiver into closer alignment. Even though the opposite is true for high types, this allows us to construct more informative equilibrium partitions—either with more elements or with elements that are more evenly spaced—than is possible in the CS model.

We are some way from concluding that noise facilitates information transmission, however. There are two negative effects of noise, which mitigate the strategic effect. First, when errors actually occur, there is a clear loss of information (the *direct effect*); second, since the receiver does not observe whether a given message was sent in error, he has to trade off the losses in each contingency (the *distortion effect*).

To analyze the trade-off between these three effects, we need a precise measure of the informativeness of an equilibrium: we follow Crawford and Sobel in using the (*ex ante*) expected payoff of the receiver for this purpose. In the uniform–quadratic case, on which we continue to focus in this section, this is equal to the negative of the residual variance of θ that the receiver expects to face after receiving his message. Further, it can be shown that (in equilibrium) $EU^S = EU^R - b^2$, so this measure also gives us a Pareto ranking of equilibria—one equilibrium is more informative than another if and only if it Pareto dominates it. In Example 1 below, we show how much of the receiver's change in payoff once noise is introduced is due to each of the three effects described above.

Two questions naturally arise regarding the welfare properties of noise equilibria. First, what does the most informative equilibrium of the noise model look like? And second, does noise increase or reduce informativeness? We consider the second question first, and prove two key results: (1) a small amount of noise is (almost) always a good thing; and (2) if the bias is large, any amount of noise is a good thing. The first result is expressed formally in Proposition 9.

4.1 *The welfare effects of noise*

4.1.1 *Low noise*

Proposition 9. *If $b < \frac{1}{2}$ and $b \neq 1/(2N^2)$ for all integers $N > 1$, there exists $\overline{\epsilon} > 0$ such that for all $\epsilon \in (0, \overline{\epsilon})$ there is an equilibrium of the noise model that is Pareto superior to all equilibria of the CS model.*

The proof of this proposition can be found in the Appendix. We construct an equilibrium of the noise model and show that it is more informative than every equilibrium of the CS model for small values of ϵ. In this equilibrium, the sender adopts a *front-loading* strategy, using up almost all of the messages in the first partition element.[12] The following example provides an illustration.

Example 1. Suppose that $b = \frac{1}{10}$. We compare the receiver's expected payoff in the Pareto optimal equilibrium of the CS model and a three-step equilibrium of the noise model, with noise $\epsilon = \frac{1}{126}$.

[12]The reader may recall from the characterization of two-step equilibrium in Section 3.1 above that front-loading (i.e. setting $m_1 = 1$) maximizes the size of the first partition element. As long as the level of noise is small, the first partition element is the smallest; increasing its size thus makes the equilibrium more informative, *ceteris paribus.*

CS model The Pareto optimal equilibrium of the CS model has two steps, with partition elements $[0,\frac{3}{10})$ and $[\frac{3}{10},1]$, and resulting $EU^R = -\frac{37}{1200} = -0.0308$.

Noise model, $\epsilon = \frac{1}{126}$ Consider the partition $\{[0,\frac{1}{25}),[\frac{1}{25},\frac{8}{25}),[\frac{8}{25},1]\}$. Suppose that the sender obeys the following strategy:

$$\begin{aligned}&\text{if } \theta \in [0,\tfrac{1}{25}), \text{ randomize uniformly on } [0,1]\setminus\{m_2,m_3\};\\ &\text{if } \theta \in [\tfrac{1}{25},\tfrac{8}{25}), \text{ send message } m_2;\\ &\text{if } \theta \in [\tfrac{8}{25},1], \text{ send message } m_3.\end{aligned}$$

Given the sender's strategy, if there is an error in message transmission, then with probability one the message received coincides with one of the messages sent by that first partition element. The receiver's best response is to choose actions according to the following strategy:

$$\begin{aligned}&\text{if } m \in [0,1]\setminus\{m_2,m_3\} \text{ is received, choose } a_1 = \tfrac{1}{10};\\ &\text{if } m = m_2 \text{ is received, choose } a_2 = \tfrac{9}{50};\\ &\text{if } m = m_3 \text{ is received, choose } a_3 = \tfrac{33}{50}.\end{aligned}$$

In each case, the action chosen is equal to the receiver's expectation of θ given his information. Notice that for the second and third partitions elements, this is simply the midpoint of the interval. This is because messages m_2 and m_3 are sent by error with probability zero, so the receiver can be certain that the sender's type is in the relevant interval. This eliminates the distortion effect except for the first (and smallest) partition element.

To check that we have an equilibrium, we need to verify that the sender's strategy is also a best response. This amounts to checking that the boundary types $\theta_1 = \frac{1}{25}$ and $\theta_2 = \frac{8}{25}$ satisfy the indifference conditions:

$$\theta_1 : \frac{1}{25} = \frac{a_1 + a_2}{2} - b = \frac{\frac{1}{10} + \frac{9}{50}}{2} - \frac{1}{10} = \frac{1}{25} \qquad \checkmark$$

$$\theta_2 : \frac{8}{25} = \frac{a_2 + a_3}{2} - b = \frac{\frac{9}{50} + \frac{33}{50}}{2} - \frac{1}{10} = \frac{8}{25} \qquad \checkmark$$

The resulting expected payoff for the receiver is $EU^R = -\frac{36}{1200}$ (see the Appendix for the calculation). As we can see, the additional information conveyed by the sender more than compensates for the loss of information through noise, resulting in a Pareto improvement compared to the equilibrium of the CS model.

Figure 2 provides a graphical illustration of these equilibria. The boundary points are shown above the unit interval, and the actions chosen in each case are given below.

How much of this overall change in EU^R is due to the three effects discussed above? To calculate the strategic effect, we compute the receiver's expected payoff if there is no

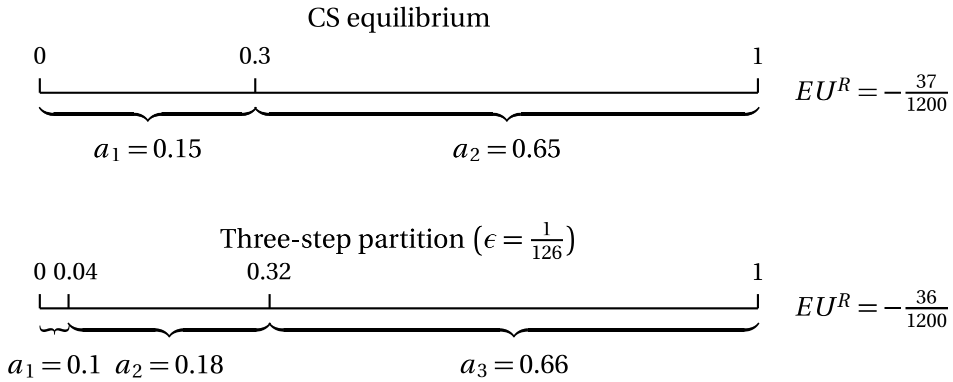

FIGURE 2. Equilibria with $b = \frac{1}{10}$.

noise but his information partition is the same as in the equilibrium of the noise model ($\{[0, \frac{1}{25}), [\frac{1}{25}, \frac{8}{25}), [\frac{8}{25}, 1]\}$; of course, this is not an equilibrium). For the direct effect, we take this value from his payoff if he had this information partition in the no-noise event, and no information in the noise event (so we are effectively assuming that he knows whether a given message was sent in error). The remaining change is due to the distortionary effect, which isolates the payoff loss resulting because the receiver cannot in fact distinguish messages sent in error from correct ones. The size of each of these effects is given as follows:

Decomposition of change in EU^R when noise is introduced

−0.0308	⟶	−0.0280	⟶	−0.0285	⟶	−0.0300
	strategic effect		direct effect		distortionary effect	
	(+0.0028)		(−0.0004)		(−0.0015)	◊

The threshold level of noise, $\overline{\epsilon}$, below which the front-loading equilibrium generates a Pareto improvement over the best equilibrium of the CS model is shown in Figure 3. High values of b are omitted for the sake of clarity; $\overline{\epsilon}$ rises from 0 to 1 as b goes from $\frac{1}{8}$ to $\frac{1}{4}$, and $\overline{\epsilon} = 1$ for $b \in [\frac{1}{4}, \frac{1}{2})$ (see also Observation 1 below). Clearly, $\overline{\epsilon}$ is a non-monotonic function of b. It turns out that whenever $b = 1/(2N^2)$ ($N = 2, 3, \ldots$), the most informative equilibrium of the CS model is Pareto optimal in a very general class of communication protocols,[13] which includes noisy talk. When the bias is equal to these values, then, noise cannot generate a Pareto improvement, and $\overline{\epsilon} = 0$. For other values of b, however, a small amount of noise can be beneficial. The further b is from these critical values, the more potential there is for a Pareto improvement, and hence the larger the value of $\overline{\epsilon}$. The peaks of the graph are at $b = 1/(2N(N-1))$ ($N = 2, 3, \ldots$).As already mentioned, the technique used to construct the equilibrium of the noise model in Example 1 is to

[13]This result is implied by Lemma 1 of Goltsman et al. (2007), discussed in more detail in Section 4.2.1 below.

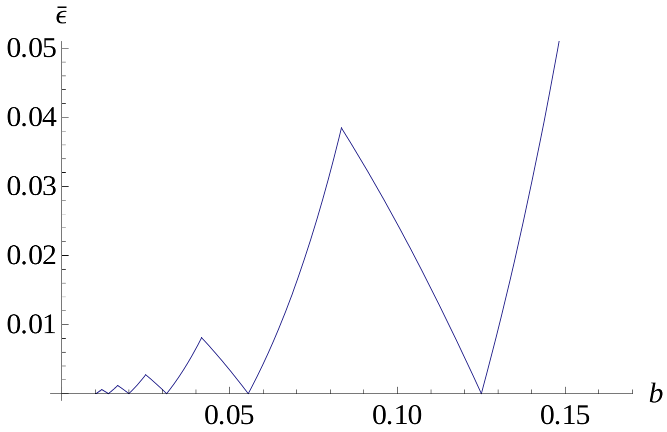

FIGURE 3. Maximum level of noise for a Pareto improvement.

have the sender employ a front-loading strategy, where the first partition element uses almost all of the message space (a generalization of this construction is used in the proof of Proposition 9 in the Appendix). This strategy is effective because if the sender's type lies in any of the *other* partition elements and her message is relayed faithfully, the receiver can be certain that there was no error. An identical result could be achieved in a framework where messages are simply lost (rather than garbled) with small probability. To see how, suppose that types in the lowest partition element do not send any messages, and all types in the other partition elements send distinct messages. If the receiver observes a message, he can be certain which partition element it came from, just as if he receives message m_2 or m_3 in Example 1; on the other hand, if he receives no message, he has to balance the probability that no message was sent (and therefore the sender's type is in the lowest partition element) against the probability that a message was sent but got lost (and so her type is higher).

The role of risk aversion We have just seen that the introduction of a small amount of noise into the information transmission process can result in welfare improvements. The source of this welfare gain is the strategic effect: the presence of noise induces the sender to reveal more information than she otherwise would. When ϵ is low, this effect dominates the direct effect of lost messages. This result is perhaps even more surprising when we consider that our agents are risk averse: payoff is a concave function of distance from their ideal actions. In fact, risk aversion helps as well as hinders. Consider the position of the receiver, faced with a message that may have been sent in error. To minimize his expected loss, he adjusts his action toward the *ex ante* expectation of θ. The size of this adjustment depends, of course, on the amount of noise, but also on the degree of risk aversion. Very risk averse agents are more concerned about the small probability of being far from the ideal action, and therefore make a larger adjustment. This implies that the receiver's actions are less responsive to the different messages sent by the sender, who thus has less incentive to exaggerate, so that more informative partitions are possible. It is easy to show, however, that noise can generate a Pareto improvement even when the payoff loss is a linear function of distance from the ideal action (in a sense, the risk neutral case). This contrasts with the results of Krishna and Morgan (2004): they construct equilibria in which multiple rounds of communication can be beneficial by leveraging the risk aversion of the sender.

4.1.2 *High bias* While Proposition 9 states that a small amount of noise can generate a Pareto improvement for almost any bias, the following observation notes that a Pareto improvement is possible for any amount of noise if the bias is high.

Observation 1. *For all $b \in [\frac{1}{4}, \frac{1}{2})$ and all $\epsilon \in (0,1)$ the noise model has an equilibrium that is Pareto superior to all equilibria of the CS model.*

To prove this result, we refer back to the characterization of the set of two-step equilibrium outcomes in Section 3.1 above. There we show that as long as $m_1 > 0$, a two-step equilibrium exists for all $b < \frac{1}{2}$ and $\epsilon \in (0,1)$. On the other hand, for $b \in [\frac{1}{4}, \frac{1}{2})$ the unique equilibrium outcome of the CS model is completely uninformative. Since the receiver is strictly better off with some information rather than none, the result follows.

In the more general framework of Section 2, the link between the sender's and receiver's payoffs is broken and equilibria cannot always be Pareto ranked. The finding that noise enables communication when it would otherwise not have been possible, however, seems fairly robust. Suppose that Crawford and Sobel's monotonicity condition (M) holds,[14] and let b^* be the lowest bias level for which the unique equilibrium outcome of the CS model is pooling (i.e. b^* is the level of bias that is just too high for communication to be possible). Then we can show that there is some $b^{**} > b^*$ such that, for $b \in [b^*, b^{**})$, there exists a two-step equilibrium of the noise model, for any level of noise.[15] This equilibrium is better for the receiver, but not necessarily for the sender, than the equilibrium of the CS model.[16]

A related result is obtained by Austen-Smith (1994) in a rather different context. In his model, the sender may or may not know the value of her type; the receiver is unable to determine whether she is informed (there is *receiver uncertainty*), and the sender is allowed to send a message if and only if she is informed (but is not required to do so). He shows that, for a given set-up, if there is an informative equilibrium of the CS model then there is an informative equilibrium of the receiver-uncertainty model; but there is a range of values of sender bias for which there is an informative equilibrium of the receiver-uncertainty model only. In this sense, receiver uncertainty, like noise, facilitates communication. From a formal standpoint, the equilibrium construction used by Austen-Smith to prove this result resembles a two-step front-loading equilibrium of the

[14] See page 1444 of Crawford and Sobel (1982), or the discussion preceding the proof of Proposition 5 in the Appendix.

[15] Details are available in a supplementary file on the journal website, http://econtheory.org/supp/263/supplement.pdf.

[16] Crawford and Sobel's Theorem 5 states that (under condition (M)) the sender strictly prefers equilibria with more steps to equilibria with fewer steps. This need not be true in the noise model since, in the noise event, sender types may not obtain their intended actions.

noise model: informed sender types in the first partition element pool with uninformed sender types, sending no message, in the same way that in a front-loading equilibrium of the noise model sender types in the first partition element can be thought of as pooling with types who suffered from the error event; on the other hand, types in the second partition element guarantee self-identification by sending *some* message (for Austen-Smith) or by sending a specific message that is received with zero probability in the error event (in our equilibrium of the noise model).

4.2 *Optimal equilibria of the noise model*

Proposition 9 and Observation 1 describe circumstances under which we can find equilibria of the noise model that Pareto dominate the best equilibrium of the CS model. But we would also like to know, for given parameter values (b and ϵ), the optimal equilibrium of the noise model. We are able to provide only a very partial answer to this question. Specifically, for given b, we are able to find the optimal equilibrium of the noise model if ϵ is a choice variable (Section 4.2.1). For arbitrary b and arbitrary ϵ, however, we do not know what optimal equilibria look like. We have been unable to solve this problem even if attention is restricted to equilibria with a given number of steps. Further, unlike in the CS model,[17] equilibria with more steps do not necessarily Pareto dominate equilibria with fewer steps. First, equilibrium partitions of the noise model with more steps may nevertheless divide the state space less evenly than equilibrium partitions with fewer steps, and hence provide less information. Second, the coding of messages is also important: in general, the more messages that are used by a given partition element, the more distortion is created, since it is harder to distinguish whether such messages were sent by error or not. For a given partition, then, a particular coding minimizes the distortion effect (perhaps having all of the messages sent by the smallest partition element). But changing the coding changes the equilibrium partition, and there might be a trade-off between the kind of coding that minimizes the distortion effect and the kind of coding that generates the finest partition (i.e. maximizes the strategic effect).

4.2.1 *The optimal level of noise* Proposition 9 says that as long as the level of noise is low enough, we can find an equilibrium of the noise model that Pareto dominates the best equilibrium of the CS model. We prove this proposition by constructing a front-loading equilibrium in which the sender types in the first element of the equilibrium partition use almost all of the message space. Now suppose that we are free to choose the level of noise. Within this class of equilibria, it is easy to compute the optimal level of noise, i.e. the level that maximizes the receiver's (and sender's) expected payoff.

We show in the proof of Proposition 9 that the receiver's expected payoff in this kind of equilibrium is given by

$$EU^R = -\frac{4b^2(N-2)(N-1)^2N + 4b(N-1)^2(2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2},$$

[17] Again, restricting attention to the uniform–quadratic case.

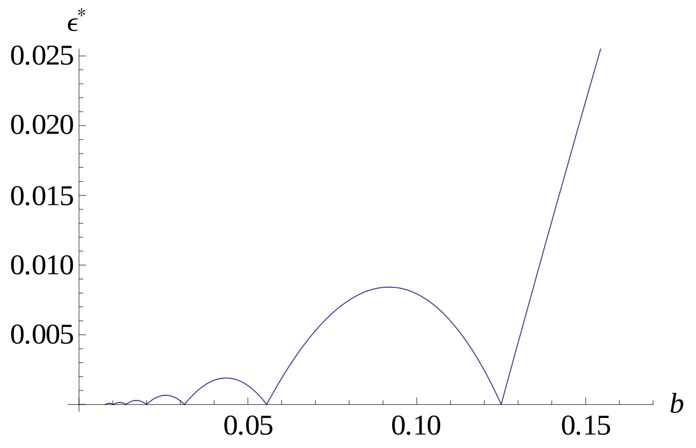

FIGURE 4. ϵ^* as a function of b.

where $N = \lceil 1/\sqrt{2b} \rceil$ is the number of steps in the equilibrium and θ_1 is the boundary type between the first and second partition elements (see page 432 below). The value of θ_1, of course, depends on the level of noise, ϵ, and is given by (11). Maximizing EU^R with respect to θ_1 we obtain

$$\theta_1^* = \frac{1-2b(N-1)^2}{2N-1}$$

and hence

$$\epsilon^* = \frac{(1-2b(N-1)^2)(1-2bN^2)}{4(N-1)N(b+b^2(N-1)N-1)}.$$

Figure 4 shows ϵ^* as a function of b. Values of b above $\frac{1}{6}$ are omitted for the sake of clarity. The function continues to rise for $b > \frac{1}{6}$, and $\epsilon^*(b) \to \frac{1}{4}$ as $b \to \frac{1}{2}$. Note that for $b = 1/(2N^2)$ ($N = 2, 3, 4, \ldots$), the optimal level of noise is $\epsilon^* = 0$—as Proposition 9 states, for these values of b no equilibrium of the noise model Pareto dominates the best equilibrium of the CS model.

Substituting the optimal value of θ_1 into the expression for the receiver's expected payoff, we get

$$EU^R = -\tfrac{1}{3}b(1-b).$$

This value of the expected payoff is exactly the same as can be achieved by the very different equilibrium construction considered by Krishna and Morgan (2004), although their construction is valid only for values of $b < \frac{1}{8}$. More significantly, Goltsman et al. (2007) show that $-\frac{1}{3}b(1-b)$ is an upper bound on the payoff that the receiver can obtain in *any* mediated equilibrium[18] (see their Lemma 1). *A fortiori*, it follows that the front-loading construction with noise level ϵ^* gives us the optimal equilibrium of the noise model.

[18]That is, in any equilibrium in which the sender can submit her message to an impartial mediator, who then passes on a recommendation to the receiver according to some pre-determined and possibly stochastic rule. Clearly such a mediator could reproduce the effect of noise in our model, so a noise equilibrium is a special case of a mediated equilibrium.

5. Comments and extensions

5.1 *Error probabilities that are correlated with messages*

In this paper we explore the impact of noise on communication, where the noise mechanism takes a very specific form: both the probability of error and what happens in the event of error are independent of the original message sent. We now consider what happens if we relax the first assumption, allowing the probability of error to vary across messages; relaxing the second assumption is left for future research.

Consider the uniform–quadratic model introduced in Section 3, except that the probability of error is a continuous function of the message, $\mathfrak{e} : [0,1] \to (0,1)$, so that when the sender sends message m, with probability $1-\mathfrak{e}(m)$ the message is faithfully transmitted and with probability $\mathfrak{e}(m)$ the received message is a draw from the uniform distribution on M. Call the resulting model the *correlated-noise model.* It turns out that in this framework, it is possible to establish a result that is analogous to Proposition 9. First we show (Lemma 1) that if there is an N-step front loading equilibrium of the noise model with noise level ϵ, then there is a N-step equilibrium of the correlated-noise model that is arbitrarily close to it, as long as ϵ is in the range of $\mathfrak{e}$. The welfare result (Proposition 10) follows easily from this lemma.

Lemma 1. *Consider the noise model with noise level ϵ and the correlated-noise model with error function $\mathfrak{e}$, where $\mathfrak{e}$ is continuous and includes ϵ in its range. Suppose that there is an N-step front-loading equilibrium of the noise model that yields expected payoff EU^R for the receiver. Then for any $\eta > 0$, there is an N-step equilibrium of the correlated-noise model that yields expected payoff EU'^R for the receiver, where $\left|EU^R - EU'^R\right| < \eta$.*

The proof of Lemma 1 can be found in the Appendix. We show that it is possible to construct an N-step front-loading equilibrium of the correlated-noise model where each interval $2,\ldots,N$ of the equilibrium partition uses a single message with noise level very close to ϵ, while the first interval randomizes over all remaining messages in such a way that whichever of these messages is received, the receiver's posterior probability of an error is the same. It turns out that this probability depends only on the errors associated with the messages sent by intervals $2,\ldots,N$, and is also very close to the corresponding probability in the N-step equilibrium of the noise model. Hence the indifference conditions for boundary types are very similar across the two models, and we can therefore find an equilibrium of the correlated-noise model with almost the same equilibrium partition, and almost the same actions induced, as in the equilibrium of the noise model. Furthermore, it is easy to see that this equilibrium induces an outcome (joint distribution over types and actions) that is very close to that of the N-step noise equilibrium: although types in the first interval send a range of messages associated with different noise levels, they induce some action a_1 (where a_1 is close to the corresponding value in the equilibrium of the noise model) with probability one whether or not there is an error; and types in each other interval i induce some action a_i (again close to the corresponding value in the equilibrium of the noise model) with probability close to $1-\epsilon$, and action a_1 otherwise. This gives the required result.

Proposition 10 follows immediately from Lemma 1.

Proposition 10. *If $b < \frac{1}{2}$ and $b \neq 1/(2N^2)$ for all integers $N > 1$, then there exists an $\overline{\epsilon} > 0$ such that if $\mathfrak{e}(m) < \overline{\epsilon}$ for some m, there is an equilibrium of the correlated-noise model that is Pareto superior to all equilibria of the CS model.*

As long as there are some messages associated with low enough error, then, we can find "good" equilibria of the correlated-noise model even if the error is very high for almost all other messages. All of these unreliable messages will be sent by the first interval of types.

Finally, it is worth noting that if the range of the error function includes ϵ^*, the optimal level of noise derived in Section 4.2.1, the same construction can be used to find an equilibrium of the correlated-noise model that yields an expected payoff for the receiver that is arbitrarily close to its upper bound in any mediated equilibrium, $-\frac{1}{3}b(1-b)$.

Proposition 11. *Consider the correlated-noise model with error function $\mathfrak{e}$, where $\mathfrak{e}$ is continuous and includes ϵ^* in its range, and let $\mathbf{EU}^R$ denote the set of equilibrium payoffs that are attainable for the receiver in this model. Then* $\sup \mathbf{EU}^R = -\frac{1}{3}b(1-b)$.

5.2 *Noisy talk with common interest*

Although the focus of this paper is the interaction between noise and divergent interests, it is instructive to consider the effects of noise in the common-interest case, where $b = 0$. Recall that an equilibrium is separating if every sender type induces a different action (in the no-noise event). Without noise, there is a separating equilibrium where the sender follows the "natural" strategy of sending message $m = \theta$ when her type is θ, and the receiver chooses action $a = m$. But this is not an equilibrium when there is noise. The reason is that under this sender strategy, the posterior probability of an error having occurred equals the prior, ϵ. Thus, following each message the receiver attributes this probability to the event that this message was the result of noise and therefore distorts his response toward the pooling response, choosing action $a = (1-\epsilon)m + \epsilon\frac{1}{2}$. A rational sender would try to offset this distortion by deviating from the rule $m = \theta$. This illustrates nicely the distortionary effect of introducing noise: even in the common-interest game sender and receiver cannot simply continue to use the strategies that "work" in the absence of noise. Our next result shows that nevertheless a separating equilibrium does exist for any value of the error probability ϵ. The fundamental idea underlying the construction of such an equilibrium is to have the sender use only a small subset of the message space. Then, whenever a message from this subset is received, the posterior probability that it was not sent by error is high. As the size of the set of used messages converges to zero, this posterior probability converges to one. Denote the common expected payoff in a separating equilibrium of the common-interest game with error probability ϵ by $\Pi(\epsilon)$. Note that the pooling payoff, Π_p, is independent of the error probability (in the uniform–quadratic case considered here, $\Pi(0) = 0$ and $\Pi_p = -\frac{1}{12}$, but it is easy to see that the proof is easily extended to the more general model introduced in Section 2.1).

Proposition 12. *The common-interest game has a separating equilibrium. In this equilibrium the receiver learns with probability one whether or not a received message was sent in error. Equilibria of this form are efficient in the common-interest game and have a common expected payoff*

$$\Pi(\epsilon) = (1-\epsilon)\Pi(0) + \epsilon\Pi_p.$$

Proof. Take any set $M^0 \subset [0,1]$ that has the same cardinality as the set $[0,1]$ and at the same time has (Lebesgue) measure zero (for example, M^0 can be the Cantor set). Since M^0 has the same cardinality as $[0,1]$, there exists a sender strategy that is a bijection from the type space $[0,1]$ into M^0. At the same time, since the error distribution has a density, sets of (Lebesgue) measure zero have probability zero; it follows from Bayes' rule that the probability of an error following a message $m \in M^0$ equals zero. Therefore, whenever he receives a message in M^0, the receiver knows that with probability one the message was not sent in error. In that case, given that the sender's strategy is a bijection, the receiver correctly infers the sender's type. Similarly, whenever he receives a message in $[0,1] \setminus M^0$, the receiver knows that with probability one the message was sent in error. Regarding efficiency, it suffices to consider receiver payoffs. Conditional on each event, noise or no-noise, the receiver maximizes his payoff. Therefore he maximizes his *ex ante* expected payoff. The fact that the expected payoff $\Pi(\epsilon)$ has the indicated form is a simple consequence of the receiver taking the separating action in the no-noise event, and the pooling action in the noise event. Finally, note that since all sender types receive their ideal actions in the no-noise event and cannot affect the action taken in the noise event, the sender has no incentive to deviate. □

As an aside, we observe that Theorem 4 of Gordon (2007) can be used to show that there is an alternative, *proportional-coding* equilibrium with an infinite partition, in which types in partition element $[\theta_{j-1}, \theta_j)$ randomize uniformly over the interval $[\theta_{j-1}, \theta_j)$. These equilibria have the intuitive property that types in a given element of the equilibrium partition use only messages in the same set. Since these equilibria do not make optimal use of the available information, however, unlike equilibria in which the set of used messages has measure zero, they are not Pareto optimal. Therefore, we have the additional observation that in the common-interest game there are multiple Pareto-ranked infinite-interval partition equilibria.

Returning to the construction used to prove Proposition 12, a referee has observed that it relies on the message space being a continuum, so we can find an uncountable set of messages that nevertheless has measure zero. In the finite case, separating equilibria may not exist if the cardinality of the message space is close to the cardinality of the type space and the level of noise is sufficiently high. But we now show that, as long as the message space is large enough, separation can be achieved through a construction analogous to that used in the infinite case.

Let K be a positive integer and consider the finite set of types $T(K) = \{\theta \in [0,1] \mid \theta = n \times 1/K \text{ for some } n \in \mathbb{N}_0\}$, each of which is equally likely. First, to show why separation may be impossible, let the message space be identical to the type space,[19] and

[19] If the message space is smaller than the type space, separation is trivially impossible.

suppose the error distribution is uniform. In a candidate separating equilibrium, the receiver's response to the message sent by the lowest type ($\theta = 0$) would be the action $a_0 = (1-\epsilon)0 + \epsilon\frac{1}{2}$, while his best response to the message sent by the second lowest type would be $a_1 = (1-\epsilon)(1/K) + \epsilon\frac{1}{2}$. For incentive compatibility, type $\theta = 1/K$ must prefer a_1 to a_0, i.e.

$$-\left(\frac{1}{K} - a_1\right)^2 \geq -\left(\frac{1}{K} - a_0\right)^2$$
$$\Rightarrow \epsilon \leq \frac{1}{K-1}.$$

If ϵ is above this threshold, then, no separating equilibrium exists (in fact, this condition is necessary and sufficient for the existence of a separating equilibrium).

If we fix K and ϵ, however, but increase the size of the message space, we can always find a message space large enough that a separating equilibrium exists. Suppose that each sender type sends exactly one (distinct) message. Then as the message space grows, the receiver's response to a message sent by type θ converges to θ for all $\theta \in T(K)$. Formally, if the receiver observes the message sent by type θ, his best response is given by

$$a_\theta = (1-\mu)\theta + \mu\tfrac{1}{2},$$

where

$$\mu = \frac{\epsilon\frac{1}{|M|}}{(1-\epsilon)\frac{1}{K+1} + \epsilon\frac{1}{|M|}}$$

is the receiver's posterior probability that the message was sent in error. As $|M| \to \infty$, $\mu \to 0$ and so $a_\theta \to \theta$. Hence for a sufficiently large message space, each sender type strictly prefers to send the message assigned to her than any message sent by the other types, or one of the unsent messages (to which the receiver's best response is the action $a = \frac{1}{2}$).

6. Conclusion

We have examined two principal barriers to communication, misaligned preferences and the possibility of misunderstandings, and their interaction. We find that while each of these factors limits communication on its own, the possibility of misunderstandings may help partially overcome the limitations due to divergent preferences. We have shown that introducing a small amount of noise into information transmission can almost always benefit communication. When noise levels continuously vary across a sufficiently large range of messages, there are equilibria that approximate optimal mediated communication. In the case of extreme biases, introducing noise may enable communication when it would otherwise not have been possible.

Appendix: Proofs and calculations

Before proving Proposition 1, we start with two lemmas.

Lemma 2. *If type θ induces actions a_1 and a_2 with $a_1 < a_2$, then there exists $\eta > 0$ such that types in $(\theta - \eta, \theta)$ induce action a_1 and types in $(\theta, \theta + \eta)$ induce action a_2.*

Proof. Concavity of the sender's payoff function in a implies that $a_1 < a^S(\theta, b) < a_2$ (where $a^S(\theta, b)$ denotes type θ's ideal action). Continuity of the sender's payoff function and the single-crossing condition ($U_{12}^S > 0$) imply that there is a nonempty open set of types $(\theta, \theta + \eta_1)$ such that for all $\theta' \in (\theta, \theta + \eta_1)$ we have $a^S(\theta, b) < a^S(\theta', b) < a_2$. Type θ''s payoff is decreasing to the right of a_2 and by single crossing, θ' prefers a_2 to all actions $a \in (-\infty, a_1)$. No action $a \in (a_1, a_2)$ is induced in equilibrium because otherwise type θ would have an incentive to deviate. This shows that all types in $(\theta, \theta + \eta_1)$ must induce action a_2. An analogous argument shows that we can find a nonempty open set $(\theta - \eta_2, \theta)$ such all types in that set induce action a_1. Choose $\eta = \min\{\eta_1, \eta_2\}$. □

Lemma 3. *In every equilibrium of the noise model, for every action a, the set of sender types that induce action a is an interval. If this interval has a nonempty interior, then all types in the interior induce only action a.*

Proof. If a is induced by only one type, the result holds trivially. If types θ and θ' with $\theta < \theta'$ both induce action a, then types $\theta'' \in (\theta, \theta')$ never induce an action $a_1 > a$, because otherwise, since $U_{12}^S(a, \theta, b) > 0$, type θ' would strictly prefer a_1 to a; similarly, types $\theta'' \in (\theta, \theta')$ never induce an action $a_0 < a$, because otherwise by single-crossing type θ would strictly prefer a_0 to a. Hence types in the interval (θ, θ') induce only action a. □

Proof of Proposition 1. The result now follows easily: Lemma 2 implies immediately that at most a countable number of types induce two actions and that the receiver's response is not altered if we have all such types switch to induce only one of these actions. With each type inducing exactly one action, Lemma 3 implies that each of these actions is induced by an interval of types. □

Proof of Proposition 2. For any interval I in the partition of types induced by the sender strategy σ, let M_I be the union of the supports of the distributions $\sigma(\cdot \mid \theta)$ over all $\theta \in I$. Since types in distinct intervals induce distinct actions, $I \neq I'$ implies $M_I \cap M_{I'} = \varnothing$. The receiver's payoff from choosing action a conditional on observing message $m \in M_I$ is given by

$$\int_0^1 U^R(a, \theta) \mu(\theta \mid m)\, d\theta.$$

Maximizing this expression with respect to a is equivalent to maximizing

$$\int_0^1 U^R(a, \theta)((1 - \epsilon)\sigma(m \mid \theta) + \epsilon g(m)) f(\theta)\, d\theta.$$

Since $a(m)$ is a common maximizer for all $m \in M_I$, it also maximizes

$$\int_{M_I}\int_0^1 U^R(a,\theta)((1-\epsilon)\sigma(m\mid\theta)+\epsilon g(m))f(\theta)\,d\theta\,dm$$
$$=\int_I U^R(a,\theta)(1-\epsilon)f(\theta)d\theta+\int_0^1 U^R(a,\theta)\epsilon\int_{M_I} g(m)dm\,f(\theta)\,d\theta.$$

Maximizing the latter expression, however, is equivalent to maximizing

$$\int_I U^R(a,\theta)\frac{(1-\epsilon)g(m)}{\int_{M_I} g(m')dm'}f(\theta)d\theta+\int_0^1 U^R(a,\theta)\epsilon g(m)f(\theta)\,d\theta,$$

which is exactly the problem that the receiver solves after receiving a message $m \in M_I$ when all types in I use the common distribution $g(m)/\int_{M_I} g(m')\,dm'$ on M_I. □

Proof of Proposition 3. First assume that there is a type θ_0 whose ideal action is the pooling action, a_p. Suppose there is a set M_0 of unused messages that has positive measure. Whenever the receiver observes a message $m_0 \in M_0$, the receiver's optimal reply $a(m_0)$ satisfies

$$a(m_0)=\arg\max_a \int_0^1 U^R(a,\theta)f(\theta)\,d\theta.$$

By assumption $a(m_0)$ is the ideal action for type θ_0. Since this type could induce this action by sending one of the unused messages, he induces it in equilibrium. Consider first any equilibrium in which the set of types, Θ_0, that induce the same action $a(m_0)$ as θ_0 has positive probability. Using $M(\Theta_0)$ to denote the set of messages used by Θ_0, we have

$$a(m_0)=\arg\max_a \int_{\Theta_0} U^R(a,\theta)\frac{(1-\epsilon)g(m)}{\int_{M(\Theta_0)} g(m')\,dm'}f(\theta)d\theta+\int_0^1 U^R(a,\theta)\epsilon g(m)f(\theta)\,d\theta.$$

Since $a(m_0)$ maximizes the second term of this expression, it also maximizes the first term. Therefore, we also have

$$a(m_0)=\arg\max_a \int_{\Theta_0} U^R(a,\theta)\frac{(1-\epsilon)g(m)}{\int_{M_0\cup M(\Theta_0)} g(m')dm'}f(\theta)d\theta+\int_0^1 U^R(a,\theta)\epsilon g(m)f(\theta)\,d\theta,$$

which is the receiver's best response if we change the sender's strategy so that types in Θ_0 randomize over $M_0 \cup M(\Theta_0)$ according to the error distribution restricted to that set. Next consider any equilibrium in which the set of types Θ_0 has probability zero. Having all these types randomize uniformly over $M_0 \cup M(\Theta_0)$ does not alter the receiver's posterior after any message and thus preserves equilibrium.

Now we deal with the case where there is no type whose ideal action is the pooling action. Since $b > 0$ and U^S is continuous, every type's ideal action must be larger than

a_p ($a^S(\theta,b) > a_p$ for all $\theta \in T$). If the equilibrium under consideration is pooling, the conclusion follows immediately. If not, then we can find two actions, a_1 and a_2, induced in equilibrium with the property that $a_1 < a_p < a_2$. It follows easily that there cannot be any unused messages: sending an unused message would induce action a_p, which is preferred by every type to action a_1. □

Proof of Proposition 4. The result is immediate if $\mathcal{M}$ and $\mathcal{M}'$ have different numbers of elements; therefore let both have N elements. Consider an equilibrium of the noise model that is adapted to $\mathcal{M}$, and let $T_1, \dots, T_N$ denote the elements of the equilibrium partition. Define

$$a(T_i, M_i) \equiv \arg\max_a \int_{T_i} U^R(a,\theta)(1-\epsilon) f(\theta)\, d\theta + \int_0^1 U^R(a,\theta)\epsilon \lambda_G(M_i) f(\theta)\, d\theta.$$

Since types in T_i randomize with the error distribution over messages in M_i, the receiver's equilibrium response a_i to a message $m \in M_i$ satisfies $a_i = a(T_i, M_i)$. If $\lambda_G(M_i') \neq \lambda_G(M_i)$, then $a(T_i, M_i') \neq a(T_i, M_i)$, unless we have

$$\arg\max_a \int_{T_i} U^R(a,\theta) f(\theta)\, d\theta = \arg\max_a \int_0^1 U^R(a,\theta) f(\theta)\, d\theta.$$

This condition, however, can be satisfied for at most one T_i, while for any $\mathcal{M}'$ that is G-distinguished from $\mathcal{M}$, there must be at least two partition elements T_j and T_k, such that the corresponding message sets satisfy $\lambda(M_j) \neq \lambda(M_j')$ and $\lambda(M_k) \neq \lambda(M_k')$. It follows immediately that $O(\mathcal{M}) \cap O(\mathcal{M}') = \varnothing$, as required. In contrast, for any $\mathcal{M}$ and $\mathcal{M}'$ that are not G-distinguished, we have $a(T_i, M_i') = a(T_i, M_i)$ for all i, and therefore any outcome of an equilibrium that is adapted to $\mathcal{M}$ can be reproduced as an outcome of an equilibrium that is adapted to $\mathcal{M}'$ and vice versa. □

Before proving Proposition 5, we provide a formal restatement of Crawford and Sobel's monotonicity condition (M). In the CS model, let $a_{CS}(\theta_{i-1}, \theta_i)$ denote the receiver's best response to a message that indicates only that the sender's type lies in (θ_{i-1}, θ_i), i.e.

$$a_{CS}(\theta_{i-1}, \theta_i) = \arg\max_{a'} \int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta) f(\theta)\, d\theta.$$

Consider the second-order difference equation

$$U^S(a_{CS}(\theta_{i-1}, \theta_i), \theta_i, b) = U^S(a_{CS}(\theta_i, \theta_{i+1}), \theta_i, b). \tag{2}$$

Then condition (M) says:

(M) Suppose $(\theta_0, \theta_1, \dots, \theta_N)$ and $(\theta_0', \theta_1', \dots, \theta_N')$ are two solutions to (2), and $\theta_0 = \theta_0'$ and $\theta_1 > \theta_1'$. Then $\theta_i > \theta_i'$ for all $i \geq 2$.

Proof of Proposition 5. Denote the boundary types of the N-step CS partition by θ_i^*, $i = 1,\dots,N-1$. Let $\hat{a}(\theta_{i-1},\theta_i,\mathscr{M},\epsilon) \equiv a(T_i,M_i)$, where $T_i = (\theta_{i-1},\theta_i)$, M_i is the ith component of $\mathscr{M}$, and $a(T_i,M_i)$ is as defined in the proof of Proposition 4 (so $\hat{a}(\theta_{i-1},\theta_i,\mathscr{M},\epsilon)$ is the receiver's best response to messages in M_i when types in (θ_{i-1},θ_i) randomize over messages in that set according to the error distribution and ϵ is the level of noise). We can view $\mathscr{M}$ as a point in the $N-1$ simplex Δ^{N-1}. Then, $\hat{a}$ is a continuous function on the set $\{(\theta_{i-1},\theta_i) \mid \theta_{i-1} \le \theta_i\} \times \Delta^{N-1} \times [0,1]$ (where in this instance (θ_{i-1},θ_i) denotes an ordered pair rather than an open interval); this follows from continuity of U^R, the Theorem of the Maximum and the fact that $\hat{a}(\theta_{i-1},\theta_i,\mathscr{M},\epsilon)$ is a singleton for all $(\theta_{i-1},\theta_i,\mathscr{M},\epsilon)$. It follows from the single-crossing condition, $U_{12}^R > 0$, that $\hat{a}(\theta_i,\theta_{i+1},\mathscr{M},\epsilon)$ is strictly increasing in its first two arguments.

Next, define

$$V(\theta_{i-1},\theta_i,\theta_{i+1},\mathscr{M},\epsilon,b) \equiv U^S(\hat{a}(\theta_i,\theta_{i+1},\mathscr{M},\epsilon),\theta_i,b) - U^S(\hat{a}(\theta_{i-1},\theta_i,\mathscr{M},\epsilon),\theta_i,b).$$

The continuity of $\hat{a}$ and of U^S implies that V is continuous on the compact set $\{(\theta_{i-1},\theta_i,\theta_{i+1}) \mid \theta_{i-1} \le \theta_i \le \theta_{i+1}\} \times \Delta^{N-1} \times [0,1] \times \{b\}$ (whatever the value of b). Therefore V is uniformly continuous on this set.

The function $V(\theta_{i-1}^*,\theta_i^*,\tau_{i+1},\mathscr{M},0,b)$ is strictly decreasing in τ_{i+1} in a neighborhood of θ_{i+1}^*. This property follows from the fact that $V(\theta_{i-1}^*,\theta_i^*,\theta_i^*,\mathscr{M},0,b) > 0$, $V(\theta_{i-1}^*,\theta_i^*,\theta_{i+1}^*,\mathscr{M},0,b) = 0$, $\hat{a}(\theta_i,\theta_{i+1},\mathscr{M},0)$ is strictly increasing in θ_{i+1}, and $U^R(a,\theta)$ is strictly concave in a. This implies that there exist τ'_{i+1} and τ''_{i+1} with $\tau'_{i+1} < \theta_{i+1}^* < \tau''_{i+1}$ such that $V(\theta_{i-1}^*,\theta_i^*,\tau'_{i+1},\mathscr{M},0,b) > 0 > V(\theta_{i-1}^*,\theta_i^*,\tau''_{i+1},\mathscr{M},0,b)$. This and the uniform continuity of V on $\{(\theta_{i-1},\theta_i,\theta_{i+1}) \mid \theta_{i-1} \le \theta_i \le \theta_{i+1}\} \times \Delta^{N-1} \times [0,1] \times \{b\}$ imply that there exists $\eta_1 > 0$ such that for all $\tau_{i-1} \in [\theta_{i-1}^* - \eta_1, \theta_{i-1}^* + \eta_1]$, for all $\tau_i \in [\theta_i^* - \eta_1, \theta_i^* + \eta_1]$, for all $\epsilon \in [0,\eta_1]$, and for all $\mathscr{M}$, we have $V(\tau_{i-1},\tau_i,\tau'_{i+1},\mathscr{M},\epsilon,b) > 0 > V(\tau_{i-1},\tau_i,\tau''_{i+1},\mathscr{M},\epsilon,b)$. Hence, the Intermediate Value Theorem implies that for all $\tau_{i-1} \in [\theta_{i-1}^* - \eta_1, \theta_{i-1}^* + \eta_1]$, for all $\tau_i \in [\theta_i^* - \eta_1, \theta_i^* + \eta_1]$, for all $\epsilon \in [0,\eta_1]$, and for all $\mathscr{M}$, there exists $\tau_{i+1}(\tau_{i-1},\tau_i,\mathscr{M},\epsilon)$ such that $V(\tau_{i-1},\tau_i,\tau_{i+1}(\tau_{i-1},\tau_i,\mathscr{M},\epsilon),\mathscr{M},\epsilon,b) = 0$.

Furthermore, from $V(\tau_{i-1},\tau_i,\tau'_{i+1},\mathscr{M},\epsilon,b) > 0 > V(\tau_{i-1},\tau_i,\tau''_{i+1},\mathscr{M},\epsilon,b)$, the fact that $\hat{a}(\tau_i,\tau_{i+1},\mathscr{M},\epsilon)$ is strictly increasing in τ_{i+1}, and the strict concavity of U^R in its first argument, it follows that $\tau_{i+1}(\tau_{i-1},\tau_i,\mathscr{M},\epsilon)$ is unique. In conjunction with the continuity of V, this implies that $\tau_{i+1}(\tau_{i-1},\tau_i,\mathscr{M},\epsilon)$ is continuous for all $\tau_{i-1} \in [\theta_{i-1}^* - \eta_1, \theta_{i-1}^* + \eta_1]$, for all $\tau_i \in [\theta_i^* - \eta_1, \theta_i^* + \eta_1]$, for all $\epsilon \in [0,\eta_1]$, and for all $\mathscr{M}$.

Iterating on i, this implies that there exists $\eta > 0$ such that for all $\mathscr{M}$, for all θ_1 with $|\theta_1 - \theta_1^*| \le \eta$ and ϵ that satisfy $\epsilon \le \eta$, there exists a solution $\theta_i(\theta_1,\mathscr{M},\epsilon)$ for $i = 0,\dots,N-1$ to the difference equation

$$U^S(\hat{a}(\theta_{i-1},\theta_i,\mathscr{M},\epsilon),\theta_i,b) = U^S(\hat{a}(\theta_i,\theta_{i+1},\mathscr{M},\epsilon),\theta_i,b)$$

with initial values $\theta_0 = 0$ and θ_1, and that the solution is continuous on this domain.

Define

$$\begin{aligned} W(\theta_1,\mathscr{M},\epsilon) \equiv{}& U^S(\hat{a}(\theta_{N-1}(\theta_1,\mathscr{M},\epsilon),1,\mathscr{M},\epsilon),\theta_{N-1}(\theta_1,\mathscr{M},\epsilon),b) \\ &- U^S(\hat{a}(\theta_{N-2}(\theta_1,\mathscr{M},\epsilon),\theta_{N-1}(\theta_1,\mathscr{M},\epsilon),\mathscr{M},\epsilon),\theta_{N-1}(\theta_1,\mathscr{M},\epsilon),b). \end{aligned}$$

The continuity of $\theta_i(\theta_1,\mathcal{M},\epsilon)$ implies that $W(\theta_1,\mathcal{M},\epsilon)$ is continuous on the compact set $[\theta_1^*-\eta,\theta_1^*+\eta]\times[0,\eta]\times\Delta^{N-1}$ and therefore uniformly continuous on that set. This implies that for all $\zeta>0$ we can find $\overline{\epsilon}>0$ such that $\epsilon<\overline{\epsilon}$ implies $|W(\theta_1,\mathcal{M},\epsilon)-W(\theta_1,\mathcal{M},0)|<\zeta$ for all $\mathcal{M}$ and all $\theta_1\in[\theta_1^*-\eta,\theta_1^*+\eta]$. Since $W(\theta_1,\mathcal{M},0)$ does not depend on $\mathcal{M}$, we have that for all $\zeta>0$ there exists $\overline{\epsilon}>0$ such that $\epsilon<\overline{\epsilon}$ implies $|W(\theta_1,\mathcal{M}',\epsilon)-W(\theta_1,\mathcal{M},0)|<\zeta$ for all $\mathcal{M},\mathcal{M}'$ and all $\theta_1\in[\theta_1^*-\eta,\theta_1^*+\eta]$.

Consider $\theta_1'\in(\theta_1^*-\eta,\theta_1^*)$, and $\theta_1''\in(\theta_1,\theta_1^*+\eta)$ such that $\theta_{N-1}(\theta_1'',\mathcal{M},0)<1$. Then condition (M) implies that

$$W(\theta_1',\mathcal{M},0)<0<W(\theta_1'',\mathcal{M},0)$$

for all $\mathcal{M}$. It follows from our earlier argument that we can find $\tilde{\epsilon}>0$ such that $\epsilon<\tilde{\epsilon}$ implies that

$$W(\theta_1',\mathcal{M},\epsilon)<0<W(\theta_1'',\mathcal{M},\epsilon)$$

for all $\mathcal{M}$. Hence, by the Intermediate Value Theorem, for all $\epsilon<\tilde{\epsilon}$ and for all $\mathcal{M}$ there exists θ_1 for which

$$W(\theta_1,\mathcal{M},\epsilon)=0.$$

It is easy to see, for this θ_1, that the boundary values $\theta_1,\theta_2(\theta_1,\mathcal{M},\epsilon),\ldots,\theta_{N-1}(\theta_1,\mathcal{M},\epsilon)$ describe an equilibrium partition that is adapted to $\mathcal{M}$.

Finally, if we denote the corresponding sender strategy by σ_ϵ, uniform continuity of $\theta_i(\theta_1,\mathcal{M},\epsilon)$ on the set $[\theta_1^*-\eta,\theta_1^*+\eta]\times[0,\eta]\times\Delta^{N-1}$ implies that for any $\delta>0$ we can choose $\tilde{\epsilon}$ such that for any $\epsilon<\tilde{\epsilon}$ we have $|\mathbb{P}(\sigma_\epsilon)-\mathbb{P}(\sigma)|<\delta$. □

Proof of Proposition 6. Suppose that there is a separating equilibrium of the noise model. Then by the single-crossing condition ($U_{12}^R(a,\theta)>0$), ω is strictly monotonic. Therefore ω is continuous except at a countable number of types. Let θ be a point of continuity of ω and suppose that $\omega(\theta)\neq\arg\max_a U^S(a,\theta,b)$. Then we can find a type θ' near θ such that both types either prefer $\omega(\theta)$ to $\omega(\theta')$ or prefer $\omega(\theta')$ to $\omega(\theta)$, violating incentive compatibility. It follows that $\omega(\theta)=\arg\max_a U^S(a,\theta,b)$ at all θ at which ω is continuous. For any $\eta>0$ we can find θ such that $1-\theta<\eta$ and ω is continuous at θ. Furthermore, we can choose η small enough to ensure that the receiver's optimal response to the message sent by type θ is less than $\arg\max_a U^R(a,\theta)$. This implies $\omega(\theta)<\arg\max_a U^R(a,\theta)<\arg\max_a U^S(a,\theta,b)=\omega(\theta)$, establishing a contradiction. □

Derivation of two-step equilibria Recall that we impose the following restrictions on the parameters: $0<b<\frac{1}{2}$; $0<\epsilon<1$; $0\leq\lambda_1\leq 1$. We are interested in two-step equilibria where types in the first interval, denoted $[0,\theta_1)$, randomize uniformly over messages in M_1, and types in the second interval, $[\theta_1,1]$, randomize uniformly over messages in M_2, where $M_1\cap M_2=\varnothing$ and $M_1\cup M_2=[0,1]$. Let λ_1 denote the measure of M_1 according to the error distribution, so that $1-\lambda_1$ is the measure of M_2. The actions chosen by the receiver on receiving messages in M_1 and M_2 are respectively

$$a_1 = \frac{(1-\epsilon)\theta_1 \frac{\theta_1}{2} + \epsilon\lambda_1 \frac{1}{2}}{(1-\epsilon)\theta_1 + \epsilon\lambda_1}$$
$$a_2 = \frac{(1-\epsilon)(1-\theta_1)\frac{\theta_1+1}{2} + \epsilon(1-\lambda_1)\frac{1}{2}}{(1-\epsilon)(1-\theta_1)+\epsilon(1-\lambda_1)}.$$

Equilibrium requires the sender to be indifferent between a_1 and a_2 when $\theta = \theta_1$, i.e.

$$\theta_1 + b = \tfrac{1}{2}(a_1 + a_2). \tag{3}$$

Furthermore, since $a_1 < a_2$ for $\theta_1 \in (0,1)$, (3) along with the condition that $0 < \theta_1 < 1$ is sufficient for equilibrium. We use $\theta_1^*(b,\epsilon,\lambda_1)$ to denote such a solution, when it exists.

Rather than analyzing the set of equilibria for various values of b, as we did in Section 3.1, it is more convenient to consider different values of λ_1. The results below establish all of the claims made in Section 3.1.

$\lambda_1 = 0$: Substituting $\lambda_1 = 0$ into the expressions for a_1 and a_2 and simplifying, we obtain

$$\tfrac{1}{2}(a_1 + a_2) = \frac{1+\theta_1 - 2(1-\epsilon)\theta_1^2}{4(1-(1-\epsilon)\theta_1)}.$$

We can then solve (3):

$$\begin{aligned}\theta_1 + b = {}& \frac{1+\theta_1 - 2(1-\epsilon)\theta_1^2}{4(1-(1-\epsilon)\theta_1)}\\ &\Rightarrow \quad (\theta_1+b)4(1-(1-\epsilon)\theta_1) = 1+\theta_1-2(1-\epsilon)\theta_1^2\\ &\Rightarrow \quad 2(1-\epsilon)\theta_1^2 - (3-4b+4b\epsilon)\theta_1 + 1 - 4b = 0\\ &\Rightarrow \quad \theta_1 = \frac{3-4b+4b\epsilon \pm \sqrt{(3-4b+4b\epsilon)^2 - 8(1-\epsilon)(1-4b)}}{4(1-\epsilon)}.\end{aligned}$$

It can be shown that the relevant solution is the smaller one, which lies strictly between 0 and 1 if and only if $b < \frac{1}{4}$. Thus a two-step equilibrium exists for $b \in (0, \frac{1}{4})$, with boundary type

$$\theta_1^*(b,\epsilon,0) = \frac{3-4b+4b\epsilon - \sqrt{(3-4b+4b\epsilon)^2 - 8(1-\epsilon)(1-4b)}}{4(1-\epsilon)}.$$

If $b \in [\frac{1}{4}, \frac{1}{2}]$, there is no two-step equilibrium.

$\lambda_1 = 1$: Substituting $\lambda_1 = 1$ into the expressions for a_1 and a_2 and solving (3), we obtain

$$\tfrac{1}{2}(a_1 + a_2) = \frac{2\epsilon + \theta_1 + 2(1-\epsilon)\theta_1^2}{4(\epsilon + (1-\epsilon)\theta_1)}.$$

We can then solve (3):

$$\theta_1+b=\frac{2\epsilon+\theta_1+2(1-\epsilon)\theta_1^2}{4(\epsilon+(1-\epsilon)\theta_1)}$$
$$\Rightarrow\quad (\theta_1+b)4(\epsilon+(1-\epsilon)\theta_1)=2\epsilon+\theta_1+2(1-\epsilon)\theta_1^2$$
$$\Rightarrow\quad 2(1-\epsilon)\theta_1^2-(1-4b(1-\epsilon)-4\epsilon)\theta_1-2\epsilon(1-2b)=0$$
$$\Rightarrow\quad \theta_1=\frac{1-4b(1-\epsilon)-4\epsilon\pm\sqrt{1+8b(1-\epsilon)+16b^2(1-\epsilon)^2+8\epsilon}}{4(1-\epsilon)}.$$

It can be shown that the relevant solution is the larger one, which lies strictly between 0 and 1 for all $b\in\left(0,\frac{1}{2}\right)$. Thus a two-step equilibrium exists for $b\in\left(0,\frac{1}{2}\right)$, with boundary type

$$\theta_1^*(b,\epsilon,1)=\frac{1-4b(1-\epsilon)-4\epsilon+\sqrt{1+8b(1-\epsilon)+16b^2(1-\epsilon)^2+8\epsilon}}{4(1-\epsilon)}.$$

$\lambda_1\in(0,1)$: When $\lambda_1\in(0,1)$, (3) reduces to a cubic and finding an analytical solution is cumbersome. However, we can show that the equation has exactly one solution that lies (strictly) between 0 and 1 for all $b\in\left(0,\frac{1}{2}\right)$. To see why, notice that at $\theta_1=0$, $\theta_1+b<\frac{1}{2}(a_1+a_2)=\frac{1}{2}$, while at $\theta_1=1$, $\theta_1+b>\frac{1}{2}(a_1+a_2)=\frac{1}{2}$. Since both θ_1+b and $\frac{1}{2}(a_1+a_2)$ are continuous in θ_1, (3) is satisfied for at least one value of $\theta_1\in(0,1)$. To show uniqueness, notice that

$$\frac{d}{d\theta_1}(\theta_1+b)=1,$$

while

$$\frac{\partial}{\partial\theta_1}\left(\tfrac{1}{2}(a_1+a_2)\right)=\tfrac{1}{2}-\frac{\epsilon(1-\lambda_1)(1-\epsilon\lambda_1)}{(1-\epsilon(\lambda_1-\theta_1)-\theta_1)^2}-\frac{\epsilon(1-\epsilon(1-\lambda_1))\lambda_1}{(\epsilon(\lambda_1-\theta_1)+\theta_1)^2}<\tfrac{1}{2}.$$

Further, since $\frac{1}{2}(a_1+a_2)$ is continuous in $\lambda_1\in(0,1)$, the equilibrium boundary type $\theta_1(b,\epsilon,\lambda_1)$ is also continuous in λ_1. To show that $\theta_1^*(b,\epsilon,\lambda_1)$ is strictly increasing in λ_1, note that (for given θ_1)

$$\frac{\partial a_1}{\partial\lambda_1}=\frac{(1-\epsilon)\epsilon(1-\theta_1)\theta_1}{2(\epsilon(\lambda_1-\theta_1)+\theta_1)^2}>0$$
$$\frac{\partial a_2}{\partial\lambda_1}=\frac{(1-\epsilon)\epsilon(1-\theta_1)\theta_1}{2(1-\epsilon(\lambda_1-\theta_1)+\theta_1)^2}>0,$$

so

$$\frac{\partial}{\partial\lambda_1}\left(\tfrac{1}{2}(a_1+a_2)\right)>0.$$

Thus when we increase λ_1, the left-hand side of (3) is unchanged, while the right-hand side shifts up, for every value of θ_1; since the right-hand side intersects the left-hand side from above, the (unique) point of intersection also increases. Intuitively, this means that allowing the first step of the equilibrium partition to use a larger proportion of the message space shifts the boundary between the two steps to the right.

Finally, we show that all boundary values between the lower bound of $\theta_1(b,\epsilon,0)$ (if $b<\frac{1}{4}$) or 0 (if $b\geq\frac{1}{4}$) and the upper bound of $\theta_1(b,\epsilon,1)$ can be achieved by an appropriate choice of λ_1. First note that $\frac{1}{2}(a_1+a_2)=\overline{a}(\lambda_1,\theta_1)$ is continuous in $\lambda_1\in[0,1]$ for all $\theta_1\in(0,1)$. So $\lim_{\lambda_1\to 0}\overline{a}(\lambda_1,\theta_1)=\overline{a}(0,\theta_1)$ and $\lim_{\lambda_1\to 1}\overline{a}(\lambda_1,\theta_1)=\overline{a}(1,\theta_1)$ for all $\theta_1\in(0,1)$, and hence $\lim_{\lambda_1\to 0}\theta_1(b,\epsilon,\lambda_1)=\theta_1(b,\epsilon,0)$ if $b<\frac{1}{4}$ and $\lim_{\lambda_1\to 1}\theta_1(b,\epsilon,\lambda_1)=\theta_1(b,\epsilon,1)$. It remains to show that $\lim_{\lambda_1\to 0}\theta_1(b,\epsilon,\lambda_1)=0$ if $b\geq\frac{1}{4}$. To see this, fix some $\hat{\theta}_1>0$, and notice that $\lim_{\lambda_1\to 0}\overline{a}(\lambda_1,\hat{\theta}_1)=\overline{a}(\lambda_1,0)$. Further,

$$\begin{aligned}\overline{a}(0,\hat{\theta}_1)&=\frac{1+\hat{\theta}_1-2(1-\epsilon)\hat{\theta}_1^2}{4(1-(1-\epsilon)\hat{\theta}_1)}\\&=\frac{1}{4}\frac{1-\hat{\theta}_1}{(1-(1-\epsilon)\hat{\theta}_1)}+\frac{\hat{\theta}_1}{2}\\&<\tfrac{1}{4}+\tfrac{1}{2}\hat{\theta}_1\\&<\hat{\theta}_1+b.\end{aligned}$$

It follows that for λ_1 sufficiently close to 0, the (unique) relevant solution to (3), $\theta_1(b,\epsilon,\lambda_1)$, is less than $\hat{\theta}_1$. Since $\hat{\theta}_1$ was chosen arbitrarily, we have the desired result.

Proof of Proposition 7. The proof is constructive. Consider an (infinite) partition of the type space of the form

$$\{\{0\},\ldots,[\theta_{-3},\theta_{-2}),[\theta_{-2},\theta_{-1}),[\theta_{-1},1]\},$$

where $\theta_{i-1}<\theta_i$ and $0<\theta_i<1$ for $i\leq -1$, and $\lim_{i\to-\infty}\theta_i=0$ (so the set does indeed partition $[0,1]$). Suppose that type $\theta=0$ sends message $m=0$, all types $\theta\in[\theta_{i-1},\theta_i)$ ($i\leq -1$) randomize uniformly over messages in $[\zeta\theta_{i-1},\zeta\theta_i)$, and types $\theta\in[\theta_{-1},1]$ randomize uniformly over messages in $[\zeta\theta_{-1},1]$ (ζ is a constant whose value will be determined later). Consider the receiver's best response to this strategy of the sender. Conditional on receiving a message $m\in[\zeta\theta_{i-1},\zeta\theta_i)$, the receiver's posterior belief that the message was received in error is given by

$$\eta=\frac{\epsilon\zeta(\theta_i-\theta_{i-1})}{(1-\epsilon)(\theta_i-\theta_{i-1})+\epsilon\zeta(\theta_i-\theta_{i-1})}=\frac{\epsilon\zeta}{(1-\epsilon)+\epsilon\zeta}.$$

Thus the receiver's optimal action, a_i, solves

$$\begin{aligned}&\max_a(1-\eta)\int_{\theta_{i-1}}^{\theta_i}-(\theta-a)^2\frac{1}{\theta_i-\theta_{i-1}}\,d\theta+\eta\int_0^1-(\theta-a)^2\,d\theta\\&\Rightarrow\quad a_i=(1-\eta)\tfrac{1}{2}(\theta_i+\theta_{i-1})+\eta\tfrac{1}{2}\ (\text{for } i\leq -1).\end{aligned}$$

On receiving a message $m\in[\zeta\theta_{-1},1]$, on the other hand, it is easy to verify that the receiver's optimal response is to choose the action

$$a_0=\frac{(1-\epsilon)(1-\theta_{-1})\frac{1}{2}(\theta_{-1}+1)+\epsilon(1-\zeta\theta_{-1})\frac{1}{2}}{(1-\epsilon)(1-\theta_{-1})+\epsilon(1-\zeta\theta_{-1})}.$$

Turning now to the sender's strategy, we need each sender boundary type θ_i to be indifferent between inducing action a_i and action a_{i+1}, i.e.

$$\theta_i + b = \tfrac{1}{2}(a_i + a_{i+1}) \text{ (for } i \leq -1).$$

Notice that each of these *indifference conditions* (except for θ_{-1}) involves θ_{i+1}, θ_i and θ_{i-1}. Solving for θ_{i+1}, we obtain the second-order difference equation

$$\theta_{i+1} = \frac{2+2\eta}{1-\eta}\theta_i - \theta_{i-1} + \frac{4b-2\eta}{1-\eta} \text{ (for } i \leq -2). \tag{4}$$

We have said nothing yet about the value of ζ. Let

$$\zeta = \frac{2b(1-\epsilon)}{\epsilon(1-2b)},$$

so that $\eta = 2b$ (notice that $\zeta > 0$ as long as $0 < b < \frac{1}{2}$). Then a solution to this difference equation is

$$\theta_{i-1} = \theta_{-1}\left(\frac{1-\sqrt{2b}}{1+\sqrt{2b}}\right)^{-i} \quad (i = \ldots, -2, -1, 0).$$

As long as we choose a value of $\theta_{-1} \in (0,1)$, we have $\theta_{i-1} < \theta_i$ and $0 < \theta_i < 1$ for $i \leq -1$, and $\lim_{i \to -\infty} \theta_i = 0$, as required.

The remaining indifference condition fixes the value of θ_{-1}:

$$\begin{aligned} \theta_{-1} + b &= \tfrac{1}{2}(a_{-1} + a_0) \\ \Rightarrow \quad \theta_{-1} &= \frac{1-\sqrt{2b}}{1+\sqrt{\epsilon}}. \end{aligned}$$

For this construction to work, we need to make sure that the sender's strategy described at the beginning of this section is well-defined, i.e. there are some messages left over for the final interval of sender types to send. This requires

$$\begin{aligned} \zeta\theta_{-1} &\leq 1 \\ \Rightarrow \quad \epsilon &\geq \frac{2b}{(1+\sqrt{2b})^2}. \end{aligned}$$

□

Note that a very similar construction can be used to demonstrate the existence of an N-step equilibrium for any *finite* $N \geq 3$ (Section 3.1 above deals with the case where $N = 2$). To see how, consider the N-step partition

$$\{[0, \theta_{1-N}), [\theta_{1-N}, \theta_{2-N}), \ldots, [\theta_{-2}, \theta_{-1}), [\theta_{-1}, 1]\}.$$

Suppose that the sender adopts the same strategy as before, with types $\theta \in [\theta_{i-1}, \theta_i)$ ($i \leq -1$) randomizing uniformly over messages in $[\zeta\theta_{i-1}, \zeta\theta_i)$ and types $\theta \in [\theta_{-1}, 1]$ randomizing uniformly over messages in $[\zeta\theta_{-1}, 1]$. The indifference conditions for boundary types $\theta_{1-N}, \ldots, \theta_{-2}$ yield the same second-order difference equation as above, (4).

Solving this equation with the boundary condition $\theta_{-N}=0$ and treating θ_{-1} as a parameter, we can compute a_{-1} (the action corresponding to the penultimate step) as a function of θ_{-1}. For any given θ_{-1}, the value of a_{-1} is lower than in the infinite case, since the steps are more spaced out. The final indifference condition, therefore, gives a lower value of θ_{-1}, so the threshold value of ϵ required for this construction to work is strictly lower than before.

Proof of Proposition 8. Consider the partition

$$\{\{\{\theta\}\}_{\theta\in[0,\theta^*]},\ldots,(\theta_{-3},\theta_{-2}],(\theta_{-2},\theta_{-1}],(\theta_{-1},1]\},$$

where $\theta_{i-1}<\theta_i$ and $0<\theta^*<\theta_i<\theta_0=1$ for $i\leq -1$, and $\lim_{i\to-\infty}\theta_i=\theta^*$. Suppose the sender adopts the strategy

- if $\theta\in[0,\theta^*]$, send message $m=s(\theta)$ where s is a strictly increasing differentiable function with $s(0)=0$
- if $\theta\in(\theta_{i-1},\theta_i]$ ($i\leq 0$), randomize uniformly over messages in $(\zeta(\theta_{i-1}-\theta^*)+s(\theta^*),\zeta(\theta_i-\theta^*)+s(\theta^*)]$, where $\zeta(1-\theta^*)+s(\theta^*)=1$.

(Note that we use intervals that are open on the left rather than on the right here merely to simplify the notation.) Now consider the receiver's best response. Suppose he receives a message $m\in[0,s(\theta^*)]$; the distribution of sent messages in this continuous portion of the message space is given by $s^{-1}(m)$ with density

$$\frac{1}{s'(s^{-1}(m))}.$$

Hence, conditional on receiving a message $m\in[0,s(\theta^*)]$, the posterior probability that m was received by error is given by[20]

$$\mu(m)\equiv\frac{\epsilon}{\epsilon+\frac{(1-\epsilon)}{s'(s^{-1}(m))}}.$$

The receiver then chooses the action a that maximizes

$$-(1-\mu(m))(a-s^{-1}(m))^2-\mu(m)\int_0^1(a-\theta)^2\,d\theta.$$

[20]Let $m\in[m',m'']\subset[0,s(\theta^*)]$ and recall that the error distribution is $G(m)=m$. Then the probability that a message was received in error, E, conditional on knowing that the message is in $[m',m'']$, equals

$$\begin{aligned}P(E\mid[m',m''])&=\frac{P([m',m'']\mid E))\epsilon}{P([m',m'']\mid E)\epsilon+P([m',m'']\mid\neg E)(1-\epsilon)}\\&=\frac{(m''-m')\epsilon}{(m''-m')\epsilon+(s^{-1}(m'')-s^{-1}(m'))(1-\epsilon)}\\&=\frac{\epsilon}{\epsilon+\frac{s^{-1}(m'')-s^{-1}(m')}{m''-m'}(1-\epsilon)}.\end{aligned}$$

Now consider the limit as $m''-m'\to 0$.

The maximum is achieved at

$$a_m = (1-\mu(m))s^{-1}(m) + \mu(m)\tfrac{1}{2}.$$

Let $\theta = s^{-1}(m)$. Clearly if the receiver's optimal response matches the sender's ideal point $(\theta + b)$, the sender has no incentive to deviate from the specified strategy (for $\theta \in [0, \theta^*]$).[21] Formally,

$$\begin{aligned} a_m &= \theta + b \\ \Rightarrow \quad (1-\mu(m))\theta + \mu(m)\tfrac{1}{2} &= \theta + b \\ \Rightarrow \quad \mu(m) &= \frac{2b}{1-2\theta}, \end{aligned}$$

which can be solved for

$$s'(\theta) = \frac{2b(1-\epsilon)}{\epsilon(1-2b-2\theta)}.$$

Using the boundary condition $s(0) = 0$, we obtain the sender's strategy for types $\theta \in [0, \theta^*]$:

$$s(\theta) = -\frac{b(1-\epsilon)}{\epsilon} \ln\left(\frac{1-2b-2\theta}{1-2b}\right).$$

Next, suppose that the message received is in the interval $(\zeta\theta_{i-1} + s(\theta^*), \zeta\theta_i + s(\theta^*)]$ $(i \le 0)$; then the receiver's optimal action is given by

$$a_i = (1-\eta)\tfrac{1}{2}(\theta_i + \theta_{i-1}) + \eta\tfrac{1}{2},$$

where η is as defined in the proof of Proposition 7. As before, it follows that boundary types must satisfy the difference equation

$$\theta_{i+1} = \frac{2+2\eta}{1-\eta}\theta_i - \theta_{i-1} + \frac{4b-2\eta}{1-\eta} \text{ (for } i \le -1\text{).}$$

We need a solution of this difference equation that converges to θ^*—this ensures that the required indifference condition for the boundary type at θ^* is satisfied. Hence θ^* must satisfy

$$\begin{aligned} \theta^* &= \frac{2+2\eta}{1-\eta}\theta^* - \theta^* + \frac{4b-2\eta}{1-\eta} \\ \Rightarrow \quad \theta^* &= \frac{\eta-2b}{2\eta} \\ \Rightarrow \quad \theta^* &= \frac{1}{2} - \frac{b(1-\epsilon(1-\zeta))}{\epsilon\zeta}. \end{aligned}$$

[21] In fact, it follows from the proof of Proposition 6 that, in any equilibrium, almost all types that fully reveal themselves must induce their ideal actions; thus there is no equilibrium in which the highest types adopt a separating strategy.

Solving for ζ, we obtain

$$\zeta = \frac{2b(1-\epsilon)}{\epsilon(1-2\theta^*-2b)},$$

so

$$\eta = \frac{2b}{1-2\theta^*}.$$

The difference equation becomes

$$\theta_{i+1} - \frac{2-4\theta^*+4b}{1-2\theta^*-2b}\theta_i + \theta_{i-1} = -\frac{8\theta^* b}{1-2\theta^*-2b}.$$

With the boundary constraint $\theta_0 = 1$, the solution is

$$\theta_i = (1-\theta^*)\left(\frac{1-2\theta^*+2b-\sqrt{4b(2-4\theta^*)}}{1-2\theta^*-2b}\right)^{-i} + \theta^*.$$

Finally, recall that we require that

$$\zeta(1-\theta^*) + s(\theta^*) = 1,$$

so all messages are used in equilibrium. Can we find a θ^* that solves this equation? Notice that $s(0)=0$, and $s'(\theta^*)$ is strictly increasing for $\theta^* \in [0, \frac{1}{2}-b)$ with $\lim_{\theta^*\to\frac{1}{2}-b} = \infty$. Further, $\zeta(1-\theta^*)$ is increasing in θ^* (as long as $b < \frac{1}{2}$). So, by continuity, we can find a solution to this equation if and only if $\zeta(1-\theta^*) < 1$ when $\theta^* = 0$. This implies that $\epsilon > 2b$. □

Calculation of EU^R for three-step equilibrium in Example 1 With an error of $\epsilon = \frac{1}{126}$, we showed that there is an equilibrium partition with elements $[0, \frac{1}{25})$, $[\frac{1}{25}, \frac{8}{25})$, and $[\frac{8}{25}, 1]$. In the event of no error, these elements induce the actions $a_1 = \frac{1}{10}$, $a_2 = \frac{9}{50}$, and $a_3 = \frac{33}{50}$ respectively. The expected payoff of the receiver is given by

$$\begin{aligned} EU^R &= (1-\epsilon)\left(\int_0^{\frac{1}{25}} -(\theta-a_1)^2\,d\theta + \int_{\frac{1}{25}}^{\frac{8}{25}} -(\theta-a_2)^2\,d\theta + \int_{\frac{8}{25}}^{1} -(\theta-a_3)^2\,d\theta\right) \\ &\qquad + \epsilon\left(\int_0^1 -(\theta-a_1)^2\,d\theta\right) \\ &= -\tfrac{36}{1200}. \end{aligned}$$

Proof of Proposition 9. Suppose that

$$\frac{1}{2N^2} < b < \frac{1}{2(N-1)^2}$$

for some integer $N > 1$. We show there exists $\overline{\epsilon} > 0$ such that for all $\epsilon \in (0, \overline{\epsilon})$, there is an N-step equilibrium of the noise model that Pareto dominates the Pareto optimal equilibrium of the CS model.[22]

[22] As an aside, it is worth noting that as ϵ tends to 0, the equilibrium constructed here tends to the most informative equilibrium of the CS model.

Let the probability of error be ϵ. Consider the partition $\{[0,\theta_1),\ldots,[\theta_{N-1},1]\}$, and suppose the sender obeys the strategy

$$\begin{aligned}
&\text{if } \theta\in[0,\theta_1], \text{ randomize uniformly on } [0,1]\setminus\{m_2,\ldots,m_N\};\\
&\text{if } \theta\in(\theta_1,\theta_2], \text{ send message } m_2;\\
&\quad\vdots\\
&\text{if } \theta\in(\theta_{N-1},1], \text{ send message } m_N.
\end{aligned}$$

The actions chosen in each step are, respectively,

$$\begin{aligned}
&\text{if } m\in[0,1]\setminus\{m_2,m_3\} \text{ is received, choose } a_1=\frac{(1-\epsilon)\theta_1\frac{1}{2}\theta_1+\epsilon\frac{1}{2}}{(1-\epsilon)\theta_1+\epsilon};\\
&\text{if } m=m_2 \text{ is received, choose } a_2=\tfrac{1}{2}(\theta_1+\theta_2);\\
&\quad\vdots\\
&\text{if } m=m_N \text{ is received, choose } a_N=\tfrac{1}{2}(\theta_{N-1}+1).
\end{aligned}$$

Solving the indifference conditions

$$\begin{aligned}
\theta_1+b&=\tfrac{1}{2}(a_1+a_2)\\
\theta_2+b&=\tfrac{1}{2}(a_2+a_3)\\
&\vdots\\
\theta_{N-1}+b&=\tfrac{1}{2}(a_{N-1}+a_N)
\end{aligned}$$

gives

$$\theta_2=\frac{2\theta_1(2b+\theta_1)+\epsilon(\theta_1-1)(4b+2\theta_1-1)}{\epsilon(\theta_1-1)+\theta_1} \tag{5}$$

$$\theta_3-\theta_2=\theta_2-\theta_1+4b \tag{6}$$

$$\vdots$$

$$1-\theta_{N-1}=\theta_{N-1}-\theta_{N-2}+4b. \tag{7}$$

Now observe that

$$\sum_{i=1}^{N-1}(\theta_{i+1}-\theta_i)=1-\theta_1 \tag{8}$$

and combining (6)–(7) we obtain

$$\sum_{i=1}^{N-1}(\theta_{i+1}-\theta_i)=\tfrac{1}{2}(N-1)(N-2)4b+(N-1)(\theta_2-\theta_1). \tag{9}$$

Equations (8) and (9) give us

$$1-\theta_1 = \tfrac{1}{2}N(N-1)4b + (N-1)(\theta_2-\theta_1)$$
$$\Rightarrow \quad \theta_2-\theta_1 = \frac{1-\theta_1}{N-1} - 2b(N-2). \tag{10}$$

Finally, solving (5) and (10) for θ_1, we find

$$\begin{aligned}\theta_1 &= \frac{1+2bN-2\epsilon N-2b\epsilon N-2bN^2+2b\epsilon N^2}{2(1-\epsilon)N}\\ &\quad + \frac{\sqrt{4\epsilon(\epsilon-1)(-1+2b(N-1))N^2+(1-2\epsilon N+2b(\epsilon-1)(N-1)N)^2}}{2(1-\epsilon)N}.\end{aligned} \tag{11}$$

Equation (5) gives us the position of the first boundary point, θ_1, in an N-step equilibrium of the noise model. Solving for ϵ, we can find the level of noise required to sustain a particular equilibrium value of θ_1:

$$\epsilon = \frac{\theta_1(-1-2bN+2bN^2+N\theta_1)}{N(1-\theta_1)(1+2b-2bN-\theta_1)}. \tag{12}$$

The expected payoff of receiver is given by

$$\begin{aligned} EU^R_{noise} &= (1-\epsilon)\left(\int_0^{\theta_1} -(\theta-a_1)^2\,d\theta + \int_{\theta_1}^{\theta_2} -(\theta-a_2)^2\,d\theta + \cdots + \int_{\theta_{N-1}}^{1} -(\theta-a_N)^2\,d\theta\right)\\ &\qquad + \epsilon\left(\int_0^1 -(\theta-a_1)^2\,d\theta\right)\\ &= (1-\epsilon)\left(-\frac{\theta_1(-2\epsilon(\theta_1-1)\theta_1^3+\theta_1^4+\epsilon^2(\theta_1-1)^2(3+\theta_1^2))}{12(\epsilon+\theta_1-\epsilon\theta_1)^2} - \frac{1}{12}\sum_{i=2}^{N}(\theta_i-\theta_{i-1})^3\right)\\ &\qquad + \epsilon\left(-\frac{\epsilon^2-2(\epsilon-1)\epsilon\theta_1+4(\epsilon-1)^2\theta_1^2-6(\epsilon-1)^2\theta_1^3+3(\epsilon-1)^2\theta_1^4}{12(\epsilon+\theta_1-\epsilon\theta_1)^2}\right).\end{aligned}$$

Solving and substituting for ϵ using (12), we can re-write the expected payoff of the receiver in terms of θ_1:

$$EU^R_{noise} = -\frac{4b^2(N-2)(N-1)^2N+4b(N-1)^2(2N-1)\theta_1+((2N-1)\theta_1-1)^2}{12(N-1)^2}.$$

To see that this equilibrium Pareto dominates the Pareto optimal equilibrium of the CS model for small ϵ, we consider two cases.

Case 1: $1/[2N(N-1)] \leqslant b < 1/[2(N-1)^2]$. The Pareto optimal equilibrium of the CS model has $N-1$ steps, with resulting expected payoff

$$EU^R_{CS} = -\frac{1}{12(N-1)^2} - \frac{b^2((N-1)^2-1)}{3}.$$

Notice that this equilibrium coincides precisely with the construction above when $\epsilon = 0$, so that $\theta_1 = 0$. By introducing a small amount of noise, we are able to squeeze an extra step into the equilibrium partition. We now compute the difference in the receiver's expected payoff in the two types of equilibrium.

$$\begin{aligned} EU^R_{noise} - EU^R_{CS} \\ &= -\frac{4b^2(N-2)(N-1)^2N + 4b(N-1)^2(2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2} \\ &\quad - \left(-\frac{1}{12(N-1)^2} - \frac{b^2((N-1)^2-1)}{3}\right) \\ &= -\frac{(2N-1)\theta_1(4b(N-1)^2 - 2 + (2N-1)\theta_1)}{12(N-1)^2} \\ &> 0 \text{ for } \theta_1 \in \left(0, \frac{2-4b(N-1)^2}{2N-1}\right) \end{aligned}$$

Substituting for θ_1, we obtain $EU^R_{noise} - EU^R_{CS} > 0$ if

$$\epsilon \in \left(0, \frac{2(1-2b(N-1)^2)(1+2b(N-1)N)}{((2N-3)^2 + 2b(2N-3)^2(N-1) - 8b^2(N-1)^3)N}\right)$$

To see that the upper bound of this interval is strictly positive, we show that both numerator and denominator are strictly positive. Consider first the numerator. Clearly $(1+2b(N-1)N) > 0$, and since $b < 1/[2(N-1)^2]$, we also have $(1-2b(N-1)^2) > 0$ as required. For the denominator, suppose first that $N = 2$ (recall that N is an integer greater than 1). Then the denominator simplifies to $2(1-2b(4b-1)) > 0$ since $b < \frac{1}{2}$. Now suppose $N \geq 3$. We can rewrite the denominator as follows:

$$\begin{aligned} N((2N-3)^2 &+ 2b(2N-3)^2(N-1) - 4b(N-1)2b(N-1)^2) \\ &> N((2N-3)^2 + 2b(2N-3)^2(N-1) - 4b(N-1)) \\ &= N((2N-3)^2 + 2b((2N-3)^2 - 2)(N-1)) \\ &> 0 \text{ as required.} \end{aligned}$$

Case 2: $1/[2N^2] < b < 1/[2N(N-1)]$. In this case, the most informative equilibrium of the CS model has N steps, with resulting expected payoff

$$EU^R_{CS} = -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3}.$$

This equilibrium coincides precisely with the construction above when $\epsilon = 0$ and $\theta_1 = (1-2b(N-1)N)/N$. By introducing a small amount of noise, we increase the size of the first (and smallest) element of the equilibrium partition. As before, we compute $EU^R_{noise} - EU^R_{CS}$.

$$
\begin{aligned}
&EU^R_{noise} - EU^R_{CS}\\
&\quad = -\frac{4b^2(N-2)(N-1)^2N + 4b(N-1)^2(2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2}\\
&\qquad\qquad - \left(-\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3}\right)\\
&\quad = \frac{(2N-1)(-1+4b^2(N-1)^2N^2 - 4b(N-1)^2N^2\theta_1 - 2N^3\theta_1^2 + N^2\theta_1(2+\theta_1))}{12(N-1)^2N^2}\\
&\quad > 0 \text{ for } \theta_1 \in \left(\frac{1-2bN(N-1)}{N}, \frac{1+2bN(N-1)}{N(2N-1)}\right).
\end{aligned}
$$

Substituting for θ_1, we obtain $EU^R_{noise} - EU^R_{CS} > 0$ if

$$
\epsilon \in \left(0, \frac{2(1+2b(N-1)N)(2bN^2-1)}{(N-1)(1+2(1-b)N)(1+2N-4bN^2)}\right).
$$

It is easy to see that the upper bound is strictly positive, completing the proof. □

Proof of Lemma 1. Suppose that the N-step equilibrium of the noise model has an equilibrium partition given by $\{[0,\theta_1), [\theta_1,\theta_2), \dots, [\theta_{i-1},\theta_i), \dots, [\theta_{N-1},1]\}$. In the model with correlated noise, consider an N-step partition $\{[0,\theta'_1), [\theta'_1,\theta'_2), \dots, [\theta'_{i-1},\theta'_i), \dots, [\theta'_{N-1},1]\}$. Let $m_2, m_3, \dots, m_N$ be a sequence of messages, with $m_i \neq m_{i'}$ for all $i \neq i'$ and $\mathfrak{e}(m_i) \geq \mathfrak{e}(m_{i+1})$ for all $i = 2, \dots, N-1$. Define $M^* \equiv \cup_{i=2}^N \{m_i\}$. The sender adopts the following strategy:

if $\theta \in [0,\theta'_1)$, randomize over $M \setminus M^*$ with a distribution that has density ϕ;
if $\theta \in [\theta'_1,\theta'_2)$, send message m_2;
⋮
if $\theta \in [\theta'_{N-1},1]$, send message m_N.

(We define ϕ shortly.) The receiver's posterior probability that the sender's type is in $[0,\theta'_1)$ conditional on receiving a message $m \in M \setminus M^*$ equals

$$
P(\theta \in [0,\theta'_1) \mid m) = \frac{((1-\mathfrak{e}(m))\phi(m) + \int_0^1 \mathfrak{e}(\lambda)\phi(\lambda)d\lambda)\theta'_1}{((1-\mathfrak{e}(m))\phi(m) + \int_0^1 \mathfrak{e}(\lambda)\phi(\lambda)\,d\lambda)\theta'_1 + \sum_{j=2}^N \mathfrak{e}(m_j)(\theta'_j - \theta'_{j-1})}.
$$

And the receiver's posterior probability that the sender's type is in $[\theta'_{i-1},\theta'_i)$ conditional on receiving a message $m \in M \setminus M^*$ equals

$$
P(\theta \in [\theta'_{i-1},\theta'_i) \mid m) = \frac{\mathfrak{e}(m_j)(\theta'_i - \theta'_{i-1})}{((1-\mathfrak{e}(m))\phi(m) + \int_0^1 \mathfrak{e}(\lambda)\phi(\lambda)\,d\lambda)\theta'_1 + \sum_{j=2}^N \mathfrak{e}(m_j)(\theta'_j - \theta'_{j-1})}.
$$

Notice that if we can ensure that these posteriors do not vary with $m \in M \setminus M^*$, then the sender is indifferent among all messages in this set. This condition in turn is satisfied if there is a constant c such that

$$(1-\mathfrak{e}(m))\phi(m)+\int_0^1 \mathfrak{e}(\lambda)\phi(\lambda)\,d\lambda = c \tag{13}$$

for all $m \in M \setminus M^*$. Integrating (13) with respect to m shows that we must have $c = 1$. The resulting integral equation is solved by the function ϕ that is defined by

$$\phi(m)=\frac{1}{\int_0^1 \frac{1-\epsilon(m)}{1-\epsilon(v)}\,dv}.$$

This implies that the receiver's posteriors do not depend on the entire shape of the error function $\mathfrak{e}$, but only on the specific values $\mathfrak{e}(m_i)$ for $i = 2,\dots,N$. (The resulting posteriors are in fact the same as in a model where messages may get lost with probabilities $\mathfrak{e}(m_i)$, $i = 2,\dots,N$ that depend on the messages sent, and the lowest interval of types refrains from sending a message.)

For all $m \in M \setminus M^*$ the receiver's best response is given by

$$\begin{aligned} a_1' &= P(\theta \in [0,\theta_1') \mid m)\frac{\theta_1'}{2}+\sum_{i=2}^{N} P(\theta \in [\theta_{i-1}',\theta_i') \mid m)\tfrac{1}{2}(\theta_{i-1}'+\theta_i') \\ &= \frac{\theta_1'}{\theta_1'+\sum_{j=2}^{N}\mathfrak{e}(m_j)(\theta_j'-\theta_{j-1}')}\frac{\theta_1'}{2}+\sum_{i=2}^{N}\frac{\mathfrak{e}(m_i)(\theta_i'-\theta_{i-1}')}{\theta_1'+\sum_{j=2}^{N}\mathfrak{e}(m_j)(\theta_j'-\theta_{j-1}')}\frac{\theta_{i-1}'+\theta_i'}{2} \end{aligned}$$

(where $\theta_N' = 1$), and for $m_i \in M^*$ the receiver's best response is

$$a_i' = \tfrac{1}{2}(\theta_{i-1}'+\theta_i').$$

Then type θ's payoff from sending message $m_i \in M^*$ equals

$$EU^S(\theta,m_i) \equiv -(1-\mathfrak{e}(m_i))(\theta+b-a_i')^2-\mathfrak{e}(m_i)(\theta+b-a_1')^2.$$

The payoff from sending some message $m_1 \in M \setminus M^*$ equals

$$EU^S(\theta,m_1) \equiv -(\theta+b-a_1')^2.$$

(Henceforth, we use m_1 to denote a generic message in $M\setminus M^*$.) Suppose that $a_2' > a_1'$ (it follows from the expression for a_i' above that $a_{i+1}' > a_i'$ for $i = 2,\dots,N-1$). Since we have assumed $\epsilon(m_i) \geq \epsilon(m_{i+1})$, if any type θ is indifferent between sending messages m_i and m_{i+1}, then any type $\theta' > \theta$ strictly prefers m_{i+1} to m_i and any type $\theta'' < \theta$ strictly prefers m_i to m_{i+1}. Intuitively, sending message m_i increases the risk of the "bad" action a_1', so for type θ to be indifferent between m_i and m_{i+1}, she must prefer action a_i' to action a_{i+1}'; for a higher type θ' this preference is weaker (and is reversed eventually), while the payoff loss from action a_1' is higher, so m_{i+1} is strictly preferred. For lower types θ'', on the other hand, the preference for a_i' over a_{i+1}' is stronger, while the payoff loss from a_1' is lower, so m_i is strictly preferred.

Therefore, if our partition is chosen such that $a_2' > a_1'$ and each boundary type θ_i is indifferent between messages m_i and m_{i+1} ($i = 1,\ldots,N-1$), we have an equilibrium. The indifference conditions for these boundary types are given by

$$EU^S(\theta_1', m_1) = EU^S(\theta_1', m_2)$$
$$\vdots$$
$$EU^S(\theta_{N-1}', m_{N-1}) = EU^S(\theta_{N-1}', m_N).$$

In the case where there are at least $N-1$ messages that yield equal error probabilities of exactly ϵ, the result follows easily. Substituting ϵ for $\mathfrak{e}(m_i)$, $i = 2,\ldots,N$, we see that these indifference conditions are identical to the corresponding conditions in the noise model; thus $\{[0,\theta_1),,\ldots,[\theta_{N-1},1]\}$ is also an equilibrium of the correlated-noise model, with the same induced actions $a_1,\ldots,a_N$.

If we cannot find $N-1$ such messages, a more intricate argument is needed. Redefine a_1' for arbitrary $(\epsilon_2,\ldots,\epsilon_N)$ as

$$a_1' \equiv \frac{\theta_1'}{\theta_1' + \sum_{j=2}^N \epsilon_j(\theta_j' - \theta_{j-1}')}\frac{\theta_1'}{2} + \sum_{i=2}^N \frac{\epsilon_i(\theta_i' - \theta_{i-1}')}{\theta_1' + \sum_{j=2}^N \epsilon_j(\theta_j' - \theta_{j-1}')}\frac{\theta_{i-1}' + \theta_i'}{2}.$$

Define

$$U^{S,i}(\theta;\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_N') \equiv -(1-\epsilon_i)(\theta + b - a_i')^2 - \epsilon_i(\theta + b - a_1')^2.$$

Define

$$\begin{aligned} &V^{S,i-1,i}(\theta_{i-1}';\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_N') \\ &\qquad \equiv U^{S,i-1}(\theta_{i-1}';\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_N') - U^{S,i}(\theta_{i-1}';\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_N'). \end{aligned}$$

Note that for $N > 2$, $V^{S,N-1,N}$ is a continuously differentiable function in an open neighborhood of $\epsilon_2 = \cdots = \epsilon_N = \epsilon$ and $\theta_i' = \theta_i, i = 1,\ldots,N$. Also, at $\epsilon_2 = \cdots = \epsilon_N = \epsilon$ and $\theta_i' = \theta_i$, $i = 1,\ldots,N$, the derivative with respect to θ_N' equals $-(1-\epsilon)(\theta_{N-1} + b - \frac{1}{2}(\theta_{N-1} + \theta_N))$, which is strictly positive. Therefore, by the Implicit Function Theorem, the equation

$$V^{S,N-1,N}(\theta_{N-1}';\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_N') = 0$$

has a local solution

$$\theta_N' = f_N(\epsilon_2,\ldots,\epsilon_N;\theta_1',\ldots,\theta_{N-1}'),$$

where f_N is continuously differentiable.

For $N-1 > 2$, substitute this solution into $V^{S,N-2,N-1}$. The resulting function is continuously differentiable in an open neighborhood of $\epsilon_2 = \cdots = \epsilon_N = \epsilon$ and $\theta_i' = \theta_i$, $i = 1,\ldots,N-1$. Its derivative at $\epsilon_2 = \cdots = \epsilon_N = \epsilon$ and $\theta_i' = \theta_i, i = 1,\ldots,N-1$ with respect to θ'_{N-1} is strictly positive and once again the Implicit Function Theorem guarantees the existence of a local solution

$$\theta'_{N-1} = f_{N-1}(\epsilon_2,\ldots,\epsilon_N;\theta'_1,\ldots,\theta'_{N-2}),$$

where the function f_{N-1} is continuously differentiable.

As long as $i>2$, we can recursively continue this procedure with $V^{S,i-1,i}$. The case of $i=2$ requires slightly more attention. Let $\boldsymbol{\epsilon}$ denote $(\epsilon,\ldots,\epsilon)$. We need to determine

$$\begin{aligned}\frac{\partial}{\partial\theta'_2}&V^{S,1,2}(\theta_1;\boldsymbol{\epsilon};\theta_1,\theta_2,f_3(\boldsymbol{\epsilon},\theta_1,\theta_2),f_4(\boldsymbol{\epsilon},\theta_1,\theta_2,f_3(\boldsymbol{\epsilon},\theta_1,\theta_2)),\ldots)\\ &=\frac{\partial}{\partial\theta'_2}(1-\epsilon)\{-(\theta_1+b-a'_1)^2+(\theta_1+b-a'_2)^2\}\\ &=-(1-\epsilon)\Big\{-2(\theta_1+b-a'_1)\frac{\partial a'_1}{\partial\theta'_2}+\big(\theta_1+b-\tfrac{1}{2}(\theta_1+\theta_2)\big)^2\Big\}.\end{aligned}$$

The boundary values $\theta'_1,\theta'_2,\ldots,\theta'_N$ (notice that θ'_0 is excluded) satisfy the difference equation

$$\theta'_{i+1}=2\theta'_i-\theta'_{i-1}+4b.$$

With θ'_1 and θ'_2 as parameters, the solution becomes

$$\theta'_i=2\theta'_1-\theta'_2+4b+(\theta'_2-\theta'_1-6b)i+2bi^2.$$

Hence, $\partial\theta'_N/\partial\theta'_2=N-1$. Therefore,

$$\frac{\partial a'_1}{\partial\theta'_2}=\frac{\partial a'_1}{\partial\theta'_N}\frac{\partial\theta'_N}{\partial\theta'_2}=\frac{\partial a'_1}{\partial\theta'_N}(N-1).$$

Finally,

$$\frac{\partial a'_1}{\partial\theta'_N}=\frac{1}{2}\left\{\frac{-\epsilon\theta_1^2}{(\theta_1+\epsilon(1-\theta_1))^2}+\frac{2\epsilon(\theta_1+\epsilon(1-\theta_1))-\epsilon^2(1-\theta_1^2))}{(\theta_1+\epsilon(1-\theta_1))^2}\right\}>0.$$

Therefore

$$\frac{\partial}{\partial\theta'_2}V^{S,1,2}(\theta_1;\boldsymbol{\epsilon};\theta_1,\theta_2,f_3(\boldsymbol{\epsilon},\theta_1,\theta_2),f_4(\boldsymbol{\epsilon},\theta_1,\theta_2,f_3(\boldsymbol{\epsilon},\theta_1,\theta_2)),\ldots)>0.$$

This means that our recursion extends to $V^{S,1,2}$. Let $\vec{\epsilon}=(\epsilon_2,\ldots,\epsilon_N)$. We have shown that we can generate a series of functions $f_2,\ldots,f_N$ that are locally continuously differentiable in their arguments and that have the property that for θ'_1 in an open neighborhood of θ_1 and $\vec{\epsilon}$ in an open neighborhood $\mathcal{O}(\boldsymbol{\epsilon})$ of $\boldsymbol{\epsilon}$ the values

$$\begin{aligned}&\theta'_1\\ &\theta'_2=f_2(\vec{\epsilon};\theta'_1)\\ &\theta'_3=f_3(\vec{\epsilon};\theta'_1,f_2(\vec{\epsilon};\theta'_1))\\ &\quad\vdots\\ &\theta'_N=f_N(\vec{\epsilon};\theta'_1,f_2(\vec{\epsilon};\theta'_1),\ldots,f_{N-1}(\vec{\epsilon};\theta'_1,\ldots,f_{N-2}(\vec{\epsilon};\theta'_1,\ldots)))\end{aligned}$$

solve the system of equations

$$V^{S,i-1,i}(\theta'_{i-1};\vec{\epsilon};\theta'_1,\ldots,\theta'_N)=0,\ i=2,\ldots,N.$$

Notice that since $\mathfrak{e}$ is continuous, regardless of how small we choose $\mathcal{O}(\boldsymbol{\epsilon})$, we can find messages m_i, $i=2,\ldots,N$ with $m_i \neq m_j$ for $i \neq j$, $(\mathfrak{e}(m_2),\ldots,\mathfrak{e}(m_N) \in \mathcal{O}(\boldsymbol{\epsilon})$, and $\mathfrak{e}(m_i) \geq \mathfrak{e}(m_{i+1})$. In the sequel consider values of $\vec{\epsilon}$ with $\vec{\epsilon}=(\mathfrak{e}(m_2),\ldots,\mathfrak{e}(m_N))$.

Then, for equilibrium it remains to find a value of θ'_1 such that

$$f_N(\vec{\epsilon};\theta'_1,f_2(\vec{\epsilon};\theta'_1),\ldots,f_{N-1}(\vec{\epsilon};\theta'_1,\ldots,f_{N-2}(\vec{\epsilon};\theta'_1,\ldots)))=1.$$

To see that such a θ'_1 exists, first note that, evaluated at $\boldsymbol{\epsilon}$, the indifference conditions for boundary types $\theta_2,\ldots,\theta_N$ are solved by

$$\begin{aligned}
f_2(\boldsymbol{\epsilon};\theta_1) &= \frac{2\theta_1(2b+\theta_1)+\epsilon(\theta_1-1)(4b+2\theta_1-1)}{\epsilon(\theta_1-1)+\theta_1}\\
f_3(\boldsymbol{\epsilon};\theta_1,\theta_2) &= 2\theta_2-\theta_1+4b\\
&\vdots\\
f_N(\boldsymbol{\epsilon};\theta_1,\ldots,\theta_N) &= 2\theta_{N-1}-\theta_{N-2}+4b.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{df_2}{d\theta_1}(\boldsymbol{\epsilon};\theta_1) &= 2+\frac{\epsilon}{(\epsilon+\theta_1-\epsilon\theta_1)^2}\\
\frac{df_3}{d\theta_1}(\boldsymbol{\epsilon};\theta_1,\theta_2) &= \frac{\partial f_3}{\partial\theta_2}\frac{df_2}{d\theta_1}+\frac{\partial f_3}{\partial\theta_1}=2\left(2+\frac{\epsilon}{(\epsilon+\theta_1-\epsilon\theta_1)^2}\right)-1\\
&= 3+\frac{2\epsilon}{(\epsilon+\theta_1-\epsilon\theta_1)^2}\\
&\vdots\\
\frac{df_N}{d\theta_1}(\boldsymbol{\epsilon};\theta_1,\ldots,\theta_N) &= N+(N-1)\frac{\epsilon}{(\epsilon+\theta_1-\epsilon\theta_1)^2}.
\end{aligned}$$

In particular, observe that $(df_N/d\theta_1)(\epsilon;\theta_1,\ldots,\theta_N)>0$. It follows that for sufficiently small $\beta>0$ and $\vec{\epsilon}$ in a sufficiently small open neighborhood of $\boldsymbol{\epsilon}$,

$$\begin{aligned}
&f_N(\vec{\epsilon};\theta_1-\beta,f_2(\vec{\epsilon};\theta_1-\beta),\ldots,f_{N-1}(\vec{\epsilon};\theta_1-\beta,\ldots,f_{N-2}(\vec{\epsilon};\theta_1-\beta,\ldots)))<1\\
&f_N(\vec{\epsilon};\theta_1+\beta,f_2(\vec{\epsilon};\theta_1+\beta),\ldots,f_{N-1}(\vec{\epsilon};\theta_1+\beta,\ldots,f_{N-2}(\vec{\epsilon};\theta_1+\beta,\ldots)))>1.
\end{aligned}$$

The existence of the required value of θ'_1 follows from the Intermediate Value Theorem.

To complete the proof, observe that $f_2(\boldsymbol{\epsilon};\theta_1)=\theta_2$, $f_3(\boldsymbol{\epsilon};\theta_1,\theta_2)=\theta_3,\ldots$, and each $f_i(\vec{\epsilon};\ldots)$ converges to $f_i(\boldsymbol{\epsilon};\ldots)$ as $\vec{\epsilon}\to\boldsymbol{\epsilon}$. □

References

Aumann, Robert J. and Sergiu Hart (2003), "Long cheap talk." *Econometrica*, 71, 1619–1660. [397]

Austen-Smith, David (1994), "Strategic transmission of costly information." *Econometrica*, 62, 955–963. [412]

Chen, Ying (2006), "Perturbed communication games with honest senders and naive receivers." Unpublished paper, Arizona State University. [398]

Crawford, Vincent P. and Joel Sobel (1982), "Strategic information transmission." *Econometrica*, 50, 1431–1451. [396, 402, 403, 412]

Forges, Françoise (1986), "An approach to communication equilibria." *Econometrica*, 54, 1375–1385. [397]

Ganguly, Chirantan and Indrajit Ray (2007), "On mediated equilibria of cheap-talk games." Unpublished paper, University of Birmingham. [397]

Goltsman, Maria, Johannes Hörner, Gregory Pavlov, and Francesco Squintani (2007), "Mediation, arbitration and negotiation." Unpublished paper, University of Western Ontario. [397, 410, 414]

Gordon, Sidartha (2007), "Informative cheap talk equilibria as fixed points." Unpublished paper, Université de Montréal. [406, 417]

Kartik, Navin (2007), "Information transmission with almost-cheap talk." Unpublished paper, University of California, San Diego. [398]

Kartik, Navin, Marco Ottaviani, and Francesco Squintani (2007), "Credulity, lies, and costly talk." *Journal of Economic Theory*, 134, 93–116. [398]

Kováč, Eugen and Tymofiy Mylovanov (2006), "Stochastic mechanisms in settings without monetary transfers: Regular case." Unpublished paper, University of Bonn. [397]

Krishna, Vijay and John Morgan (2004), "The art of conversation: Eliciting information from experts through multi-stage communication." *Journal of Economic Theory*, 117, 147–179. [397, 412, 414]

Morgan, John and Phillip C. Stocken (2003), "An analysis of stock recommendations." *RAND Journal of Economics*, 34, 183–203. [399]

Myerson, Roger B. (1986), "Multistage games with communication." *Econometrica*, 54, 323–358. [397]

Myerson, Roger B. (1991), *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts. [397]

Olszewski, Wojciech (2004), "Informal communication." *Journal of Economic Theory*, 117, 180–200. [399]

Shannon, Claude E. (1948), "A mathematical theory of communication." *Bell System Technical Journal*, 27, 379–423 and 623–656. [396]

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