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The use of parabolic arcs in matching curved boundaries by point transformations for some higher order triangular elements

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ABSTRACT

This paper is concerned with curved boundary triangular elements having one curved side and two straight sides. The curved elements considered here are the 6-node (quadratic), 10-node (cubic), 15-node (quartic) and 21-node (quintic) triangular elements. On using the isoparametric coordinate transformation, these curved triangles in the global (x, y) coordinate system are mapped into a standard triangle: $\{(\xi,\eta)/0 \leqslant \xi, \eta \leqslant 1, \xi+\eta \leqslant 1\}$ in the local coordinate system (ξ, η). Under this transformation curved boundary of these triangular elements is implicitly replaced by quadratic, cubic, quartic and quintic arcs. The equations of these arcs involve parameters, which are the coordinates of points on the curved side. This paper deduces relations for choosing the parameters in quartic and quintic arcs in such a way that each arc is always a parabola which passes through four points of the original curve, thus ensuring a good approximation. The point transformations which are thus determined with the above choice of parameters on the curved boundary and also in turn the other parameters in the interior of curved triangles will serve as a powerful subparametric coordinate transformation for higher order curved triangular elements with one curved side and two straight sides.

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1. Introduction

The finite element method applied to problems involving a enclosed region R^2, elements with straight sides, usually triangles or quadrilaterals are perfectly satisfactory, if the original domain has a polygonal boundary and suitable basis functions defined on these elements are easy to construct. However, when the problem domain is curved, elements with at least one curved side are desirable. This is also the case when curved material interfaces are present in the region. The curved element was introduced into structural analysis by Ergatoudis et al. [1] and reference to it can be found in [2–6]. Mitchell [7] describes three approaches to this problem. One of these involves a transformation of the entire domain onto some standard shape and hence is really a global method as opposed to the standard finite element approach which is local. The other two methods, the isoparametric method and the direct method are local in nature. In the direct method, the basis functions are constructed to match the curved boundaries and integrations are carried out directly in the original plane. This method has been developed with some success by Wachpress [8–10] and Mcleod and Mitchell [11] for triangular elements. The main difficulty with this procedure is that the basis functions in the triangles adjacent to the curved boundary are, in all but a few special cases, no longer polynomials and so the numerical work in these triangles is correspondingly more involved. The major disadvantage of these methods lies in the fact that the basis functions are usually rational functions making the integrations much more difficult. The direct methods have the advantage of being able to match curves more accurately than isoparametric methods. The isoparametric method has advantage of simplicity in defining of transformation and in the fact that the basis functions are polynomials which make the numerical integration easier. In the isoparametric method a triangle with one curved side and two straight sides in global (x, y) space is mapped into a standard triangle i.e. $\{(\xi,\eta)/0 \leqslant \xi, \eta \leqslant 1, \xi+\eta \leqslant 1\}$ in the local parametric space (ξ, η). When the isoparametric coordinates are used to deal with curved boundaries in the finite element method, the original boundary is implicitly replaced by parabolic, cubic, quartic, quintic, etc., arcs. The equations of these arcs involve parameters which are the coordinates of points on the curved side. Mcleod and Mitchell [12] determine equations of parabolic and cubic curves using isoparametric coordinate transformations. Further, they also present a simple and systematic procedure to choose the parameters of the cubic curves so that the implicit equations of the curves always represent the

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parabola passing through four points of the original curves and so is a reasonable approximation to it. The development is put to practical use in the recent works of Rathod and Karim [13,14]. It is the purpose of this paper to find equations for point transformation of quartic and quintic arcs using isoparametric coordinate transformations and also to choose the parameters (coordinates of the points on the curved side) in a systematic way so that the implied curves are always a parabola passing through four points (quartic and quintic arcs) of the original curves.

2. Point transformations for triangular elements with one curved boundary

We consider the triangular elements in which one of the sides is curved and the other two sides are straight as shown in Figs. 1–4. The Lagrange interpolants for the field variable u (say) governing the physical problem are

$$u = \sum_{i=1}^{(n+1)(n+2)/2} N_i^{(n)}(\xi,\eta)u_i^e \quad (n = 2,3,4,5) \tag{1}$$

where $n = 2$ refers to quadratic, $n = 3$ refers to cubic, $n = 4$ refers to quartic and $n = 5$ refers to quintic order triangular elements, and $N_i^{(n)}(\xi,\eta)$ refers to the convential triangular element shape functions of order n at the node i. These are listed in Appendix A. Hence the transformation formulae between the physical (Cartesian) and the local (natural) coordinate system are

$$t = \sum_{i=1}^{(n+1)(n+2)/2} N_i^{(n)}(\xi,\eta)t_i \quad (t = x,y) \tag{2}$$

It is well known that the nodes along the straight sides 3–1 and 3–2 in Figs. 1–4 are always equi-spaced, except for certain special finite elements, like those required near the crack tip singularity in fracture mechanics. Now if we use the standard formulae on dividing a line segment in a given ratio from the plane analytic geometry to the straight sides 3–1 and 3–2, then the Eqs. (2) reduces to

$$\begin{aligned} m^{(n)}t(\xi,\eta) &= m^{(n)}t_3 + m^{(n)}(t_1 - t_3)\xi + m^{(n)}(t_2 - t_3)\eta + a_{11}^{(n)}(\underline{t})\xi\eta \\ &\quad + H(n-3)\sum_{\substack{i+j=n \\ (i\neq j)}} a_{ij}^{(n)}\xi^i\eta^j \\ &\quad (1 \leqslant i,j \leqslant n-1, n = 3,4,5), \quad (t = x,y) \end{aligned} \tag{3a}$$

where, $\underline{t}$ is nodal values of the triangular element and $H(n-3)$ is the well-known Heaviside step function or unit step function and it has

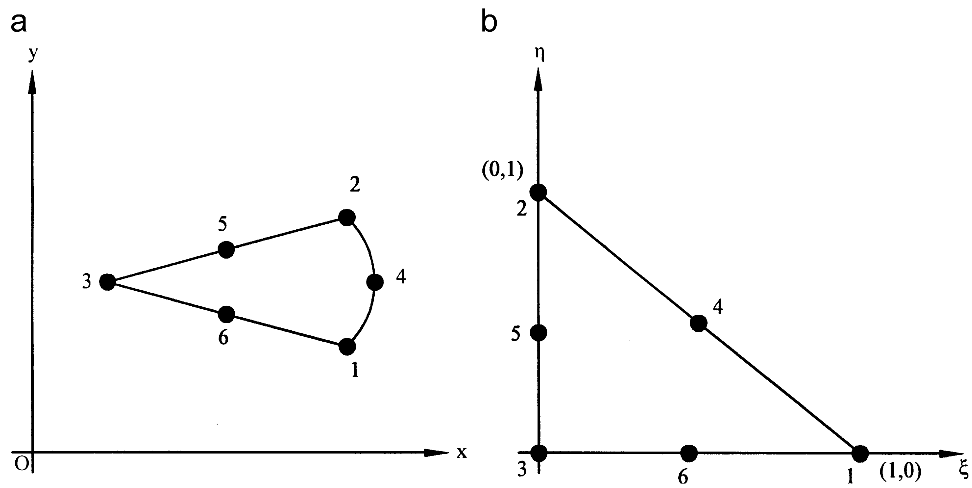

Fig. 1. Mapping of a 6-node quadratic curve triangle into isosceles triangle: (a) unmapped quadratic triangle, (b) mapped quadratic triangle.

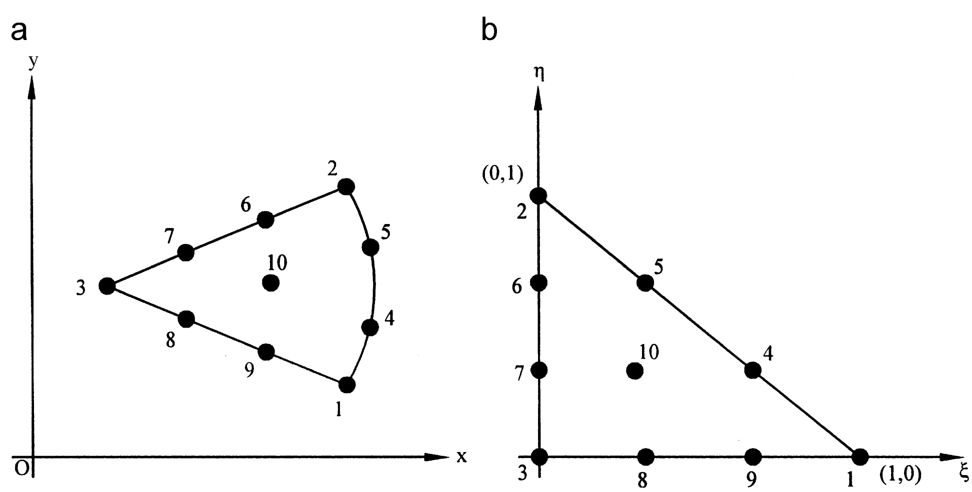

Fig. 2. Mapping of a 10-node cubic triangle into isosceles triangle: (a) unmapped cubic triangle, (b) mapped cubic triangle.

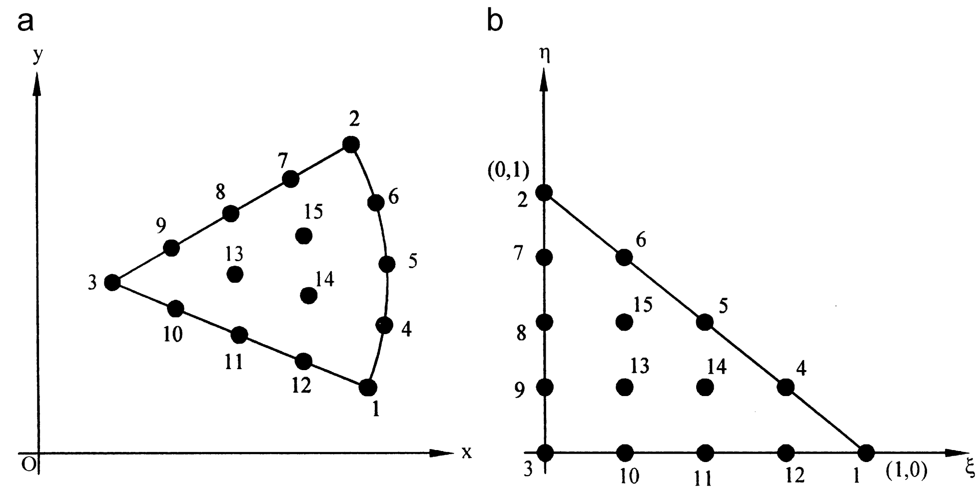

Fig. 3. Mapping of a 15-node quartic curve triangle into isosceles triangle: (a) unmapped quartic triangle, (b) mapped quartic triangle.

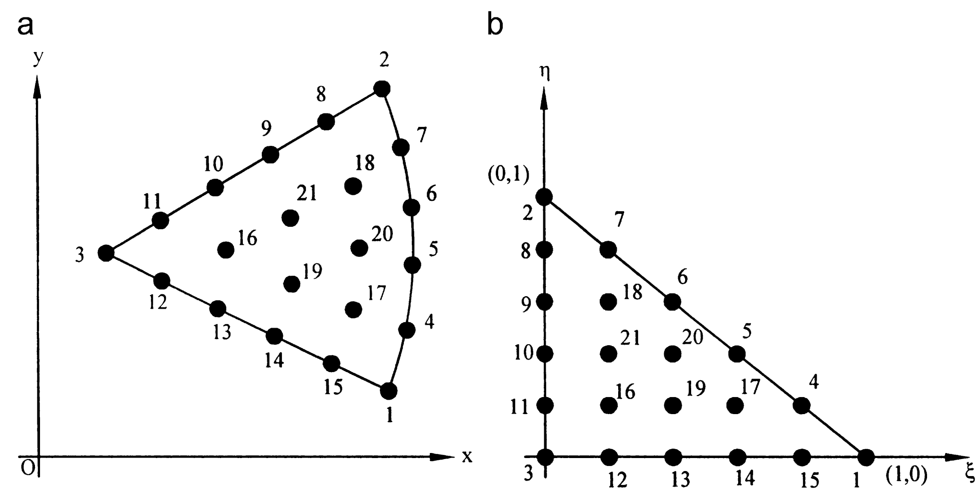

Fig. 4. Mapping of a 21-node quintic curve triangle into right isosceles triangle: (a) unmapped quintic triangle, (b) mapped quintic triangle.

the meaning for the present as

$$H(n-3)=\begin{cases}0, & n<3 \text{ i.e. } n=2\\ 1, & n\geqslant 3 \text{ i.e. } n=3,4,5\end{cases} \tag{3b}$$

$m^{(2)}=1$ for Quadratic curved triangular element
$m^{(3)}=2$ for Cubic curved triangular element
$m^{(4)}=3$ for Quartic curved triangular element
$m^{(5)}=24$ for Quintic curved triangular element (3c)

and the coefficients

$$\begin{aligned}&(a_{11}^{(n)}(\underline{t}), n=2(1)5)\\ &(a_{21}^{(n)}(\underline{t}), a_{12}^{(n)}(\underline{t}), n=3(1)5)\\ &(a_{31}^{(n)}(\underline{t}), a_{22}^{(n)}(\underline{t}), a_{13}^{(n)}(\underline{t}), n=4(1)5)\\ &(a_{41}^{(5)}(\underline{t}), a_{32}^{(5)}(\underline{t}), a_{23}^{(5)}(\underline{t}), a_{14}^{(5)}(\underline{t}))\end{aligned} \tag{3d}$$

are listed in Appendix B.

3. Triangles with one parabolic boundary

In the previous section, we have seen that the point transformation for the curved triangle with one curved side is expressed by Eqs. (3a)–(3d). This transformation will reduce to two parametric equations of the degree n ($n=2,3,4,5$) in local variate ξ or η along the curved boundary for which $\xi+\eta=1$. We would now like to approximate the curved boundary of the triangle by a parabolic arc i.e. by two parametric equations for x and y by a quadratic polynomial in ξ or η. This is possible only if we neglect the higher order terms in Eq. (3a) i.e. the terms $\sum_{\substack{i+j=n\\(i\neq j)}} a_{ij}^{(n)}\xi^i\eta^j$. Hence we may assume without loss of generality that the point transformation over the curved triangle is given by

$$t(\xi,\eta)=t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+A_{11}^{(n)}\xi\eta, \quad t=x,y \ (n=2,3,4,5) \tag{4}$$

where $A_{11}^{(n)}=a_{11}^{(n)}/m^{(n)}$ and $m^{(n)}$ are integral constant which are already defined in Eq. (3c).

4. Explicit form of point transformations and Jacobians

We note that Eq. (4) reduces to a pair of parametric equations for x and y along the curved boundary and they are quadratic polynomials, either in ξ or η (parametric variates, $0\leqslant\xi,\eta\leqslant 1$). Let us assume that the given curved boundary can be approximated by a general conic

[14], that is, the equation, (say)

$$f(x,y) = p_{00} + p_{10}x + p_{01}y + p_{20}x^2 + p_{11}xy + p_{02}y^2 = 0 \quad (5)$$

We have also from Eq. (4) the parametric equation along the curved boundary is of the form (say):

$$x(\xi, 1-\xi) = r_0(\underline{t}) + r_1(\underline{t})\xi + r_2(\underline{t})\xi^2$$
$$y(\xi, 1-\xi) = s_0(\underline{t}) + s_1(\underline{t})\xi + s_2(\underline{t})\xi^2 \quad (6a)$$

If we substitute from Eq. (6a) into Eq. (5), then on the curved boundary f has the form:

$$f(\xi, 1-\xi) = f_0 + f_1\xi + f_2\xi^2 + f_3\xi^3 + f_4\xi^4 = 0 \quad (6b)$$

Clearly Eq. (6b) is a polynomial in ξ, of degree four, since it has to pass through the end points of the curved boundary, $\xi = 0, 1$ are definitely two of its roots. The other two roots in $0 < \xi < 1$, determine two intermediate points on the curved boundary. Thus, we can only determine the curved boundary by a parabolic arc which passes through two intermediate points in $0 < \xi < 1$ and two end points at $\xi = 0$ and 1. If we have more than two intermediate points on the parabolic arc of this curved boundary, then they will be all expressible in terms of the two intermediate points which only lie on the original curved boundary. We shall now determine the relations among the nodal points along the curved boundary, if the curved triangle has more than four nodes along the curved boundary.

Lemma. *Let the point transformation for the curved triangle with one parabolic curved boundary side and two straight sides are expressible as*

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + A_{11}^{(n)}(\underline{t})\xi\eta \quad (7a)$$

where

$$A_{11}^{(n)} = \frac{a_{11}^{(n)}}{m^{(n)}}, \quad t = x, y \ (n = 2, 3, 4, 5) \quad (7b)$$

then it can be shown that

(i) *Quadratic case* $(n = 2)$:

$$A_{11}^{(2)} = a_{11}^{(2)} = [4t_4 - 2(t_1 + t_2)] \quad (8)$$

(ii) *Cubic case* $(n = 3)$:

$$A_{11}^{(3)} = \frac{a_{11}^{(3)}}{2} = \frac{9}{4}[(t_4 + t_5) - (t_1 + t_2)] \quad (9)$$

(iii) *Quartic case* $(n = 4)$:

$$A_{11}^{(4)} = \frac{a_{11}^{(4)}}{3} = \frac{8}{3}[(t_4 + t_6) - (t_1 + t_2)]$$
$$t_5 = \tfrac{1}{6}[4(t_4 + t_6) - (t_1 + t_2)] \quad (10)$$

(iv) *Quintic case* $(n = 5)$:

$$A_{11}^{(5)} = \frac{a_{11}^{(5)}}{24} = \begin{cases} \frac{75}{24}\{(t_4 + t_7) - (t_1 + t_2)\} \\ \text{OR} \\ \frac{50}{24}\{(t_5 + t_6) - (t_1 + t_2)\} \end{cases} \quad (11)$$

and

$$(t_4 + t_7) = \tfrac{2}{3}(t_5 + t_6) + \tfrac{1}{3}(t_1 + t_2)$$
$$(t_5 + t_6) = \tfrac{3}{2}(t_4 + t_7) - \tfrac{1}{2}(t_1 + t_2) \quad (12)$$

Proof. The proof follows from the foregoing analysis of point transformations to match the parabolic arc discussed in Section 3 of the paper and alternatively it also follows from the global to local transformation of coordinates and geometric considerations.

5. Analysis of point transformations

In the previous section, we have considered triangular elements of order two to five with two straight sides and one curved side. These triangles are spanned by a total of 6, 10, 15 and 21 nodes and each of these have 3, 4, 5 and 6 nodes, respectively, along the curved side. The physical (global/cartesian) and reference (local/natural) coordinates of any node i are (x_i, y_i) and (ξ_i, η_i), respectively. The global coordinates (x, y) and the local coordinates (ξ, η) under the subparametric coordinate transformation which map these curved triangles into-isosceles right triangles are as shown in Figs. 1(b)–4(b) and they are related by Eqs. (3a)–(3d) as derived in the previous section. The quadratic and cubic point transformations have been the subject of intensive research in the earlier works by several authors [11–14].

The parametric equations of the curved side in Figs. 1(a)–4(a) can be obtained by substituting $\eta = 1 - \xi$ in Eqs. (3a)–(3d). This leads to equations of the form:

$$m^{(n)}t(\xi, 1-\xi) = \alpha_0^{(n)}(\underline{t}) + \alpha_1^{(n)}(\underline{t})\xi + \alpha_2^{(n)}(\underline{t})\xi^2 + \cdots + \alpha_k^{(n)}(\underline{t})\xi^n$$
$$(n = 2, 3, 4, 5), \quad (t = x, y) \quad (13)$$

$(\alpha_k^{(n)}(\underline{t}), k = 0, 1, 2, \ldots, n)$ can be obtained from $a_{ij}^{(n)}(\underline{t})$ values as listed in Appendix B. Let us now analyze each of the Eqs. (3a) for $n = 2, 3, 4, 5$ one by one.

Quadratic case $(n = 2)$: In this case, the curved side of the triangle is spanned by the coordinates $(t_1, t_4, t_2, t = x, y)$.

Hence on the curved side, we obtain the following equation on substituting $\eta = 1 - \xi$ in Eq. (3a)

$$t(\xi, 1-\xi) = \alpha_0^{(2)}(\underline{t}) + \alpha_1^{(2)}(\underline{t})\xi + \alpha_2^{(2)}(\underline{t})\xi^2, \quad t = x, y \quad (14a)$$

Clearly Eq. (14a) describes the parametric form of equation of a parabola passing through the points $(t_i, i = 1, 4, 2)$ and

$$t = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + (4t_4 - 2t_1 - 2t_2)\xi\eta, \quad (t = x, y) \quad (14b)$$

Cubic case $(n = 3)$:

In this case, the curved side of the triangle is spanned by the coordinates $(t_i,\ i = 1, 4, 5, 2)$. Hence on the curved side, we obtain the following equation on substituting $\eta = 1 - \xi$ in Eq. (3a):

$$2t(\xi, 1-\xi) = \alpha_0^{(2)}(\underline{t}) + \alpha_1^{(2)}(\underline{t})\xi + \alpha_2^{(2)}(\underline{t})\xi^2 + \alpha_3^{(3)}(\underline{t})\xi^3, \quad t = x, y \quad (15)$$

Clearly the parametric equation (15) describes a cubic curve passing through the points $(t_i,\ i = 1, 4, 5, 2)$. Since a cubic curve must possess a double point, which may result in a cusp or a loop in the curve, it is in general undesirable as an approximation to a simple smooth curve [7]. However, the choice for location of points $(t_i,\ i = 1, 4, 5, 2)$ to make the cubic curve to reduce to a unique parabola can be achieved by setting [11]:

$$\alpha_3^{(3)}(\underline{t}) = 0, \quad (t = x, y) \quad (16)$$

That is to set: $a_{12}^{(3)}(\underline{t}) - a_{21}^{(3)}(\underline{t}) = 0$, and this implies

$$t_4 - t_5 = \tfrac{1}{3}(t_1 - t_2), \quad (t = x, y) \quad (17)$$

In addition, if we set

$$a_{12}^{(3)}(\underline{t}) + a_{21}^{(3)}(\underline{t}) = 0 \quad (18)$$

then the choice

$$t_{10} = \tfrac{1}{12}(t_1 + t_2 + 4t_3 + 3t_4 + 3t_5), \quad (t = x, y) \quad (19)$$

and the transformation formulae Eq. (3a) reduces to

$$2t = 2t_3 + 2(t_1 - t_3)\xi + 2(t_2 - t_3)\eta + a_{11}^{(3)}(\underline{t})\xi\eta, \quad (t = x, y) \quad (20)$$

where

$$a_{11}^{(3)}(\underline{t})=\frac{9}{2}(t_4+t_5-t_1-t_2), \quad (t=x,y) \tag{21}$$

and

$$t=t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+\frac{9}{4}(t_4+t_5-t_1-t_2)\xi\eta, \quad (t=x,y) \tag{22}$$

Quartic case ($n=4$):

In this case, the curved side of the triangle is spanned by the coordinates (t_i, $i=1,4,5,6,2$). The point transformation for this case can be obtained from Eq. (3a). Hence, on the curved side, we obtain

$$3t(\xi,1-\xi)=\alpha_0^{(4)}(\underline{t})+\alpha_1^{(4)}(\underline{t})\xi+\alpha_2^{(4)}(\underline{t})\xi^2+\alpha_3^{(4)}(\underline{t})\xi^3+\alpha_4^{(4)}(\underline{t})\xi^4 \tag{23}$$

Now the choice for the location of points (t_i, $i=4,5,6$) to make the quartic curve to a unique parabola, can be achieved by setting:

$$\alpha_3^{(4)}(\underline{t})=0, \quad \alpha_4^{(4)}(\underline{t})=0 \tag{24}$$

Now, Eq. (24) can be explicitly written as

$$\begin{aligned}&-a_{21}^{(4)}(\underline{t})+a_{12}^{(4)}(\underline{t})+a_{31}^{(4)}(\underline{t})-2a_{22}^{(4)}(\underline{t})+3a_{13}^{(4)}(\underline{t})=0\\&-a_{31}^{(4)}(\underline{t})+a_{22}^{(4)}(\underline{t})-a_{13}^{(4)}(\underline{t})=0\end{aligned} \tag{25}$$

Using the explicit relations for the coefficients $a_{ij}^{(4)}(\underline{t})$ as listed in Appendix B, in Eq. (25) above, we obtain

$$t_4-t_6=\frac{1}{2}(t_1-t_2), \quad 4(t_4+t_6)-6t_5=(t_1+t_2) \tag{26}$$

Using Eq. (13), we have

$$\begin{aligned}a_{31}^{(4)}(\underline{t})&=\frac{1}{2}(a_{22}^{(4)}(\underline{t})-a_{12}^{(4)}(\underline{t})+a_{21}^{(4)}(\underline{t}))\\a_{13}^{(4)}(\underline{t})&=\frac{1}{2}(a_{22}^{(4)}(\underline{t})+a_{12}^{(4)}(\underline{t})-a_{21}^{(4)}(\underline{t}))\end{aligned} \tag{27}$$

Using Eq. (27) in Eq. (3a), we obtain

$$\begin{aligned}3t&=3t_3+3(t_1-t_3)\xi+3(t_2-t_3)\eta+a_{11}^{(4)}(\underline{t})\xi\eta\\&\quad+a_{21}^{(4)}(\underline{t})\left(\xi^2\eta-\frac{\xi^3\eta}{2}+\frac{\xi\eta^3}{2}\right)+a_{12}^{(4)}(\underline{t})\left(\xi\eta^2+\frac{\xi^3\eta}{2}-\frac{\xi\eta^3}{2}\right)\\&\quad+a_{22}^{(4)}(\underline{t})\left(\xi^2\eta^2+\frac{\xi^3\eta}{2}+\frac{\xi\eta^3}{2}\right)\end{aligned} \tag{28}$$

Now choose t_{13}, t_{14} and t_{15} such that $a_{21}^{(4)}(\underline{t})=0$, $a_{12}^{(4)}(\underline{t})=0$ and $a_{22}^{(4)}(\underline{t})=0$, so that the above Eq. (28) simplifies to the quadratic form:

$$3t=3t_3+3(t_1-t_3)\xi+3(t_2-t_3)\eta+a_{11}^{(4)}(\underline{t})\xi\eta \tag{29}$$

From Appendix B, on using explicit relations of $a_{21}^{(4)}(\underline{t}), a_{12}^{(4)}(\underline{t})$ and $a_{22}^{(4)}(\underline{t})$ and further on account of Eq. (26), we suppose that the location of all the points on the curved side is known. Hence, we obtain the three additional equations from

$a_{21}^{(4)}(\underline{t})=0, \quad a_{12}^{(4)}(\underline{t})=0 \quad$ and $\quad a_{22}^{(4)}(\underline{t})=0 \quad$ as

$$\begin{aligned}&14t_{13}+2t_{14}+10t_{15}=t_5+2t_6-t_1-4t_3\\&-14t_{13}+10t_{14}+2t_{15}=t_5+2t_4-t_1-4t_3\\&4t_{13}-2t_{14}-2t_{15}=t_3-t_5\end{aligned} \tag{30}$$

The solution to the system of Eqs. (26) and (30) has three solutions which depend on the relations:

$$t_4-t_6=\frac{1}{2}(t_1-t_2) \tag{31a}$$

$$8t_4-6t_5=(3t_1-t_2) \tag{31b}$$

$$6t_5-8t_6=(t_1-3t_2) \tag{31c}$$

The first solution can be obtained from the relation of Eq. (31a) which will locate t_1, t_4, t_6 and t_2 on the original curved boundary and then t_5 can be determined by the relation:

$$t_5=\frac{1}{6}(-3t_1+t_2+8t_4) \tag{32a}$$

and the t_5 thus determined may or may not lie on curved boundary. Thus all points along the curved boundary are known in this manner. The second solution can be obtained in a similar manner from Eq. (31b) and the relation

$$t_6=\frac{1}{8}(-t_1+3t_2+6t_5) \tag{32b}$$

The third solution can be determined from Eq. (31c) and the relation

$$t_4=\frac{1}{2}(t_1-t_2+2t_6) \tag{33}$$

The interior points t_{13}, t_{14} and t_{15} can be then determined either Eq. (30) or Eq. (29), which is further expressible as

$$t=t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+\frac{8}{3}[(t_4+t_6)-(t_1+t_2)]\xi\eta \tag{34}$$

Once the location of the points on the boundary is known, we have from Eq. (34), we can determine the interior points from the following:

$$t_{13}=t(1/4,1/4), \quad t_{14}=t(1/2,1/4) \quad \text{and} \quad t_{15}=t(1/4,1/2) \tag{35}$$

Quintic case ($n=5$):

In this case, the curved side of the triangle is spanned by the coordinates (t_i, $i=1,4,5,6,7,2$). The point transformation for this case can be obtained from Eq. (3a). Hence, on the curved side, we obtain the following equation on substituting $\eta=1-\xi$ in Eq. (3a):

$$\begin{aligned}24t(\xi,1-\xi)&=\alpha_0^{(5)}(\underline{t})+\alpha_1^{(5)}(\underline{t})\xi+\alpha_2^{(5)}(\underline{t})\xi^2+\alpha_3^{(5)}(\underline{t})\xi^3+\alpha_4^{(5)}(\underline{t})\xi^4\\&\quad+\alpha_5^{(5)}(\underline{t})\xi^5\end{aligned} \tag{36}$$

Now, the choice for the location of point's t_i, $i=4,5,6,7$ to make the above quintic curve to reduce to a unique parabola can be achieved by setting:

$$\alpha_3^{(5)}(\underline{t})=0, \quad \alpha_4^{(5)}(\underline{t})=0, \quad \alpha_5^{(5)}(\underline{t})=0 \tag{37}$$

Now, Eq. (37) can be explicitly written as

$$\begin{aligned}&-a_{21}^{(5)}(\underline{t})+a_{12}^{(5)}(\underline{t})+a_{31}^{(5)}(\underline{t})-2a_{22}^{(5)}(\underline{t})+3a_{13}^{(5)}(\underline{t})+a_{32}^{(5)}(\underline{t})\\&\quad-3a_{23}^{(5)}(\underline{t})+6a_{14}^{(5)}(\underline{t})=0\\&-a_{31}^{(5)}(\underline{t})+a_{22}^{(5)}(\underline{t})-a_{13}^{(5)}(\underline{t})+a_{41}^{(5)}(\underline{t})-2a_{32}^{(5)}(\underline{t})+3a_{23}^{(5)}(\underline{t})-4a_{14}^{(5)}(\underline{t})=0\\&-a_{41}^{(5)}(\underline{t})+a_{32}^{(5)}(\underline{t})-a_{23}^{(5)}(\underline{t})+a_{14}^{(5)}(\underline{t})=0\end{aligned} \tag{38}$$

We can equivalently express the Eq. (38) as

$$\begin{aligned}&(a_{41}^{(5)}(\underline{t})-a_{14}^{(5)}(\underline{t}))-(a_{32}^{(5)}(\underline{t})-a_{23}^{(5)}(\underline{t}))=0\\&(a_{12}^{(5)}(\underline{t})-a_{21}^{(5)}(\underline{t}))+(a_{13}^{(5)}(\underline{t})-a_{31}^{(5)}(\underline{t}))+(a_{23}^{(5)}(\underline{t})-a_{32}^{(5)}(\underline{t}))=0\\&a_{22}^{(5)}(\underline{t})-(a_{31}^{(5)}(\underline{t})+a_{13}^{(5)}(\underline{t}))-\frac{3}{2}(a_{41}^{(5)}(\underline{t})+a_{14}^{(5)}(\underline{t}))\\&\quad+\frac{1}{2}(a_{32}^{(5)}(\underline{t})+a_{23}^{(5)}(\underline{t}))=0\end{aligned} \tag{39}$$

Using the explicit relations for the coefficients $a_{ij}^{(5)}(\underline{t})$ as listed in Appendix B, in Eq. (39) above, we obtain the solution:

$$t_4-t_7=\frac{3}{5}(t_1-t_2) \tag{40a}$$

$$t_5-t_6=\frac{1}{5}(t_1-t_2) \tag{40b}$$

$$3(t_4+t_7)-2(t_5+t_6)=(t_1+t_2), \quad (t=x,y) \tag{40c}$$

We now suppose that on using the above Eqs. (40a)–(40c), all the coordinate points along the curved boundary of the quintic triangular element are known. We have at least two sets of solutions emerging from the above equations. In the first solution, we can assume t_4 and t_7 to lie on the original curved boundary and the other two points t_5 and t_6 may lie slightly off the curved boundary. In the second solution, we can assume t_5 and t_6 to lie on the original curved boundary and then the other two point's t_4 and t_7 may lie off the curved boundary.

We shall now proceed to determine the interior points $t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}$ of the quintic curved triangular element. It is also clear by now that these interior points can be easily determined from Eqs. (7a–b) and Eq. (11).

Using Eq. (39) in Eq. (3a), we can write

$$\begin{aligned}24t(\xi,\eta)=&24t_3+24(t_1-t_3)\xi+24(t_2-t_3)\eta+a_{11}^{(5)}(\underline{t})\xi\eta\\&+\tfrac{1}{2}(\xi^2\eta+\xi\eta^2)(a_{21}^{(5)}(\underline{t})+a_{12}^{(5)}(\underline{t}))\\&+\tfrac{1}{2}(\xi^3\eta+\xi\eta^3+2\xi^2\eta^2)(a_{31}^{(5)}(\underline{t})+a_{13}^{(5)}(\underline{t}))\\&+\tfrac{1}{2}(\xi^4\eta+\xi\eta^4+3\xi^2\eta^2)(a_{41}^{(5)}(\underline{t})+a_{14}^{(5)}(\underline{t}))\\&+\tfrac{1}{2}(\xi^2\eta^3+\xi^3\eta^2-\xi^2\eta^2)(a_{32}^{(5)}(\underline{t})+a_{23}^{(5)}(\underline{t}))\\&+\tfrac{1}{2}(-\xi\eta^2+\xi^2\eta+\xi^2\eta^3-\xi^3\eta^2-\xi^4\eta+\xi\eta^4)(a_{21}^{(5)}(\underline{t})-a_{12}^{(5)}(\underline{t}))\\&+\tfrac{1}{2}(-\xi\eta^3+\xi^3\eta+\xi^2\eta^3-\xi^3\eta^2-\xi^4\eta+\xi\eta^4)(a_{31}^{(5)}(\underline{t})\\&-a_{13}^{(5)}(\underline{t}))\end{aligned} \tag{41}$$

Now, we choose the interior points $t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}$ so that the quintic curve passing through the points $t_1, t_4, t_5, t_6, t_7, t_2$ degenerates into a unique parabola, this requires from the above Eq. (41) that

$$\begin{aligned}&a_{21}^{(5)}(\underline{t})+a_{12}^{(5)}(\underline{t})=0, \quad a_{31}^{(5)}(\underline{t})+a_{13}^{(5)}(\underline{t})=0, \quad a_{41}^{(5)}(\underline{t})+a_{14}^{(5)}(\underline{t})=0\\&a_{32}^{(5)}(\underline{t})+a_{23}^{(5)}(\underline{t})=0, \quad a_{21}^{(5)}(\underline{t})-a_{12}^{(5)}(\underline{t})=0\\&a_{31}^{(5)}(\underline{t})-a_{13}^{(5)}(\underline{t})=0\end{aligned} \tag{42}$$

Now on using Eq. (42) in Eq. (41), we obtain

$$24t=24t_3+24(t_1-t_3)\xi+24(t_2-t_3)\eta+a_{11}^{(5)}(\underline{t})\xi\eta \tag{43}$$

It can be shown that the relations of Eq. (42), viz

$$a_{21}^{(5)}(\underline{t})-a_{12}^{(5)}(\underline{t})=0, \quad a_{31}^{(5)}(\underline{t})-a_{13}^{(5)}(\underline{t})=0$$

lead us to

$$\begin{aligned}&7(t_1-t_2)+11(t_4-t_7)+2(t_5-t_6)-60(t_{17}-t_{18})\\&\quad+120(t_{19}-t_{21})=0\\&(t_1-t_2)-3(t_4-t_7)-(t_5-t_6)+16(t_{17}-t_{18})\\&\quad-27(t_{19}-t_{21})=0\end{aligned} \tag{44}$$

Now from Eqs. (40a–b) and Eq. (44), we obtain

$$t_{17}-t_{18}=\tfrac{2}{5}(t_1-t_2), \quad t_{19}-t_{21}=\tfrac{1}{5}(t_1-t_2) \tag{45}$$

The remaining relations of Eq. (42), viz

$$\begin{aligned}&a_{21}^{(5)}(\underline{t})+a_{12}^{(5)}(\underline{t})=0, \quad a_{31}^{(5)}(\underline{t})+a_{13}^{(5)}(\underline{t})=0\\&a_{41}^{(5)}(\underline{t})+a_{14}^{(5)}(\underline{t})=0 \quad \text{and} \quad a_{32}^{(5)}(\underline{t})+a_{23}^{(5)}(\underline{t})=0\end{aligned}$$

further lead us to the following set of linear equations:

$$\begin{aligned}&188t_{16}+36t_{20}+38(t_{17}+t_{18})-114(t_{19}+t_{21})=k_1\\&96t_{16}+12t_{20}+32(t_{17}+t_{18})-66(t_{19}+t_{21})=k_2\\&4t_{16}+0t_{20}+2(t_{17}+t_{18})-3(t_{19}+t_{21})=k_3\\&12t_{16}+6t_{20}+2(t_{17}+t_{18})-9(t_{19}+t_{21})=k_4\end{aligned} \tag{46a}$$

where

$$\begin{aligned}&k_1=(t_1+t_2)+44t_3+13(t_4+t_7)\\&k_2=(t_1+t_2)+20t_3+9(t_4+t_7)\\&k_3=\tfrac{1}{10}[(t_1+t_2)+8t_3+5(t_4+t_7)]\\&k_4=\tfrac{1}{10}[-5(t_1+t_2)+20t_3+15(t_4+t_7)]\end{aligned} \tag{46b}$$

The solution to the linear system of Eq. (46a–b) is

$$\begin{aligned}&t_{16}=\tfrac{1}{40}[3(t_1+t_2)+24t_3+5(t_4+t_7)]\\&t_{17}+t_{18}=\tfrac{1}{20}[(t_1+t_2)+8t_3+15(t_4+t_7)]\\&t_{19}+t_{21}=\tfrac{1}{10}[(t_1+t_2)+8t_3+5(t_4+t_7)]\\&t_{20}=\tfrac{1}{10}[-(t_1+t_2)+2t_3+5(t_4+t_7)]\end{aligned} \tag{47}$$

Now, we present the two sets of solutions emerging from Eqs. (40a–c) as explained earlier.

First solution: We determine the points t_4 and t_7 to lie on the original curved boundary by using Eq. (40a). Then the points t_5 and t_6 can be determined by Eq. (40b) and Eq. (40c). This gives us

$$t_5=\tfrac{3}{4}(t_4+t_7)-\tfrac{1}{20}(3t_1+7t_2) \tag{48a}$$

$$t_6=\tfrac{3}{4}(t_4+t_7)-\tfrac{1}{20}(7t_1+3t_2) \tag{48b}$$

Second solution: We determine the points t_5 and t_6 to lie on the original curved boundary by using Eq. (45b). Then the points t_4 and t_7 can be determined by Eq. (40a) and Eq. (40c). This gives us

$$t_4=\tfrac{1}{3}(t_5+t_6)+\tfrac{1}{15}(7t_1-2t_2) \tag{49a}$$

$$t_7=\tfrac{1}{3}(t_5+t_6)+\tfrac{1}{15}(-2t_1+7t_2) \tag{49b}$$

The interior points on the curved triangle in either case can be determined by Eqs. (45) and (47). This gives us

$$\begin{aligned}&t_{16}=\tfrac{3}{40}[(t_1+t_2)+\tfrac{3}{5}t_3+\tfrac{1}{8}(t_4+t_7)]\\&t_{17}=\tfrac{1}{40}[(9t_1-7t_2)+\tfrac{1}{5}t_3+\tfrac{3}{8}(t_4+t_7)]\\&t_{18}=\tfrac{1}{40}[(-7t_1+9t_2)+\tfrac{1}{5}t_3+\tfrac{3}{8}(t_4+t_7)]\\&t_{19}=\tfrac{1}{20}[(3t_1-t_2)+\tfrac{2}{5}t_3+\tfrac{1}{4}(t_4+t_7)]\\&t_{20}=\tfrac{-1}{10}[(t_1+t_2)+\tfrac{1}{5}t_3+\tfrac{1}{2}(t_4+t_7)]\\&t_{21}=\tfrac{1}{20}[(-t_1+3t_2)+\tfrac{2}{5}t_3+\tfrac{1}{4}(t_4+t_7)]\end{aligned} \tag{50}$$

Explicit form of the point transformations:

Theorem. *The point transformation for the curved triangular elements with one curved side and two straight sides can be expressed in terms of the four points* (t_i, $i=1,2,3,4$), ($t=x,y$) *as*:

$$\begin{aligned}t(\xi,\eta)=&t_3+(t_1-t_3)\xi+(t_2-t_3)\eta\\&+\frac{n}{(n-1)}[nt_4-((n-1)t_2+t_1)]\xi\eta\end{aligned} \tag{51}$$

where $n=2,3,4,5$ *for the quadratic, cubic, quartic and quintic curved triangular elements, respectively*.

Proof. This follows from Lemma and the linear relation between the nodal coordinates along the curved boundary derived in the previous sections. □

Explicit form of the Jacobians: By using the transformation Eq. (51), the Jacobian $J(\xi,\eta)$ can be expressed as

$$J(\xi,\eta)=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi},\quad J(\xi,\eta)=\alpha_0+\alpha_1\xi+\alpha_2\eta \tag{52}$$

where

$$\alpha_0=(x_1-x_2)(y_2-y_3)-(x_2-x_3)(y_1-y_3)$$

$$\alpha_1=(x_1-x_3)A_{11}^{(n)}(\underline{y})-(y_1-y_3)A_{11}^{(n)}(\underline{x})$$

$$\alpha_2=(y_2-y_3)A_{11}^{(n)}(\underline{x})-(x_2-x_3)A_{11}^{(n)}(\underline{y})$$

$$A_{11}^{(n)}(\underline{t})=\frac{n}{(n-1)}[nt_4-((n-1)t_2+t_1)]$$

$$t=x,y\quad(n=2,3,4,5) \tag{53}$$

6. Application example

6.1. Determination of points over the curved triangle

To determine the application of derived solutions of curved boundary triangular elements, we consider a domain consisting of the quarter ellipse defined by

$$\frac{x^2}{36}+\frac{y^2}{4}=1$$

The location of points along the curved boundary, which reduce the isoparametric transformation to parametric equations of the form: $t=\alpha_0^{(n)}(\underline{t})+\alpha_1^{(n)}(\underline{t})\xi+\alpha_2^{(n)}(\underline{t})\xi^2$, $(n=2,3,4,5)$ is discussed in the previous sections of this paper in full detail. Further, the location of the points in the interior of the curved triangle which reduce the isoparametric transformations from cubic to quintic order to the quadratic transformation is: $t(\xi,\eta)=t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+A_{11}^{(n)}(\underline{t})\xi\eta$, $(n=3,4,5)$ under the subparametric concept, is also fully described in the previous sections. The determination of points over the curved triangle i.e. the points along the curved boundary of the triangle and the points located in the interior of the curved triangle is of utmost importance for us to proceed with the application of higher order curved triangular elements under the subparametric transformation. Hence, we have tabulated these points for the quartic and quintic order curved triangular elements in the various tables listed in this paper, viz, Table 1a–c (quartic element) and Table 2a–c (quintic element). We have also included a table of coordinates in Table 3 for cubic triangle as derived in [14].

6.2. Determination of arc length for the curved triangle

Calculating the length of a given curve between two end points is useful in many applications. Hence, to demonstrate further application of the derived quadratic transformation of curved triangular elements (quadratic, cubic, quartic and quintic), we propose to determine the arc length of the quarter ellipse (as triangular element). We have shown that the parametric equations along the curved boundary are: $t(\xi,1-\xi)=\alpha_0^{(n)}(\underline{t})+\alpha_1^{(n)}(\underline{t})\xi+\alpha_2^{(n)}(\underline{t})\xi^2$, $(n=2,3,4,5)$ i.e.

$$x(\xi)=\alpha_0^{(n)}(x)+\alpha_1^{(n)}(x)\xi+\alpha_2^{(n)}(x)\xi^2$$

$$y(\xi)=\alpha_0^{(n)}(y)+\alpha_1^{(n)}(y)\xi+\alpha_2^{(n)}(y)\xi^2,\quad(n=2,3,4,5) \tag{54}$$

We can find the arc lengths from the above equation as:

$$s=\text{arc length}=\int_0^1\sqrt{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2}\,d\xi \tag{55}$$

Table 1

Nodal points-i	x-coordinate point x_i	y-coordinate point y_i	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
(a) *First solution*[a]			
4	5.468626967	0.822875655	0.999999999
5	4.291502623	1.430500873	1.023166375
6	2.468626967	1.822875655	0.999999999
13	1.822875656	0.607625218	0.184604203
14	3.645751311	0.715250436	0.497104202
15	2.145751311	1.215250437	0.497104202
(b) *Second solution*[b]			
4	5.527746982	0.777746981	0.999999999
5	4.370329309	1.370329308	0.999999999
6	2.527746982	1.777746981	0.967582326
13	1.842582327	0.592582327	0.182097054
14	3.685164655	0.685164654	0.494597054
15	2.185164655	1.185164654	0.483791163
(c) *Third solution*[c]			
4	5.333240943	0.842582327	0.967582326
5	4.110987924	1.456776436	0.999999999
6	2.333240943	1.842582327	0.999999999
13	1.777746981	0.614194109	0.182097054
14	3.555493962	0.728388218	0.483791163
15	2.055493962	1.228388218	0.494597054

[a] Table of the coordinate points $((x_i,y_i),i=4,5,6)$ along the curved boundary and the coordinate points $((x_i,y_i),i=13,14,15)$ in the interior of the *curved quartic triangle*: $\{(x,y)/x=0,y=0,x^2/36+y^2/4\leqslant 1\}$.

[b] Table of the coordinate points $((x_i,y_i),i=4,5,6)$ along the curved boundary and the coordinate points $((x_i,y_i),i=13,14,15)$ in the interior of the *curved quartic triangle*: $\{(x,y)/x=0,y=0,x^2/36+y^2/4\leqslant 1\}$.

[c] Table of the coordinate points $((x_i,y_i),i=4,5,6)$ along the curved boundary and the coordinate points $((x_i,y_i),i=13,14,15)$ in the interior of the *curved quartic triangle*: $\{(x,y)/x=0,y=0,x^2/36+y^2/4\leqslant 1\}$.

Table 2

i	x_i	y_i	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
(a) *First solution*[a]			
4	5.641874854	0.680624847	0.999999999
5	4.862811813	1.220937271	1.029531364
6	3.662811813	1.620937271	1.029531364
7	2.041874542	1.880624847	0.999999999
16	1.410468636	0.470156211	0.110523481
17	4.231405907	0.610468635	0.590523431
18	1.831405907	1.410468635	0.590523431
19	2.820937271	0.540312423	0.294031242
20	3.241874542	1.080624847	0.583875030
21	1.620937271	0.940312423	0.294031242
(b) *Second solution*[b]			
4	5.6	0.666666666	0.982222222
5	4.8	1.2	1
6	3.6	1.6	1
7	2.0	1.866666666	0.982222222
16	1.4	0.466666666	0.108888888
17	4.2	0.6	0.58
18	1.8	1.4	0.58
19	2.8	0.533333333	0.288888888
20	3.2	1.066666666	0.568888888
21	1.6	0.933333333	0.288888888

[a] Table of the coordinate points $((x_i,y_i),i=4,5,6,7)$ along the curved boundary and the coordinate points $((x_i,y_i),i=16,17,18,19,20,21)$ in the interior of the *curved quintic triangle*: $\{(x,y)/x=0,y=0,x^2/36+y^2/4\leqslant 1\}$.

[b] Table of the coordinate points $((x_i,y_i),i=4,5,6,7)$ along the curved boundary and the coordinate points $((x_i,y_i),i=16,17,18,19,20,21)$ in the interior of the *curved quintic triangle*: $\{(x,y)/x=0,y=0,x^2/36+y^2/4\leqslant 1\}$.

We have described the parametric equations along the curved boundary of the ellipse (under subparametric point transformation and usual isoparametric point transformation) and the computed values of the arc length in Table 4a,b for the various curved triangular elements (quadratic, cubic, quartic and quintic order elements). We can also note that the theoretical value of the arc length s of a

Table 3

Nodal points-i	x-coordinate point x_i	y-coordinate point y_i	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
4	5.123105626	1.041035209	1
5	3.123105626	1.707701875	0.999999999
10	2.561552813	0.853850937	0.364530711

Table of the coordinate points $((x_i, y_i),\ i = 4, 5)$ along the curved boundary and the coordinate point (x_{10}, y_{10}) in the interior of the *curved cubic triangle*:

$\{(x, y)/x = 0, y = 0, x^2/36 + y^2/4 \leqslant 1\}$.

quarter ellipse: $x^2/a^2 + y^2/b^2 = 1$ is given by the series expression

$$s = a\int_0^{\pi/2}\sqrt{1-e^2\cos^2\theta}\,d\theta, \quad e^2 = 1-\frac{b^2}{a^2}$$
$$= \frac{a\pi}{2}\left[1-\sum_{n=1}^{\infty}\left\{\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\right\}^2\frac{e^{2n}}{(2n-1)}\right] \quad (56a)$$

Now we have for ellipse of the application example of this section, namely:

$$\frac{x^2}{36}+\frac{y^2}{4}=1, \quad e^2=\frac{8}{9}, \quad a=6, \quad b=2$$

and the value of s is

$$s = 3\pi\left[1-\sum_{n=1}^{14}\left\{\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\right\}^2\frac{e^{2n}}{(2n-1)}\right]$$
$$s = 6.688222104 \quad (56b)$$

We have then compared the theoretical value of arc length s as found in Eq. (56b) with finite element approximation of s (expressed as an integral): (i) by straight forward application of numerical integration and the usual isoparametric mapping and then (ii) by straight forward application of numerical/analytical integration and the subparametric mapping proposed in the present paper. These findings are given in Tables 4a and b, respectively.

6.3. *Determination of center of gravity (Centroid) of curved triangular element*

Mass property calculations are one of the earliest engineering applications implemented into CAD/CAM systems. One of these properties is the centroid of an area bounded by a curve. Hence, to demonstrate the further application of the derived quadratic transformation formula of curved triangular elements we propose to determine the centroid of the quarter ellipse (as a curved triangular element). Let us consider the area A of one quadrant of the ellipse: $x^2/a^2+y^2/b^2=1$ then the centroid $(\bar{x}, \bar{y})$ of the area A is given by

$$\bar{x} = \iint_A x\,dx\,dy\Big/\iint_A dx\,dy, \quad \bar{y} = \iint_A y\,dx\,dy\Big/\iint_A dx\,dy \quad (57a)$$

$$\iint_A x\,dx\,dy = \int_0^a\left(\int_0^{b/a\sqrt{a^2-b^2}} x\,dy\right)dx = \frac{ba^2}{3}$$
$$\iint_A y\,dx\,dy = \int_0^a\left(\int_0^{\frac{b}{a}\sqrt{a^2-b^2}} y\,dy\right)dx = \frac{ab^2}{3}$$
$$\iint_A dx\,dy = \int_0^a\left(\int_0^{\frac{b}{a}\sqrt{a^2-b^2}} dy\right)dx = \frac{\pi ab}{4} \quad (57b)$$

From Eqs. (57a,b), we obtain

$$\bar{x} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{4a}{3\pi} \quad (58)$$

For the application example, we have $a = 6$, $b = 2$ and from Eq. (58):

$$\bar{x} = 2.546479089, \quad \bar{y} = 0.848826363, \quad \iint_A x\,dx\,dy = 24$$
$$\iint_A y\,dx\,dy = 8, \quad \iint_A dx\,dy = 9.424777961 \quad (59)$$

We shall now use the subparametric point transformations and explicit form of Jacobian derived in Eqs. (57)–(59) to obtain the above physical quantities (theoretical):

$$\iint_A dx\,dy = \frac{\alpha_0}{2}+\frac{(\alpha_1+\alpha_2)}{6} \quad (60a)$$

$$\iint_A t\,dx\,dy = \alpha_0\left[\frac{t_3}{2}+\frac{(t_1-t_3)}{6}+\frac{(t_2-t_3)}{6}+\frac{A_{11}^{(n)}(\underline{t})}{24}\right]$$
$$+\alpha_1\left[\frac{t_3}{6}+\frac{2(t_1-t_3)}{24}+\frac{(t_2-t_3)}{24}+\frac{2A_{11}^{(n)}(\underline{t})}{120}\right]$$
$$+\alpha_2\left[\frac{t_3}{6}+\frac{(t_1-t_3)}{24}+\frac{2(t_2-t_3)}{24}+\frac{2A_{11}^{(n)}(\underline{t})}{120}\right] \quad (60b)$$

where, $t = x, y$ and $n = 2(1)\,5$ for quadratic, cubic, quartic and quintic order curved triangle. We can then obtain the required integrals, viz, $\iint_A x\,dx\,dy, \iint_A y\,dx\,dy$ from Eq. (60b).

We have then compared the theoretical values of $\bar{x}, \bar{y}$ and that of the centroid found by two methods and these findings are tabulated in Tables 4a and b.

7. Conclusions

This paper concerns the use of isoparametric coordinate transformation to deal with the curved boundaries in the finite element method. This involves the transformation of each triangle in global/physical coordinate system (x, y) with one curved side and two straight sides into a standard triangle: $\{(\xi,\eta)/0 \leqslant \xi, \eta \leqslant 1, \xi+\eta \leqslant 1\}$ in the local or natural coordinate system (ξ, η). Isoparametric coordinate transformation for each curved triangle is obtained through point transformation of global (x, y) coordinates and so the original curves are implicitly replaced by parabolic, cubic, quartic and quintic curves depending on the degree of parametric coordinates. It is shown in this paper to find equations of quartic and quintic curves in terms of isoparametric coordinate transformations and to choose the coordinate points on the curved sides in a systematic way so that the implied curve is always a parabola passing through four points of the original curved boundary and so is a reasonable approximation to it. We have also shown that the point transformations are expressible as

$$t(\xi,\eta) = t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+\frac{n}{(n-1)}[nt_4-\{(n-1)t_2+t_1\}]\xi\eta$$
$$(t = x, y), \quad (n = 2, 3, 4, 5)$$

and the Jacobian required in the evaluation of integrals is also easily expressed as

$$J = \alpha_0+\alpha_1\xi+\alpha_2\eta$$

Finally we have considered an application example, which consists of the quarter ellipse:

$$\left\{(x, y)/x = 0, y = 0, \frac{x^2}{36}+\frac{y^2}{4} = 1\right\}$$

We take this as a curved triangle in the physical coordinate system (x, y). We have demonstrated the use of point transformations to determine the points along the curved boundary of the triangle and

Table 4

Triangle order/ discretisation on type	Location of points on boundary curve	Parametric equations of the curved boundary $x(\xi)$	$y(\xi)$	Arc length $s=\int_0^1\sqrt{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2}\,d\xi$
(a)				
Quadratic triangle				
First solution	$x_4=\sqrt{3.6}$, $y_4=\sqrt{3.6}$	$1.589466384\xi+4.4105336\xi^2$	$2+1.589466384\xi-3.589466384\xi^2$	6.520633016
Second solution	$x_4=3\sqrt{3}$, $y_4=\sqrt{2}$	$10.970562748\xi-4.970562748\xi^2$	$2-0.34314575\xi-1.656854249\xi^2$	6.643436878
Third solution	$x_4=3$, $y_4=\sqrt{3}$	6ξ	$2+0.92820323\xi-2.92820323\xi^2$	6.524585318
Fourth solution	$x_4=3\sqrt{3}$, $y_4=1$	$14.784609691\xi-8.784609691\xi^2$	$2-2\xi$	6.985197369
Cubic triangle	$x_4=5.123105626$, $y_4=1.041035209$, $x_5=3.123105626$, $y_5=1.707701875$	$11.053975317\xi-5.053975317\xi^2$	$2-0.315341559\xi-1.684658441\xi^2$	6.656076937
Quartic triangle				
First solution	$x_4=5.468626967$, $y_4=0.822875655$, $x_5=4.291502623$, $y_5=1.430500873$, $x_6=2.468626967$, $y_6=1.822875655$	$11.166010491\xi-5.166010491\xi^2$	$2-0.277996507\xi-1.722003493\xi^2$	6.673543003
Second solution	$x_4=5.527746982$, $y_4=0.777746982$, $x_5=4.370329309$, $y_5=1.370329309$, $x_6=2.527746982$, $y_6=1.777746981$	$11.481317237\xi-5.481317237\xi^2$	$2-0.518682763\xi-1.481317237\xi^2$	6.666616598
Third solution	$x_4=5.333240943$, $y_4=0.842582327$, $x_5=4.110987924$, $y_5=1.456776436$, $x_6=2.333240943$, $y_6=1.842582327$	$10.443951696\xi-4.443951696\xi^2$	$2-0.172894256\xi-1.827105744\xi^2$	6.566192725
Quintic triangle				
First solution	$x_4=5.641874854$, $y_4=0.680624847$, $x_5=4.862811813$, $y_5=1.220937271$, $x_6=3.662811813$, $y_6=1.620937271$, $x_7=2.041874542$, $y_7=1.880624847$	$11.261715888\xi-5.261715888\xi^2$	$2-0.2460947065\xi-1.753905294\xi^2$	6.688909768
Second solution	$x_4=5.6,\ y_4=2/3$, $x_5=4.8,\ y_5=1.2$, $x_6=3.6,\ y_6=1.6$, $x_7=2.0$, $y_7=1.866666666$	$11\xi-5\xi^2$	$2-(1/3)\xi-(5/3)\xi^2$	6.647862862
				Exact arc length $s=6.688222104$
(b)				
Cubic triangle	$x_4=4,\ y_4=1.490711985$, $x_5=2,\ y_5=1.885618083$	6ξ	$2-0.73764118\xi+2.40640886\xi^2$ $-3.66876767\xi^3$	6.572261275
Quartic triangle	$x_4=4.5,\ y_4=1.322875656$, $x_5=3,\ y_5=1.732050808$, $x_6=1.5,\ y_6=1.936491673$	6ξ	$2+0.5879269\xi-5.3482509\xi^2$ $+9.642078\xi^3-6.881754\xi^4$	6.604479265
Quintic triangle	$x_4=4.8,\ y_4=1.2$, $x_5=3.6,\ y_5=1.6$, $x_6=2.4,\ y_6=1.83303027$, $x_7=1.2,\ y_7=1.959591794$	6ξ	$2-0.502628\xi+4.261812\xi^2-18.575792\xi^3$ $+26.679724\xi^4-13.863112\xi^5$	6.620567528
				Exact arc length $s=6.688222104$

(a) Parametric equations $x(\xi), y(\xi)$ of the curved boundary and arc length for the triangle: $\{(x,y)/x=0, y=0, x^2/36+y^2/4\leqslant 1\}$: (Present theory (i.e., subparametric mapping)).
(b) Parametric equations $x(\xi), y(\xi)$ of the curved boundary and arc length for the triangle: $\{(x,y)/x=0, y=0, x^2/36+y^2/4\leqslant 1\}$: (Isoparametric mapping).

also the points in the interior of the curved triangle. These findings are tabulated in Tables 1 and 2. We have next demonstrated the use of point transformation to determine the arc length of the curved boundary and this is summarized in Tables 4a and b. An additional demonstration that uses the point transformation and the Jacobian is considered. We have thus evaluated certain integrals, for example,

Table 5

Triangle order/ discretisation type	Location of points on boundary curve	Explicit form of parametric equations $x=a_{10}\xi+a_{11}\xi\eta$ $y=b_{10}\eta+b_{11}\xi\eta$	Explicit form of Jacobian $J=\alpha_0+\alpha_1\xi+\alpha_2\eta$	$\bar{x}$ (centroid) $\bar{y}$ (centroid)	
(a)					
Quadratic triangle	$x_4=\sqrt{3.6}$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	8.1192885118
First solution	$y_4=\sqrt{3.6}$	$a_{11}=-4.41053361$	$\alpha_1=21.5367983$	I_x	18.0715731
		$b_{01}=2$	$\alpha_2=-8.821067232$	I_y	6.88
		$b_{11}=3.58946638$		$\bar{x}$	2.225758214
				$\bar{y}$	0.847364887
Second solution	$x_4=3\sqrt{3}$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.313708499
	$y_4=\sqrt{2}$	$a_{11}=4.970562748$	$a_1=9.941125498$	I_x	23.58882251
		$b_{01}=2$	$\alpha_2=9.941125498$	I_y	7.8627417
		$b_{11}=1.656854249$		$\bar{x}$	2.532635105
				$\bar{y}$	0.844211701
Third solution	$x_4=3$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	8.92320323
	$y_4=\sqrt{3}$	$a_{11}=0$	$\alpha_1=17.56921938$	I_x	20.78460969
		$b_{01}=2$	$\alpha_2=0$	I_y	7.785640646
		$b_{11}=2.92820323$		$\bar{x}$	2.327972287
				$\bar{y}$	0.872027713
Fourth solution	$x_4=3\sqrt{3}$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	8.92320323
	$y_4=1$	$a_{11}=8.784609691$	$\alpha_1=0$	I_x	23.35692194
		$b_{01}=2$	$\alpha_2=17.56921938$	I_y	6.92820323
		$b_{11}=0$		$\bar{x}$	2.61608314
				$\bar{y}$	0.775990762
Cubic triangle	$x_4=5.123105626$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.369316877
	$y_4=1.041035209$	$a_{11}=5.053975317$	$\alpha_1=10.10795063$	I_x	23.81079506
	$x_5=3.123105626$	$b_{01}=2$	$\alpha_2=10.10795063$	I_y	7.937894674
	$y_5=1.707701875$	$b_{11}=1.684658441$		$\bar{x}$	2.541358711
				$\bar{y}$	0.847222351
Quartic triangle	$x_4=5.468626967$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.44400699
First solution	$y_4=0.822875655$	$a_{11}=5.166010491$	$\alpha_1=10.33202098$	I_x	24.1111986
	$x_5=4.291502623$	$b_{01}=2$	$\alpha_2=10.33202098$	I_y	8.037066197
	$y_5=1.430500873$	$b_{11}=1.722003493$		$\bar{x}$	2.553068695
	$x_6=2.468626967$			$\bar{y}$	0.851022898
	$y_6=1.822875655$				
Second solution	$x_4=5.527746982$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.308422982
	$y_4=0.777746982$	$a_{11}=5.481317237$	$\alpha_1=8.887903422$	I_x	23.73872054
	$x_5=4.370329309$	$b_{01}=2$	$\alpha_2=10.96263447$	I_y	7.7985505381
	$y_5=1.370329309$	$b_{11}=1.481317237$		$\bar{x}$	2.55024085
	$x_6=2.527746982$			$\bar{y}$	0.837790181
	$y_6=1.777746981$				
Third solution	$x_4=5.333240943$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.308422975
	$y_4=0.842582327$	$a_{11}=4.443951696$	$\alpha_1=10.96263446$	I_x	23.39551612
	$x_5=4.110987924$	$b_{01}=2$	$\alpha_2=8.887903392$	I_y	7.912906838
	$y_5=1.456776436$	$b_{11}=1.827105744$		$\bar{x}$	2.513370544
	$x_6=2.333240943$			$\bar{y}$	0.850080283
	$y_6=1.842582327$				
Quintic triangle	$x_4=5.641874854$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.50781059
First solution	$y_4=0.680624847$	$a_{11}=5.261715888$	$\alpha_1=10.52343176$	I_x	24.36914204
	$x_5=4.862811813$	$b_{01}=2$	$\alpha_2=10.52343178$	I_y	8.123047347
	$y_5=1.220937271$	$b_{11}=1.75390529$		$\bar{x}$	2.563065577
	$x_6=3.662811813$			$\bar{y}$	0.854355192
	$y_6=1.620937271$				
	$x_7=2.041874542$				
	$y_7=1.880624847$				
Second solution	$x_4=5.6$, $y_4=2/3$	$a_{10}=6$	$\alpha_0=12$	$\bar{A}$	9.333333333
	$x_5=4.8$, $y_5=1.2$	$a_{11}=5$	$\alpha_1=10$	I_x	23.66666666
	$x_6=3.6$, $y_6=1.6$	$b_{01}=2$	$\alpha_2=10$	I_y	7.888888888
	$x_7=2.0$	$b_{11}=5/3$		$\bar{x}$	2.535714286
	$y_7=1.866666666$			$\bar{y}$	0.845238095

Exact values: $\bar{A}=9.424777961$, $I_x=24$, $I_y=8$, $\bar{x}=2.546479089$, $\bar{y}=0.848826363$

Triangle order	Location of points	Explicit form of parametric equations	Explicit form of Jacobian	$\bar{x}$ (centroid) $\bar{y}$ (centroid)	
(b)					
Cubic triangle	$x_4=4$	$x=6\xi$	$J=12-37.16091194\xi$	$\bar{A}$	9.096742596
	$y_4=1.490711985$	$y=2\eta-6.193485324\xi\eta$	$+66.74767092\xi^2$	I_x	23.12461092
	$x_5=2$,	$+11.12461182\xi^2\eta$	$+89.47012968\xi\eta$	I_y	7.377903058
	$y_5=1.885618083$	$+7.4555844139\xi\eta^2$		$\bar{x}$	2.542075988
				$\bar{y}$	0.811048898

Table 5 (*Continued*)

Triangle order	Location of points	Explicit form of parametric equations	Explicit form of Jacobian	$\bar{x}$ (centroid) $\bar{y}$ (centroid)	
Quartic triangle	$x_4=4.5$ $y_4=1.322875656$ $x_5=3$ $y_5=1.732050808$ $x_6=1.5$ $y_6=1.93649167$	$x=6\xi$ $y=2\eta+9.644828983\xi\eta$ $-38.04483392\xi^2\eta$ $-25.680546464\xi\eta^2$ $+46.851251712\xi^2\eta^2$ $+35.109361323\xi^3\eta$ $+18.623644715\xi\eta^3$	$J=12+57.8689739\xi$ $-228.2690035\xi^2$ $-308.1665575\xi\eta$ $+210.6561679\xi^3$ $+232.5034189\xi\eta^2$ $+714.3943562\xi^2\eta$ $-280.8748907\xi^3\eta$ $+148.9891577\xi\eta^3$	$\bar{A}$ I_x I_y $\bar{x}$ $\bar{y}$	9.272290945 23.053301581 7.933060468 2.486257357 0.855566809
Quintic triangle	$x_4=4.8$ $y_4=1.2$ $x_5=3.6$ $y_5=1.6$ $x_6=2.4$ $y_6=1.83303027$ $x_7=1.2$ $y_7=1.959591794$	$x=6\xi$ $y=2\eta-13.218408\xi\eta$ $+84.021464\xi^2\eta$ $+57.050476\xi\eta^2$ $-223.910981\xi^2\eta^2$ $-166.66666667\xi^3\eta$ $-89.156544\xi\eta^3$ $+104.1666667\xi^4\eta$ $+46.821848\xi\eta^4$ $+208.333333\xi^3\eta^2$ $+164.851634\xi^2\eta^3$	$J=6[2-13.218408\xi$ $+84.021464\xi^2$ $+114.10095\xi\eta$ $-166.66667\xi^3$ $-267.46963\xi\eta^2$ $-447.82196\xi^2\eta$ $+104.16667\xi^4$ $+187.28739\xi\eta^3$ $+416.6667\xi^3\eta$ $+494.55489\xi^2\eta^2]$	$\bar{A}$ I_x I_y $\bar{x}$ $\bar{y}$	9.304387916 23.36086391 7.923785998 2.510736238 0.85161819
Exact values: $\bar{A}=9.424777961$, $I_x=24$, $I_y=8$, $\bar{x}=2.546479089$, $\bar{y}=0.848826363$					

(a) Table of explicit form of subparametric point transformations and the Jacobian for the curved triangle $A:\{(x,y)/x=0, y=0, x^2/36+y^2/4\leqslant 1\}$ and also the values of integrals $\bar{A}=\iint_A \mathrm{d}x\,\mathrm{d}y$, $I_x=\iint_A x\,\mathrm{d}x\,\mathrm{d}y$, $I_y=\iint_A y\,\mathrm{d}x\,\mathrm{d}y$, $\bar{x}=I_x/\bar{A}=$ *x-centroid*, $\bar{y}=I_y/\bar{A}=$ *y-centroid*.
(b) Table of explicit form of isoparametric point transformations and the Jacobian for the curved triangle $A:\{(x,y)/x=0, y=0, x^2/36+y^2/4\leqslant 1\}$ and also the values of integrals $\bar{A}=\iint_A \mathrm{d}x\,\mathrm{d}y$, $I_x=\iint_A x\,\mathrm{d}x\,\mathrm{d}y$, $I_y=\iint_A y\,\mathrm{d}x\,\mathrm{d}y$, $\bar{x}=I_x/\bar{A}=$ *x-centroid*, $\bar{y}=I_y/\bar{A}=$ *y-centroid*.

$\iint_A t^\alpha\,\mathrm{d}x\,\mathrm{d}y$, $(t=x,y,\alpha=0,1)$ and found the physical quantities like area and centroid of the curved triangular elements. These findings are tabulated in Tables 5a and b. We hope that this study gives us the required impetus in the use of higher order curved triangular elements under the subparametric coordinate transformation.

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Appendix A

A.1. Quadratic ($n=2$)

$$N_1^{(2)}=[-\xi+2\xi^2],\quad N_2^{(2)}=[-\eta+2\eta^2]$$
$$N_3^{(2)}=[1-3\xi-3\eta+2\xi^2+4\xi\eta+2\eta^2]$$
$$N_4^{(2)}=4\xi\eta,\quad N_5^{(2)}=[4\eta-4\xi\eta-4\eta^2],\quad N_6^{(2)}=[4\xi-4\xi\eta-4\xi^2]$$

A.2. Cubic ($n=3$)

$$N_1^{(3)}=\tfrac{1}{2}[2\xi-9\xi^2+9\xi^3],\quad N_2^{(3)}=\tfrac{1}{2}[2\eta-9\eta^2+9\eta^3]$$
$$N_3^{(3)}=\tfrac{1}{2}[2-11\xi-11\eta+18\xi^2+36\xi\eta+18\eta^2-9\xi^3-27\xi^2\eta-27\xi\eta^2-9\eta^3]$$
$$N_4^{(3)}=\tfrac{9}{2}[-\xi\eta+3\xi^2\eta],\quad N_5^{(3)}=\tfrac{9}{2}[-\xi\eta+3\xi\eta^2]$$
$$N_6^{(3)}=\tfrac{9}{2}[-\eta+\xi\eta+4\eta^2-3\xi\eta^2-3\eta^3]$$
$$N_7^{(3)}=\tfrac{9}{2}[2\eta-5\xi\eta-5\eta^2+3\xi^2\eta+6\xi\eta^2+3\eta^3]$$
$$N_8^{(3)}=\tfrac{9}{2}[2\xi-5\xi\eta-5\xi^2+3\xi\eta^2+6\xi^2\eta+3\xi^3]$$
$$N_9^{(3)}=\tfrac{9}{2}[-\xi+\xi\eta+4\xi^2-3\xi^2\eta-3\xi^3],\quad N_{10}^{(3)}=27[\xi\eta-\xi^2\eta-\xi\eta^2]$$

A.3. Quartic ($n=4$)

$$N_1^{(4)}=\tfrac{1}{3}[-3\xi+22\xi^2-48\xi^3+32\xi^4]$$
$$N_2^{(4)}=\tfrac{1}{3}[-3\eta+22\eta^2-48\eta^3+32\eta^4]$$
$$N_3^{(4)}=\tfrac{1}{3}[3-25\xi-25\eta+70\xi^2+140\xi\eta+70\eta^2-80\xi^3-240\xi^2\eta-240\xi\eta^2-80\eta^3+32\xi^4+128\xi^3\eta+192\xi^2\eta^2+128\xi\eta^3+32\eta^4]$$
$$N_4^{(4)}=\tfrac{1}{3}[16\xi\eta-96\xi^2\eta+128\xi^3\eta]$$
$$N_5^{(4)}=\tfrac{1}{3}[12\xi\eta-48\xi^2\eta-48\xi\eta^2+192\xi^2\eta^2]$$
$$N_6^{(4)}=\tfrac{1}{3}[16\xi\eta-96\xi\eta^2+128\xi\eta^3],$$
$$N_7^{(4)}=\tfrac{1}{3}[16\eta-16\xi\eta-112\eta^2+96\xi\eta^2+224\eta^3-128\xi\eta^3-128\eta^4]$$
$$N_8^{(4)}=\tfrac{1}{3}[-36\eta+84\xi\eta+228\eta^2-48\xi^2\eta-432\xi\eta^2-384\eta^3+192\xi^2\eta^2+384\xi\eta^3+192\eta^4]$$
$$N_9^{(4)}=\tfrac{1}{3}[48\eta-208\xi\eta-208\eta^2+288\xi^2\eta+576\xi\eta^2+288\eta^3-128\xi^3\eta-384\xi^2\eta^2-384\xi\eta^3-128\eta^4]$$
$$N_{10}^{(4)}=\tfrac{1}{3}[48\xi-208\xi\eta-208\xi^2+288\xi\eta^2+576\xi^2\eta+288\xi^3-128\xi\eta^3-384\xi^2\eta^2-384\xi^3\eta-128\xi^4]$$
$$N_{11}^{(4)}=\tfrac{1}{3}[-36\xi+84\xi\eta+228\xi^2-48\xi\eta^2-432\xi^2\eta-384\xi^3+192\xi^2\eta^2+384\xi^3\eta+192\xi^4]$$
$$N_{12}^{(4)}=\tfrac{1}{3}[16\xi-16\xi\eta-112\xi^2+96\xi^2\eta+224\xi^3-128\xi^3\eta-128\xi^4]$$
$$N_{13}^{(4)}=\tfrac{1}{3}[288\xi\eta-672\xi^2\eta-672\xi\eta^2+384\xi^3\eta+768\xi^2\eta^2+384\xi\eta^3]$$
$$N_{14}^{(4)}=\tfrac{1}{3}[-96\xi\eta+480\xi^2\eta+96\xi\eta^2-384\xi^3\eta-384\xi^2\eta^2]$$
$$N_{15}^{(4)}=\tfrac{1}{3}[-96\xi\eta+96\xi^2\eta+480\xi\eta^2-384\xi\eta^3-384\xi^2\eta^2]$$

A.4. Quintic $(n=5)$

$$N_1^{(5)} = \tfrac{1}{24}[24\xi - 250\xi^2 + 875\xi^3 - 1250\xi^4 + 625\xi^5]$$

$$N_2^{(5)} = \tfrac{1}{24}[24\eta - 250\eta^2 + 875\eta^3 - 1250\eta^4 + 625\eta^5]$$

$$N_3^{(5)} = \tfrac{1}{24}[24 - 274\xi - 274\eta + 1125\xi^2 + 2250\xi\eta + 1125\eta^2 - 2125\xi^3 - 6375\xi^2\eta - 6375\xi\eta^2 - 2125\eta^3 + 1875\xi^4 + 7500\xi^3\eta + 11250\xi^2\eta^2 + 7500\xi\eta^3 + 1875\eta^4 - 625\xi^5 - 3125\xi^4\eta - 6250\xi^3\eta^2 - 6250\xi^2\eta^3 - 3125\xi\eta^4 - 625\eta^5]$$

$$N_4^{(5)} = \tfrac{1}{24}[-150\xi\eta + 1375\xi^2\eta - 3750\xi^3\eta + 3125\xi^4\eta]$$

$$N_5^{(5)} = \tfrac{1}{24}[-100\xi\eta + 750\xi^2\eta + 500\xi\eta^2 - 3750\xi^2\eta^2 - 1250\xi^3\eta + 6250\xi^3\eta^2]$$

$$N_6^{(5)} = \tfrac{1}{24}[-100\xi\eta + 750\xi\eta^2 + 500\xi^2\eta - 3750\xi^2\eta^2 - 1250\xi\eta^3 + 6250\xi^2\eta^3]$$

$$N_7^{(5)} = \tfrac{1}{24}[-150\xi\eta + 1375\xi\eta^2 - 3750\xi\eta^3 + 3125\xi\eta^4]$$

$$N_8^{(5)} = \tfrac{1}{24}[-150\eta + 150\xi\eta + 1525\eta^2 - 1375\xi\eta^2 - 5125\eta^3 + 3750\xi\eta^3 + 6875\eta^4 - 3125\xi\eta^4 - 3125\eta^5]$$

$$N_9^{(5)} = \tfrac{1}{24}[400\eta - 900\xi\eta - 3900\eta^2 - 3750\xi^2\eta^2 + 7750\xi\eta^2 + 12\,250\eta^3 + 500\xi^2\eta - 18\,750\xi\eta^3 + 6250\xi^2\eta^3 - 15\,000\eta^4 + 12\,500\xi\eta^4 + 6250\eta^5]$$

$$N_{10}^{(5)} = \tfrac{1}{24}[-600\eta + 2350\xi\eta + 5350\eta^2 + 18\,750\xi^2\eta^2 - 17\,750\xi\eta^2 - 14\,750\eta^3 - 3000\xi^2\eta + 1250\xi^3\eta + 33\,750\xi\eta^3 + 16\,250\eta^4 - 18\,750\xi^2\eta^3 - 6250\xi^3\eta^2 - 18\,750\xi\eta^4 - 6250\eta^5]$$

$$N_{11}^{(5)} = \tfrac{1}{24}[600\eta - 3850\xi\eta - 3850\eta^2 + 17\,750\xi\eta^2 + 8875\eta^3 + 8875\xi^2\eta - 26\,250\xi\eta^3 + 18\,750\xi^2\eta^3 - 26\,250\xi^2\eta^2 - 8750\xi^3\eta - 8750\eta^4 + 12\,500\xi^3\eta^2 + 12\,500\xi\eta^4 + 3125\xi^4\eta + 3125\eta^5]$$

$$N_{12}^{(5)} = \tfrac{1}{24}[600\xi - 3850\xi\eta - 3850\xi^2 + 17\,750\xi^2\eta + 8875\xi^3 + 8875\xi\eta^2 - 26\,250\xi^3\eta + 18\,750\xi^3\eta^2 - 26\,250\xi^2\eta^2 - 8750\xi\eta^3 - 8750\xi^4 + 12\,500\xi^2\eta^3 + 12\,500\xi^4\eta + 3125\xi\eta^4 + 3125\xi^5]$$

$$N_{13}^{(5)} = \tfrac{1}{24}[-600\xi + 2350\xi\eta + 5350\xi^2 + 18\,750\xi^2\eta^2 - 17\,750\xi^2\eta - 14\,750\xi^3 - 3000\xi\eta^2 + 1250\xi\eta^3 + 33\,750\xi^3\eta - 18\,750\xi^3\eta^2 - 6250\xi^2\eta^3 - 18\,750\xi^4\eta + 16\,250\xi^4 - 6250\xi^5]$$

$$N_{14}^{(5)} = \tfrac{1}{24}[400\xi - 900\xi\eta - 3900\xi^2 - 3750\xi^2\eta^2 + 7750\xi^2\eta + 12\,250\xi^3 + 500\xi\eta^2 - 18\,750\xi^3\eta + 6250\xi^3\eta^2 - 15\,000\xi^4 + 12\,500\xi^4\eta + 6250\xi^5]$$

$$N_{15}^{(5)} = \tfrac{1}{24}[-150\xi + 150\xi\eta + 1525\xi^2 - 1375\xi^2\eta - 5125\xi^3 + 3750\xi^3\eta + 6875\xi^4 - 3125\xi^4\eta - 3125\xi^5]$$

$$N_{16}^{(5)} = \frac{1}{24}[6000\xi\eta - 23\,500\xi\eta^2 - 23\,500\xi^2\eta + 60\,000\xi^2\eta^2 + 30\,000\xi\eta^3 + 30\,000\xi^3\eta - 37\,500\xi^3\eta^2 - 37\,500\xi^2\eta^3 - 12\,500\xi\eta^4 - 12\,500\xi^4\eta]$$

$$N_{17}^{(5)} = \tfrac{1}{24}[1000\xi\eta - 1000\xi\eta^2 - 8500\xi^2\eta + 7500\xi^2\eta^2 + 20\,000\xi^3\eta - 12\,500\xi^3\eta^2 - 12\,500\xi^4\eta]$$

$$N_{18}^{(5)} = \tfrac{1}{24}[1000\xi\eta - 1000\xi^2\eta - 8500\xi\eta^2 + 7500\xi^2\eta^2 + 20\,000\xi\eta^3 - 12\,500\xi^2\eta^3 - 12\,500\xi\eta^4]$$

$$N_{19}^{(5)} = \tfrac{1}{24}[-3000\xi\eta + 6750\xi\eta^2 + 21\,750\xi^2\eta - 41\,250\xi^2\eta^2 - 3750\xi\eta^3 - 37\,500\xi^3\eta + 37\,500\xi^3\eta^2 + 18\,750\xi^2\eta^3 + 18\,750\xi^4\eta]$$

$$N_{20}^{(5)} = \tfrac{1}{24}[750\xi\eta - 4500\xi\eta^2 - 4500\xi^2\eta + 26\,250\xi^2\eta^2 + 3750\xi\eta^3 + 3750\xi^3\eta - 18\,750\xi^3\eta^2 - 18\,750\xi^2\eta^3]$$

$$N_{21}^{(5)} = \tfrac{1}{24}[-3000\xi\eta + 6750\xi^2\eta + 21\,750\xi\eta^2 - 41\,250\xi^2\eta^2 - 3750\xi^3\eta - 37\,500\xi\eta^3 + 37\,500\xi^2\eta^3 + 18\,750\xi^3\eta^2 + 18\,750\xi\eta^4]$$

Appendix B

B.1. Quartic $(n=4)$

$$a_{11}^{(4)} = [-22t_1 - 22t_2 - 96t_3 + 16t_4 + 12t_5 + 16t_6 + 288t_{13} - 96t_{14} - 96t_{15}]$$

$$a_{21}^{(4)} = [0t_1 + 48t_2 + 192t_3 - 96t_4 - 48t_5 - 672t_{13} + 480t_{14} + 96t_{15}]$$

$$a_{12}^{(4)} = [48t_1 + 0t_2 + 192t_3 - 48t_5 - 96t_6 - 672t_{13} + 96t_{14} + 480t_{15}]$$

$$a_{31}^{(4)} = [0t_1 - 32t_2 - 96t_3 + 128t_4 + 384t_{13} - 384t_{14}]$$

$$a_{22}^{(4)} = [0t_1 + 0t_2 - 192t_3 + 192t_5 + 768t_{13} - 384t_{14} - 384t_{!5}]$$

$$a_{13}^{(4)} = [-32t_1 + 0t_2 - 96t_3 + 128t_6 + 384t_{13} - 384t_{15}]$$

B.2. Quintic $(n=5)$

$$a_{11}^{(5)} = [-250t_1 - 250t_2 - 1750t_3 - 150t_4 - 100t_5 - 100t_6 - 150t_7 + 6000t_{16} + 1000t_{17} + 1000t_{18} - 3000t_{19} + 750t_{20} - 3000t_{21}]$$

$$a_{21}^{(5)} = [0t_1 + 875t_2 + 5500t_3 + 1375t_4 + 750t_5 + 500t_6 - 23\,500t_{16} - 8500t_{17} - 1000t_{18} + 21\,750t_{19} - 4500t_{20} + 6750t_{21}]$$

$$a_{12}^{(5)} = [0t_1 + 875t_2 + 5500t_3 + 500t_5 + 750t_6 + 1375t_7 - 23\,500t_{16} - 1000t_{17} - 8500t_{18} + 6750t_{19} - 4500t_{20} + 21\,750t_{21}]$$

$$a_{31}^{(5)} = [0t_1 - 1250t_2 - 6250t_3 - 3750t_4 - 1250t_5 + 30\,000t_{16} + 20\,000t_{17} - 37\,500t_{19} + 3750t_{20} - 3750t_{21}]$$

$$a_{22}^{(5)} = [0t_1 + 0t_2 - 11\,250t_3 - 3750t_5 - 3750t_6 + 60\,000t_{16} + 7500t_{17} + 7500t_{18} - 41\,250t_{19} + 26\,250t_{20} - 41\,250t_{21}]$$

$$a_{13}^{(5)} = [-1250t_1 + 0t_2 - 6250t_3 - 1250t_4 - 3750t_5 + 30\,000t_{16} + 20\,000t_{18} - 3750t_{19} + 3750t_{20} - 37\,500t_{21}]$$

$$a_{41}^{(5)} = [0t_1 + 625t_2 + 2500t_3 + 3125t_4 - 12\,500t_{16} - 12\,500t_{17} + 18\,750t_{19}]$$

$$a_{32}^{(5)} = [0t_1 + 0t_2 + 6250t_3 + 6250t_5 - 37\,500t_{16} - 12\,500t_{17} + 37\,500t_{19} - 18\,750t_{20} + 18\,750t_{21}]$$

$$a_{23}^{(5)} = [0t_1 + 0t_2 + 6250t_3 + 6250t_6 - 37\,500t_{16} - 12\,500t_{18} + 18\,750t_{19} - 18\,750t_{20} + 37\,500t_{21}]$$

$$a_{14}^{(5)} = [625t_1 + 0t_2 + 2500t_3 + 3125t_7 - 12\,500t_{16} - 12\,500t_{18} + 18\,750t_{21}]$$

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