



# Effects of radiation and heat source on MHD flow of a viscoelastic liquid and heat transfer over a stretching sheet

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## Abstract

We study the MHD flow and also heat transfer in a viscoelastic liquid over a stretching sheet in the presence of radiation. The stretching of the sheet is assumed to be proportional to the distance from the slit. Two different temperature conditions are studied, namely (i) the sheet with prescribed surface temperature (PST) and (ii) the sheet with prescribed wall heat flux (PHF). The basic boundary layer equations for momentum and heat transfer, which are non-linear partial differential equations, are converted into non-linear ordinary differential equations by means of similarity transformation. The resulting non-linear momentum differential equation is solved exactly. The energy equation in the presence of viscous dissipation (or frictional heating), internal heat generation or absorption, and radiation is a differential equation with variable coefficients, which is transformed to a confluent hypergeometric differential equation using a new variable and using the Rosseland approximation for the radiation. The governing differential equations are solved analytically and the effects of various parameters on velocity profiles, skin friction coefficient, temperature profile and wall heat transfer are presented graphically. The results have possible technological applications in liquid-based systems involving stretchable materials.

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## 1. Introduction

An interesting fluid mechanical application is found in polymer extrusion processes, where the object on passing between two closely placed solid blocks is stretched into a liquid region. The stretching imparts a unidirectional orientation to the extrudate, thereby improving its mechanical properties [1].

The liquid is basically meant to cool the stretching sheet whose property as a final product depends greatly on the rate at which it is cooled. It is imperative therefore to consider two important aspects in this physically interesting problem:

- (i) Proper choice of cooling liquid.
- (ii) Regulation of the flow of the cooling liquid, due to the stretching sheet, to achieve a desired rate of cooling appropriate for successfully arriving at a sought final product.

The cooling liquid in earlier times was chosen to be the abundantly available water, but this has the

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drawback of rapidly quenching the heat leading to sudden solidification of the stretching sheet. From the standpoint of desirable properties of the final product (solidified stretching sheet) water does not seem to be the ideal cooling liquid. A careful examination of the needs in the system suggests that it is advantageous to have a controlled cooling system. An electrically conducting polymeric liquid seems to be a good candidate for such an application situation because its flow can be regulated by external means through a magnetic field. Further, this arrangement does not involve any moving parts and does not tamper with the flow that we are investigating theoretically. The problem is a prototype for many other practical problems also, akin to the polymer extrusion process (Fig. 1), like

- drawing, annealing and tinning of copper wires,

- continuous stretching, rolling and manufacturing of plastic film and artificial fibers,
- extrusion of a material and heat-treated materials that travel between feed and wind-up rollers or on conveyor belts.

The delicate nature of the problem dictates the fact that the magnitude of the stretching rate has to be small. This also ensures that the stretching material released between the two solid blocks into the liquid continues to be a plane surface rather than a curved one. Mathematical manageability is therefore at its best in the problem.

A number of works are presently available that follow the pioneering classical works of Sakiadis [2], Tsou et al. [3] and Crane [4]. The following Table lists some relevant works that pertain to viscoelastic cooling liquids:

| Author/s                    | Type of visco-elastic liquid                   | Nature of temperature boundary condition | Remarks                      |
|-----------------------------|--|--|------------------------------|
| Rajagopal et al. [5]        | Second-order liquid                            | —  | Heat transfer not considered |
| Andersson et al. [6]        | Walters' liquid B                              | —  | Heat transfer not considered |
| Siddappa and Subhash [7]    | Walters' liquid B                              | —  | Heat transfer not considered |
| Rajagopal et al. [8]        | Second-order liquid                            | —  | Heat transfer not considered |
| McLeod and Rajagopal [9]    | Second-order liquid                            | —  | Heat transfer not considered |
| Bujurke et al. [10]         | Second-order liquid                            | PST                                      | —                            |
| Char and Chen [11]          | Walters' liquid B                              | PHF                                      | —                            |
| Dandapat and Gupta [12]     | Second-order liquid                            | PST                                      | —                            |
| Chang [13]                  | Second-order liquid                            | —  | Heat transfer not considered |
| Rollins and Vajravelu [14]  | Second-order liquid                            | Variable PST and variable PHF            | —                            |
| Andersson and Dandapat [15] | Second-order liquid                            | —  | Heat transfer not considered |
| Lawrence and Rao [16]       | Second-order liquid                            | —  | —                            |
| Andersson [17]              | Walters' liquid B                              | —  | Heat transfer not considered |
| Kelly et al. [18]           | Walters' liquid B                              | —  | Heat transfer not considered |
| Maneschy et al. [19]        | Second-order liquid                            | —  | Heat transfer not considered |
| Bhatnagar et al. [20]       | Oldroyd-B liquid                               | —  | Heat transfer not considered |
| Lawrence and Rao [21]       | Walters' liquid B                              | —  | Heat transfer not considered |
| Subhash and Veena [22]      | Walters' liquid B                              | PST and PHF                              | —                            |
| Subhash et al. [23]         | Weak electrically conducting Walters' liquid B | PST and PHF                              | —                            |
| Sonth et al. [24]           | Walters' liquid B                              | PST and PHF                              | —                            |

(see also references therein).

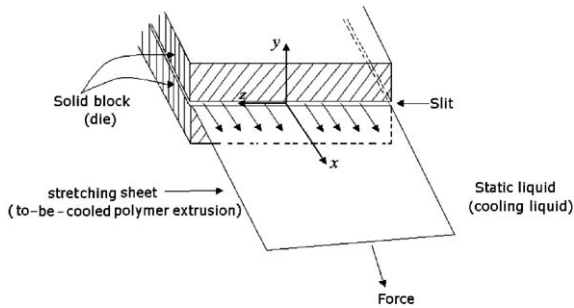


Fig. 1. Schematic of a polymer extrusion process.

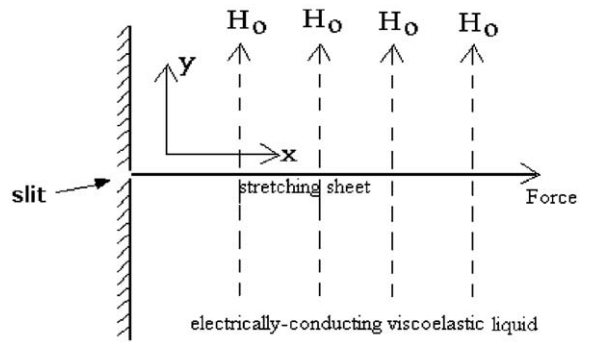


Fig. 2. Schematic of the two-dimensional stretching sheet problem.

In most of the investigations involving heat transfer, we observe that either the constant prescribed surface temperature (PST) or constant prescribed wall heat flux (PHF) boundary condition is assumed. It is a well-known fact that constant PST and PHF assumed by many are difficult to realize (see [25]). Also if the final product that is obtained after cooling needs to be non-uniform in terms of properties warranted by an application, then the physically realistic “variable PHF” is the appropriate temperature boundary condition.

Heat generation or absorption may become important in weak-electrically conducting polymeric liquids due to the non-isothermal situation they are in and also due to the presence of cation/anion salts dissolved in them. An example of such a liquid is polyethylene oxide.

In all the stretching sheet problems (both hydrodynamic and hydromagnetic) mentioned earlier, radiation effect has not been considered. We know that the radiation effect is important under many non-isothermal situations. If the entire system involving the polymer extrusion process is placed in a thermally controlled environment, then radiation could become important. The knowledge of radiation heat transfer in the system can perhaps lead to a desired product with a sought characteristic. Radiation effect on viscoelastic flows has been considered by Raptis [26], and Raptis and Perdikis [27]. In this paper, we consider the effect of radiation and temperature-dependent heat source on the MHD viscoelastic flow and convective heat transfer over a stretching sheet, with variable PST/ PHF.

## 2. Mathematical formulation and solution

We consider two-dimensional motion in the  $xy$ -plane on the stretching sheet, as the flow in any parallel plane is identical due to the assumptions discussed in the earlier section. Further, we discuss the motion above the stretching sheet, as the flow on the under side is essentially similar. The  $x$ -axis is taken along the plate in the direction of its motion and  $y$ -axis perpendicular to it (see Fig. 2). We consider the flow of an incompressible and electrically conducting visco-elastic Walters’ liquid B model past the flat and impermeable stretching sheet. The liquid is confined to the half-space  $y > 0$  above the sheet. By applying two equal and opposite forces along the  $x$ -axis the sheet is being stretched with a speed proportional to the distance from the origin  $x = 0$ . The assumptions are such that they facilitate the use of boundary layer theory (see [28]).

The axial and transverse velocities  $u$  and  $v$  for the problem at hand are governed by the following ordinary differential equation by virtue of the similarity transformation [29]:

$$f''' - f'^2 + ff'' = Qf' + k_1\{2f'f''' - f''^2 - ff''''\}, \quad (1)$$

where  $u = cx f'(\eta)$  and  $v = -\sqrt{c\nu} f(\eta)$  velocity components  $x$  and  $y$ -directions, respectively,  $\rho$  is the density of the liquid,  $\mu$  is the limiting viscosity at small rates of shear,  $k_0$  is the first moment of the distribution function of relaxation times,  $\sigma$  is the electrical conductivity of the liquid and prime denotes differentiation with respect to  $\eta$ . The non-dimensional param-

eters appearing in the equation are defined below:

$$Q = \left( H_0^2 \frac{\sigma}{c\rho} \right), \quad \text{Chandrasekhar number } (\sqrt{Q} \text{ is called Hartmann number),}$$

$$k_1 = \left( \frac{ck_0}{\mu} \right), \quad \text{viscoelastic parameter.}$$

We note here that the equation for the stretching sheet problem involving a second-order liquid can be obtained from Eq. (1) by replacing  $k_1$  with  $-k_1$ . In deriving Eq. (1) it has been assumed that the fluid has weak electrical conductivity. Since the cooling fluid is poorly conducting, any charge that might be created gets accumulated on the extrusion and is not a serious factor because of the not-so-strong dynamics that is prevalent around the sheet.

The assumed boundary conditions are:

$$f(0) = 0, \quad f'(0) = 1, \quad (2)$$

$$f'(\infty) \rightarrow 0 \quad \text{and} \quad f''(\infty) = 0. \quad (3)$$

The condition we have assumed on  $u$  and  $v$  signify that there is a linear stretching of the sheet in the  $x$ -direction and the stretching is such that it induces only a weak transverse velocity component. Further, it is assumed that the stretching of the sheet does not induce dynamics at distances far away from the sheet. In other words, fluid dynamics is restricted to the immediate non-isothermal neighborhood of the sheet. It is also important to mention here that Eq. (1) is higher order than the Navier–Stokes equation and in general would require would need additional boundary conditions. It is because, we are working in an unbounded domain we are able to add an asymptotic condition. This issue cannot be overemphasized as a lot of errors have been made on this account, singular perturbation problems being treated as regular. This issue is discussed at length in Rajagopal and Gupta [30], Rajagopal and Kaloni [31], Rajagopal [32,33].

From the mathematics point of view, Eq. (1) is quite interesting because one can reduce this fourth-order differential equation to a third-order equation. Indeed, using the transformation  $f' = Y$ , one can write

$$f'' = \frac{\partial Y}{\partial \eta} = \frac{\partial Y}{\partial f} \frac{\partial f}{\partial \eta} = \dot{Y} Y.$$

Similarly, we can write

$$f''' = Y^2 \ddot{Y} + Y \dot{Y}^2$$

and

$$f'''' = 2Y^2 \dot{Y} \ddot{Y} + Y \dot{Y}^3 + Y^3 \ddot{Y} + 2Y^2 \dot{Y} \ddot{Y}.$$

Substituting the above expressions for  $f'$ ,  $f''$ ,  $f'''$  and  $f''''$  into Eq. (1), we get a third-order ordinary differential equation which may be solved numerically or may be identified as being some well-known classical equation. In the present paper, we adopt a different approach of seeking an exact solution of Eq. (1) satisfying Eqs. (2) and (3).

Analyzing the nature of  $f(\eta)$  at  $\eta=0$  and as  $\eta \rightarrow \infty$ , it seems appropriate to take the following form for  $f$ :

$$f(\eta) = A_1 + B_1 \exp[-\beta\eta].$$

On using the boundary conditions (2) and (3),  $A_1$  and  $B_1$  can be determined and the solution turns out to be (see [9,12,34])

$$f = \frac{1 - \exp[-\beta\eta]}{\beta}, \quad \beta > 0. \quad (4)$$

At this point we call attention to the paper by Chang [13]. He assumed the following form of  $f(\eta)$ :

$$f(\eta) = A_0 + B_0 \exp[-\beta\eta] \cos(\omega\eta),$$

where  $A_0$ ,  $B_0$  and  $\omega$  are to be determined. We note that this form also conforms to the nature of  $f(\eta)$  at  $\eta=0$  and as  $\eta \rightarrow \infty$  provided the following conditions hold:

$$A_0 = -B_0 = \frac{1}{\beta}$$

and three other conditions involving  $\beta$  and  $\omega$ . This proves the existence of another solution of Eq. (1), in addition to the first one (Eq. (4)) derived earlier. We may thus infer that the solution of Eq. (1) is not unique and find ourselves in a situation of having to make a decision on the appropriateness of one of the solutions. This question was addressed by Lawrence and Rao [21] who advocated, with proper physical reasoning, that the solution (4) is the more realistic one compared to the second solution. We abide by their counsel and further the present analysis with this realistic solution.

Substitution of Eq. (4) into Eq. (1) reveals that Eq. (4) is a solution of the non-linear differential equation (1) if

$$\beta = \sqrt{\frac{1+Q}{1-k_1}} \quad (1 \leq \beta < \infty). \tag{5}$$

We note here that  $\beta$  is related to an important non-dimensional quantity as will be seen in the next section.

Using Eq. (4) in the expressions for  $u$  and  $v$  we obtain

$$\begin{aligned} u &= cx \exp[-\beta\eta] \quad \text{and} \\ v &= -\sqrt{cv} \left( \frac{1 - \exp[-\beta\eta]}{\beta} \right), \end{aligned} \tag{6a,b}$$

where  $\beta$  is given by Eq. (5). Having obtained the velocity distribution we now move on to find the skin friction coefficient at the stretching sheet.

### 3. Skin friction

The wall shearing stress  $\tau_w$  on the surface of the stretching sheet can be easily calculated from the expression:

$$\tau_w = -\mu \left( \frac{\partial u}{\partial y} \right)_{y=0}. \tag{7}$$

Substituting Eq. (6a) in Eq. (7), we get

$$\tau_w = \mu cx \beta \sqrt{\frac{c}{v}}. \tag{8}$$

The local skin-friction coefficient or frictional drag coefficient is

$$C_f = \frac{\tau_w}{\mu cx \sqrt{c/v}} = \beta, \tag{9}$$

where  $\beta$  is given by Eq. (5). In the next section we discuss the heat transport in the aforementioned forced convective flow due to a stretching sheet.

### 4. Heat transfer

The governing boundary layer heat transport equation with viscous dissipation, temperature-dependent

internal heat generation and radiation is

$$\begin{aligned} &cx \exp[-\beta\eta] \frac{\partial T}{\partial x} - \sqrt{cv} \left( \frac{1 - \exp[-\beta\eta]}{\beta} \right) \frac{\partial T}{\partial y} \\ &= \alpha^* \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p} \left( \frac{\partial u}{\partial y} \right)^2 \\ &+ \frac{Q^*}{\rho C_p} (T - T_\infty) - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}, \end{aligned} \tag{10}$$

where  $T$  is the temperature of the liquid,  $\alpha^*$  is the thermal diffusivity,  $C_p$  specific heat at constant pressure and  $Q^*$  uniform heat source. In writing Eq. (10), we are aware that we are making a rather serious assumption that the thermodynamic quantities associated with a viscoelastic fluid are the same as a Newtonian fluid and this in general is not true.

By using Rosseland approximation for radiation (see [35]), the radiative heat flux  $q_r$  is given by

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial(T^4)}{\partial y}, \tag{11}$$

where  $\sigma^*$  is the Stefan–Boltzmann constant and  $k^*$  is the mean absorption coefficient.

We now expand  $T^4$  in a Taylor series about  $T_\infty$  as follows:

$$\begin{aligned} T^4 &= T_\infty^4 + 4T_\infty^3(T - T_\infty) \\ &+ 6T_\infty^2(T - T_\infty)^2 + \dots \end{aligned}$$

Neglecting higher-order terms in the above equation beyond the first degree in  $(T - T_\infty)$ , we get

$$T^4 \cong -3T_\infty^4 + 4T_\infty^3 T. \tag{12}$$

By employing Eqs. (11) and (12), Eq. (10) becomes

$$\begin{aligned} &cx \exp[-\beta\eta] \frac{\partial T}{\partial x} - \sqrt{cv} \left( \frac{1 - \exp[-\beta\eta]}{\beta} \right) \frac{\partial T}{\partial y} \\ &= \left( \alpha^* + \frac{16\sigma^* T_\infty^3}{3\rho C_p k^*} \right) \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p} \left( \frac{\partial u}{\partial y} \right)^2 \\ &+ \frac{Q^*}{\rho C_p} (T - T_\infty). \end{aligned} \tag{13}$$

From the above equation it is apparent that the effect of radiation is to enhance the thermal diffusivity.

The thermal boundary conditions for solving Eq. (13) depend on the type of heating process under consideration. We consider two different heating processes, namely

- (i) PST, and
- (ii) PHF.

#### 4.1. PST

The prescribed power law surface temperature is considered to be a power of  $x$  in the form

$$T = T_w = T_\infty + A \left( \frac{x}{l} \right)^s \quad \text{at } y = 0,$$

$$T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty, \quad (14a,b)$$

where  $A$  is a constant,  $l$  is the characteristic length,  $T_w$  is the wall (sheet) temperature,  $s$  is the variable heat flux index and  $T_\infty$  is the constant temperature far away from the sheet. We now define a non-dimensional temperature  $\theta(\eta)$  as

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad (15)$$

where

$$T - T_\infty = A \left( \frac{x}{l} \right)^s \theta(\eta) \quad \text{and} \quad T_w - T_\infty = A \left( \frac{x}{l} \right)^s.$$

Substitution of Eq. (15) in the energy equation (13) leads to the following equation:

$$(1 + N_R)\theta'' + \frac{Pr}{\beta}(1 - \exp[-\beta\eta])\theta' - Pr(s \exp[-\beta\eta] - \alpha)\theta = -Pr E(xl)^{s-2} \beta^2 \exp[-2\beta\eta],$$

where prime denotes differentiation with respect to  $\eta$  and the non-dimensional parameters are defined

as given below:

$$N_R = \left( \frac{16\sigma^* T_\infty^3}{3kk^*} \right), \quad \text{radiation number,}$$

$$Pr = \left( \frac{\nu}{\alpha^*} \right), \quad \text{Prandtl number,}$$

$$\alpha = \left( \frac{Q^*}{c\rho C_p} \right), \quad \text{heat source/sink parameter,}$$

$$E = \left( \frac{c^2 l^2}{AC_p} \right), \quad \text{Eckert number.}$$

Obviously, we get an  $x$ -independent similarity equation from the above when  $s = 2$  and this yields

$$(1 + N_R)\theta'' + \frac{Pr}{\beta}(1 - \exp[-\beta\eta])\theta' - Pr(2 \exp[-\beta\eta] - \alpha)\theta = -Pr E \beta^2 \exp[-2\beta\eta]. \quad (16)$$

The boundary condition in terms of  $\theta$  can be obtained from Eqs. (14) and (15) as

$$\theta = 1 \quad \text{at } \eta = 0,$$

$$\theta \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (17)$$

Eq. (16) is linear in  $\theta$  and we now transform the same into a confluent hypergeometric equation by using the transformation

$$\xi = -R \exp[-\beta\eta], \quad (18)$$

where  $R = Pr/\beta^2$ . Substituting Eq. (18) into Eq. (16), we get

$$(1 + N_R)\xi\ddot{\theta} + [4(1 + N_R) - R - \xi]\dot{\theta} + \left( 2 + \frac{R\alpha}{\xi} \right)\theta = -Pr E \frac{\xi}{R^2}, \quad (19)$$

where overdot denotes differentiation with respect to  $\xi$ .

The boundary conditions in Eq. (17), in terms of  $\xi$  translate to

$$\theta(\xi = -R) = 1 \quad \text{and} \quad \theta(0) = 0. \quad (20)$$

The solution of Eq. (19) satisfying the conditions (20) in terms of Kummer’s function (see [32]) is

$$\theta(\xi) = \frac{\{1 + Pr E[4(1 + N_R) - 2R + R\alpha]^{-1}\} \left(\frac{-\xi}{R}\right)^{\left(\frac{\lambda_1+d_1}{2}\right)} F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, \xi\right]}{F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, -R\right]} - Pr E[4(1 + N_R) - 2R + R\alpha]^{-1} \left(\frac{\xi}{R}\right)^2, \tag{21}$$

where

$$d_1 = \sqrt{\lambda_1^2 - 4\lambda_2}, \quad \lambda_1 = \frac{R}{1 + N_R} \quad \text{and} \quad \lambda_2 = \alpha\lambda.$$

The solution in Eq. (21) can be written in terms of  $\eta$  as

$$\theta(\eta) = \frac{\{1 + Pr E[4(1 + N_R) - 2R + R\alpha]^{-1}\} \exp\left[-\beta\left(\frac{\lambda_1+d_1}{2}\right)\eta\right] F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, -R \exp[-\beta\eta]\right]}{F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, -R\right]} - Pr E[4(1 + N_R) - 2R + R\alpha]^{-1} \exp[-2\beta\eta]. \tag{22}$$

The non-dimensional wall temperature gradient derived from Eq. (22) is

$$\dot{\theta}(0) = \frac{\{1 + Pr E[4(1 + N_R) - R + R\alpha]^{-1}\}}{F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, -R\right]} \times \left\{ \begin{aligned} & -\beta\left(\frac{\lambda_1+d_1}{2}\right) F\left[\frac{\lambda_1+d_1-4}{2}, d_1 + 1, -R\right] \\ & + \frac{R\beta}{2} \left(\frac{\lambda_1+d_1-4}{d_1+1}\right) F\left[\frac{\lambda_1+d_1-2}{2}, d_1 + 2, -R\right] \end{aligned} \right\} + 2\beta Pr E[4(1 + N_R) - 2R + R\alpha]^{-1} \tag{23}$$

and the local heat flux can be expressed as

$$q_w = -k \left(\frac{\partial T}{\partial x}\right)_{y=0} = -kA \sqrt{\frac{c}{v}} \left(\frac{x}{l}\right)^2 \dot{\theta}(0).$$

The expressions in Eqs. (22) and (23) are numerically evaluated for several values of the parameters  $E, k_1, N_R, Pr, Q$  and  $\alpha$ , and the results are discussed in the last section. We now move on to discuss the case of a temperature boundary condition involving a PHF.

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#### 4.2. PHF

The power law heat flux on the wall surface is considered to be a power of  $x$  in the form

$$\begin{aligned} -k \frac{\partial T}{\partial y} &= q_w = D \left(\frac{x}{l}\right)^s \quad \text{at } y = 0, \\ T &\rightarrow T_\infty \quad \text{as } y \rightarrow \infty, \end{aligned} \tag{24}$$

where  $D$  is a constant and  $k$  is the thermal conductivity. We now define a non-dimensional temperature  $g(\eta)$  as

$$g(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{25}$$

where

$$T - T_\infty = \frac{D}{k} \left(\frac{x}{l}\right)^s \sqrt{\frac{v}{c}} g(\eta) \tag{26}$$

and

$$T_w - T_\infty = \frac{D}{k} \left(\frac{x}{l}\right)^s \sqrt{\frac{v}{c}}.$$

In spite of the fact that  $g(\eta)$  in Eq. (25) is the same as  $\theta(\eta)$  defined in Eq. (15) for PST case, we prefer to use a different notation for the PHF case. Substitution of Eq. (25) in the energy equation (13) leads to the following equation:

$$(1 + N_R)g'' + \frac{Pr}{\beta}(1 - \exp[-\beta\eta])g' - Pr(s \exp[-\beta\eta] - \alpha)\theta = -Pr E_s (xl)^{s-2} \beta^2 \exp[-2\beta\eta].$$

Obviously, we get an  $x$ -independent similarity equation from the above when  $s = 2$  and this yields

$$(1 + N_R)g'' + \frac{Pr}{\beta}(1 - \exp[-\beta\eta])g' - Pr(2 \exp[-\beta\eta] - \alpha)g = -Pr E_s \beta^2 \exp[-2\beta\eta]. \tag{27}$$

The boundary conditions in terms of  $g$  can be obtained from Eqs. (24) and (25) as

$$g'(0) = -1 \quad \text{and} \quad g(\infty) = 0, \tag{28}$$

where  $E_s = (E/D)\sqrt{c/v}$  scaled Eckert number, prime denotes differentiation with respect to  $\eta$  and all other parameters are as defined in the PST case, but wherever  $A$  is involved in the equations of PST case it is to be replaced by  $D$  of PHF. Substituting Eq. (18) into Eqs. (27) and (28), we get

$$(1 + N_R)\xi\ddot{g} + [4(1 + N_R) - R - \xi]\dot{g} + \left(2 + \frac{R\alpha}{\xi}\right)g = -\frac{Pr E_s}{R^2}\xi, \tag{29}$$

$$\dot{g}(-R) = -\frac{1}{R\beta} \quad \text{and} \quad g(0) = 0, \tag{30}$$

where overdot denotes differentiation with respect to  $\xi$ . Eq. (29) is a confluent hypergeometric equation and the solution for  $g$  satisfying Eq. (30) is obtained in

terms of Kummer's function (see [36]) as

$$g(\xi) = \left[ \frac{1}{\beta} + \frac{2Pr E_s}{(4(1 + N_R) - 2R + R\alpha)} \right] \times \left\{ \frac{\lambda_1 + d_1}{2} F \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -R \right] - R\dot{F} \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -R \right] \right\}^{-1} \times \left( \frac{-\xi}{R} \right)^{\left( \frac{\lambda_1 + d_1}{2} \right)} \times F \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -\xi \right] - \frac{Pr E_s}{(4(1 + N_R) - 2R + R\alpha)} \left( \frac{\xi}{R} \right)^2, \tag{31}$$

where the function  $\dot{F}$  satisfies the relationship

$$\dot{F}[a, b, z] = \frac{a}{b} F[a + 1, b + 1, z]$$

and the other terms are as defined earlier. In terms of  $\eta$ , the expression for  $g$  is

$$g(\eta) = \left[ \frac{1}{\beta} + \frac{2Pr E_s}{[4(1 + N_R) - 2R + R\alpha]} \right] \times \left\{ \frac{\lambda_1 + d_1}{2} F \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -R \right] - R F' \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -R \right] \right\}^{-1} \times \exp \left[ -\beta \left( \frac{\lambda_1 + d_1}{2} \right) \eta \right] \times F \left[ \frac{\lambda_1 + d_1 - 4}{2}, d_1 + 1, -R \exp(-\beta\eta) \right] - \frac{Pr E_s \exp(-2\beta\eta)}{[4(1 + N_R) - 2R + R\alpha]}. \tag{32}$$

The wall temperature  $T_w$  is obtained from Eq. (26) as

$$T_w - T_\infty = \frac{D}{k} \left( \frac{x}{l} \right)^2 \sqrt{\frac{v}{c}} g(0). \tag{33}$$

### 5. Results and discussion

In the paper, we investigate the MHD boundary-layer flow and heat transfer in a viscoelastic liquid



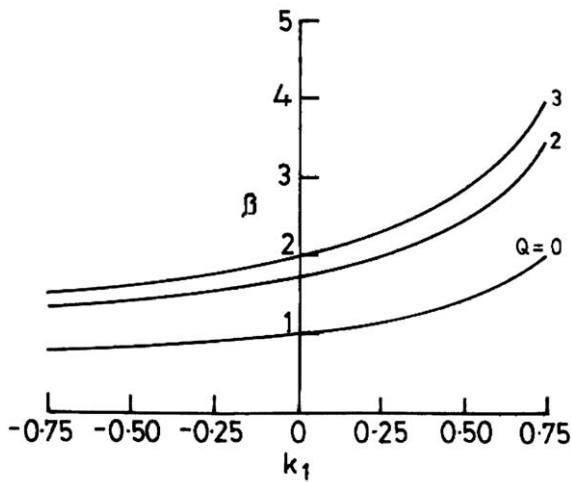
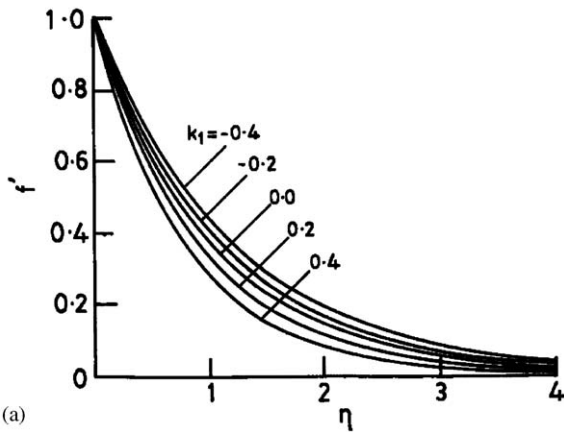
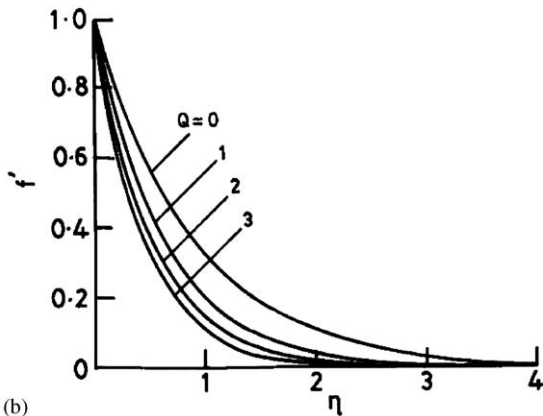


Fig. 3. Plot of  $\beta$  vs. visco-elastic parameter ( $k_1$ ) for different values of Chandrasekhar number ( $Q$ ).



(a)



(b)

Fig. 4. (a) Plot of axial velocity component  $f'(\eta)$  vs.  $\eta$  with  $Q=0$  and for different values of  $k_1$  and (b) plot of  $f'(\eta)$  vs.  $\eta$  with  $k_1=0.2$  and for different values of  $Q$ .

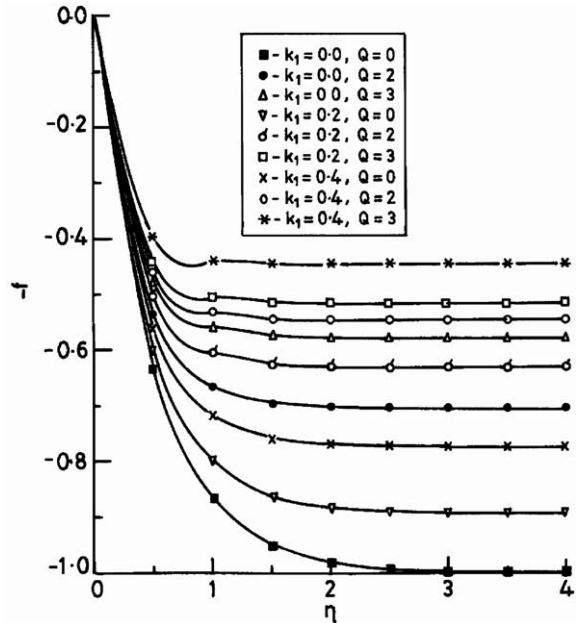


Fig. 5. Plot of transverse velocity component  $f(\eta)$  vs.  $\eta$  for different values of  $k_1$  and  $Q$ .

over a stretching sheet in the presence of radiation. The study encompasses within its realm both Walters' liquid B and second-order liquid. Similarity solution is used to obtain the velocity distribution which is governed by a non-linear differential equation. Heat transfer in the presence of radiation is studied in the above boundary layer flow due to a stretching sheet. Negative values of  $k_1$  give us the results of a second order liquid and positive values of  $k_1$  give us the results of a Walters' liquid B model. The velocity, both transverse as well as axial, is a decreasing function of  $\eta$  as it is an exponential function with negative argument. It is clear from Eq. (4) that  $\beta$ , which is a function of the viscoelastic parameter  $k_1$  and Chandrasekhar number  $Q$ , contributes to the slope of the above exponentially decreasing velocity profiles. Thus  $\beta$  is an important parameter in the present study. From Fig. 3 it is evident that  $\beta$  is an increasing function of  $k_1$  and  $Q$  thus implying that increasing  $k_1$  and  $Q$  gives us steeper gradients in the axial and transverse velocity profiles. This result is borne out in Figs. 4 and 5. Also it is apparent from these figures that the transverse velocity profile decays faster than the axial velocity profile for increasing values of  $k_1$  and  $Q$ .

Table 1  
Value of  $f'(\eta)$  for different values of  $\eta$  and  $k_1$  and  $Q = 0.0$

| $k_1$  | $\eta$    |           |           |           |           |           |           |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
|        | 0.28      | 0.64      | 1.11      | 2.47      | 4.70      | 6.32      | 8.38      |
| -0.8   | 0.68683   | 0.42370   | 0.22554   | 0.03637   | 0.00183   | 0.00021   | 0.00002   |
| -0.6   | 0.70175   | 0.44506   | 0.24560   | 0.04396   | 0.00262   | 0.00034   | 0.00003   |
| -0.4   | 0.71798   | 0.46895   | 0.26892   | 0.05379   | 0.00385   | 0.00056   | 0.00005   |
| -0.2   | 0.73585   | 0.49605   | 0.29643   | 0.06682   | 0.00581   | 0.00098   | 0.00011   |
| -0.05  | 0.75057   | 0.51910   | 0.32065   | 0.07958   | 0.00809   | 0.00156   | 0.00018   |
|        | (0.74870) | (0.51640) | (0.32050) | (0.07950) | (0.00800) | (0.00150) | (0.0001)  |
| -0.01  | 0.75473   | 0.52562   | 0.32774   | 0.08355   | 0.00888   | 0.00174   | 0.00022   |
|        | (0.75290) | (0.52320) | (0.32800) | (0.08370) | (0.00890) | (0.01700) | (0.0002)  |
| -0.005 | 0.75526   | 0.52645   | 0.32865   | 0.08406   | 0.00898   | 0.00177   | 0.00022   |
|        | (0.75350) | (0.52410) | (0.32900) | (0.08420) | (0.00900) | (0.00180) | (0.00020) |
| 0.0    | 0.75578   | 0.52729   | 0.32955   | 0.08455   | 0.00909   | 0.00179   | 0.00022   |
| 0.005  | 0.75631   | 0.52813   | 0.33047   | 0.08510   | 0.00920   | 0.00182   | 0.00023   |
| 0.01   | 0.75684   | 0.52898   | 0.33139   | 0.08563   | 0.09312   | 0.00185   | 0.00023   |
| 0.05   | 0.76116   | 0.53591   | 0.33895   | 0.09004   | 0.01024   | 0.00211   | 0.00028   |
| 0.2    | 0.77846   | 0.56415   | 0.37053   | 0.10978   | 0.01494   | 0.00351   | 0.00056   |
| 0.4    | 0.80502   | 0.60912   | 0.42325   | 0.14759   | 0.02623   | 0.00748   | 0.00152   |
| 0.6    | 0.83771   | 0.66713   | 0.49558   | 0.20968   | 0.05117   | 0.01837   | 0.00450   |
| 0.8    | 0.88231   | 0.75110   | 0.60871   | 0.33138   | 0.12222   | 0.05923   | 0.02357   |

(Values inside parenthesis are those of Rajagopal et al. [5]).

It is apt to note here that the similarity equation (1) is an important differential equation mathematically in the sense that the order of the differential equation can be reduced by one on using the transformation  $f' = Y$  as discussed in Section 2 of this paper. The parameter  $\beta$  is also important due to fact that it is nothing but the skin friction coefficient at the stretching sheet. From Fig. 3, we may conclude that viscoelasticity and applied magnetic field work in unison in increasing the skin friction coefficient. This fact can be elicited by seeking recourse to the Einstein formula for viscosity of suspensions, viz.,  $\mu = \mu_0[1 + 2.5\varphi]$ , where  $\varphi$  is the concentration of the suspended particles which imparts non-Newtonian characteristics to suspensions (see [37]). The above formula explains the enhanced viscosity of suspensions compared to the carrier liquids without suspended particles. The effect of magnetic field is to provide rigidity to the electrically conducting liquid. The observation on the skin friction coefficient for increasing values of  $k_1$  and  $Q$  is therefore not surprising.

We also note here that due to the assumption of a homogeneous liquid, the temperature does not alter the velocity profile and hence we do not see the

effects of viscous dissipation, internal heat generation and radiation on the velocity profiles. These effects are noticed only on the temperature profile due to one way coupling between temperature and velocity.

The axial velocity distribution is tabulated in Table 1 for different values of  $\eta$  and  $k_1$  for the hydrodynamic case with the intention of comparing the results with the work of Rajagopal et al. [5] who solved the problem using a regular perturbation technique. From Table 1, it is clear that our results coincide with those of Rajagopal et al. [5] up to the second decimal digit. It can be shown by stability analysis of the Taylor–Gortler type that the above-boundary layer flow over a stretching sheet considered in the paper is stable.

In the forced flow over a stretching sheet discussed earlier we now analyse the heat transport in the presence of viscous dissipation, internal heat generation and radiation. The viscous dissipation renders the heat equation inhomogeneous and radiation enhances the effect of thermal conductivity. The effect of internal heat generation (source/sink) is to dampen or enhance the heat transport in a linear fashion. The governing

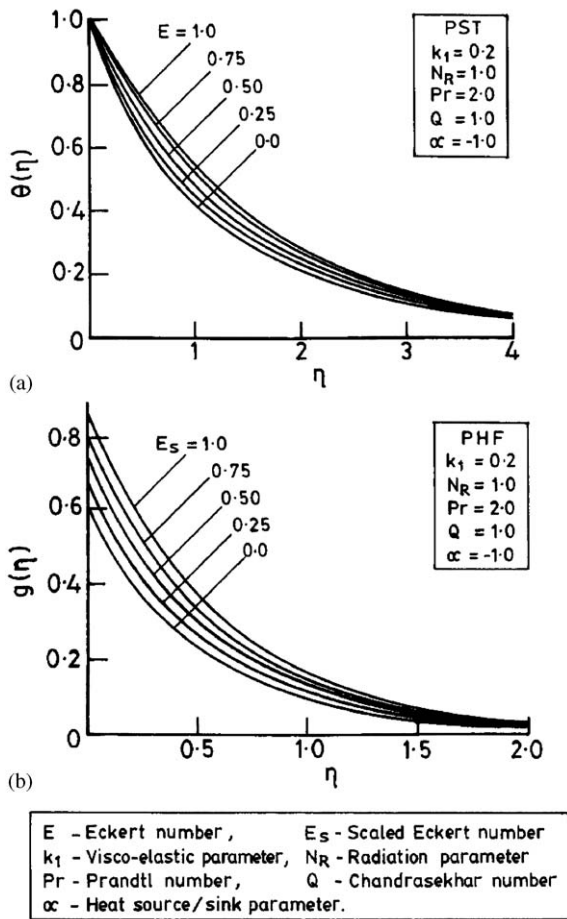


Fig. 6. (a) Plot of temperature distribution  $\theta(\eta)$  vs.  $\eta$  for different values of  $E$  and (b) plot of temperature distribution  $g(\eta)$  vs.  $\eta$  for different values of  $E_s$ .  $E$ —Eckert number,  $E_s$ —scaled Eckert number,  $k_1$ —visco-elastic parameter,  $N_R$ —radiation parameter,  $Pr$ —Prandtl number,  $Q$ —Chandrasekhar number,  $\alpha$ —heat source/sink parameter.

differential equation for heat transport in the presence of radiation is a variable coefficients inhomogeneous differential equation. In arriving at the governing equation use has been made of the Rosseland approximation for the radiative heat flux. Both the PST and PHF boundary conditions are used for solving the heat transport equation. Figs. 6–10 are plots of the temperature distribution for different values of the parameters  $E$  ( $E_s$ ),  $k_1$ ,  $N_R$ ,  $Pr$  and  $Q$ . Fig. 6 indicates that effect of increasing  $E$  ( $E_s$ ) is to enhance the temperature at any point. This is true of both PST and PHF cases.

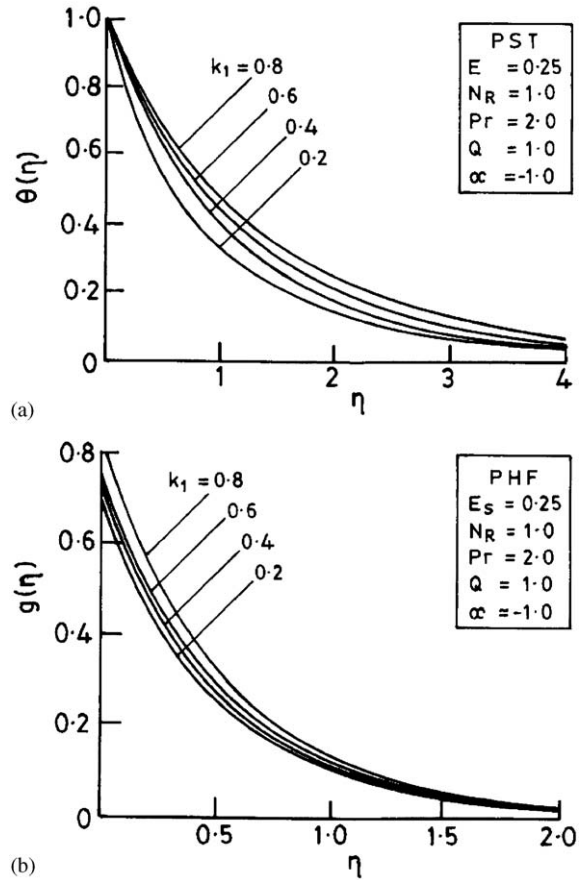


Fig. 7. (a) Plot of temperature distribution  $\theta(\eta)$  vs.  $\eta$  for different values of  $k_1$  and (b) plot of temperature distribution  $g(\eta)$  vs.  $\eta$  for different values of  $k_1$ .

On comparing the temperature distribution of the PST and PHF cases it is apparent that the PST boundary condition succeeds in keeping the viscoelastic cooling liquid warmer than in the case when PHF boundary condition is applied. It may therefore be inferred that the PHF boundary condition is better suited for faster cooling of the stretching sheet. Qualitatively the effects of the viscoelastic parameter  $k_1$ , the radiation parameter  $N_R$  and Chandrasekhar number  $Q$  on the temperature are similar to that of  $E$ . In contrast to the effect of  $E$ ,  $k_1$ ,  $N_R$  and  $Q$  on  $\theta$ , the effect of increasing  $Pr$  is to decrease the magnitude of  $\theta(\eta)$ . In other words, it means that the thermal boundary layer thickness is a function of all the above parameters. These results are depicted in Figs. 6–10.

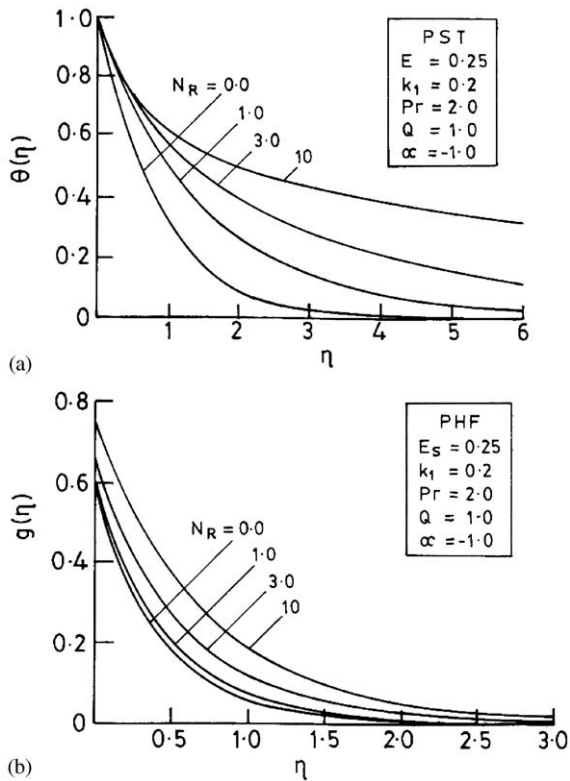


Fig. 8. (a) Plot of temperature distribution  $\theta(\eta)$  vs.  $\eta$  for different values of  $N_R$  and (b) plot of temperature distribution  $g(\eta)$  vs.  $\eta$  for different values of  $N_R$ .

Like the local skin friction coefficient for the velocity it is equally important that we consider the analog of this for the temperature that happens to be the wall temperature gradient  $-\dot{\theta}(0)$  (PST). On looking at the results of Figs. 6–10 in conjunction with those of Table 2, we note that the parameters  $E$ ,  $k_1$ ,  $N_R$ ,  $Pr$  and  $Q$  have opposing influence on the skin friction coefficient and the wall temperature gradient. In Table 2 we have extracted information for the PST case on the wall temperature gradient. Clearly in this case, the effect of increasing the strength of the heat sink is to decrease the wall temperature gradient and the opposite behaviour is seen for a heat source. In the case of the PHF boundary condition, the values of the wall temperature  $g(0)$  as a function of all the parameters of the problem have also been tabulated in Table 2. The variation of  $g(0)$  with all parameters is on expected

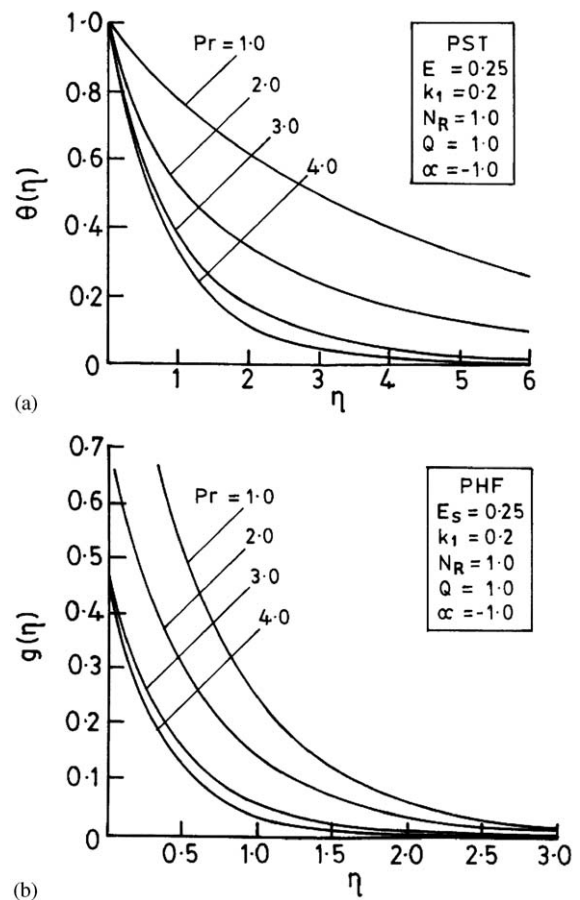


Fig. 9. (a) Plot of temperature distribution  $\theta(\eta)$  vs.  $\eta$  for different values of  $Pr$  and (b) plot of temperature distribution  $g(\eta)$  vs.  $\eta$  for different values of  $Pr$ .

lines except that of  $Pr$ . It seems that there is a critical value of  $Pr$ , viz.,  $Pr_c$ , beyond which the wall temperature increases with increase in  $Pr$ .

## 6. Conclusion

1. The PHF boundary condition is better suited for effective cooling of the stretching sheet.
2. Viscoelastic liquids with negligible viscous dissipation must be chosen for a cooling liquid. Further highly viscous liquids with mild viscoelasticity are ideally suited as a coolant. However, Table 2 suggests that one will have

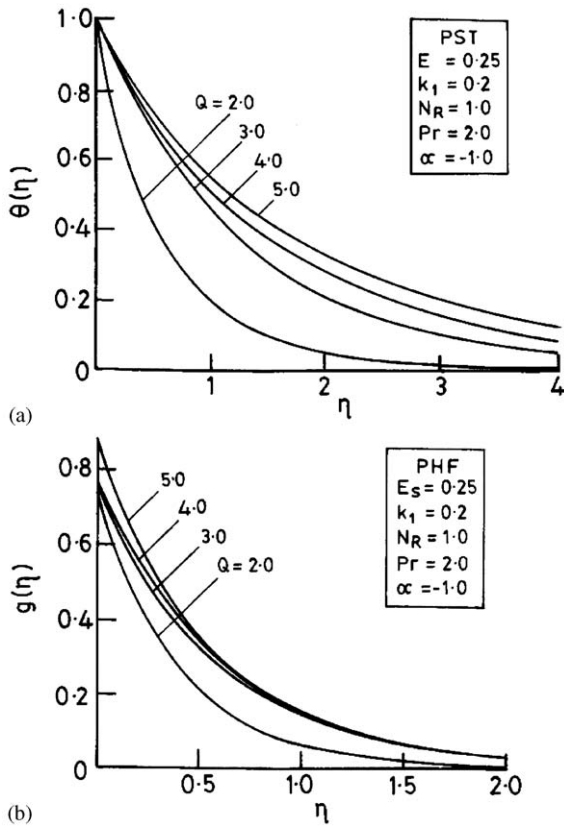


Fig. 10. (a) Plot of temperature distribution  $\theta(\eta)$  vs.  $\eta$  for different values of  $Q$  and (b) plot of temperature distribution  $g(\eta)$  vs.  $\eta$  for different values of  $Q$ .

to exercise caution in choosing high viscosity liquids. This is due to the fact that as  $Pr$  increases beyond  $Pr_c$  then the wall temperature is increased.

3. In arriving at an appropriate polymer extrusion it is desirable that the operating temperatures are as low as possible to ensure minimum radiation.
4. The strength of the applied magnetic field should be as low as is possible to realize.
5. Several earlier works form a limiting case of the present study:

(a)  $\lim_{\substack{Q \rightarrow 0 \\ N_R \rightarrow 0}}$  {Our results on both PST and PHF}  $\rightarrow$  {Results of Subhash and Veena [22]}

Table 2

Values of wall temperature gradient  $-\dot{\theta}(0)$  (PST case) and wall temperature  $g(0)$  (PHF case) for different values of  $E$  ( $E_s$ ),  $k_1$ ,  $N_R$ ,  $Pr$ ,  $Q$  and  $\alpha$

| $E^a$ | $k_1$ | $N_R$ | $Pr$ | $Q$ | $\alpha$ | $-\dot{\theta}(0)$ | $g(0)$  |
|-------|-------|-------|------|-----|----------|--------------------|---------|
| 0.0   |       |       |      |     |          | 1.77006            | 0.63291 |
| 0.02  | 0.2   | 5.0   | 4.0  | 1.0 | 0.05     | 1.76473            | 0.63677 |
|       |       |       |      |     |          | 1.63671            | 0.72957 |
|       |       |       |      |     |          | 1.71087            | 0.71591 |
| 0.02  | 0.2   | 5.0   | 4.0  | 1.0 | 0.05     | 1.76473            | 0.63677 |
|       |       |       |      |     |          | 1.80387            | 0.57094 |
|       |       |       |      |     |          | 2.14202            | 0.63430 |
| 0.02  | 0.2   | 5.0   | 4.0  | 1.0 | 0.05     | 1.76473            | 0.63677 |
|       |       | 10    |      |     |          | 1.7088             | 0.63057 |
|       |       |       | 2.0  |     |          | 1.28704            | 0.74872 |
| 0.02  | 0.2   | 5.0   | 4.0  | 1.0 | 0.05     | 1.76473            | 0.63677 |
|       |       |       | 6.0  |     |          | 2.02431            | 0.66082 |
|       |       |       |      |     | 0.0      | 1.54899            | 0.87011 |
| 0.02  | 0.2   | 5.0   | 4.0  | 2.0 | 0.05     | 1.81435            | 0.55554 |
|       |       |       |      | 3.0 |          | 1.81543            | 0.53257 |
|       |       |       |      |     | -0.1     | 1.80633            | 0.63565 |
|       |       |       |      |     | -0.05    | 1.79452            | 0.63735 |
| 0.02  | 0.2   | 5.0   | 4.0  | 1.0 | 0.0      | 1.78148            | 0.63825 |
|       |       |       |      |     | 0.05     | 1.76473            | 0.63677 |
|       |       |       |      |     | 0.1      | 1.74889            | 0.63549 |

<sup>a</sup>( $E_s$  for PHF case).

(b)  $\lim_{\substack{Q \rightarrow 0 \\ N_R \rightarrow 0 \\ E \rightarrow 0}}$  {Our results on both PST and PHF}  $\rightarrow$  {Results of Rollins and Vajravelu [14]}.

6. On replacing the Chandrasekhar number  $Q$  by the porous parameter  $Da^{-1}$  ( $Da$ : Darcy number) we get the results of porous media problem.
7. In the absence of radiation and viscous dissipation, the results of the problem for heat sink yield the results of the analogous isothermal problem for species concentration with first-order chemical reaction.

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## References

- [1] E.G. Fisher, *Extrusion of Plastics*, third ed., Newnes-Butterworld, London, 1976.
- [2] B.C. Sakiadis, Boundary layer behavior on continuous solid surface: II The boundary layer on a continuous flat surface, *A.I.Ch.E. Journal* 7 (2) (1961) 221.
- [3] F.K. Tsou, E.M. Sparrow, R.J. Goldstein, Flow and heat transfer in the boundary layer on a continuous moving surfaces, *Int. J. Heat Mass Transfer* 10 (1967) 219.
- [4] L.J. Crane, Flow past a stretching plate, *Z. Angew. Math. Phys.* 21 (1970) 645.
- [5] K.R. Rajagopal, T.Y. Na, A.S. Gupta, Flow of a viscoelastic fluid over a stretching sheet, *Rheol. Acta* 23 (1984) 213.
- [6] H.I. Andersson, K.H. Bech, B.S. Dandapat, MHD flow of power-law over a stretching sheet, *Int. J. Non-Linear Mech.* 27 (1992) 929.
- [7] B. Siddappa, A.M. Subhash, Non-Newtonian flow past a stretching plate, *Z. Angew. Math. Phys.* 36 (1985) 890.
- [8] K.R. Rajagopal, T.Y. Na, A.S. Gupta, A non-similar boundary layer on a stretching sheet in non-Newtonian fluid with uniform free stream, *J. Math. Phys. Sci.* 21 (1987) 189.
- [9] B.J. McLeod, K.R. Rajagopal, On the uniqueness of the flow of a Navier–Stokes fluid due to stretching boundary, *Arch. Mech. Anal.* 98 (1987) 385.
- [10] N.M. Bujurke, S.N. Biradar, P.S. Hiremath, Second order fluid flow past a stretching sheet with heat transfer, *Z. Angew. Math. Phys.* 38 (1987) 653.
- [11] M.I. Char, C.K. Chen, Temperature field in non-Newtonian flow over a stretching plate with variable heat flux, *Int. J. Heat Mass Transfer* 31 (1988) 917.
- [12] B.S. Dandapat, A.S. Gupta, Flow and heat transfer in a viscoelastic fluid over a stretching sheet, *Int. J. Non-Linear Mech.* 24 (3) (1989) 215.
- [13] W.D. Chang, The non-uniqueness of the flow of a viscoelastic fluid over a stretching sheet, *Quart. Appl. Math.* 47 (1989) 365.
- [14] D. Rollins, K. Vajravelu, Heat transfer in a second order fluid over a continuous stretching surface, *Acta Mech.* 89 (1991) 167.
- [15] H.I. Andersson, B.S. Dandapat, Flow of power law fluid over a stretching sheet, *Stability Appl. Anal. Cont. Media* 1 (1991) 339.
- [16] S.P. Lawrence, N.B. Rao, Heat transfer in the flow of a viscoelastic fluid over a stretching sheet, *Acta Mech.* 93 (1992) 53.
- [17] H.I. Andersson, MHD flow of a viscoelastic fluid past a stretching surface, *Acta Mech.* 95 (1992) 227.
- [18] D. Kelly, K. Vajravelu, L. Andrews, Analysis of heat mass transfer of a viscoelastic, electrically conducting fluid past a continuous stretching sheet, *Nonlinear Anal.* 36 (1999) 767.
- [19] C.E. Maneschy, M. Massoudi, K.R. Rajagopal, Flow of a second grade fluid over a porous elastic sheet due to stretching, *J. Math. Phys. Sci.* 27 (6) (1993) 353.
- [20] R.K. Bhatnagar, K.R. Rajagopal, A.S. Gupta, Flow of on Oldroyd B model due to a stretching sheet in the presence of a free stream velocity, *Int. J. Non-Linear Mech.* 30 (1995) 391.
- [21] S.P. Lawrence, N.B. Rao, The non-uniqueness of MHD flow of viscoelastic fluid past a stretching sheet, *Acta Mech.* 112 (1995) 223.
- [22] A.M. Subhash, P. Veena, Viscoelastic fluid flow and heat transfer in a porous medium over a stretching sheet, *Int. J. Non-Linear Mech.* 33 (1998) 531.
- [23] A.M. Subhash, A. Joshi, R.M. Sonth, Heat Transfer in MHD viscoelastic fluid flow over a stretching surface, *Z. Angew. Math. Mech.* 10 (2001) 691.
- [24] R.M. Sonth, S.K. Khan, A.M. Subhash, K.V. Prasad, Heat and mass transfer in a viscoelastic fluid flow over an accelerating surface with heat source/sink and viscous dissipation, *Heat Mass Transfer* 38 (2002) 213.
- [25] J.K. Platten, J.C. Legros, *Convection in Liquids*, Springer, Berlin, 1984.
- [26] A. Raptis, Radiation and viscoelastic flow, *Int. Commun. Heat Mass Transfer* 26 (6) (1999) 889.
- [27] A. Raptis, C. Perdikis, Viscoelastic flow by the presence of radiation, *Z. Angew. Math. Mech.* 78 (1998) 277.
- [28] K.R. Rajagopal, A.S. Gupta, A.S. Wineman, On a boundary layer theory for non-Newtonian fluids, *Lett. Appl. Eng. Sci.* 18 (1980) 875.
- [29] D.W. Beard, K. Walters, Elastico-viscous boundary-layer flows I. Two-dimensional flow heat stagnation point, *Proc. Cambridge Philos. Soc.* 60 (1964) 667.
- [30] K.R. Rajagopal, A.S. Gupta, Exact solution for the flow of a non-Newtonian fluid past an infinite porous plate, *Meccanica* 19 (1984) 158.
- [31] K.R. Rajagopal, P.N. Kaloni, Some remarks on boundary conditions for the flows of fluids of the differential type, *Proceedings of the Canadian Applied Mathematics Society*, Barnaby, Hemisphere Press, Washington, DC, 1989.
- [32] K.R. Rajagopal, On the creeping flow of a second order fluid, *J. Non-Newtonian Fluid Mech.* 14 (1984) 239.
- [33] K.R. Rajagopal, On boundary conditions for fluids of the differential type, in: A. Sequiera (Ed.), *Navier–Stokes Equations and Related Non-linear Problems*, Plenum Press, New York, 1995.
- [34] W.C. Troy, E.A. II Overman, G.B. Ermentrout, J.P. Keener, Uniqueness of flow of a second-order fluid past a stretching sheet, *Quart. Appl. Math.* 45 (1987) 753.
- [35] M.Q. Brewster, *Thermal Radiative Transfer Properties*, Wiley, Canada, 1992.
- [36] M. Abramowitz, F. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1980.
- [37] K.R. Rajagopal, L. Tao, *Mechanics of Mixtures*, World Scientific, Singapore, 1995.