Abstract

A gravitational version of the S function for 2-component massless spinor fields or Weyl fields is introduced in the manifestly covariant operator formalism of quantum gravity. This function is defined as a bilocal operator that relates to the 4D anti-commutation relation between a Weyl field and its Hermitian conjugate. Some properties of this function are investigated.

1. Introduction

The 4D anti-commutation relation between a free Dirac field $\psi(x)$ with mass $m$ and its Dirac conjugate $\bar{\psi}(y)$ is given by

$$\{\psi(x), \bar{\psi}(y)\} = iS(x - y; m),$$  \hspace{1cm} (1.1)

in the Minkowski spacetime [1]. Here, $S(x - y; m)$ is the S function for Dirac fields defined by

$$(i\gamma^\mu \partial^\mu - m)S(x - y; m) = 0,$$  \hspace{1cm} (1.2)

$$S(x - y; m)|_{x_0 = y_0} = -i\gamma^0 \delta^3(x - y),$$  \hspace{1cm} (1.3)

with the use of the ordinary gamma matrices $\gamma^\mu$. Abe and Nakanishi [2] introduced an extended version of the S function in their method for solving
the manifestly covariant operator formalism of quantum electrodynamics. In this method, the electromagnetic field and the electron one are expanded in powers of $e^2$ where $e$ denotes the electromagnetic coupling constant. The zeroth-order of the S function, $S^{(0)}(x, y)$, is defined by the following q-number Cauchy problem:

$$\{i\gamma^\mu [\partial^x - iA^{(0)}(x)] - m\}S^{(0)}(x, y) = 0,$$

$$S^{(0)}(x, y)|_{x^0 = y^0} = -i\gamma^0 \delta^3(x - y).$$

Here $A^{(0)}(x)$ is the zeroth order of the electromagnetic field.

Comparing Eqs. (1.2) and (1.3) with Eqs. (1.4) and (1.5), and taking account of the vierbein formalism of quantum gravity [1], we defined a quantum-gravity version of the S function $S(x, y; m)$ [3] for Dirac fields interacting with the gravitational field $h_\mu^a(x) (a = 0, 1, 2, 3)$. Namely, we form the following q-number Cauchy problem:

$$ih(x)\gamma^\mu(x)[\partial^x + \omega^\mu(x)]S(x, y; m) - mh(x)S(x, y; m) = 0,$$

$$S(x, y; m)|_{x^0 = y^0} = -i\frac{\gamma^0(x)}{h(x)g^{00}(x)} \delta^3(x - y).$$

Here, $h = \det h_\mu^a$, $\gamma^\mu \equiv h^\mu_\sigma \tilde{\gamma}_a$ with the flat-space gamma matrices $\tilde{\gamma}_a (a = 0, 1, 2, 3)$,

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\eta_{ab}, \quad \eta_{ab} = \text{diag}(+1, -1, -1, -1),$$

and

$$g_{\mu\nu} = \eta_{ab} h_\mu^a h_\nu^b.$$  

As in Ref. [3], we use Greek small letters for $GL(4)$ indexes and italic small letters for internal Lorentz ones. The symbol $\omega_\mu$ in (1.6) is defined by

$$\omega_\mu \equiv \frac{1}{2} \omega_\mu^{ab} \tilde{\sigma}_{ab},$$

where $\tilde{\sigma}_{ab} = \{\tilde{\gamma}_a, \tilde{\gamma}_b\}$. In order to treat Weyl fields in this system, we extract a corresponding Lagrangian density for Weyl fields and to investigate it. For this purpose, we analogously use the properties of the quantum-gravity S function for Dirac fields.
with

\[ \bar{\sigma}_{ab} = \frac{1}{4}(\bar{\gamma}_a \bar{\gamma}_b - \bar{\gamma}_b \bar{\gamma}_a), \]

(1.11)

\[ \omega_{ab}^{\mu} = \frac{1}{2}[h^{p a}(\partial_{\mu} h_{p}^b - \partial_{p} h_{\mu}^b) - h^{p b}(\partial_{\mu} h_{p}^a - \partial_{p} h_{\mu}^a) + h_{\mu}^c h^{p a} h^{c b}(\partial_{\sigma} h_{\rho c} - \partial_{\rho} h_{\sigma c})]. \]

(1.12)

Here \( \omega_{ab}^{\mu} \) denotes the spin connection [1].

If \( m = 0 \), then we can decompose Dirac fields into two 2-component spinor fields. These are called Weyl fields [4]. So, we need to treat Weyl fields in the vierbein formalism of quantum gravity, and consequently to have a quantum-gravity version of the S function for Weyl fields.

The purpose of the present paper is to introduce the quantum-gravity S function for Weyl fields and to investigate it. For this purpose, we analogously use the properties of the quantum-gravity S function for Dirac fields defined by Eqs. (1.6) and (1.7).

This paper is organized as follows. In the next section, we briefly review the manifestly covariant operator formalism for Weyl fields interacting with the gravitational field. In Sect. 3, we introduce the quantum-gravity S function for Weyl fields. In Sect. 4, we investigate its transformation properties. The last section is devoted to discussion.

2. Covariant operator formalism for Weyl fields

We consider a quantum system in which Weyl fields interact with the gravitational field. Let us call it the quantum coupled Einstein–Weyl system.

2.1. Lagrangian density for Weyl fields

In order to treat Weyl fields in this system, we extract a corresponding Lagrangian density from the following one [1] for a massless Dirac field.
\[ \mathcal{L}_D = \frac{i}{2} [\hbar \bar{\psi} \gamma^\mu (\partial_\mu + \omega_\mu) \psi - \psi (\bar{\partial}_\mu - \omega_\mu) \gamma^\mu \psi \hbar] , \quad (2.1) \]

where \( \bar{\psi} \equiv \psi^\dagger \gamma_0 \).

Massless Dirac fields are equivalent to pairs of two corresponding Weyl fields. Therefore, we express \( \psi(x) \) and \( \tilde{\gamma}_a \) as follows [4]:

\[ \psi(x) \equiv \begin{pmatrix} \xi_A(x) \\ \eta^B(x) \end{pmatrix} , \quad (2.2) \]

and

\[ \tilde{\gamma}_a \equiv \begin{pmatrix} 0 & (\tilde{\sigma}_a)^A_B \\ (\bar{\sigma}_a)^\dot{A}\dot{B} & 0 \end{pmatrix} . \quad (2.3) \]

Here, \( \xi_A \) and \( \eta^B \) are Weyl fields, \( \tilde{\sigma}_a \) and \( \bar{\sigma}_a \) consist of the unit matrix and the Pauli spin matrices,

\[ (\tilde{\sigma}_a)^A_B \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3)^A_B , \quad (2.4) \]

\[ (\bar{\sigma}_a)^\dot{A}\dot{B} \equiv (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)^\dot{A}\dot{B} . \quad (2.5) \]

We use capital Roman letters, \( A, B, C \), etc., for 2-component spinor indexes, and raise or lower them by

\[ \epsilon^{AB} = \epsilon_{AB} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (2.6) \]

Of course, \( \tilde{\sigma}_a \) is a complex conjugate of \( \bar{\sigma}_a \):

\[ (\tilde{\sigma}_a)^\dot{A}\dot{B} = [(\bar{\sigma}_a)^A_B]^* , \quad (2.7) \]

with (2.6).

Inserting (2.3) to (1.8), we obtain

\[ (\tilde{\sigma}_a \tilde{\sigma}_b + \bar{\sigma}_b \bar{\sigma}_a)_A^B = 2\eta_{ab} \delta_A^B , \quad (2.8) \]

\[ (\bar{\sigma}_a \tilde{\sigma}_b + \tilde{\sigma}_b \bar{\sigma}_a)_\dot{A}\dot{B} = 2\eta_{ab} \delta_\dot{A}_\dot{B} . \quad (2.9) \]
Combining (1.11) and (2.3), we write
\[ \hat{\sigma}_{ab} = \begin{pmatrix} (\hat{S}_{ab})^B_A & 0 \\ 0 & (\hat{S}_{ab})^A_B \end{pmatrix} \] (2.10)
with
\[ (\hat{S}_{ab})^B_A \equiv \frac{1}{4}((\sigma_a \sigma_b - \sigma_b \sigma_a)^B_A, \] (2.11)
\[ (\hat{S}_{ab})^A_B \equiv \frac{1}{4}((\sigma_a \sigma_b - \sigma_b \sigma_a)^A_B = -[(\hat{S}_{ab})^\dagger]^A_B. \] (2.12)

Equations (2.8), (2.9), (2.11), and (2.12) yield
\[ (\hat{\sigma}_a \hat{S}_{bc} - \hat{S}_{bc} \hat{\sigma}_a)_{A\dot{B}} = \eta_{ab}(\hat{\sigma}_c)_{A\dot{B}} - \eta_{ac}(\hat{\sigma}_b)_{A\dot{B}}; \] (2.13)
\[ (\hat{\sigma}_a \hat{S}_{bc} - \hat{S}_{bc} \hat{\sigma}_a)^{A\dot{B}} = \eta_{ab}(\hat{\sigma}_c)^{A\dot{B}} - \eta_{ac}(\hat{\sigma}_b)^{A\dot{B}}, \] (2.14)
and
\[ (\hat{S}_{ab} \hat{S}_{cd} - \hat{S}_{cd} \hat{S}_{ab})^B_A \]
\[ = \eta_{ad}(\hat{S}_{bc})^B_A - \eta_{ac}(\hat{S}_{bd})^B_A + \eta_{bc}(\hat{S}_{ad})^B_A - \eta_{bd}(\hat{S}_{ac})^B_A, \] (2.15)
\[ (\hat{S}_{ab} \hat{S}_{cd} - \hat{S}_{cd} \hat{S}_{ab})^A_B \]
\[ = \eta_{ad}(\hat{S}_{bc})^A_B - \eta_{ac}(\hat{S}_{bd})^A_B + \eta_{bc}(\hat{S}_{ad})^A_B - \eta_{bd}(\hat{S}_{ac})^A_B. \] (2.16)

Inserting Eqs. (2.2) and (2.3) to Eq. (2.1), we obtain the sum of two Lagrangian densities:
\[ \mathcal{L}_D = \mathcal{L}_\xi + \mathcal{L}_\eta, \] (2.17)
where
\[ \mathcal{L}_\xi \equiv \frac{i}{2}[h^a \xi^\dagger h^{\mu a} \hat{\sigma}_a (\partial_\mu + \omega_\mu) \xi - \xi^\dagger (\partial_\mu - \bar{\omega}_\mu) h^{\mu a} \hat{\sigma}_a \xi h]; \] (2.18)
\[ \mathcal{L}_\eta \equiv \frac{i}{2}[h^a \eta^\dagger h^{\mu a} \hat{\sigma}_a (\partial_\mu + \bar{\omega}_\mu) \eta - \eta^\dagger (\partial_\mu - \omega_\mu) h^{\mu a} \hat{\sigma}_a \eta h]. \] (2.19)
2.2. Properties of field operators

Now we use $\mathcal{L}^\xi$ in (2.18) because it is enough to treat one of the pair of $\xi$ and $\eta$. In order to take $\xi$ as the canonical variable, we replace $\mathcal{L}^\xi$ by

$$\tilde{\mathcal{L}}_W \equiv i\hbar \xi^\dagger h^{\mu a} \hat{\sigma}_a (\partial_\mu + \omega_\mu) \xi.$$  
(2.22)

The discarded term is a total divergence,

$$\mathcal{L}^\xi - \tilde{\mathcal{L}}_W = -\frac{i}{2} \partial_\mu (h^{\mu a} \hat{\sigma}_a),$$
(2.23)

with the use of

$$\partial_\mu (h^{\mu a}) \cdot \hat{\sigma}_a = h^{\mu a} (\hat{\sigma}_a \omega_\mu - \overline{\omega}_\mu \hat{\sigma}_a).$$
(2.24)

The field equation,

$$i\hbar h^{\mu a} \hat{\sigma}_a (\partial_\mu + \omega_\mu) \xi = 0,$$
(2.25)

is derived from $\tilde{\mathcal{L}}_W$. Modifying this equation, we see

$$\dot{\xi} = -\frac{\hbar}{g^{00}} \hat{\sigma}_a (h^{0a} \hat{\sigma}_b \partial_k + h^{ab} \hat{\sigma}_b \omega_\mu) \xi.$$  
(2.26)

The canonical conjugate of $\xi$ is defined by

$$\pi^A_\xi = \frac{\partial \tilde{\mathcal{L}}_W}{\partial \dot{\xi}^A} = -i\hbar (\xi^1)_B h^{0a} (\hat{\sigma}_a)^B,$$
(2.27)

via the “left” functional derivative with respect to $\xi$. The equal-time canonical anti-commutation relation is set as follows:

$$\{\pi_\xi^A, \xi_B^t\} = -i \delta^A_B \delta^3,$$
(2.28)
where $\delta^3$ denotes the spatial delta function $\Pi_k^{3} \delta(x^k - y^k)$. In the above
and hereafter, a prime attached to a spacetime function means that its
argument is not $x^\lambda$ but $y^\lambda$ where it is understood $x^0 = y^0$. Using the
field equation (2.25), the canonical conjugate (2.27), and the equal-time
canonical anti-commutation relation (2.28), we obtain
\[
\{\xi_A, (\xi^\dagger)_B\} = \frac{h^{0a}}{h g^{00}} (\dot{\sigma}_a)_{AB} \delta^3.
\] (2.29)

The action integral of $\hat{\mathcal{L}}_W$ in (2.22) is invariant under the gravitational
BRST transformation. The corresponding charge [1] is defined by
\[
Q_G \equiv \int d^3x \ h^{0\nu}(b_\rho \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho),
\] (2.30)
where $b_\rho$ is the gravitational B-field and $c^\rho$ is the gravitational Faddeev–
Popov ghost field. The anti-commutation relation between $Q_G$ and the
Weyl field $\xi$ is given by
\[
\{iQ_G, \xi\} = i \int d^3y \ h' g^{00'} \ [\dot{b}'_\rho, \xi] c^{0'}. \quad \text{(2.31)}
\]
Inserting the commutator,
\[
[\dot{b}'_\rho, \xi] = -\frac{i\kappa}{h g^{00}} \partial_\rho \xi \cdot \delta^3, \quad \text{(2.32)}
\]
to the right-hand side of (2.31), we obtain the gravitational BRST transfor-
mation of $\xi$. Here, $\kappa$ is Einstein’s gravitational constant. The commutator
(2.32) is derived from the combination of (2.26) and the following commu-
tators [1]:
\[
[ h^{\mu a}, b'_\rho] = \frac{i\kappa}{h g^{00}} h^{0a} \delta^{\mu}_\rho \delta^3, \quad \text{(2.33)}
\]
\[
[ g^{\mu\nu}, b'_\rho] = \frac{i\kappa}{h g^{00}} (g^{\mu0} \delta^{\nu}_\rho + g^{\nu0} \delta^{\mu}_\rho) \delta^3, \quad \text{(2.34)}
\]
\[
[\omega^{ab}_\mu, b'_\rho] = -\frac{i\kappa}{h g^{00}} \delta^{0}_\mu \omega^{ab}_\rho \delta^3. \quad \text{(2.35)}
\]
In addition, the Lagrangian density $\tilde{L}_W$ is invariant under the internal Lorentz BRST transformation. The corresponding charge [1] is defined by

$$Q_L \equiv \int d^3x h^0 \mu [s_{ab}(D_{\mu}t)^{ab} - \partial_\nu s_{ab} \cdot t^{ab} + i\partial_\nu \bar{t}_{ab} \cdot t^{bc} t^c] ,$$

(2.36)

with

$$(D_{\mu}t)^{ab} \equiv \partial_\mu t^{ab} + \omega^{ac}_\mu t_c^b - \omega^{bc}_\mu t^a .$$

(2.37)

Here, $s_{ab}$ is the internal Lorentz B-field, and $t^{ab}$ and $\bar{t}_{ab}$ are the internal Lorentz Faddeev–Popov ghost and anti-ghost fields; these fields are anti-symmetric with respect to the indexes $a$ and $b$. The anti-commutation relation between $Q_L$ and $\xi$ is given by

$$\{iQ_L, \xi\} = i\int d^3y h^0 [s'_{ab}, \xi] t^{ab} .$$

(2.38)

Inserting the commutator,

$$[s'_{ab}, \xi] = -\frac{i}{2hg^{00}} S_{ab} \xi \delta^3 ,$$

(2.39)

to the right-hand side of (2.38), we obtain the internal Lorentz BRST transformation of $\xi$. The commutator (2.39) is derived from the combination of (2.26) and the following commutator [1]:

$$[\omega^{ab}_\mu, s'_{cd}] = \frac{i\delta^0_\mu}{2hg^{00}} (\delta^a_c \delta^b_d - \delta^b_c \delta^a_d) \delta^3 .$$

(2.40)

The BRST charges, $Q_G$ and $Q_L$, give us the subsidiary conditions,

$$Q_G|_{\text{phys}} = 0, \quad Q_L|_{\text{phys}} = 0 ,$$

(2.41)

to define the physical subspace of the indefinite-metric Hilbert space.

### 3. Quantum-gravity S function for Weyl fields

In order to define the $S$ function for Weyl fields, we form a q-number Cauchy problem for it. This is analogous to the set of Eqs. (1.6) and
(1.7) for \( S(x, y; m) \); Eq. (1.6) relates to the gravitational Dirac equation, and Eq. (1.7) is the quantum-gravity version of the condition (1.3) for the ordinary S function.

Let us have two equations to define the quantum-gravity S function \( S_{AB}(x, y) \) for \( \xi \). Comparing the field equation (2.25) for \( \xi(x) \) with the gravitational Dirac equation for \( \psi(x) \) from \( L_D \) in (2.1), we firstly have a form of

\[
i h(x) h^{\mu a}(x) (\bar{\sigma}_a)^C_D \{ \delta^A_D \partial^x_\mu + [\omega_\mu(x)]_D^A \} S_{AB}(x, y) = 0. \tag{3.1}
\]

Comparing the equal-time anti-commutation relation (2.29) with that between \( \psi(x) \) and \( \bar{\psi}(y) \) [3], we secondly have a form of

\[
S_{AB}(x, y)|_0 = -i \frac{h^{\mu a}(x)}{h(x) g^{00}(x)} (\bar{\sigma}_a)^A_{AB} \delta^3, \tag{3.2}
\]

where the symbol \( |_0 \) denotes to set \( x^0 = y^0 \). Using the set of Eqs. (3.1) and (3.2), we form the q-number Cauchy problem for the quantum-gravity S function \( S_{AB}(x, y) \). This function is a bilocal operator and does not depend upon \( x - y \) alone, because of the same reason for the quantum-gravity D function \( D(x, y) [5]. \)

We regard the Hermitian conjugate of \( S_{AB}(x, y) \) as follows:

\[
[S_{AB}(x, y)]^\dagger = -S_{BA}(y, x). \tag{3.3}
\]

This is analogous to that of the ordinary S function for Dirac fields [3]. Thus, we have

\[
S_{\dot{A}B}(x, y) \{ \bar{\epsilon}_\mu^\dot{A} B^\dot{C} \partial^x_\mu - [\bar{\omega}_\mu(y)]^B_{\dot{C}} \} (\bar{\sigma}_a)^C_D h^{\mu a}(y) h(y) \partial^y_\mu = 0. \tag{3.4}
\]

Equations (3.1) and (3.4) denote that the index \( A \) of \( S_{AB}(x, y) \) relates to the spinor property at the point \( x \) while the index \( \dot{B} \) does to that at the point \( y \).
Modifying Eqs. (3.1) and (3.4), we see
\[
\partial_0^\nu S_{AB}(x, y)|_0 = i \frac{h^{0a}}{g^{00}} (\sigma^a_0 \sigma_0) C \left[ h^{bC} D \partial^\nu \left( \frac{h^{0c}}{g^{00}} \delta^3 \right) + h^{\mu b} (\omega_\mu)_C D \frac{h^{0c}}{g^{00}} \delta^3 \right] (\sigma_c)_D B ,
\]
and
\[
S_{AB}(x, y) \partial_0^\nu y|_0 = i (\sigma^a_0)_{AC} \left[ \left( \frac{h^{0a}}{g^{00}} \delta^3 \right) \partial^\nu \delta_D h^{lb}(y) - \frac{h^{0a}}{g^{00}} (\omega_\mu)_C D h^{\mu b} \delta^3 \right] (\sigma_b \sigma_c)_D B \left( \frac{h^{0c}}{g^{00}} \right) ,
\]
respectively. These are parallel with (2.26).

We can solve (2.25) in terms of an integral representation,
\[
\xi_A(x) = \int d^3 y J_A^0(x, y) ,
\]
with
\[
J_A^\lambda(x, y) \equiv i S_{AB}(x, y) h(y) h^{\lambda a}(y) (\sigma_a)_B C \xi_C(y) .
\]
Here note that the index \( \lambda \) relates to the vector property at the point \( y \).

The bilocal “current” \( J_A^\lambda(x, y) \) is conserved with respect to \( y \) by virtue of Eqs. (2.24) and (3.4). Therefore, the right-hand side of (3.7) is independent of \( y^0 \) and reduces to \( \xi_A(x) \) by setting \( y^0 = x^0 \) via (3.2).

Inserting (3.7) to the right-hand side of (3.8), we have
\[
\xi_A(x) = \int d^3 z \int d^3 y i S_{AB}(x, y) h(y) h^{0a}(y) (\sigma_a)_B C \times i S_{CD}(y, z) h(z) h^{0b}(z) (\sigma_b)_D E \xi_E(z) .
\]
Comparing (3.7) and (3.9), we find an integral representation,
\[
S_{AB}(x, z) = \int d^3 y K_{AB}^0(x, y, z) .
\]
with

\[ K_{AB}^\lambda(x, y, z) \equiv i S_{AC}(x, y)h(y)h^{\lambda a}(y)(\bar{\sigma}_a)\hat{C}P S_{DB}(y, z). \]  

(3.11)

The nonlocal “current” \( K_{AB}^\lambda(x, y, z) \) is conserved with respect to \( y \) by virtue of Eqs. (2.24), (3.1), and (3.4). Of course, the right-hand side of (3.10) is independent of \( y^0 \) and reduces to \( S_{AB}^x(x, z) \) by setting \( y^0 = x^0 \) via (3.2).

Using Eqs. (2.29) and (3.7), we obtain the 4D anti-commutation relation between \( \xi_A(x) \) and \((\xi^\dagger)_B(y)\) as follows:

\[ \{\xi_A(x), (\xi^\dagger)_B(y)\} = i S_{AB}(x, y) + R_{AB}(x, y; \xi^\dagger), \]  

(3.12)

where \( R_{AB}(x, y; \xi^\dagger) \) contains a commutator between \( S_{AB}(x, y) \) and \((\xi^\dagger)_C(z)\).

4. Transformation properties

In the quantum coupled Einstein–Weyl system, there are the affine, the gravitational BRST, and the internal Lorentz BRST symmetries. We investigate the transformation properties of the quantum-gravity S function \( S_{AB}(x, y) \) with respect to these three symmetries.

4.1. Affine transformation

Let \( \hat{P}_\lambda \) and \( \hat{M}_\lambda^\kappa \) be the translation generator and the \( GL(4) \) one [1], respectively. In what follows, we show that the affine transformation of \( S_{AB}(x, y) \) is given by

\[ [i\hat{P}_\lambda, S_{AB}(x, y)] = (\partial^\xi + \partial^\rho)S_{AB}(x, y), \]  

(4.1)

\[ [i\hat{M}_\lambda^\kappa, S_{AB}(x, y)] = (x^\kappa \partial^\xi + y^\kappa \partial^\rho)S_{AB}(x, y). \]  

(4.2)

As in Ref. [3], we prove only (4.2) since \( \hat{P}_\lambda \) can formally be regarded as \( \hat{M}_\lambda^5 \) with \( x^5 = y^5 = 1 \), and \( \delta^5_\mu = \delta^5_\nu = 0 \).
We define the difference between both sides of (4.2) as follows:

\[ \mathcal{A}^{\kappa AB}(x, y) \equiv [i\mathcal{M}^\kappa_\lambda, \mathcal{S}_{AB}(x, y)] - \left(x^\kappa \partial^x_\lambda + y^\kappa \partial^y_\lambda\right)\mathcal{S}_{AB}(x, y). \]  

(4.3)

Then we form a Cauchy problem for \( \mathcal{A}^{\kappa AB}(x, y) \): we apply the operator \( i\hbar^{\mu a} \hat{\sigma}_a \left( \partial_\mu + \omega_\mu \right) \) and the condition \( x^0 = y^0 \) to (4.3). Using Eqs. (2.24), (3.1), (3.2), and (3.4), we obtain

\[ i\hbar^{\mu a}(x)(\hat{\sigma}_a)^AB \{ \delta_B^C \partial^x_\mu + [\omega_\mu(x)]_B^C \} \mathcal{A}^{\kappa CD}(x, y) \]

\[ = -i [i\mathcal{M}^\kappa_\lambda, i\hbar^{\mu a}(x)](\hat{\sigma}_a)^AB \{ \delta_B^C \partial^x_\mu + [\omega_\mu(x)]_B^C \} \mathcal{S}_{CD}(x, y) \]

\[ -i h^{\mu a}(x)(\hat{\sigma}_a)^AB [i\mathcal{M}^\kappa_\lambda, [\omega_\mu(x)]_B^C ] \mathcal{S}_{CD}(x, y) \]

\[ + i x^\kappa \partial^x_\lambda h^{\mu a}(x) \cdot (\hat{\sigma}_a)^AB \{ \delta_B^C \partial^x_\mu + [\omega_\mu(x)]_B^C \} \mathcal{S}_{CD}(x, y) \]

\[ + i x^\kappa h^{\mu a}(x)(\hat{\sigma}_a)^AB \partial^x_\lambda [\omega_\mu(x)]_B^C \cdot \mathcal{S}_{CD}(x, y) \]

\[ - i h^{\kappa a}(x)(\hat{\sigma}_a)^AB \partial^x_\lambda \mathcal{S}_{BD}(x, y), \]  

(4.4)

and

\[ \mathcal{A}^{\kappa AB}(x, y)|_0 \]

\[ = \left[ i\mathcal{M}^\kappa_\lambda, -i \frac{h^0_a}{\hbar g^{00}} (\hat{\sigma}_a)^{AB} \delta^2 \right] \]

\[ + i \left[ x^\kappa \partial^x_\lambda \left( \frac{h^0_a}{\hbar g^{00}} \right) - \delta^{\kappa}_\lambda h^{00}_a + \frac{\delta^0_\lambda}{\hbar g^{00}} \left( 2 \frac{h^0_a g^{00} \delta^0_\lambda}{g^{00}} - h^{\kappa a} \right) \right] (\hat{\sigma}_a)^{AB} \delta^2. \]

(4.5)

The right-hand sides of Eqs. (4.4) and (4.5) vanish by virtue of the following commutators:

\[ [i\mathcal{M}^\kappa_\lambda, h] = x^\kappa \partial_\lambda h + \delta^\kappa_\lambda h, \]

(4.6)

\[ [i\mathcal{M}^\kappa_\lambda, g^{\rho \sigma}] = x^\kappa \partial_\lambda g^{\rho \sigma} - \delta^\rho_\lambda g^{\kappa \sigma} - \delta^\sigma_\lambda g^{\rho \kappa}, \]

(4.7)

\[ [i\mathcal{M}^\kappa_\lambda, h^{\mu a}] = x^\kappa \partial_\lambda h^{\mu a} - \delta^\mu_\lambda h^{\kappa a}, \]

(4.8)

\[ [i\mathcal{M}^\kappa_\lambda, \omega^{ab}_\mu] = x^\kappa \partial_\lambda \omega^{ab}_\mu + \delta^\kappa_\mu \omega^{ab}_\lambda; \]

(4.9)
these are derived from the affine transformation of $h_\mu^a$ [1],

$$[\hat{i} M^\kappa, h_\mu^a] = x^\kappa \partial_\lambda h_\mu^a + \delta_\mu^\kappa h_\lambda^a.$$  \hspace{1cm} (4.10)

Thus, we find

$$\mathcal{A}^\kappa_{\lambda AB}(x, y) = 0.$$  \hspace{1cm} (4.11)

Hence the set of Eqs. (4.1) and (4.2) is proved. Equation (3.3) and the Hermitian conjugate of (4.11) yield

$$[i\hat{M}^\kappa, \mathcal{S}_{AB}(x, y)]^\dagger = -(x^\kappa \partial_\lambda^x + y^\kappa \partial_\lambda^y)\mathcal{S}_{BA}(y, x).$$  \hspace{1cm} (4.12)

On the basis of Eqs. (4.1), (4.2), (4.6), and (4.8), the affine transformation of the integral representation (3.7) for $\xi_A$ is given by

$$[i\hat{M}^\kappa, \xi_A(x)] = x^\kappa \partial_\lambda \xi_A(x)$$

$$+ \int d^3 y \{ \partial_\lambda^y [y^\kappa J_A^0(x, y)] - \delta_\lambda^0 J_A^\kappa(x, y) \}.$$  \hspace{1cm} (4.13)

In the right-hand side, the integral term vanish since the “current” $J_A^\kappa(x, y)$ is conserved with respect to $y$.

### 4.2. Gravitational BRST transformation

We next show that the gravitational BRST transformation of $\mathcal{S}_{\dot{A}B}(x, y)$ is given by

$$[iQ_G, \mathcal{S}_{\dot{A}B}(x, y)] = -\kappa [e^\rho(x)\partial_\rho \mathcal{S}_{\dot{A}B}(x, y) + \mathcal{S}_{\dot{A}B}(x, y)\tilde{\mathcal{S}}_{\dot{A}B} \cdot e^\rho(y)].$$  \hspace{1cm} (4.14)

We define the difference between both sides of (4.14) as follows:

$$\mathcal{G}_{\dot{A}B}(x, y) \equiv [iQ_G, \mathcal{S}_{\dot{A}B}(x, y)]$$

$$+ \kappa [e^\rho(x)\partial_\rho \mathcal{S}_{\dot{A}B}(x, y) + \mathcal{S}_{\dot{A}B}(x, y)\tilde{\mathcal{S}}_{\dot{A}B} \cdot e^\rho(y)].$$  \hspace{1cm} (4.15)
Then we form a Cauchy problem for $G_{AB}(x, y)$: we apply the operator $i h^{\mu a}(\bar{\sigma}_a (\partial_\mu + \omega_\mu)$ and the condition $x^0 = y^0$ to (4.15). Using Eqs. (2.24), (3.1), (3.2), and (3.4), we obtain

$$i h^{\mu a}(x)(\bar{\sigma}_a)^{\hat{A}B}\{\delta_B^C \partial_\mu^x + [\omega_\mu(x)]_B^C\}G_{CD}(x, y)$$

$$= -i [iQ_G, h^{\mu a}(x)](\bar{\sigma}_a)^{\hat{A}B}\{\delta_B^C \partial_\mu^x + [\omega_\mu(x)]_B^C\}S_{CD}(x, y)$$

$$-i h^{\mu a}(x)(\bar{\sigma}_a)^{\hat{A}B}[iQ_G, [\omega_\mu(x)]_B^C]S_{CD}(x, y)$$

$$-i \kappa c^\rho(x)\partial_\rho h^{\mu a}(x)\cdot(\bar{\sigma}_a)^{\hat{A}B}\{\delta_B^C \partial_\mu^x + [\omega_\mu(x)]_B^C\}S_{CD}(x, y)$$

$$-i h^{\mu a}(x)(\bar{\sigma}_a)^{\hat{A}B}\partial_\rho[\omega_\mu(x)]_B^C \cdot S_{CD}(x, y)$$

$$+i \kappa \partial_\mu c^\rho(x) \cdot h^{\mu a}(x)(\bar{\sigma}_a)^{\hat{A}B}\partial_\rho^x S_{BD}(x, y), \quad (4.16)$$

and

$$G_{AB}(x, y)|_0 = [iQ_G, -i \frac{h^{0a}}{h g^{00}}(\bar{\sigma}_a)^{\hat{A}B}\delta^3]$$

$$- i \kappa \frac{h^{0a}}{h g^{00}}\left\{c^\rho(\partial_\rho h^{0a} - \partial_\mu c^\mu \cdot h^{0a}$$

$$-h^{0a} [\partial_\rho(c^\mu h^0) - 2 \partial_\mu c^\mu \cdot h^0]\right\}(\bar{\sigma}_a)^{\hat{A}B}\delta^3. \quad (4.17)$$

The right-hand sides of Eqs. (4.16) and (4.17) vanish by virtue of the following commutators:

$$[iQ_G, h] = -\kappa \partial_\mu(c^\mu h), \quad (4.18)$$

$$[iQ_G, g^{\lambda \mu}] = \kappa(\partial_\mu c^\mu \cdot g^\lambda + \partial_\nu c^\lambda \cdot g^{\mu \nu} - c^\nu \partial_\nu g^\lambda \mu), \quad (4.19)$$

$$[iQ_G, h^{\mu a}] = \kappa(\partial_\mu c^\mu \cdot h^{\mu a} - c^\nu \partial_\nu h^{\mu a}), \quad (4.20)$$

$$[iQ_G, \omega^{ab}_\mu] = -\kappa(\partial_\mu c^\nu \cdot \omega^{ab}_\nu + c^\nu \partial_\nu \omega^{ab}_\mu); \quad (4.21)$$

these are derived from the gravitational BRST transformation of $h^{\mu a}$ [1],

$$[iQ_G, h^{\mu a}] = -\kappa(\partial_\mu c^\rho \cdot h^{\mu a} + c^\rho \partial_\rho h^{\mu a}). \quad (4.22)$$
Thus, we find

\[ \mathcal{G}_{\dot{A}\dot{B}}(x, y) = 0. \quad (4.23) \]

Hence (4.14) is proved. Equation (3.3) and the Hermitian conjugate of (4.23) yield

\[ [iQ_G, \mathcal{S}_{\dot{A}\dot{B}}(x, y)]^\dagger = \kappa [c^\rho(y)\partial^\mu_\rho \mathcal{S}_{\dot{B}\dot{A}}(y, x) + \mathcal{S}_{\dot{B}\dot{A}}(y, x)\partial^\mu_\rho \cdot c^\rho(x)]. \quad (4.24) \]

On the basis of Eqs. (4.14), (4.18), and (4.20), the gravitational BRST transformation of the integral representation (3.7) for \( \xi_A \) is given by

\[ \{iQ_G, \xi_A(x)\} = -\kappa c^\rho(x)\partial^\mu_\rho \xi_A(x) \]

\[ + \kappa \int d^3 y \{\partial^\mu_\rho [\mathcal{J}_A^0(x, y)c^\rho(y)] - \mathcal{J}_A^\rho(x, y)\partial^\mu_\rho c^0(y)\}. \quad (4.25) \]

In the right-hand side, the integral term vanish since the “current” \( \mathcal{J}_A^\rho(x, y) \) is conserved with respect to \( y \).

### 4.3. Internal Lorentz BRST transformation

In addition, we show that the internal Lorentz transformation of \( \mathcal{S}_{\dot{A}\dot{B}}(x, y) \) is given by

\[ [iQ_L, \mathcal{S}_{\dot{A}\dot{B}}(x, y)] = -\frac{1}{2} [t^{ab}(x)(\mathcal{S}_{ab})_A^C \mathcal{S}_{\dot{C}\dot{B}}(x, y) - \mathcal{S}_{\dot{A}\dot{C}}(x, y)(\mathcal{S}_{ab})_B^C t^{ab}(y)]. \quad (4.26) \]

We define the difference between both sides of (4.26) as follows:

\[ \mathcal{V}_{\dot{A}\dot{B}}(x, y) = [iQ_L, \mathcal{S}_{\dot{A}\dot{B}}(x, y)] \]

\[ + \frac{1}{2} [t^{ab}(x)(\mathcal{S}_{ab})_A^C \mathcal{S}_{\dot{C}\dot{B}}(x, y) - \mathcal{S}_{\dot{A}\dot{C}}(x, y)(\mathcal{S}_{ab})_B^C t^{ab}(y)]. \quad (4.27) \]
Then we form a Cauchy problem for $\mathcal{V}_{AB}(x, y)$: we apply the operator $i\hbar^{\mu a} \overset{\circ}{\sigma}_a (\partial_\mu + \omega_\mu)$ and the condition $x^0 = y^0$ to (4.27). Using Eqs. (2.24), (2.37), (3.1), (3.2), and (3.4), we obtain

$$i \hbar^{\mu a}(x)(\overset{\circ}{\sigma}_a)^{\hat{A}\hat{B}} \{ \delta_B C \partial^\mu \partial^\mu + [\omega_\mu(x)]_B^C \} \mathcal{V}_{CD}(x, y)$$

$$= -i \left[iQ_L, \hbar^{\mu a}(x)\right](\overset{\circ}{\sigma}_a)^{\hat{A}\hat{B}} \{ \delta_B C \partial^\mu \partial^\mu + [\omega_\mu(x)]_B^C \} \mathcal{S}_{CD}(x, y)$$

$$-i \hbar^{\mu a}(x)(\overset{\circ}{\sigma}_a)^{\hat{A}\hat{B}} [iQ_L, [\omega_\mu(x)]_B^C] \mathcal{S}_{CD}(x, y)$$

$$+ i t^{ab} \hbar^{\mu a}(x)(\overset{\circ}{\sigma}_b)^{\hat{A}\hat{B}} \{ \delta_B C \partial^\mu \partial^\mu + [\omega_\mu(x)]_B^C \} \mathcal{S}_{CD}(x, y)$$

$$+ \frac{i}{2} \hbar^{\mu a}(x)(\overset{\circ}{\sigma}_a)^{\hat{A}\hat{B}} [D^a t^b (x)]^{cd} (\overset{\circ}{\sigma}_b)^{CD} \mathcal{S}_{CD}(x, y),$$

(4.28)

and

$$\mathcal{V}_{AB}(x, y)|_0 = \left[iQ_L, -i \frac{\hbar^0 a}{h g^{00}} (\overset{\circ}{\sigma}_a)_{AB} \delta^3 \right] + it^{ab} \frac{\hbar^0 a}{h g^{00}} (\overset{\circ}{\sigma}_b)_{AB} \delta^3.$$

(4.29)

The right-hand sides of Eqs. (4.28) and (4.29) vanish by virtue of the following commutators:

$$[iQ_L, h] = 0,$$

(4.30)

$$[iQ_L, h^{\mu a}] = -t^{ab} h^{\mu b},$$

(4.31)

$$[iQ_L, \omega^{ab}] = (D_\mu t)^{ab};$$

(4.32)

these are derived from the internal Lorentz BRST transformation of $h^{\mu a}$ [1],

$$[iQ_L, h^{\mu a}] = -t^{ab} h^{\mu b}.$$  \hspace{1cm} (4.33)

Thus, we find

$$\mathcal{V}_{AB}(x, y) = 0.$$  \hspace{1cm} (4.34)

Hence (4.26) is proved. Equation (3.3) and the Hermitian conjugate of (4.34) yield

$$[iQ_L, \mathcal{S}_{AB}(x, y)]^\dagger = \frac{1}{2} [t^{ab}(y)(\mathcal{S}_{ab})_B^C \mathcal{S}_{CD}(y, x) - \mathcal{S}_{BC}(y, x)(\mathcal{S}_{ab})_A^C t^{ab}(x)],$$  \hspace{1cm} (4.35)
via (2.12).

Applying Eqs. (4.26), (4.30), and (4.31) to the anti-commutation relation between $Q_L$ and the integral representation (3.7) for $\xi_A$, we obtain

$$\{iQ_L, \xi_A(x)\} = -\frac{t^{ab}(x)}{2}(S_{ab})^B_A \xi_B(x). \quad (4.36)$$

5. Discussion

In the present paper, we have briefly reviewed the manifestly covariant operator formalism for Weyl fields in quantum Einstein gravity. We have introduced a quantum-gravity S function $S_{AB}(x,y)$ for Weyl fields as a function satisfying a q-number Cauchy problem given by Eqs. (3.1) and (3.2). Our treatment of $S_{AB}(x,y)$ has been analogous to that of $S(x,y;m)$ in Ref. [3]. We have shown that the affine, the gravitational BRST, and the internal Lorentz BRST transformations of $S_{AB}(x,y)$ are consistent with Eqs. (3.1), (3.2), and (3.7).

As the Weyl field, we have taken $\xi(x)$ rather than $\eta(x)$ in (2.2). Of course, one can select $\eta(x)$ with the use of $\mathcal{L}_{\eta}$ in (2.19), and consequently introduce the quantum-gravity S function $S^{AB}(x,y)$ for it. This S function is defined by the following q-number Cauchy problem:

$$ih(x)h^{\mu a}(x)(\hat{\sigma}_a)_{CD}\{\delta^D_A, \partial_{\mu}^x + [\hat{\omega}_\mu(x)]_A^D\}S^{AB}(x,y) = 0, \quad (5.1)$$

$$S^{AB}(x,y)|_0 = -i\frac{h^{0a}(x)}{h(x)g^{00}(x)}(\hat{\sigma}_a)^{AB}\delta^3. \quad (5.2)$$

The treatment of $S^{AB}(x,y)$ is parallel with that of $S_{AB}(x,y)$.

Since $S_{AB}(x,y)$ is not a function of $x - y$ alone, we must distinguish $\partial^y$ from $-\partial^x$. On the other hand, if a physical vacuum $|0\rangle$ is translationally invariant,

$$\hat{P}_\lambda|0\rangle = 0, \quad (5.3)$$
then the vacuum expectation value of (4.1) is given by

\[
\langle 0 | [ i \hat{P}_{\lambda}, S_{AB}(x, y) ] | 0 \rangle = ( \partial^x_{\lambda} + \partial^y_{\lambda} ) \langle 0 | S_{AB}(x, y) | 0 \rangle = 0 , \tag{5.4}
\]

These mean that \( \partial^y_{\lambda} \) is equivalent to \( -\partial^x_{\lambda} \).

References