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Hotelling's Beach with Linear and Quadratic Transportation Costs: Existence of Pure Strategy Equilibria

Alain Egli

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Schanzeneckstrasse 1
Postfach 8573
CH-3001 Bern, Switzerland
http://www.vwi.unibe.ch

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University of Bern

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Abstract

In Hotelling type models consumers have the same transportation cost function. We deviate from this assumption and introduce two consumer types. Some consumers have linear transportation costs, while the others have quadratic transportation costs. If at most half the consumers have linear transportation costs, a subgame perfect equilibrium in pure strategies exists for all symmetric locations. Furthermore, no general principle of differentiation holds. With two consumer types, the equilibrium pattern ranges from maximum to intermediate differentiation. The degree of product differentiation depends on the fraction of consumer types.

Keywords: Hotelling, Horizontal Product Differentiation, Equilibrium

JEL-Classification Numbers: D34, L13, R32

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*Universität Bern, Volkswirtschaftliches Institut, Abteilung für Wirtschaftstheorie, Schanzeneckstrasse 1, Postfach 8573, CH-3001 Bern, Switzerland, alain.egli@vwi.unibe.ch.
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1 Introduction

Product differentiation is a major marketing tool. Firms use product differentiation to soften price competition. In his seminal paper, Hotelling (1929) introduced a very appealing model of horizontal product differentiation to circumvent the discontinuous consumer behavior proposed by Bertrand. The Hotelling approach models product differentiation by introducing firm locations and consumer addresses. Consumers have different addresses. An address represents a consumer's ideal good or most preferred sales location. The distance between a firm's location and a consumer's address indicates how close the good actually produced is to the consumers' ideal good. Consumers who buy a less-than-ideal good incur a disutility; or, in Hotelling's term, transportation costs.

The literature views Hotelling's original model as a two-stage location-then-price game. Two firms compete for demand with a location choice in the first stage, and with prices in the second. However, the two-stage location-then-price game has a drawback. D'Aspremont, Gabszewicz, and Thisse (1979) show that no subgame perfect equilibrium in pure strategies exists if transportation costs are linear in distance. This non-existence occurs because demand functions are discontinuous and hence profit functions are neither continuous nor quasi-concave.

Existence of equilibrium in Hotelling type models depends on the basic assumptions and a number of parameters. Brenner (2001) provides a nice survey about the determinants of equilibrium existence and product differentiation. For example, various authors consider firms locating on a circumference, different number of firms, restricted reservation prices, non-uniform consumer densities over space, collusive behavior, or choice of the pricing policy. The most influential modification comes from d'Aspremont, Gabszewicz, and Thisse: quadratic transportation costs. With quadratic transportation costs an equilibrium in pure strategies exists for any of the firms' locations.

We make a related modification. Our modification consists in introducing two types of consumers. Besides varying tastes, consumers differ in the assessment of the distance between ideal and actual good. For assessment of distance we use linear and quadratic transportation cost functions, as these types are well known and widely used in literature. Some consumers have linear transportation costs. The other consumers have quadratic transportation costs. This specification represents a hybrid between Hotelling's original formulation and the modification of d'Aspremont, Gabszewicz, and Thisse.

Let us motivate different consumer types using Hotelling's cider example. We can view the firms' locations as the degree of sourness in the cider they offer. Consumers differ in the degree of sourness they desire. Now, consider consumers who prefer the most sour cider possible. All these consumers have the same address. If they consume the sweetest cider possible the distance between their preferred and their consumed good is the same. But it is possible that these consumers do not attach the same importance to the distance. Consumers value the distance between ideal and consumed good differently. Or, consider consumers whose ideal polo shirt brand is *Lacoste*. If these consumers wear a polo shirt from *Quicksilver*, say, they incur a disutility. Although the difference between *Lacoste* and *Quicksilver* is fix, the disutility may vary among consumers. The disutility varies because consumers assess the difference differently.

With our modification we remain very close to Hotelling's model. But we find pure strategy equilibrium existence for any symmetric locations if at most half the consumers have a disutility linear in distance. By contrast, the same existence problem as in Hotelling's original model arises if more than half the consumers have linear transportation costs. No equilibrium in pure strategies exists for all symmetric locations.

Previous studies with modifications of Hotelling's model reject a general principle of differentiation[1]. We also reject a general principle of differentiation. With two consumer types, differentiation between firms' goods depends on the fraction of the respective types. However, maximum differentiation is frequent. Firms locate at the extremes in product space for fractions of consumers with linear transportation costs between zero and one third. When the fraction of consumers with linear transportation costs exceeds one third, firms move towards each other. Equilibrium locations are interior solutions. If the number of consumers with linear transportation costs is high (approximately 0.86) the equilibrium distance between firms increases again. This increase is due to restrictions for location spaces that we impose to solve the non-existence problem. Firms must keep a minimal required distance. For large fractions of consumers with linear transportation costs firms locate as close to each other as the minimal required distance allows. The minimal required distance between firms is increasing in the fraction of consumers with linear transportation costs. Hence, product differentiation also increases.

The paper is organized as follows: In section 2 we set up Hotelling's model with two consumer types. Next, in section 3, we derive the demand functions and the equilibrium. In section 4 we discuss the equilibrium outcome. Finally, we conclude in section 5.

[1]See, e.g., Böckem (1994), Economides (1986), Hinloopen and van Marrewijk (1999), and Wang and Yang (1999)

2 The Model

Consider two firms, 1 and 2, each selling one good. The goods are identical except for a one dimensional characteristic. This characteristic represents for example the sweetness of cider or a firm's brand. Firms choose the amount of characteristic by locating on a line with length one. Each firm's location $q_i \in [0, 1]$ measures the amount of characteristic embodied in the good. We assume that firm 1 locates to the left of firm 2, i.e., $q_1 < q_2$. Firm i sells its good at mill price p_i. Let us also assume, for simplicity, that both firms produce at zero fixed and marginal costs.

Suppose there is a continuum of consumers with total mass one. All consumers have the same gross valuation r for exactly one unit of the good. The valuation r is sufficiently high such that in equilibrium all consumers buy from one of the firms. So, valuation r is never binding and the market always covered.

Each consumer knows her individually preferred amount of characteristic embodied in the good. Denote a consumer's most preferred amount of characteristic by the address θ. If a consumer buys a good with a different-than-ideal characteristic, she suffers a disutility. This disutility is the distance between q and θ weighted by the utility loss per unit distance t. Per unit distance costs t measure consumers' sensitivity to product differentiation. Thus, a consumer with address θ pays the mill price p and transportation costs $t|q - \theta|$ when buying a good with characteristic q. We call the mill price plus the transportation costs the generalized price.

Up to this point we follow Hotelling's original model. Our modification consists in modelling two types of consumers. A fraction $\alpha \in [0, 1]$ of consumers incur linear transportation costs. The other fraction $(1 - \alpha)$ of consumers have quadratic transportation costs. We denote a consumer's address who has linear transportation costs by θ_l. Similarly, we denote a consumer's address who has convex transportation costs by θ_c. Addresses for consumers with linear transportation costs are uniformly distributed on $[0, 1]$ with density α. Analogously, addresses for consumers with quadratic transportation costs are uniformly distributed on the unit interval with density $(1 - \alpha)$.

A consumer with linear transportation costs and address θ_l who buys a good with characteristic q at price p has utility

$$u_l(\theta_l, q, p) = r - t|q - \theta_l| - p.$$

A consumer with convex transportation costs and address θ_c who buys a good with characteristic q at price p has utility

$$u_c(\theta_c, q, p) = r - t(q - \theta_c)^2 - p.$$

We study a two-stage price-then-location game. In the first stage firms simultaneously choose locations bearing in mind the subsequent price equilibrium. Given their locations, firms simultaneously set prices in the second stage. To solve the game we use the solution concept of subgame perfect Nash equilibrium. For both stages we look for equilibria in pure strategies.

3 The Equilibrium

3.1 Demand Specification

For the sake of clarity we derive the demand functions before we determine firms' equilibrium behavior. Each consumer buys from the firm which offers her the least generalized price. First, consider consumers with linear transportation costs. Given firms' locations and their prices all consumers with address that satisfies

$$-t(\theta_l - q_1) - p_1 \geq -t(q_2 - \theta_l) - p_2$$

buy firm 1's good. The consumer who is indifferent between buying firm 1's and firm 2's good has address $\hat{\theta}_l = [t(q_1 + q_2) + p_2 - p_1] / (2t)$. Consumers with address $\theta_l \leq \hat{\theta}_l$ buy from firm 1.

Now, consider consumers with convex transportation costs. All consumers with address such that

$$-t(q_1 - \theta_c)^2 - p_1 \geq -t(q_2 - \theta_c)^2 - p_2$$

prefer to buy firm 1's good. Therefore, the indifferent consumer has address $\hat{\theta}_c = [t(q_2^2 - q_1^2) + p_2 - p_1] / (2t(q_2 - q_1))$. All consumers with address $\theta_c \leq \hat{\theta}_c$ shop at firm 1.

Implicitly, we assume that $\hat{\theta}_l$ and $\hat{\theta}_c$ lie between q_1 and q_2. It turns out that this is implied by existence of pure strategy equilibria in the price game.

Using the distributional assumptions for the addresses firm 1 faces the demand function

$$\begin{aligned} D_1 &= Prob[\theta_l \leq \hat{\theta}_l] + Prob[\theta_c \leq \hat{\theta}_c] \\ &= \alpha\hat{\theta}_l + (1 - \alpha)\hat{\theta}_c. \end{aligned}$$

Similarly, the demand function for firm 2's good is

$$D_2 = \alpha(1 - \hat{\theta}_l) + (1 - \alpha)(1 - \hat{\theta}_c).$$

3.2 The Firms' Equilibrium Behavior

To find the subgame perfect equilibrium we solve the location-then-price game by backwards induction. In the second stage we look for a Bertrand-Nash equilibrium in prices. That is, firm i takes locations and p_j as given and chooses p_i to maximize profits $\pi_i = p_i D_i$. The firms maximization problems are

$$\begin{aligned}
\max_{p_1} \pi_1 =& \max_{p_1} p_1 \left[\alpha(t(q_1+q_2)+p_2-p_1) \right. \\
& \left. +(1-\alpha)(t(q_2^2-q_1^2)+p_2-p_1)/(q_2-q_1) \right] /(2t), \\
\max_{p_2} \pi_2 =& \max_{p_2} p_2 \left[\alpha(2t-t(q_1+q_2)-p_2+p_1) \right. \\
& \left. +(1-\alpha)(2t(q_2-q_1)-t(q_2^2-q_1^2)-p_2+p_1)/(q_2-q_1) \right] /(2t).
\end{aligned}$$

The F.O.Cs. for the firms' maximization problems yield their price reaction functions:

$$\begin{aligned}
p_1(p_2) =& p_2/2 + t(q_2^2-q_1^2)/\left(2(1-\alpha(1-q_2+q_1))\right), \\
p_2(p_1) =& p_1/2 + t(2(q_2-q_1)+q_1^2-q_2^2)/(2(1-\alpha(1-q_2+q_1))).
\end{aligned}$$

Note that $\partial^2 \pi_i / \partial p_i^2 < 0$ for all α by the assumption $q_1 < q_2$. Both profit functions are strictly concave in prices and the second order conditions are satisfied. It follows that the F.O.Cs. yield the optimal price reaction functions.

The reaction functions are linearly increasing functions of the other firm's price. Therefore, we can solve the system of equations given by the reaction functions to calculate the Bertrand-Nash equilibrium prices. The firms' equilibrium prices in the second stage, given their locations, are

$$\begin{aligned}
p_1^*(q_1,q_2) =& \quad t(2+q_1+q_2)(q_2-q_1)/(3(1-\alpha(1-q_2+q_1))), \\
p_2^*(q_1,q_2) =& \quad t(4-q_1-q_2)(q_2-q_1)/(3(1-\alpha(1-q_2+q_1))).
\end{aligned}$$

So far we neglected the possibility that firms can sell to consumers in the other firm's hinterland. In Hotelling's original model a firm can lower its price and attract the consumers in the rival's back yard too. D'Aspremont, Gabszewicz, and Thisse show that the firms start undercutting each other's price if they are located too closely. This undercutting process does not converge to an equilibrium in pure strategies. For Hotelling's model with convex transportation costs d'Aspremont, Gabszewicz, and Thisse show that no undercutting process occurs. These findings suggest that in our model a process of price cuts also occurs for consumers with linear transportation costs. Firm i can lower its price so that it sells to all consumers with linear transportation costs. However, the findings also suggest that the firms do not undercut each other to attract additional consumers with convex transportation costs.

Indeed, an undercutting process does not occur with respect to consumers with convex transportation costs (see Appendix A). But it can be profitable for the firms to serve all consumers with linear transportation costs. In this case firm i lowers its price so that the consumer located at the same point where firm j is located purchases from firm i. Thus, firm i serves the entire market share α. For a given $p_j^*(q_1, q_2)$ firm i undercuts with the highest possible price $\overline{p}_i$ such that it just serves all consumers with linear transportation costs (see Appendix A).

If firm 1 undercuts with the price $\overline{p}_1$, given $p_2^*(q_1, q_2)$, the indifferent consumer has address $\hat{\theta}_c^1 = (q_2^2 - q_1^2 + (p_2^*(q_1, q_2) - \overline{p}_1)/t)/(2(q_2 - q_1))$. Similarly for firm 2. Given $p_1^*(q_1, q_2)$ and close locations, firm 2 undercuts with the price $\overline{p}_2$. The indifferent consumer has address $\hat{\theta}_c^2 = (q_2^2 - q_1^2 + (\overline{p}_2 - p_1^*(q_1, q_2))/t)/(2(q_2 - q_1))$.

The firms undercut each other if $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$ are not globally profit-maximizing. Then, the same problem as in Hotelling's original model arises. For $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$ to constitute Bertrand-Nash equilibrium prices, the firms must not undercut. Following d'Aspremont, Gabszewicz and Thisse, the firms do not undercut each other if

$$p_1^*(q_1, q_2)D_1 \geq (p_2^*(q_1, q_2) - t(q_2 - q_1))(\alpha + (1-\alpha)\hat{\theta}_c^1), \quad (1)$$

$$p_2^*(q_1, q_2)D_2 \geq (p_1^*(q_1, q_2) - t(q_2 - q_1))(\alpha + (1-\alpha)(1-\hat{\theta}_c^2)), \quad (2)$$

with the demand for good 1 and good 2

$$D_1 = (2 + q_1 + q_2)/6, \quad D_2 = (4 - q_1 - q_2)/6,$$

and the indifferent consumers

$$\hat{\theta}_c^1 = (q_1 + q_2 + 1)/2, \quad \hat{\theta}_c^2 = (q_1 + q_2 - 1)/2.$$

Note that $\hat{\theta}_c^1$ and $\hat{\theta}_c^2$ are the indifferent consumers' addresses for $\overline{p}_i = p_j^*(q_1, q_2) - t(q_2 - q_1)$.

At this point we focus on symmetric locations. Hence, $q_1 + q_2 = 1$. It follows that the indifferent consumers with quadratic transportation costs are $\hat{\theta}_c^1 = 1$ and $\hat{\theta}_c^2 = 0$. This means, the undercutting firm serves the entire market by charging $\overline{p}_i$. At the undercutting price $\overline{p}_i$ and with symmetric locations both conditions (1) and (2) simplify to:

$$1/2 \geq (3\alpha + 3\alpha q_1 - 3\alpha q_2)/3.$$

For an equilibrium to exist, the distance between the firms must satisfy $q_2 - q_1 \geq \underline{d}(\alpha) = (2\alpha - 1)/(2\alpha)$. We call $\underline{d}(\alpha)$ the minimum required distance.

It is important to discuss the minimum required distance $\underline{d}(\alpha) = (2\alpha - 1)/(2\alpha)$ in more detail. We discuss the minimum required distance for interior locations because $\underline{d}(\alpha)$ is at most 1/2, e.g., $\underline{d} = 1/2$ if $\alpha = 1$. For $\alpha \leq 1/2$ the required distance is never greater than zero. Consequently, firms never find undercutting profitable. Let us give an intuition why undercutting is not profitable for $\alpha \leq 1/2$. With symmetric locations firms' prices are the same, i.e., $p_1^*(q_1, q_2) = p_2^*(q_1, q_2)$. To gain the entire market, firm i reduces its price by $t(q_2 - q_1)$. But the higher demand comes at the expense of a price reduction $t(q_2 - q_1)$ for consumers that already buy from firm i. This expense is high if the price reduction is high relative to the price $p_i^*(q_1, q_2)$. The price is increasing in α as $\partial p_i^*(q_1, q_2)/\partial\alpha > 0$ shows. Hence, the smaller α, the higher is the price reduction compared to $p_i^*(q_1, q_2)$. For $\alpha \leq 1/2$ a gain in market share does not compensate for the loss due to a lower price. The price reduction $t(q_2 - q_1)$ is too large relative to firm i's price to make undercutting profitable. However, with an increasing α the price reduction becomes smaller relative to $p_i^*(q_1, q_2)$. Undercutting becomes more attractive. For an equilibrium to exist the firms must be further away from each other. For $\alpha > 1/2$ the minimum required distance is positive. The minimum required distance $\underline{d}(\alpha)$ is an increasing function of α. Hence, the higher α, the greater must be $\underline{d}$. The polar case $\alpha = 1$ is Hotelling's original model and the firms must be located outside the quartiles for an equilibrium in pure strategies.

We state the findings from the discussion of the minimum required distance in Lemma 1 and 2.

Lemma 1 *In Hotelling's location-then-price game with two types of consumers and $q_1 + q_2 = 1$ a pure-strategy Bertrand-Nash equilibrium always exists for $\alpha \leq 1/2$. The price equilibrium is given by $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$.*

Lemma 2 *In Hotelling's location-then-price game with two types of consumers and $q_1+q_2 = 1$ a pure-strategy Bertrand-Nash equilibrium for $\alpha > 1/2$ exists iff $q_2 - q_1 \geq \underline{d}(\alpha)$. If a price equilibrium exists, it is given by $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$.*

Lemma 2 has a crucial impact on equilibrium existence in the whole two-stage location-then-price game. According to Lemma 2, no price equilibrium in pure strategies exists for $\alpha > 1/2$ and location combinations which violate $q_2 - q_1 \geq \underline{d}(\alpha)$. For these location combinations firms cannot know their payoffs because no price equilibrium exists. Without knowledge of their payoffs, firms do not have the basis for a rational location decision. Therefore, we must restrict firms' location spaces in case $\alpha > 1/2$.

The restriction of firms' location spaces is symmetric around the center because we focus on symmetric locations. A symmetric restriction means that firms cannot locate closer to the center than half the minimum required distance. Firm 1 to the left and firm 2 to the right of the center. For firm 1 the restricted strategy space is $q_1 \in [0, (1 - \underline{d}(\alpha))/2]$. By symmetry, firm 2 chooses locations $q_2 \in [(1 + \underline{d}(\alpha))/2, 1]$.

We now turn to the first stage in the location-then-price game. In the first stage, firms simultaneously choose their locations. Firm i maximizes profits π_i with respect to its location q_i. Substituting the equilibrium prices p_1^* and p_2^* dependent on locations into the firms' profit functions, firms' maximization problems are

$$\max_{q_1} \quad t(2 + q_2 + q_1)^2 (q_2 - q_1)/(18(1 - \alpha(1 - q_2 + q_1))),$$
$$\max_{q_2} \quad t(4 - q_1 - q_2)^2 (q_2 - q_1)/(18(1 - \alpha(1 - q_2 + q_1))).$$

Differentiating firms' profits with respect to locations yields the following F.O.Cs.:

$$\frac{\partial \pi_1}{\partial q_1} = \frac{t(2 + q_1 + q_2)}{18(1 - \alpha + \alpha q_2 - \alpha q_1)^2} \underbrace{[2\alpha(q_1 - q_2)^2 + (1 - \alpha)(q_2 - 3q_1 - 2)]}_{A_1} = 0,$$
$$\frac{\partial \pi_2}{\partial q_2} = \frac{t(4 - q_1 - q_2)}{18(1 - \alpha + \alpha q_2 - \alpha q_1)^2} \underbrace{\left[(1 - \alpha)(4 + q_1 - 3q_2) - 2\alpha(q_1 - q_2)^2\right]}_{A_2} = 0.$$

A closer look at the F.O.Cs. shows that the relevant terms for firms' optimal locations are A_1 for firm 1 and A_2 for firm 2. Solving $A_i = 0$ for q_i yields firm i's optimal location as reaction function $q_i(q_j)$ of the other firm j's location. The equation $A_i = 0$ is quadratic in q_i and yields two solutions

$$q_1(q_2) = \left(4\alpha q_2 + 3(1-\alpha) \pm \sqrt{16\alpha q_2(1-\alpha) + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),$$
$$q_2(q_1) = \left(4\alpha q_1 - 3(1-\alpha) \pm \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha q_1(1-\alpha)}\right)/(4\alpha).$$

Easy algebra shows that the first solution for firm 1's location reaction function implies $q_1(q_2) \geq q_2$. Similarly, the second solution for firm 2's optimal location yields $q_2(q_1) \leq q_1$. Hence, the economically meaningful reaction function for firm 1 is the second solution and for firm 2 the first solution. To keep track of, we restate the firms' location reaction functions:

$$q_1(q_2) = \left(4\alpha q_2 + 3(1-\alpha) - \sqrt{16\alpha q_2(1-\alpha) + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),$$
$$q_2(q_1) = \left(4\alpha q_1 - 3(1-\alpha) + \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha q_1(1-\alpha)}\right)/(4\alpha).$$

The intersection of the reaction functions gives a closed form solution for an interior Nash equilibrium in locations (that is, one where $0 < q_1 < q_2 < 1$). To show the existence of an interior Nash equilibrium we need the reaction curves behavior. A detailed discussion of the reaction curves is relegated to Appendix B. Here, we report the reaction functions' main characteristics and depict them in Figure 1. Figure 1 displays all location combinations in q_1-q_2-space. The line $q_1 = q_2$ separates the q_1-q_2-space into two regions. In the region to the left of $q_1 = q_2$ lie all location combinations with $q_1 > q_2$. The right region contains all combinations with $q_1 < q_2$. Therefore, the reaction functions must lie in the region to the right of the line $q_1 = q_2$. Firm 1's reaction function is strictly convex. Firm 2's reaction function is strictly concave. Both reaction functions have slopes smaller than one. For

$q_1+q_2<1$ firm 1's reaction function has a smaller slope than firm 2's reaction function. If the firms' locations are symmetric the reaction curves have the same slope. For $q_1+q_2>1$ firm 1's reaction function is steeper than firm 2's.

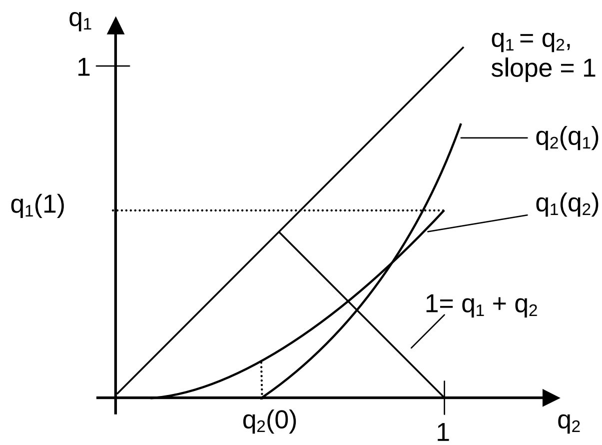

Figure 1: Intersection of the reaction functions for $\alpha > 1/3$

The following, rather tedious arguments describe when the reaction functions intersect. Consider the values $q_2(0)$ and $q_1(1)$. Firm 1's reaction function evaluated at $q_2(0)$ can be equal to, smaller, or greater than 0: $q_1(q_2(0)) \lesseqqgtr 0$. Analogously, firm 2's reaction function evaluated at $q_1(1)$ can be equal to, smaller, or greater than 1: $q_2(q_1(1)) \lesseqqgtr 1$.

For $\alpha > 1/3$ we have $q_1(q_2(0)) > 0$ and $q_2(q_1(1)) < 1$ as depicted in figure 1. From firm 1's viewpoint its reaction function lies to the left of firm 2's for $q_2(0)$. By contrast, firm 1's reaction function lies to the right of firm 2's for $q_1(1)$. Hence, for $\alpha > 1/3$ the reaction functions intersect. An interior Nash equilibrium in locations exists.

For $\alpha = 1/3$ we obtain $q_1(1) = 0$ and $q_2(0) = 1$. The reaction functions and the intersection coincide with the corner point identified by the location pair $(q_1 = 0, q_2 = 1)$.

It remains to consider firms' location choices for $\alpha < 1/3$. In these cases, we have $q_1(1) < 0$ and $q_2(1) > 1$: the reaction functions are not in the strategy spaces, i.e., $q_1 \notin [0, q_2)$ and $q_2 \notin (q_1, 1]$. Hence, no intersection between the reaction functions exists.

We are now ready to determine the firms' optimal location choices. For $\alpha > 1/3$ an interior Nash equilibrium in locations exists and is given by the system of equations containing firms' reaction functions. Solving the system of equations for firm 1's location yields two solutions:

$$q_1^* = (1 + \alpha \pm \sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha).$$

The solution $q_1 = (1 + \alpha + \sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha)$ is not in the strategy space. In particular,

$$(1 + \alpha + \sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha) > \begin{cases} 1, & \text{for } 0 \leq \alpha \leq 1/2, \\ (1 - \underline{d}(\alpha))/2, & \text{for } 1/2 < \alpha \leq 1. \end{cases}$$

Therefore, we can exclude this first solution. Plugging q_1^* in firm 2's reaction function yields its optimal location:

$$\begin{aligned} q_2^* = &\Big(4\alpha - 2 - \sqrt{(1-\alpha)(5\alpha+1)} \\ &+ \sqrt{(1-\alpha)(19\alpha + 5 + 4\sqrt{(1-\alpha)(5\alpha+1)})}\Big)/(4\alpha) \\ = &\left(4\alpha - 2 - \sqrt{1-\alpha}\left(\sqrt{5\alpha+1} - \sqrt{(\sqrt{1-\alpha} + 2\sqrt{5\alpha+1})^2}\right)\right)/(4\alpha) \\ = &\left(4\alpha - 2 + \sqrt{1-\alpha}(\sqrt{1-\alpha} + \sqrt{5\alpha+1})\right)/(4\alpha) \\ = &\left(3\alpha - 1 + \sqrt{(1-\alpha)(5\alpha+1)}\right)/(4\alpha). \end{aligned}$$

For $\alpha = 1/3$ the firms' reaction functions coincide in the corner $(0, 1)$. Easy calculations show that firm 1 chooses $q_1 = 0$ given $q_2 = 1$. Firm 2's optimal location is $q_2 = 1$ given $q_1 = 0$. Indeed, the location pair ($q_1^* = 0, q_2^* = 1$) is a Nash equilibrium in locations. Firms choose maximum differentiation.

Last, what is the firms' optimal behavior if $\alpha < 1/3$? We know that $A_1 < 0$ and $A_2 > 0$ for $\alpha < 1/3$. It follows that $\partial\pi_1/\partial q_1 < 0$ for firm 1 and $\partial\pi_2/\partial q_2 > 0$ for firm 2. Consequently, each firm increases its profits by moving away as far as possible from the other. Thus, the principle of maximum differentiation also holds for $\alpha < 1/3$.

Figure 2 shows firms' location choices by the solid lines. The shaded area

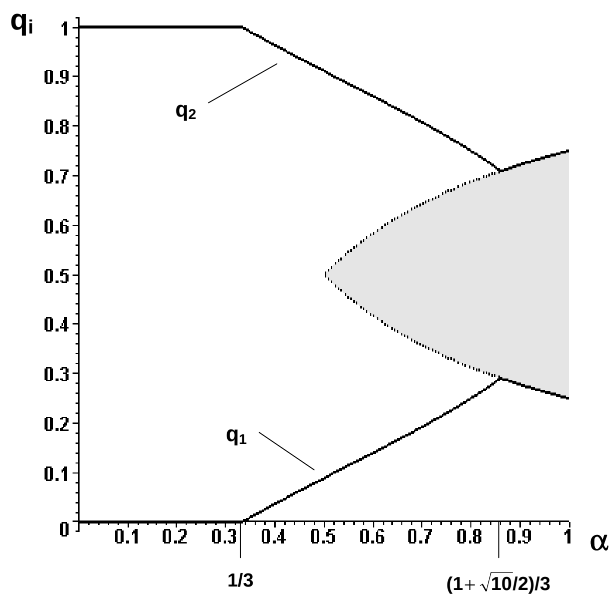

Figure 2: Firms' equilibrium locations

represents the restriction in location spaces. Firms choose symmetric interior locations around the center for $\alpha > 1/3$, i.e., $q_1^* + q_2^* = 1$. Furthermore, the distance $d^* = (\alpha - 1 + \sqrt{(1-\alpha)(5\alpha+1)})/(2\alpha)$ between firms' equilibrium locations is a decreasing function of α. With a higher α the firms increase their profits by moving towards each other. But we restrict firms' location spaces for $\alpha > 1/2$. Both firms must maintain half the minimum required distance $\underline{d}(\alpha)$: $q_1^* \leq (1-\underline{d}(\alpha))/2$ and $q_2^* \geq (1+\underline{d}(\alpha))/2$. These conditions boil down to

$$0 \leq 6\alpha^2 - 4\alpha - 1$$

for both firms. Obviously, the condition is satisfied for $1/3 < \alpha < (1+\sqrt{10}/2)/3$. In this range, firms' optimal locations are given by the solution to the system of equations containing firms' reaction functions. For $\alpha > (1+\sqrt{10}/2)/3$ firms move as close as possible to the center as strategy spaces allow: $q_1^* = (1-\underline{d}(\alpha))/2$ and $q_2^* = (1+\underline{d}(\alpha))/2$. Because $\underline{d}$ is increasing in α the restriction of location spaces forces firms further apart for $\alpha > (1+\sqrt{10}/2)/3$.

We summarize the firms' behavior in the location stage with Lemma 3:

Lemma 3 *In Hotelling's location-then-price game with two types of consumers firms choose locations*

$$q_1^* = \begin{cases} 0, & \text{for } \alpha \leq 1/3, \\ (1+\alpha-\sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha), & \text{for } 1/3 < \alpha \leq (1+\sqrt{10}/2)/3, \\ 1/(4\alpha), & \text{for } (1+\sqrt{10}/2)/3 < \alpha, \end{cases}$$

and

$$q_2^* = \begin{cases} 1, & \text{for } \alpha \leq 1/3, \\ (3\alpha-1+\sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha), & \text{for } 1/3 < \alpha \leq (1+\sqrt{10}/2)/3, \\ 1-1/(4\alpha), & \text{for } (1+\sqrt{10}/2)/3 < \alpha. \end{cases}$$

We may summarize our findings and describe the equilibrium in Proposition 1.

Proposition 1 *In the Hotelling two-stage location-then-price game with fraction α of consumers with linear transportation costs and fraction $1-\alpha$*

of consumers with quadratic transportation costs we find the following equilibria:

$\cdot$ *if* $\alpha \leq 1/3$ *(i.e.,* α *is small) firms choose locations* $q_1^* = 0$ *and* $q_2^* = 1$*. Firms set the same price* $p_1^* = p_2^* = t$ *and earn profits* $\pi_1^* = \pi_2^* = t/2$*,*

$\cdot$ *if* $1/3 < \alpha \leq (1 + \sqrt{10}/2)/3$ *(i.e.,* α *is intermediate) firms choose locations given by Lemma 3. Firms set the same price* $p_1^* = p_2^* = p^* = t(1-\alpha-\sqrt{(5\alpha+1)(1-\alpha)})/(\alpha(\alpha-1-\sqrt{(5\alpha+1)(1-\alpha)}))$ *and earn profits* $\pi_1^* = \pi_2^* = p^*/2$*,*

$\cdot$ *if* $(1+\sqrt{10}/2)/3 < \alpha$ *(i.e.,* α *is large) firms choose locations* $q_1^* = 1/(4\alpha)$ *and* $q_2^* = 1 - 1/(4\alpha)$*. Firms set the same price* $p_1^* = p_2^* = t(2\alpha - 1)/\alpha$ *and earn profits* $\pi_1^* = \pi_2^* = t(2\alpha - 1)/(2\alpha)$*.*

4 Discussion

We begin the discussion with the degree of price competition. Our specification for the degree of price competition refers to the cross-price sensitivity of demand. The cross-price sensitivity is the amount of consumers firm i gains or loses as firm j changes its price[2]. In our model, the cross-price sensitivity is equal to the own-price sensitivity multiplied by -1. Thus, our definition for the degree of price competition η is:

$$\eta = \partial D_i/\partial p_j = -\partial D_i/\partial p_i = \left(1 - \alpha(1-d)\right)/(2dt), \quad i = 1, 2,$$

where $d = q_2 - q_1$. With this definition the measure for the degree of price competition is on the positive real axis. A higher η indicates more intense price competition. Note that the degree of price competition η does not

[2]Brenner (2001) uses the cross-price sensitivity of demand as a measure for the degree of price competition to highlight the relationship between transportation cost functions and equilibrium existence.

account for undercutting effects. But we focus on pure strategy equilibria and restrict location spaces. Undercutting is ruled out. Therefore, we proceed the discussion about η without considering an undercutting process.

The degree of price competition η depends on parameters t and α as well as on distance d. First, price competition intensifies if t decreases, ceteris paribus. Firms' prices are lower, the lower is t. This is characteristic for Hotelling-type models, since t represents consumers' sensitivity to product differentiation. Consumers attach less importance to product differentiation when t is low. When t approaches zero, the model approaches Bertrand competition with homogeneous goods.

Secondly, the degree of price competition is decreasing in α, given t and d: $\partial\eta/\partial\alpha = -(1-d)/(2dt) \leq 0$. Price competition becomes less intense, the higher the fraction of consumers with linear transportation costs. When deciding about buying good 1 or good 2, consumers compare the utility from consuming good 1 with the utility from consuming good 2. This utility comparison reduces to a comparison of the difference in transportation costs with the price difference. Consumers with linear transportation costs (l-consumers) buy good 1 if

$$\underbrace{t(q_1 + q_2 - 2\theta_l)}_{\substack{\text{difference in}\\ \text{transportation costs}}} \geq \underbrace{p_1 - p_2}_{\substack{\text{price}\\ \text{difference}}}.$$

Consumers with quadratic transportation costs (c-consumers) buy good 1 if

$$\underbrace{t(q_1 + q_2 - 2\theta_c)(q_2 - q_1)}_{\text{difference in transportation costs}} \geq \underbrace{p_1 - p_2}_{\substack{\text{price}\\ \text{difference}}}.$$

Now, consider consumers with the same address but being of a different type, i.e., $\theta_l = \theta_c$. For both consumer types the price difference is the same. By contrast, the difference in transportation costs is greater for l-consumers than for c-consumers:

$$q_1 + q_2 - 2\theta_l > (q_1 + q_2 - 2\theta_c)(q_2 - q_1),$$

$$1 > q_2 - q_1,$$

except when $d = 1$. If $d = 1$ the difference is the same for both consumer types. Consumers with $\theta_l = \theta_c$ perceive the price difference relative to the difference in transportation costs equally. A price change, and hence a change in the price difference, has the same effect on consumers' buying decision, independent of their type. However, the difference in transportation costs is greater for l-consumers than for c-consumers if $d < 1$. Relative to transportation costs, l-consumers care less for a price change than c-consumers. A price change has a weaker effect on l-consumers. If α increases more consumers care less for the price relative to travel distance. The degree of price competition decreases.

We can confirm the observation $\partial\eta/\partial\alpha < 0$ by considering Hotelling's original model and the modified version of d'Aspremont, Gabszewicz, and Thisse. These models are the two polar cases $\alpha = 1$ and $\alpha = 0$ in our work. In Hotelling's original model the degree of price competition is $1/(2t)$. In the model of d'Aspremont, Gabszewicz, and Thisse the degree of price competition is $1/(2dt)$. Because $1/(2dt) > 1/(2t)$ for $d < 1$, price competition in the polar case $\alpha = 0$ is more intense than in the opposite polar case $\alpha = 1$. With an increasing α we move from more to less intense competition. The degree of price competition decreases. For $d = 1$ the degree of price competition is the same in both polar cases. It is straightforward, then, that the degree of price competition is independent of α: $\partial\eta/\partial\alpha = 0$.

The third factor that affects the degree of price competition is the distance $d = q_2 - q_1$. Keeping t and α constant, the degree of competition is decreasing in d, i.e., $\partial\eta/\partial d = (-1 + \alpha)/(2d^2 t) \leq 0$. By moving towards each other the firms offer less differentiated goods. For consumers, less differentiation leads to better substitutability between goods. Price competition increases.

Figure 3 illustrates the equilibrium degree of price competition η^* for various t by solid lines. The dotted line is the equilibrium distance d^* between firms.

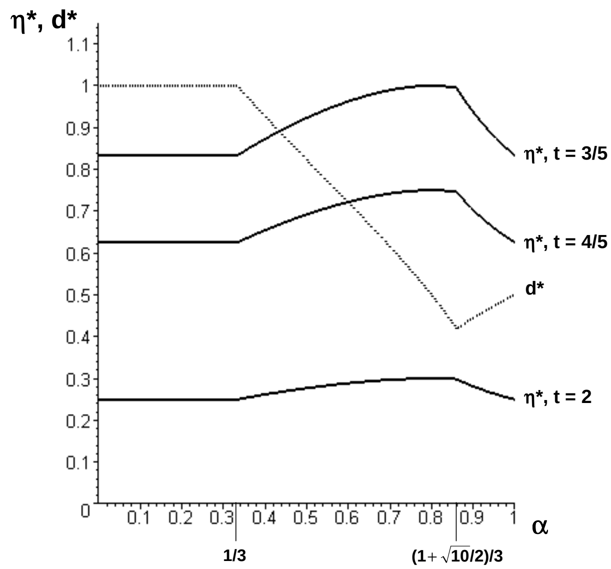

Figure 3: The degree of price competition for various t and the distance between firms in equilibrium

In the range $\alpha \leq 1/3$ firms choose maximum differentiation. The degree of price competition is constant. For $1/3 < \alpha \leq (1 + \sqrt{10}/2)/3$ both firms move towards the center. The distance d^* and product differentiation decrease. Because products are less differentiated price competition is more intense. As soon as $\alpha > (1 + \sqrt{10}/2)/3$ the degree of price competition decreases. Two effects that work in the same direction relax price competition. With an increasing α we move closer to Hotelling's original model. As argued above the degree of price competition is lower the closer we are to Hotelling's original model. The second effect stems from the restriction of location spaces. For $\alpha > (1 + \sqrt{10}/2)/3$ firms move away from each other because they must keep the minimum required distance $\underline{d}$. Because $\partial \underline{d}(\alpha)/\partial\alpha > 0$ the distance between firms increases. Firms offer more differentiated goods. More differentiated goods soften price competition.

Let us now discuss firms' location choices in the second stage. Proposition 1 and Figure 2 show that no general principle of differentiation exists in our model. Differentiation depends on the fraction of l- and c-consumers in the way intuitively expected. The more consumers with linear transportation costs, the closer we are to Hotelling's model and the closer firms move to each other. However, maximum differentiation is frequent for the range $\alpha \leq 1/3$. It seems that maximum differentiation is quite robust.

Two now standard opposite effects are responsible for firms' location choices. On the one hand, firms differentiate their goods to weaken price competition. This is the price effect. Because a larger distance between firms reduces the degree of price competition firms want to move away from each other. On the other hand, firms move inwards in the product space to capture a larger market share. This centripetal force is the demand effect. The relative strength of those effects determines the location pattern in equilibrium.

The price effect dominates the demand effect if the fraction of consumers with linear transportation costs is small. In this case, the principle of maximum differentiation holds. By contrast, maximum differentiation is not the equilibrium outcome for intermediate and large α. The reason is that the demand effect does not depend on α while the price effect does. With an increasing α the degree of price competition decreases. Relative to the demand effect the price effect becomes weaker. The price effect does not overcompensate the demand effect anymore. Firms balance the trade-off between price and demand effect increasingly in favor of the latter. Since the trade-off is increasingly in favor of the demand effect, firms move towards each other for intermediate α. For large α, the demand effect still becomes stronger. But again, the restricted location spaces lead to increased product differentiation.

Last, we turn to the condition that ensures an equilibrium. More precisely, what is the maximum fraction of consumers with linear transportation costs such that an equilibrium in pure strategies exists for all symmetric locations. The answer is short and given by Lemmas 1 and 2: $\alpha \leq 1/2$. At most half the consumers can have linear transportation costs. Otherwise, no pure-strategy price equilibrium exists in the second stage for all symmetric locations. Without price equilibrium for some location patterns firms are not able to evaluate their profits in the first stage. No (pure-strategy) equilibrium to the two-stage location-then-price game exists.

As it is well-known, non-existence of equilibrium in Hotelling's original model arises because the profit functions are not quasi-concave. The same problem occurs for $\alpha > 1/2$ in our model. Profit functions are neither continuous nor quasi-concave. Firms can undercut the opponent's price to capture the entire market. At this undercutting price the profit functions, as well as the demand functions, are discontinuous. For $\alpha > 1/2$ and sufficiently close locations undercutting is profitable. Profit functions have an upward discontinuity. In this case, firms' profit functions are not quasi-concave. Firms start undercutting each other. The undercutting process results in discontinuous best reply functions. Unfortunately, these discontinuous best reply functions do not lead to a price equilibrium in pure strategies.

5 Conclusions

Consumers may assess deviations from buying a less-than-ideal good differently. To allow for such different assessment we introduce two consumer types in Hotelling's model of product differentiation. A fraction α of consumers have linear transportation costs. The other fraction $(1-\alpha)$ of consumers have quadratic transportation costs.

As expected, we cannot support a general principle of differentiation. But maximum differentiation seems to be quite robust. In the subgame perfect Nash equilibrium firms choose maximum differentiation if at most one third of the consumers have linear transportation costs. With an increasing fraction of consumers who have a disutility linear in distance the agglomeration force becomes stronger. Firms move closer to each other.

The fraction of consumers with linear transportation costs also affects equilibrium existence. A subgame perfect equilibrium does not exist for any symmetric locations and any fraction of consumers with linear transportation costs. Only if at most half the consumers have linear transportation costs an equilibrium in the price subgame exists. A price equilibrium no longer exists for any symmetric locations if more than half the consumers have linear transportation costs. The same non-existence problem as in Hotelling's model occurs.

To circumvent the non-existence problem we impose location restrictions on firms. Firms must keep the minimal required distance such that a pure-strategy price equilibrium exists. This minimal required distance must go from zero to one half.

Appendix A

We now proof that the firms set an undercutting price $\overline{p}_i = p_j - t(q_2 - q_1)$ given selling to all consumers with linear transportation costs is profitable. To serve all consumers with linear transportation costs firm i can at most charge the price $\overline{p}_i$. With a higher price the consumer located at the same point where firm j is does not purchase from firm i. Because the consumer with address $\theta_l = q_j$ does not buy firm i's good not all consumers with linear transportation costs buy firm i's good. Therefore, we can restrict attention to prices $p_i \leq \overline{p}_i$.

Consider firm 1 that undercuts firm 2. If firm 1 sets a price $p_1 \leq \overline{p}_1$ it sells to all consumers with linear transportation costs. At an undercutting price p_1 firm 1 bags profits

$$\pi_1 = p_1 \left(\alpha + (1-\alpha) \frac{t(q_2^2 - q_1^2) + p_2 - p_1}{2t(q_2 - q_1)} \right).$$

Firm 1's profits change with p_1 according to

$$\frac{\partial \pi_1}{\partial p_1} = \alpha + (1-\alpha) \frac{t(q_2^2 - q_1^2) + p_2 - 2p_1}{2t(q_2 - q_1)}.$$

For $\overline{p}_1 = p_2 - t(q_2 - q_1)$ this derivative is

$$\left. \frac{\partial \pi_1}{\partial p_1} \right|_{p_1 = \overline{p}_1} = \alpha + (1-\alpha) \frac{t(q_2^2 - q_1^2) - p_2 + 2t(q_2 - q_1)}{2t(q_2 - q_1)}.$$

Now, for a given $p_2 = p_2^*(q_1, q_2)$ the derivative $\partial \pi_1 / \partial p_1$ evaluated at $\overline{p}_1 = p_2^*(q_1, q_2) - t(q_2 - q_1)$ is positive:

$$\left. \frac{\partial \overline{\pi}_1}{\partial \overline{p}_1} \right|_{p_1 = \overline{p}_1} \lesseqqgtr 0$$

$$2t\alpha(q_2 - q_1) + (1-\alpha)t(q_2^2 - q_1^2) \lesseqqgtr (1-\alpha)(p_2^*(q_1, q_2) - 2t(q_2 - q_1))$$

$$2 + (1-\alpha)(q_1 + q_2) \lesseqqgtr \frac{(1-\alpha)(4 - q_1 - q_2)}{3(1 - \alpha + \alpha q_2 - \alpha q_1)}$$

$$6\alpha(q_2 - q_1) + 3(1-\alpha)(q_1 + q_2)(1 - \alpha + \alpha q_2 - \alpha q_1) > -(1-\alpha)(2 + q_1 + q_2).$$

Taking the second derivative of π_1 with respect to p_1 shows that firm 1's undercutting profits are a strictly concave function:

$$\frac{\partial \pi_1^2}{\partial^2 p_1} = -1/(t(q_2 - q_1)).$$

Because firm 1's profit function is concave the derivative $\partial \pi_1 / \partial p_1$ is positive for all $p_1 \leq p_2^*(q_1, q_2) - t(q_2 - q_1)$. Firm 1's profits are an increasing function of the price: the higher the price, the higher the profits. Thus, firm 1 chooses the highest possible undercutting price $\overline{p}_1 = p_2^*(q_1, q_2) - t(q_2 - q_1)$, provided that it is profitable to attract all consumers with linear transportation costs.

Going through the same calculations for firm 2 yields an analogous result. Given that serving all consumers with linear transportation costs is profitable, firm 2 undercuts with a price $\overline{p}_2 = p_1^*(q_1, q_2) - t(q_2 - q_1)$.

Appendix B

Maximizing firms' profits with respect to their locations yields the reaction functions:

$$q_1(q_2) = \left(4\alpha q_2 + 3(1-\alpha) - \sqrt{16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),$$
$$q_2(q_1) = \left(4\alpha q_1 - 3(1-\alpha) + \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1}\right)/(4\alpha).$$

Denote the term in the square root in firm i's reaction function by φ_i. Simple inspection of φ_i shows that it is non-negative. For φ_1 in firm 1's reaction function this is

$$\varphi_1 = 16\alpha(1-\alpha)q_2 + \underbrace{9 - 2\alpha - 7\alpha^2}_{\geq 0} \geq 0, \quad \forall \alpha \in [0, 1].$$

To see that φ_2 in firm 2's reaction function is also non-negative we first observe that it negatively depends on q_1. Hence, if φ_2 is non-negative for

$q_1 = 1$, non-negativity holds for all $q_1 \leq 1$. The problem boils down to checking if φ_2 is non-negative for $q_1 = 1$. Indeed, for $q_1 = 1$, φ_2 is not smaller than zero:

$$\varphi_2 = 9 + 14\alpha - 23\alpha^2 - 16\alpha + 16\alpha^2 \geq 0, \quad \forall \alpha \in [0, 1].$$

Both reaction functions are positively sloped:

$$\begin{aligned}
\frac{\partial q_1(q_2)}{\partial q_2} =& 1 - \frac{2(1-\alpha)}{\sqrt{16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2}} > 0, \\
\frac{\partial q_2(q_1)}{\partial q_1} =& 1 - \frac{2(1-\alpha)}{\sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1}} > 0.
\end{aligned}$$

Moreover, the first derivatives show that the slopes are never greater than one. Let us also compare these slopes:

$$\begin{aligned}
\partial q_1(q_2)/\partial q_2 &\lesseqqgtr \partial q_2(q_1)/\partial q_1 \\
1 - \frac{2(1-\alpha)}{\sqrt{16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2}} &\lesseqqgtr 1 - \frac{2(1-\alpha)}{\sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1}} \\
16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2 &\lesseqqgtr 9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1 \\
q_1 + q_2 &\lesseqqgtr 1.
\end{aligned}$$

The comparison of the slopes shows that firm 1's reaction function is less steeper for $q_1 + q_2 < 1$. For symmetric locations, that is $q_1 + q_2 = 1$, firms' reaction functions exhibit the same slope. If $q_1 + q_2 > 1$ firm 1's reaction function is steeper than firm 2's reaction function.

Because the reaction functions are non-linear we need the second derivatives to make further conclusions about their behavior:

$$\begin{aligned}
\frac{\partial q_1^2(q_2)}{\partial^2 q_2} =& 16\alpha(1-\alpha)^2 \varphi_1^{-3/2} \geq 0, \\
\frac{\partial q_2^2(q_1)}{\partial^2 q_1} =& -16\alpha(1-\alpha)^2 \varphi_2^{-3/2} \leq 0.
\end{aligned}$$

Therefore, firm 1's reaction function is strictly convex in q_2. Firm 2's reaction function is strictly concave in q_1.

Next, we evaluate the functions' values at the endpoints of the strategy space.

$$
\begin{aligned}
q_1(q_2 = 0) &= \left(3(1-\alpha) - \sqrt{9 - 2\alpha - 7\alpha^2}\right)/(4\alpha) \leq 0,\\
q_1(q_2 = 1) &= \left(3 + \alpha - \sqrt{9 + 14\alpha - 23\alpha^2}\right)/(4\alpha) \leq 1,\\
q_2(q_1 = 0) &= \left(-3(1-\alpha) + \sqrt{9 + 14\alpha - 23\alpha^2}\right)/(4\alpha) \geq 0,\\
q_2(q_1 = 1) &= \left(7\alpha - 3 + \sqrt{9 - 2\alpha - 7\alpha^2}\right)/(4\alpha) \geq 1.
\end{aligned}
$$

Finally, we compare $q_2(q_1(q_2 = 1))$ with 1 and $q_1(q_2(q_1 = 0))$ with 0. Firm 2's reaction function evaluated at $q_1(q_2 = 1)$ is:

$$
\begin{aligned}
q_2(q_1(q_2 = 1)) = \Big[& 4\alpha - \sqrt{(23\alpha + 9)(1-\alpha)} \\
& + \sqrt{(1-\alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1-\alpha)})}\Big]/(4\alpha).
\end{aligned}
$$

The comparison shows that $q_2(q_1(q_2 = 1)) \lesseqqgtr 1$ dependent on α:

$$
\begin{gathered}
\left[4\alpha - \sqrt{(23\alpha + 9)(1-\alpha)} + \sqrt{(1-\alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1-\alpha)})}\right]/(4\alpha) \lesseqqgtr 1\\
\sqrt{(1-\alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1-\alpha)})} \lesseqqgtr \sqrt{(23\alpha + 9)(1-\alpha)}\\
19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1-\alpha)} \lesseqqgtr 23\alpha + 9\\
\sqrt{(23\alpha + 9)(1-\alpha)} \lesseqqgtr \alpha + 3\\
\alpha(1 - 3\alpha) \lesseqqgtr 0.
\end{gathered}
$$

For $\alpha < 1/3$ firm 2's reaction function yields a value greater than 1 evaluated at $(q_1(q_2 = 1))$. If $\alpha = 1/3$ and $(q_1(q_2 = 1))$ firm 2's optimal location is 1. For $\alpha > 1/3$ firm 2's reaction function takes a value less than 1 evaluated at $(q_1(q_2 = 1))$.

Similarly, $q_1(q_2(q_1 = 0)) \lesseqqgtr 0$:

$$(\sqrt{(23\alpha+9)(1-\alpha)} - \sqrt{(1-\alpha)(19\alpha - 3 + 4\sqrt{(23\alpha+9)(1-\alpha)})})/(4\alpha) \lesseqqgtr 0$$
$$\sqrt{(23\alpha+9)(1-\alpha)} \lesseqqgtr \sqrt{(1-\alpha)(19\alpha - 3 + 4\sqrt{(23\alpha+9)(1-\alpha)})}$$
$$0 \lesseqqgtr \alpha(1-3\alpha).$$

For $\alpha < 1/3$ firm 1's optimal location is smaller than 0 evaluated at $q_2(q_1 = 0)$. For $\alpha = 1/3$ firm 1's reaction function takes the value 0. If $\alpha > 1/3$ and $q_2(q_1 = 0)$ firm 1's optimal location is $q_1 > 0$.

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