General equilibrium with private state verification

João Correia-da-Silva[1] and **Carlos Hervés-Beloso**[2]

Abstract. We study general equilibrium with private and incomplete state verification. Trade is agreed *ex ante*, that is, before private information is received. It is useful to define a *list of bundles* as a derivative good that gives an agent the right to receive one of the bundles in the list. Enforceable trade agreements can be described by P_i-measurable plans of lists of bundles, instead of P_i-measurable plans of bundles as in Radner (1968). In equilibrium, the price of a list coincides with the price of the cheapest bundle in the list, and it is always the cheapest bundle of the list that is delivered. This property leads to a system of linear inequalities which are *deliverability* constraints on the choice set. We investigate existence of equilibrium in the case in which preferences are P_i-measurable. If there is a perfectly informed trader in the economy, existence of equilibrium is guaranteed.

Keywords: General equilibrium, Differential information, Verifiability, Uncertain delivery, Lists of bundles, Rational expectations.

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[1]CEMPRE and Faculdade de Economia, Universidade do Porto, Portugal.

[2]RGEA, Facultad de Económicas, Universidad de Vigo, Spain.

1 Introduction

In chapter 7 of his "Theory of Value", Debreu (1959) shows how to extend the general equilibrium model to the case of trade under uncertainty with public state verification. All that is needed is to consider a generalized notion of commodity that also includes in its description the state of nature on which its delivery is contingent (Arrow, 1953). The model becomes equivalent to the model without uncertainty (Arrow and Debreu, 1954; McKenzie, 1959): prices for the contingent commodities are announced, and agents choose the consumption plan that they prefer (specifying a consumption bundle for each of the possible states of nature), among those that satisfy their budget restriction; after trade agreements are made, the state of nature is publicly announced and agents receive the consumption bundle that corresponds to the announced state.

We are interested in studying the implications of differential information, in the form of private and incomplete state verification. While keeping the basic structure of the model, we assume that each agent is only able to verify (in a court of law, for contracts to be enforced) that the state of nature belongs to a certain set. The ability to verify the occurrence of events (information) is exogenous and differs across agents.

The consequence of incomplete verification is that if an agent has bought different bundles for delivery in two states and is not able to verify whether the true state is one or the other, then he has to accept delivery of any of the two bundles. This is a natural generalization of the classical model, in which state verification is complete.[3]

To study this economic setting, we consider that objects of choice are lists of bundles such that the agents have the right to receive one of the bundles in the list (they have to accept any of the alternatives).[4] Contracts in which lists are traded are pervasive. A plane ticket is a list, and there are many other examples.

A plane ticket gives you the right to travel if the plane is available at the date of departure, and, if the plane is not available, the right to stay in a hotel and travel on the next plane. But you cannot verify whether the plane is available or not. If, at the

[3]A related line of research initiated by Radner (1968) is based on the idea that the consequence of incomplete information is that an agent must consume the same in states of nature that he cannot distinguish. The corresponding notion of the core was introduced by Yannelis (1991). Several developments can be found in the volume edited by Glycopantis and Yannelis (2005).

[4]This concept builds on Arrow's (1953) notion of contingent goods. A contingent bundle is obviously a contingent list of bundles with a single element.

date of traveling, the airline announces that the plane is not available, you may have no alternative other than to accept staying in a hotel and traveling on the next day.

Some car insurance contracts give you the right to use another car temporarily, in case of accident or malfunction. But the substitute car is left undefined in the contract. It is only stipulated that the car should belong to a certain class. It may be red or yellow, have radio or not, etc.

When you order a pizza, it is actually a list of bundles. The pizza may have more or less mozzarella, more or less tomato, be made with olive oil or vegetable oil, have a thin or thick crust, etc. Goods that you order are usually defined imprecisely.

More formally, consider an agent that cannot verify in a court of law whether the state of nature is 1 or 2, but nevertheless has bought $A1$ (delivery of A in state 1) and $B2$ (delivery of B in state 2). Then: if state 1 occurs, the agent can receive A or B. When receiving B in state 1, the agent cannot prove in a court of law that the contract has been violated (state 2 could be the actual state and B the contracted delivery). For the same reason: if state 2 occurs, the agent can also receive the same bundles, A or B.

Observe that the set of alternatives that may be delivered, $\{A, B\}$, is the same in the set of states that the agent cannot distinguish, $\{1, 2\}$. Something that is constant across states of nature that the agent cannot distinguish is said to be "measurable with respect to private information". Technically, with P_i denoting an agent's information partition, a function that is constant in elements of the σ-algebra generated by P_i is designated as "P_i-measurable".

We could restrict our attention to plans of lists that are measurable with respect to each agent's private information, since, as exemplified above, any non-measurable choice can be converted into a measurable one that is equivalent. Essentially, there may be a difference between what an agent buys and what an agent gets (whenever the agent is unable to prove that what he got is different from what he was entitled to receive). Buying a non-measurable consumption plan (A in state 1 and B in state 2), the agent obtains a P_i-measurable plan of lists ($A \vee B$ in state 1 and $A \vee B$ in state 2). It is important to understand that this P_i-measurability property of lists is not a restriction on trade, but the consequence of incomplete state verification on the enforceability of trade agreements.

We have introduced this model of general equilibrium with private and incomplete state verification in two previous papers (2007, 2008). All trade is agreed *ex ante*, that is, before private information is received. Prices of the contingent lists are announced, and agents choose the plan of contingent lists that they prefer among those that belong to their budget set. After receiving their private information, agents are able to verify in which set of their information partition lies the true state of nature, and receive one of the alternatives in the list that they bought for delivery in these states. Notice that agents cannot choose which of the alternatives is delivered. On the contrary, they have to accept any of the alternatives.

The selection of the bundle to be delivered to each agent must satisfy some restrictions. First, each agent must receive an alternative that is present in the list that corresponds to the actual state of nature, or to a state of nature that is undistinguishable (in the sense that the agent cannot prove in a court of law that it is not the actual state of nature). This means that, in equilibrium, no agent can prove that his contract has been violated. Second, these deliveries must constitute a feasible allocation. These restrictions leave some degrees of freedom, giving rise to a natural question: which of the alternatives should an agent expect to receive?

If agents expect to receive the worst possible bundle in a list, there exists an equilibrium in which these expectations are fulfilled. This is a *prudent expectations equilibrium* (2007). Agents act very defensively, selecting alternatives with the same utility for delivery in states that they cannot distinguish. They insure themselves completely against being deceived. Even if they are deceived, it implies no utility loss.

A more general notion is that of a *subjective expectations equilibrium* (2008). If agents have *subjective expectations*, their beliefs about the probabilities of delivery of the different alternatives in a list depend on the prices that they observe (perfectly or imperfectly), and on the alternatives specified in the list.

In this paper, we study the case in which agents know the model of the economy, and form their expectations accordingly (Muth, 1961).

We find that, in equilibrium: (1) the price of a plan of state-contingent lists (specifying a list for each state of nature) is equal to the price of the cheapest consumption plan (specifying an alternative to be delivered in each state of nature) that satisfies the requirements of the plan of state-contingent lists; and (2) the alternative that is selected for delivery is the cheapest alternative.

Rational agents expect, then, to receive the cheapest alternative in each state of nature.[5] Observing the prices of all the contingent commodities and of all the lists, agents can predict which bundle is going to be selected for delivery (the cheapest) in each state of nature. In case of a tie, agents expect to receive the alternative that they prefer (a similar assumption is made in the mechanism design literature: in case of indifference, agents tell the truth).

Knowing the consumption bundle that results from buying each of the lists, agents can, instead of choosing among lists, choose directly among these resulting consumption bundles. These bundles are those that satisfy a system of linear inequalities, which are endogenous deliverability restrictions. Consider an agent who does not distinguish between states s and t. For a state-contingent consumption plan, (x^s, x^t), to be deliverable, it must be such that $p^s \cdot x^s \leq p^s \cdot x^t$ and $p^t \cdot x^t \leq p^t \cdot x^s$. If these deliverability conditions are not satisfied, then the agent will not receive x^s in state s and x^t in state t (because these would not be the cheapest alternatives in the corresponding states). An agent with *rational expectations* chooses among plans which are deliverable in this sense (denoted $x \in C_i(p)$).

This deliverable choice set depends, therefore, on prices and on each agent's private information. The choice set of each agent is the intersection of the budget set and the deliverable set, $B_i(p) \cap C_i(p)$. If the correspondence from prices to the choice set were continuous, equilibrium existence would be guaranteed. It has closed graph, therefore, in a bounded economy, $B_i(p) \cap C_i(p)$ is upper hemicontinuous. But $C_i(p)$ is not lower hemicontinuous.[6] This property fails when prices in some state are null or when prices in states s and t, with $t \in P_i(s)$, are collinear.[7]

We give a simple example of non-existence of equilibrium caused by null prices. In the presence of differential information, prices may be null, even if state-contingent preferences are strictly monotonic. There may be some state in which resources are abundant, but such that no agent can verify that it has occurred. As a result, no agent is willing to buy commodities contingent on the occurrence of this state.

[5] Prices differ across states, thus, the cheapest bundle should also differ (and this implies that the consumption plan is not P_i-measurable).

[6] The intersection of continuous correspondences may not be continuous, anyway (Aliprantis and Border, 2007).

[7] With agents having preferences that are P_i-measurable, collinearity does not prevent existence of equilibrium. In this case, it can be shown that (having convex preferences) agents choose the same bundle for delivery in both states, implying that the deliverability restrictions are satisfied in equality.

Introducing a perfectly informed agent removes this problem, because this agent can verify the occurrence of any state. This agent may have an arbitrarily small endowment, resembling the ϵ-agent in the model of Dubey, Geanakoplos and Shubik (2005). To impose a lower bound in prices, we also consider that this agent has strictly increasing utility. The main result in this paper establishes existence of equilibrium in an economy with this additional trader.

This paper is a contribution to the theory of general equilibrium with differential information. The central paper in this literature is the pioneering work of Prescott and Townsend (1984a, 1984b), who extended the general equilibrium model to economies in which agents have private information about their preferences. In their work, an allocation is a lottery over consumption plans, and prices are linear in probabilities (not linear in consumption[8]). The same criticism applies to the core equivalence results of Forges, Heifetz and Minelli (2001). Here prices are linear in consumption, which seems to be more in the spirit of general equilibrium theory.

By thinking of assets as pools, Dubey, Geanakoplos and Shubik (2005), Bisin and Gottardi (1999) and Minelli and Polemarchakis (2000) explore the relationship between individual actions and the payoffs of assets.[9] Our setup is closely related with the "*Hidden Information Economy*" of Bisin and Gottardi (1999), but the way of formalizing uncertainty and information is quite different. In their model, uncertainty is only about endowments, and the aggregate endowment actually becomes public information. We consider uncertainty about endowments and preferences and allow agents to retain private information about the aggregate endowment. In the model of Bisin and Gottardi (1999), the outcome of trade depends on a set of messages sent by agents. Here, agents must prove that events have occurred to enforce delivery, knowing that their information is not transmitted to other agents (aggregation of information is exogenously barred). Finally, Bisin and Gottardi (1999) consider trade both before and after agents receive their information. We restrict trade to be made *ex ante*, that is, before agents receive their information, as in Radner (1968) and Yannelis (1991). The inclusion of spot markets that open after agents receive their information is left for future research.

Recently, Zame (2007) developed a very comprehensive model with a continuum of agents, in which the set of firms and the contracts that appear are also determined

[8] This was analyzed by Jerez (2005).

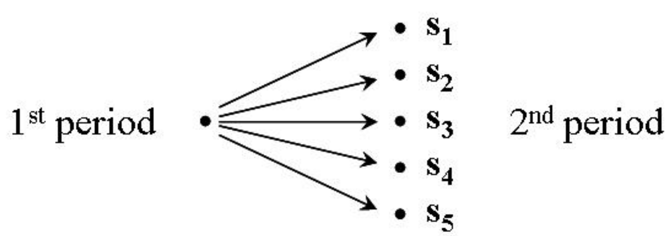
[9] In our paper, lists may also be seen as assets with many possible payoffs, with delivery rates being equilibrating variables.

endogenously at equilibrium. Other recent contributions in which agents also face incentives to make more or less effort were made by Prescott and Townsend (2006) and by Rustichini and Siconolfi (2007). Our scope is more limited: we study the case of pure exchange with a finite number of agents. In this setting, Forges, Minelli and Vohra (2002) offered a survey on the core. We provide a price-equilibrium counterpart.

The paper is organized as follows: in section 2 we motivate the paper; in section 3, the consequences of private information are analyzed; sections 4 and 5 deal with preferences over lists and prices of lists, respectively; in section 6 we present and characterize equilibrium; in section 7 we establish existence, in the presence of a small but perfectly informed trader; and section 8 concludes with some remarks. In appendix, we: (1) collect all the proofs, (2) give an example of non-existence (without the informed trader), and (3) study continuity of the deliverability correspondence.

2 The questions

Our point of departure is the classical general equilibrium model of trade under uncertainty with public state verification (Debreu, 1959, chapter 7). Uncertainty consists of a choice of nature among a finite number of possible states, $\Omega = \{1, ..., S\}$. To each state of nature corresponds a complete description of the environment (Savage, 1972), that is, the endowments and the preferences of each and every agent. Before the choice of nature, knowing their state-dependent preferences and their state-dependent endowments, agents make state-contingent trade agreements (trade is *ex ante*). After the choice of nature, the state of nature is publicly announced and the corresponding state-contingent trades are made.



This context is dealt with by considering a generalized notion of commodity (Arrow, 1953). Besides being defined by their physical properties and by their location in space and time, commodities are also distinguished by the state of nature in which they are made available. For example, instead of talking about consumption of good A in state 1 and consumption of good B in state 2, we talk about consuming good $A1$ and good $B2$.

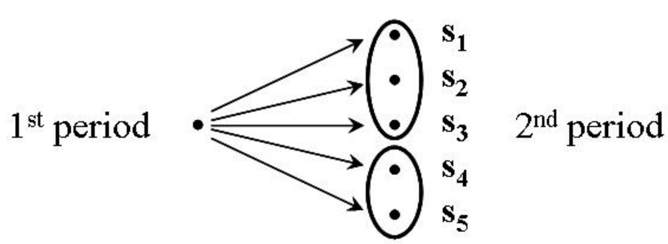
In the 1^{st} period, each agent i: attributes subjective prior probabilities to the possible states of nature, $\mu_i = (\mu_i^1, ..., \mu_i^S) \in \Delta^S$; has preferences over consumption plans that are represented by an expected utility function, $U_i(x_i) = \sum_{s=1}^{S} \mu_i^s u_i^s(x_i^s)$; observes prices for delivery in each state of nature, $p = (p^1, ..., p^S) \in \Delta^{SL}$; trades its state-contingent endowments, $e_i = (e_i^1, ..., e_i^S) \in \mathbb{R}_{++}^S$ for a consumption plan, $x_i = (x_i^1, ..., x_i^S) \in \mathbb{R}_+^{SL}$, that maximizes expected utility among the possibilities that belong to the budget set, $B_i(p) = \left\{ x_i \in \mathbb{R}_+^{SL} : \ \sum_{s=1}^{S} p^s \cdot x_i^s \leq \sum_{s=1}^{S} p^s \cdot e_i^s \right\}$.

In the 2^{nd} period: the state of nature, s, is publicly announced; each agent i delivers the endowments, e_i^s, and receives the consumption bundle, $x_i^s \in \mathbb{R}_+^L$, that correspond to this state of nature.

An equilibrium of this Arrow-Debreu-McKenzie economy with uncertainty is composed by a price system and an allocation, (p^*, x^*), such that: taking prices, p^*, as given, each agent i maximizes utility in his budget set, $x_i^* \in arg \max_{x_i \in B_i(p^*)} U_i(x_i)$; and the allocation, $x^* = (x_1^*, ..., x_n^*)$, is feasible, $\sum_i x_i^* \leq \sum_i e_i$.

What happens if, in the 2^{nd} period, agents receive different information? What happens if, instead of becoming public information, the state of nature is only incompletely and differentially revealed to each of the agents? This is the question that we address in this paper.

Let the information that agent i receives be described by a partition of the set of states of nature, P_i. If the state that occurs is s, the agent is informed that the state of nature belongs to the corresponding set of the partition, $P_i(s)$. If state t belongs to the same set of the partition, $t \in P_i(s)$, then agent i cannot *distinguish* state t from state s. Agents are endowed with what Laffont (1986) described as *fixed information structures without noise.*



To deal with this kind of differential information, Radner (1968) postulated that agents should only be interested in contracts that are contingent upon events that they can observe. In states of nature that an agent does not distinguish, the same bundle would be delivered (and consumed). With this restriction, the model of Arrow-Debreu-McKenzie could be reinterpreted to cover the case of private information.

The consumption set was restricted to $\mathbb{R}_{+}^{SL} \cap P_i$, meaning that if $t \in P_i(s)$, then $x_i^t = x_i^s$. An agent had to consume the same bundle in states of nature that he could not distinguish. It seemed that this single modification was enough to capture the consequences of differential information.

An equilibrium of the Radner economy is composed by a price system and an allocation, (p^*, x^*), such that: taking prices, p^*, as given, each agent i maximizes utility in his choice set, $x_i^* \in arg \max_{x_i \in B_i(p^*) \cap P_i} U_i(x_i)$; and the allocation, $x^* = (x_1^*, ..., x_n^*)$, is feasible, $\sum_i x_i^* \leq \sum_i e_i$. The similarity with the Arrow-Debreu-McKenzie equilibrium is striking.

Before presenting a critique of this solution, and an alternative concept, we want to make more precise the notion of information that we consider in this paper.

Consider a tree that falls in a distant forest. An agent may not even be aware of the existence of this tree. This is *unawareness*. Beyond this state of pure ignorance, we can define three hierarchic levels of information. First, an agent can be *aware* that the tree may have fallen or not, and attribute subjective probabilities to this event. A second level of information could be to *know* whether the tree fell or not. Finally, a third level would be the ability to *prove* that the tree fell or that it did not.

Having made a contract for the contingent delivery of goods, an agent may need to prove that an event has occurred to enforce delivery. This is what we assume. The meaning of the partition, P_i, is that, if state s occurs, agent i can prove that the state of nature belongs to $P_i(s)$, and uses this and only this information to enforce delivery.

A related line of research focused on the revelation of information by prices (Radner, 1979; Allen, 1981). But with trade taking place *ex ante*, an agent cannot infer the information of the other agents because, at the moment of trade, the other agents still haven't received their information.[10] After the opening of markets in the second period, agents may be able to infer the information of others. But we assume that the information

[10]The only thing that agent i could possibly infer is the priors, μ_j, of the other agents (it may be more acceptable to assume, then, a common prior).

obtained through these inferences cannot be used (in a court of law, for example) to enforce contracts.

Our main objection to the model of Radner (1968) is that agents should not be restricted to consume the same bundle in undistinguished states of nature. The example that follows shows that this restriction is too strong.

Consider an economy with two agents. Agent A is endowed with two units of '*sugar*', in all states of nature, $\Omega = \{s_1, s_2\}$, while agent B has uncertain endowments: two units of '*tea*' in state s_1 and two units of '*coffee*' in state s_2:

$$e_A^{s_1} = e_A^{s_2} = (2, 0, 0), \;\; e_B^{s_1} = (0, 2, 0) \;\text{ and }\; e_B^{s_2} = (0, 0, 2).$$

The preferences of the agents are the same, and do not depend on the state of nature. The goods '*tea*' and '*coffee*' are perfect substitutes, which agents like to drink with '*sugar*':

$$u_A^{s_1} = u_A^{s_2} = u_B^{s_1} = u_B^{s_2} = \sqrt{(x_{tea} + x_{cof})x_{sug}}.$$

Agent A cannot distinguish the two states, which are equiprobable:

$$P_A = \{s_1, s_2\} \;\text{ and }\; P_B = \{\{s_1\}, \{s_2\}\}.$$

With the restriction of consuming the same in undistinguished states of nature, there is *no trade*. To see this, observe that agent A would like to consume some '*tea*' in state s_1. But this would imply equal consumption in state s_2, and there is no '*tea*' in state s_2 (only '*coffee*'...).

In a real-life situation, the two agents could make the following agreement (valid for both states of nature): agent A would deliver one unit of '*sugar*' in exchange for one unit of '*tea*' or one unit of '*coffee*'. Agent A would get the right to receive a '*tea or coffee*', or, to put it another way, would get the right to consume $(1, 1, 0)$ or $(1, 0, 1)$.

Both agents would end up consuming $(1, 1, 0)$ in state s_1 and $(1, 0, 1)$ in state s_2. This *contract for uncertain delivery* allows the agents to attain an optimal outcome.[11]

Agents would buy what we call a *list of bundles*: a derivative good that gives the right to receive one of the bundles in the list. To guarantee delivery of a precise bundle, an agent must buy a list with a single alternative (notice that this concept builds on Arrow's (1953) notion of contingent goods, which are lists with a single alternative).

To model an economy with uncertain delivery, in which agents trade lists of bundles

[11]For other examples and a more detailed explanation, see our previous work (2007, 2008).

(instead of bundles), we must face some questions:

(1) What are the consequences of private state verification?

(2) What is the utility of a list of bundles?

(3) What is the price of a list of bundles?

3 The consequences of private state verification

As mentioned above, in the model of Radner (1968), the consequence of not distinguishing between two states is a restriction of having to consume the same in both states: $t \in P_i(s) \Rightarrow x_i^t = x_i^s$.

We do not restrict trades in this way. Agents are allowed to buy different rights for delivery in states that they do not distinguish. But, if an agent buys different rights for delivery in two states and is not able to verify whether the true state is one or the other, then the agent has to accept delivery of any of the two.

Consider an agent that cannot prove in a court of law whether the true state is s or t, but that, nevertheless, has contracted for the delivery of bundle x in state s and bundle y in state t. When receiving bundle y in state s (or bundle x in state t), the agent cannot prove that the contract is being violated. Then: if state s occurs, the agent can receive x or y; and if state t occurs, the agent can also receive the same bundles, x or y. Notice that the set of alternatives that may be delivered, $\{x, y\}$, is the same in states that the agent cannot distinguish, $\{s, t\}$.[12]

The same reasoning applies to lists. Suppose that the agent has contracted for the delivery of some alternative in the list $\tilde{x}$ in state s and some alternative in the list $\tilde{y}$ in state t. Then: if state s occurs, the agent can receive a bundle $z \in \tilde{x}$ or a bundle $z \in \tilde{y}$; and if state t occurs, the agent can also receive a bundle $z \in \tilde{x}$ or a bundle $z \in \tilde{y}$. Observe that the set of alternatives that may be delivered, $\tilde{x} \cup \tilde{y}$, is the same in the states that the agent cannot distinguish, $\{s, t\}$.

The condition that describes enforceability is not $x_i^s \in \tilde{x}_i^s$ (which means that the bun-

[12] Something that is constant across states of nature that the agent cannot distinguish is said to be "measurable with respect to private information". More technically, with P_i denoting an agent's information partition, a function that is constant in elements of the σ-algebra generated by P_i can be designated as "P_i-measurable".

dle that is delivered in state s, x_i^s, belongs to the list that was contracted for delivery in state s, $\tilde{x}_i^s$). This would be equivalent to assuming that contracts are always kept. They are not, because, in state s, agent i can only enforce delivery of a bundle that belongs to $\bigcup_{t\in P_i(s)} \tilde{x}_i^t$. The adequate condition to describe enforceability is: $x_i^s \in \bigcup_{t\in P_i(s)} \tilde{x}_i^t$.

Notice that if an agent buys the same lists for delivery in the states that he cannot distinguish, then $\bigcup_{t\in P_i(s)} \tilde{x}_i^t = \tilde{x}_i^s$, and the enforceability condition becomes $x_i^s \in \tilde{x}_i^s$. Buying the same bundle for delivery in states that are not distinguished is a sufficient condition for the contract to be enforceable (but not a necessary condition).

Formally:

(i) a state-contingent list (a list for delivery in state s) is a finite, non-empty, subset of $\mathbb{R}_+^L$, denoted $\tilde{x}_i^s \in \mathbb{F}(\mathbb{R}_+^L)$;[13]

(ii) a plan of lists is a vector of state-contingent lists $\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S$, specifying a list for delivery in each of the possible states of nature.[14]

(iii) a P_i-measurable plan of lists is a vector of state-contingent lists such that $t \in P_i(s) \Rightarrow \tilde{x}_i^t = \tilde{x}_i^s$, denoted $\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S \cap P_i$.

We define a transformation, M_i, to describe the consequences of incomplete information. If an agent buys a plan of lists $\tilde{x}_i$, the plan of lists $M_i(\tilde{x}_i) = [M_i^1(\tilde{x}_i), ..., M_i^S(\tilde{x}_i)]$ represents what the agent gets, that is, the alternatives that the agent may receive, in each state of nature. In state s, the agent may receive any of the bundles in the set $\bigcup_{t\in P_i(s)} \tilde{x}_i^t$. Therefore, M_i^s is defined as:

$$M_i^s : \ (\mathbb{F}(\mathbb{R}_+^L))^S \longrightarrow \mathbb{F}(\mathbb{R}_+^L) \ ;$$

$$M_i^s(\tilde{x}_i) = \bigcup_{t\in P_i(s)} \tilde{x}_i^t \ .$$

It should be clear that the condition that describes enforceability may be written as $x_i \in M_i(\tilde{x}_i)$. We point out that if agent i buys a list, $\tilde{x}_i$, that is not P_i-measurable, the agent gets a list, $M_i(\tilde{x}_i)$, that is P_i-measurable by construction.

In the model of Radner (1968), the consequence of incomplete information is a restriction of the choice set to P_i-measurable plans of consumption bundles. Here the

[13] Everywhere below, $\mathbb{F}(\cdot)$ denotes the set of finite and non-empty subsets.

[14] It is equivalent to consider that objects of choice are: plans of lists of consumption bundles (there is one list for each state, and an alternative in a list is a consumption bundle); or, alternatively, lists of plans of consumption bundles (there is one list for all states, and an alternative in the list is a consumption plan).

consequences are less severe. An agent can enforce delivery of P_i-measurable lists (which include all P_i-measurable consumption plans), and this does not imply P_i-measurability of the resulting consumption plan.

The main conclusion of this section is that an agent that buys a list, $\tilde{x}_i = (\tilde{x}_i^1, ..., \tilde{x}_i^S)$, may receive, in each state $s \in \Omega$, an element of the list $M_i^s(\tilde{x}_i) = \bigcup_{t \in P_i(s)} \tilde{x}_i^t$.

4 Preferences over lists

When buying a list, a rational agent has expectations about what will be the resulting consumption. These expectations, together with the preference ordering over consumption plans, induce a preference ordering over plans of state-contingent lists.

We start by making standard assumptions about preferences over consumption plans. Later we will derive preferences over plans of state-contingent lists from preferences over consumption plans.

Preferences over consumption plans are represented by an expected utility function, $U_i(x_i) = \sum_{s=1}^{S} \mu_i^s u_i^s(x_i^s)$, where μ_i^s is the subjective probability that agent i attributes to the occurrence of state s, and $u_i^s : \mathbb{R}_+^L \to \mathbb{R}$ is a particular representation of the preferences of agent i over bundles when s is the state of nature.

Assumption 4.1.

Preferences over consumption plans are represented by an expected utility function, $U_i(x_i) = \sum_{s=1}^{S} \mu_i^s u_i^s(x_i^s)$, *where each state-dependent utility function,* u_i^s, *is continuous, concave and weakly increasing.*[15]

In economies with uncertain delivery, agents choose a plan of lists, and therefore we need an objective function defined over plans of lists. Preferences over plans of lists depend on prices of lists, $\tilde{p} \in \mathcal{P}$, because rational agents see prices as a signal of the alternative that will be delivered:

$\tilde{U}_i : \ (\mathbb{F}(\mathbb{R}_+^L))^S \times \mathcal{P} \longrightarrow \mathbb{R}.$

[15] By weakly increasing, we mean that $x \gg y \Rightarrow u_i^s(x) > u_i^s(y)$.

We allow for prices to be interpreted as a signal of the alternatives that will be delivered, but it is a signal that we will only be able to understand when we characterize equilibrium (Section 6). The precise relationship between prices and deliveries will be established then. Right now, we give an example to convey the basic idea. Suppose that you want to rent a car and the agency offers you a list that includes a cheap 'Fiat' and an expensive 'Ferrari'. Wouldn't you expect to receive the 'Fiat'?

When buying a list $\tilde{x}_i$, a rational agent is aware that the possible deliveries are $M_i(\tilde{x}_i)$. Therefore, he attributes the same utility to the lists $\tilde{x}_i$ and $M_i(\tilde{x}_i)$.[16]

Assumption 4.2.

$\forall(\tilde{x}_i, p) \in (\mathbb{F}(\mathbb{R}^L_+))^S \times \Delta^{SL}: \; \tilde{U}_i(\tilde{x}_i, \tilde{p}) = \tilde{U}_i(M_i(\tilde{x}_i), \tilde{p}).$

Knowing the utility of the P_i-measurable plans of lists, we can obtain, using only this assumption, the utility of all the plans of lists that are not P_i-measurable.

We also make an assumption of *no satiation.* Agents select a list in the frontier of the budget set.

Assumption 4.3.

Let $\tilde{x}_i \in arg \max_{\tilde{z}_i \in \tilde{B}_i(\tilde{p})} \tilde{U}_i(\tilde{z}_i, \tilde{p})$. *Then:* $\tilde{p}(\tilde{x}_i) = p \cdot e_i$.

These are the starting assumptions that we make on preferences over lists. We will find, later, that agents always receive the cheapest bundle of a list. Therefore, it will make sense to assume that they attribute to a list the utility of the cheapest bundle in the list.

5 Prices of lists

In economies with uncertain delivery, it is necessary to define prices on the space of plans of lists:

[16] Recall that if $\tilde{x}_i$ is P_i-measurable, then $\tilde{x}_i = M_i(\tilde{x}_i)$.

$\tilde{p}: \ (\mathbb{F}(\mathbb{R}^L_+))^S \longrightarrow \mathbb{R}_+.$

We will dedicate most of this section to finding properties of price systems that are compatible with the absence of arbitrage, and that, therefore, are necessarily properties of an equilibrium price system.

5.1 Buying and selling lists of bundles

Suppose that an agent buys (separately) a list that delivers '*tea*' or '*coffee*' and a list that delivers '*toast*' or '*cookie*'. The agent will receive one of four alternatives: '*tea and toast*', '*tea and cookie*', '*coffee and toast*' or '*coffee and cookie*'. More generally, an agent that buys the lists $\tilde{x}_i$ and $\tilde{y}_i$ may receive any alternative in the list $\tilde{z}_i$, defined as:

$$\tilde{z}_i = \tilde{x}_i \oplus \tilde{y}_i = \{z_i \in \mathbb{R}^{SL}_+ : \ \exists (x_i, y_i) \in (\tilde{x}_i, \tilde{y}_i) \ s.t. \ z_i = x_i + y_i\}.$$

Buying two or more lists is equivalent to buying a single list with more alternatives.

Agents are also allowed to sell lists. An agent that sells a list has to deliver (in the future) one of the alternatives in the list. It is the seller that chooses the alternative to deliver, thus selling a list is different from buying a list with negative quantities (in this case, it would be the buyer that would select the alternative).

Consider an agent that buys list $\tilde{x}_i$ and sells list $\tilde{y}_i$. The agent will receive $x_i \in \tilde{x}_i$ and deliver $y_i \in \tilde{y}_i$. We assume that, in each state s, the agent delivers a best response to each possible received bundle (in case of a tie for the best response, we use a selection, $\mathcal{S}$, which is irrelevant for the results). A perfectly informed agent plans to deliver, in each state s, the alternative y_i^s, defined as:

$$y_i^s(\tilde{y}_i^s, x_i^s) = \mathcal{S}\left(arg \max_{y \in \tilde{y}_i^s} U_i^s(x_i^s - y) \right).$$

And thus obtain an equivalent list, $\tilde{z}_i = (\tilde{z}_i^1, ..., \tilde{z}_i^S)$, defined as:

$$\tilde{z}_i^s = \tilde{x}_i^s \ominus_i^s \tilde{y}_i^s = \{z \in \mathbb{F}(\mathbb{R}^L_+) : \ \exists x_i^s \in \tilde{x}_i^s \ s.t. \ z = x_i^s - y_i^s(\tilde{y}_i^s, x_i^s)\};$$

$$\tilde{z}_i = \tilde{x}_i \ominus_i \tilde{y}_i = (\tilde{z}_i^1, ..., \tilde{z}_i^S).$$

An agent with incomplete information (that buys $\tilde{x}_i$ and sells $\tilde{y}_i$) faces a further difficulty. In state s, the agent may receive an element of $\tilde{x}_i^t$ and be forced to deliver an element of $\tilde{y}_i^t$, with $t \in P_i(s)$.[17] The agent obtains, therefore, the list $M_i(\tilde{z}_i)$, with each $\tilde{z}_i^s$ defined

[17] Another option could be to consider that the agent could receive an element of $\tilde{x}_i^t$, with $t \in P_i(s)$, and be forced to deliver an element of $\tilde{y}_i^{t'}$, with $t' \in P_i(s)$. This would make trade even more difficult

as above: $\tilde{z}_i^s = \tilde{x}_i^s \ominus \tilde{y}_i^s$.

We impose a restriction on short sales. Agent i can only buy $\tilde{x}_i$ and sell $\tilde{y}_i$ such that $M_i(\tilde{z}_i) \in (\mathbb{F}(\mathbb{R}_+^L))^S$ (possible net deliveries are nonnegative). Thus, the agent can always keep the contract for delivery of an element of the list $\tilde{y}_i$. We rule out the possibility of default.

Notice that if all possible net deliveries are positive for the fully informed agent they are also positive for an agent i with incomplete information. This is true because we assume that the vector of initial endowments is P_i-measurable ($t \in P_i(s) \Rightarrow e_i^t = e_i^s$). Information does not affect the restriction on short selling:

$\tilde{z}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S \Leftrightarrow M_i(\tilde{z}_i) \in (\mathbb{F}(\mathbb{R}_+^L))^S$.

5.2 Arbitrage

An arbitrage is a trade that involves a gain and no possibility of a loss. In our context, it would consist of buying a list, $\tilde{x}_i$, and selling another, $\tilde{y}_i$, such that: (i) some income is retained; (ii) all possible net deliveries are positive.

Definition 5.1.

An arbitrage opportunity consists of a pair of lists, $(\tilde{x}_i, \tilde{y}_i)$, such that:

(i) $\tilde{p}(\tilde{x}_i) < \tilde{p}(\tilde{y}_i)$*;*

(ii) $\forall x_i \in \tilde{x}_i, \exists y_i \in \tilde{y}_i : \ x_i - y_i \geq 0$ *(that is,* $\tilde{x}_i \ominus_i \tilde{y}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S$*).*

The information of the agent is irrelevant to this definition of arbitrage. This is so because if all possible net deliveries are positive for the fully informed agent they are also positive for an agent i with incomplete information. Information does not enlarge the possibilities of arbitrage.

If there is an arbitrage opportunity, then: in the first period, agent i buys list $\tilde{x}_i$ and sells list $\tilde{y}_i$ (retaining some rent); in the second period, the agent receives $x_i^t \in \tilde{x}_i^t$, with $t \in P_i(s)$, and delivers $y_i^t \in \tilde{y}_i^t$ such that $y_i^t \leq x_i^t$.

for uniformed agents.

This implies that the budget restriction disappears. Instead of selecting a list $\tilde{w}_i$, the agent can, additionally, buy $\tilde{x}_i$ and sell $\tilde{y}_i$. The agent retains some rent and receives the same or more goods.

For a price system, $\tilde{p}$, to be an equilibrium price system, there cannot exist any arbitrage opportunities. No-arbitrage is a necessary (but not sufficient) equilibrium condition.

5.3 No-arbitrage prices

A necessary condition for absence of arbitrage is that prices must be additive, in the sense made precise below. All proofs are collected in Appendix 1.

Proposition 5.1.

Absence of arbitrage opportunities implies that:

$$\forall \tilde{x}_i, \tilde{y}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S : \ \tilde{p}(\tilde{x}_i \oplus \tilde{y}_i) = \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i).$$

Proposition 5.1 says that equilibrium prices of lists are additive. If a list that guarantees delivery of '*coffee*' or '*tea*' costs 3, and a list that guarantees delivery of a '*toast*' or a '*cookie*' costs 5, then a list that guarantees delivery of '*coffee and toast*' or '*coffee and cookie*' or '*tea and toast*' or '*tea and cookie*' must cost 8 (it is equivalent to buy the two lists separately or to buy them bundled together).

With prices being additive, agents only buy a single list. There is no point in deviating and buying two lists instead of a single one.

In the classical theory, a basic assumption on the price systems is that it does not matter for an agent to buy a single bundle or to buy its constituents in separate (prices are linear):

(i) $\forall x, y \in \mathbb{R}_+^{SL} : \ p(x+y) = p(x) + p(y);$

(ii) $\forall x \in \mathbb{R}_+^{SL}, \lambda \in \mathbb{R} : \ p(\lambda x) = \lambda p(x).$

The classical assumption (i) is a particular case of Proposition 5.1 (they are equivalent for lists with a single element). We make the classical assumption (*ii*) just for lists with a single element, that is, bundles. We do not restrict prices of lists to be scalable in this sense.

Assumption 5.1.

Given any list with a single element, $x \in \mathbb{R}^{SL}_+$, and any positive scalar, $\lambda \geq 0$:

$\tilde{p}(\lambda x) = \lambda \tilde{p}(x)$.

As a consequence of Proposition 5.1 and Assumption 5.1, the restriction of any price system, $\tilde{p}$, to the space of consumption plans can be represented by a vector of prices of the SL state-contingent commodities, $p \in \Delta^{SL}$, such that the price of a bundle is the inner product between the vector of prices and the vector of quantities:

$$\forall x \in \mathbb{R}^{SL}_+,\ \tilde{p}(x) = p \cdot x, \text{ with } p \in \Delta^{SL} = \left\{ p \in \mathbb{R}^{SL}_+ :\ \sum_{s=1}^{S} \sum_{l=1}^{L} p^{sl} = 1 \right\}.$$

The budget set of agent i is:

$$\tilde{B}_i(\tilde{p}) = \{\tilde{x}_i \in (\mathbb{F}(\mathbb{R}^L_+))^S :\ \tilde{p}(\tilde{x}_i) \leq \tilde{p}(e_i) = p \cdot e_i\}.$$

It is useful to define the function $\tilde{p}^s$ as the price of a list for delivery that is contingent on the occurrence of state s:

$$\tilde{p}^s :\ \mathbb{F}(\mathbb{R}^L_+) \longrightarrow \mathbb{R}_+;$$

$$\tilde{p}^s(\tilde{x}^s) = \tilde{p}(0, ..., \tilde{x}^s, ..., 0).$$

Observe that a plan of state-contingent lists, $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^S)$, is also the sum of the state-contingent lists: $\tilde{x} = \tilde{x}^1 \oplus ... \oplus \tilde{x}^S$. By Proposition 5.1, no arbitrage implies that the price of a plan of state-contingent lists is equal to the sum of the prices of the state-contingent lists:

$$\tilde{p}(\tilde{x}) = \sum_{s=1}^{S} \tilde{p}^s(\tilde{x}^s).$$

If a list, $\tilde{x}$, is cheaper than a list that contains it, $\tilde{y} \supset \tilde{x}$, then there is an arbitrage opportunity. An agent can buy $\tilde{x}$ and sell $\tilde{y} \supset \tilde{x}$, retaining some rent. In state s, the agent can use the goods received, $x^s \in \tilde{x}^s$, to keep the contract for delivery of $\tilde{y}^s$ (because $x^s \in \tilde{y}^s$).

Proposition 5.2.

Absence of arbitrage opportunities implies that: $\tilde{x} \subseteq \tilde{y} \ \Rightarrow\ \tilde{p}(\tilde{y}) \leq \tilde{p}(\tilde{x})$.

A corollary is that if a list, $\tilde{x}$, is more expensive than one of its alternatives, $x \in \tilde{x}$, then there exists an *arbitrage opportunity*. Agent i will buy the bundle x and sell the list $\tilde{x}$, receiving x and delivering the same $x \in \tilde{x}$. The agent gets a null delivery, $x - x$, but retains some rent. The agent is never maximizing, because it is always beneficial to scale up this arbitrage trade (buy $x + x + ...$ and sell $\tilde{x} \oplus \tilde{x} \oplus ...$).

Corollary 5.1.

Absence of arbitrage opportunities implies that: $\forall x \in \tilde{x} : \ \tilde{p}(\tilde{x}) \leq p \cdot x$.

Another corollary is that the list $M_i(\tilde{x})$, that describes what an agent gets when he buys the list $\tilde{x}$, cannot be more expensive than $\tilde{x}$.

Corollary 5.2.

$\tilde{p}(M_i(\tilde{x})) \leq \tilde{p}(\tilde{x})$.

This implies that agents do not mind being restricted to select P_i-measurable lists. They are never worse off by selecting $M_i(\tilde{x}_i)$ instead of $\tilde{x}_i$ (utility is the same, and the price may be lower).

The price of a list that is chosen, $\tilde{x}_i$, must be equal to the price of the actual list of possible deliveries that the agent obtains, $M_i(\tilde{x}_i)$. Another consequence is that the delivered bundle, $x_i \in M_i(\tilde{x}_i)$, cannot be cheaper than the list that the agent buys. In the next section we show that (in equilibrium) the price of the delivered bundle is equal to the price of the list that the agent buys.

6 Equilibrium

6.1 Concept

We consider a finite number of agents ($i = 1, ..., n$), commodities, ($l = 1, ..., L$), and states of nature ($\Omega = \{1, ..., S\}$, indexed by s and also by t when necessary).

The economy extends over two time periods, $\tau \in \{0, 1\}$, with uncertainty about which state of nature will occur in the second period. Trade agreements are made at $\tau = 0$ (all trade is *ex ante*).

Taking prices, $\tilde{p}$, as given, agents trade their state-contingent endowments, $e_i \in \mathbb{R}_+^{SL} \cap P_i$, for a plan of state-contingent lists, $\tilde{x}_i = (\tilde{x}_i^1, \tilde{x}_i^2, ..., \tilde{x}_i^S)$, specifying the bundles that may be delivered to them in each state of nature. Their objective is to maximize expected utility, $\tilde{U}_i(\tilde{x}_i, \tilde{p}) = \sum_{s=1}^S \mu_i^s \tilde{u}_i^s(\tilde{x}_i^s, \tilde{p})$.

At $\tau = 1$, agents receive their private information, trade agreements are enforced, and consumption takes place. If state s occurs, agent i should receive a bundle $x_i^s \in \tilde{x}_i^s$, but can only enforce delivery of a bundle $x_i^s \in M_i^s(\tilde{x}_i) = \bigcup_{t \in P_i(s)} \tilde{x}_i^t$.

Below is a preliminary definition of the concept of general equilibrium of an economy with uncertain delivery, in which agents trade lists of bundles instead of bundles.

Definition 6.1.

An equilibrium of the economy with uncertain delivery, $(\tilde{x}^*, x^*, \tilde{p}^*)$, is composed by: state-contingent plans of lists, $\tilde{x}^* = (\tilde{x}_1^*, ..., \tilde{x}_n^*)$; an allocation, $x^* = (x_1^*, ..., x_n^*)$; and a price system, $\tilde{p}^*$. These are such that, for every agent i:

(1) The plan of lists, $\tilde{x}_i^*$, maximizes expected utility, $\tilde{U}_i(\tilde{x}_i^*, \tilde{p}^*)$, in the agent's budget set, $\tilde{B}_i(\tilde{p}^*) = \left\{\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S : \tilde{p}^*(\tilde{x}_i) \leq p^* \cdot e_i\right\}$.

(2) In each state of nature, $s \in \Omega$, the bundle that is delivered is an alternative that the agent has to accept, $x_i^{s*} \in \bigcup_{t \in P_i(s)} \tilde{x}_i^{t*}$, that is, $x_i^* \in M_i(\tilde{x}_i^*)$.

(3) The allocation, x^*, is feasible. That is, $\sum_i x_i^* \leq \sum_i e_i$.

(4) The utility of the list is correctly anticipated: $\tilde{U}_i(\tilde{x}_i^*, \tilde{p}^*) = U_i(x_i^*)$.

6.2 Delivery of the cheapest alternative

In equilibrium, lists cannot be more expensive than any of the alternatives (Corollary 5.1). But can a list be strictly cheaper than any of the alternatives?

We show below that the price of a list that is chosen in equilibrium must be equal to the price of the alternative that is delivered.

Proposition 6.1.

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Recall that the following are true:

(i) [enforceability] $x_i^* \in M_i(\tilde{x}_i^*), \ \forall i;$

(ii) [pricing of lists] $\tilde{p}^*(\tilde{x}_i^*) = \tilde{p}^*(M_i(\tilde{x}_i^*)) \leq p^* \cdot x_i^*, \ \forall i;$

(iii) [no satiation] $\tilde{p}^*(\tilde{x}_i^*) = p^* \cdot e_i, \ \forall i;$

(iv) [feasibility] $\sum_i x_i^* \leq \sum_i e_i.$

Then, for each $i = 1, ..., n$*:*

(1) $\tilde{p}^*(\tilde{x}_i^*) = \min_{x \in M_i(\tilde{x}_i^*)} \{p^* \cdot x\};$

(2) $x_i^* \in arg \min_{x \in M_i(\tilde{x}_i^*)} \{p^* \cdot x\}.$

Given a plan of lists that is chosen in equilibrium, $\tilde{x}_i^*$, the cheapest consumption plan in $M_i(\tilde{x}_i^*)$ plays a crucial role. It is the plan that is delivered, and the price of the plan of lists is the price of this cheapest consumption plan.

6.3 Rational preferences

Knowledge of Proposition 6.1 induces rational agents to expect to receive the cheapest of the bundles in a list.

A difficulty is that there may be a tie for the cheapest bundle. We assume that, in this case, agents expect to receive the bundle with the highest utility among the cheapest bundles. This is in the spirit of the mechanism design literature, where incentive compatibility conditions only need to be satisfied in equality (in case of indifference, the agent selects the action that is preferred by the principal).

This tie-breaking assumption makes existence of equilibrium more difficult, because if agents do not actually receive this alternative with the highest utility, the economy will not be in equilibrium (see Definition 6.1, point 4). Agents would not be anticipating correctly the utility of a list.

If a rational agent did not expect to receive the bundle with the highest utility (among the cheapest bundles in the list), he would prefer to modify the list very slightly, in order to have a single cheapest bundle. We could never have an equilibrium in which an agent selected the list $x \vee y$ with $p \cdot x = p \cdot y$, $U(x) > U(y)$ and $\tilde{U}(x \vee y) < U(x)$. The agent would prefer a list $x^- \vee y$ in which x is replaced by a similar, cheaper, x^-. The list

$x^- \vee y$ would lead to delivery of x^- (the strictly cheaper alternative), which has almost the same utility as x, implying that $\tilde{U}(x^- \vee y) = U(x^-) > \tilde{U}(x \vee y)$.

Consider the cheapest consumption plans, at prices p, in the list $M_i(\tilde{x}_i)$, denoted $\tilde{Y}_i(\tilde{x}_i, p)$. Select an alternative among those that have the highest utility, and denote it by $Y_i(\tilde{x}_i, p)$ (the arbitrary selection operator is denoted $\mathcal{S}$).

$$\tilde{Y}_i : \ (\mathbb{F}(\mathbb{R}^L_+))^S \times \Delta^{SL} \longrightarrow (\mathbb{F}(\mathbb{R}^L_+))^S;$$

$$\tilde{Y}_i(\tilde{x}_i, p) = \{x \in \mathbb{R}^{SL}_+ : \ x \in arg \min_{z \in M_i(\tilde{x}_i)} \{p \cdot z\}\};$$

$$Y_i : \ (\mathbb{F}(\mathbb{R}^L_+))^S \times \Delta^{SL} \longrightarrow \mathbb{R}^L_+;$$

$$Y_i(\tilde{x}_i, p) = \mathcal{S}\left[x \in \mathbb{R}^{SL}_+ : \ \max_{x_i \in \tilde{Y}_i(\tilde{x}_i, p)} U_i(x_i)\right].$$

The utility of a plan of lists is equated to the utility of this cheapest plan of bundles (expected delivery). As a result, the preferences of rational agents over lists only depend on p, and not on its extension to lists, $\tilde{p}$.

Assumption 6.1.

$$\tilde{U}_i(\tilde{x}_i, \tilde{p}) = U_i(Y_i(\tilde{x}_i, p)).$$

The problem of agent i can be written as: $\max_{\tilde{x}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{x}_i, \tilde{p}) = \max_{\tilde{x}_i \in B_i(\tilde{p})} U_i(Y_i(\tilde{x}_i, p))$.

Recall that if a plan, $\tilde{x}_i$, maximizes utility in the budget set of agent i, then the plan $M_i(\tilde{x}_i)$, which is P_i-measurable, also does. The utility is the same and the price of $M_i(\tilde{x}_i)$, by Corollary 5.2, is not higher. Agents can maximize by accessing only P_i-measurable plans of lists.

$$\tilde{x}_i \in arg \max_{\tilde{z}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{z}_i, \tilde{p}) \Rightarrow M_i(\tilde{x}_i) \in arg \max_{\tilde{z}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{z}_i, \tilde{p}).$$

We could restrict our attention to lists that are measurable with respect to the agent's private information. A natural refinement of the equilibrium set is to impose P_i-measurability of the lists chosen by the agents.

Proposition 6.2.

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Then:

$(M(\tilde{x}^*), x^*, \tilde{p}^*)$ is also an equilibrium of the economy with uncertain delivery.

6.4 Essential equilibria

Lists that are not chosen in equilibrium may be strictly cheaper than the cheapest bundle in the list. In this case, we can raise the price of this list to equal the price of the cheapest bundle, remaining in equilibrium. If these lists were not bought by the agents before the price raise, then they would remain not being bought after their price goes up.

We designate by *essential price systems* those that are such that the prices of lists coincide with the price of the cheapest alternative contained in the list.

Definition 6.2.

The price system $\tilde{p}$ is an essential price system if and only if:

$$\forall \tilde{z} \in (\mathbb{F}(\mathbb{R}^L_+))^S : \ \tilde{p}(\tilde{z}) = \min_{z \in \tilde{z}} \{p \cdot z\}.$$

Observe that, if $\tilde{p}$ is an essential price system:

$$\tilde{p}^s(\tilde{x}^s_i) = \min_{z^s_i \in \tilde{x}^s_i} \{p^s \cdot z^s_i\};$$

$$\tilde{p}(\tilde{x}_i) = \sum_{s=1}^{S} \tilde{p}^s(\tilde{x}^s_i) = \sum_{s=1}^{S} \min_{z^s_i \in \tilde{x}^s_i} \{p^s \cdot z^s_i\}.$$

The budget restriction faced by agent i becomes:

$$\tilde{B}_i(p) = \left\{ \tilde{x}_i \in (\mathbb{F}(\mathbb{R}^L_+))^S : \ \sum_{s=1}^{S} \min_{z^s_i \in \tilde{x}^s_i} \{p^s \cdot z^s_i\} \leq \sum_{s=1}^{S} p^s \cdot e^s_i = p \cdot e_i \right\}.$$

A further refinement of the equilibrium set is to impose that the price system is essential and to remove the irrelevant alternatives in the lists (those that do not affect the price of the list and that are never delivered), making $\tilde{x}^* = M(x^*)$. We designate such equilibria as *essential equilibria*.

Definition 6.3.

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium such that:

i) $\tilde{x}^* = M(x^*)$, that is, $\tilde{x}^*_i = M_i(x^*_i)$, for $i = 1, ..., n$;

ii) $\tilde{p}^*$ is an essential price system, defined by $\tilde{p}(\tilde{z}) = \min_{z \in \tilde{z}} \{p \cdot z\}$, $\forall \tilde{z} \in (\mathbb{F}(\mathbb{R}^L_+))^S$.

Then, we say that the pair (x^*, p^*) is an essential equilibrium of the economy with uncertain delivery.

For every equilibrium of the economy with uncertain delivery, $(\tilde{x}^*, x^*, \tilde{p}^*)$, there exists an essential equilibrium, (x^*, p^*), that is equivalent in the sense that:

- the allocation is the same, x^*;

- prices of consumption plans coincide ($p^* \cdot z = \tilde{p}^*(z),\ \forall z \in \mathbb{R}_+^{SL}$);

- selected lists, $M_i(x_i^*)$, do not contain irrelevant alternatives, i.e., alternatives that are never delivered.

Proposition 6.3.

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Recall that prices of singleton lists are denoted by p^*. Let $\tilde{q}^*(\tilde{z}) = \min_{z \in \tilde{z}}\{p^* \cdot z\}$. Then:

i) $\forall \tilde{y}^*$ s.t. $x^* \subseteq \tilde{y} \subseteq M(\tilde{x}^*)$: $(\tilde{y}^*, x^*, \tilde{q}^*)$ is also an equilibrium.

ii) (x^*, p^*) is an essential equilibrium.[18]

6.5 Deliverability

Suppose that an agent bought a plan of singleton lists for delivery in two possible states of nature, $\tilde{x} = (x^s, x^t)$. If the agent can distinguish states s and t, then $M(\tilde{x}) = (x^s, x^t)$, and thus delivery of x^s in state s and x^t in state t is guaranteed. If the agent cannot distinguish the two states, then we have $M(\tilde{x}) = (x^s \vee x^t, x^s \vee x^t)$. As a result: in state s, the agent receives the cheapest of the two alternatives according to p^s; and, in state t, the cheapest according to p^t.

An agent that buys $\tilde{x}$ always receives one of the cheapest bundles in $M_i(\tilde{x})$, therefore, the bundle that is delivered in state s cannot be more expensive (according to prices for delivery in state s) than any of the bundles that are delivered in states $t \in P_i(s)$:

$$\forall t \in P_i(s) : \ p^s \cdot x^s \leq p^s \cdot x^t.$$

The plans that are deliverable depend on prices, and this dependence is described by the deliverability correspondence defined below.

[18] It should be clear that q^* coincides with p^*.

Definition 6.4.

$$C_i : \Delta^{SL} \twoheadrightarrow \mathbb{R}^{SL}_{+};$$

$$C_i(p) = \left\{ x_i \in \mathbb{R}^{SL}_{+} : \ \forall s \in \Omega, \ p^s \cdot x_i^s = \min_{t \in P_i(s)} \{p^s \cdot x_i^t\} \right\}.$$

A consumption plan is deliverable, $x_i \in C_i(p)$, if and only if there exists a P_i-measurable list of which x_i is the cheapest alternative. It is enough to check whether or not x_i is the cheapest alternative in the list $M_i(x_i)$.

We can formulate the problem of the agent, equivalently, as a choice over lists or as a choice over consumption plans. The following propositions make this precise.

Proposition 6.4.

Let $\tilde{p}$ be an essential price system, $\tilde{x}_i \in arg \max_{\tilde{x}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{x}_i, \tilde{p})$, and $x_i = Y_i(\tilde{x}_i, p)$. Then:

$$x_i \in arg \max_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i).$$

Proposition 6.5.

Let $\tilde{p}$ be an essential price system, and $x_i \in arg \max_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i)$. Then:

$$M_i(x_i) \in arg \max_{\tilde{x}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{x}_i, \tilde{p}).$$

This equivalence leads us to reformulate the notion of essential equilibrium.

Proposition 6.6.

The pair (x^*, p^*) is an essential equilibrium of the economy with uncertain delivery if and only if:

(1) Each agent's choice is optimal, $x_i^* \in arg \max_{x_i \in B_i(p^*) \cap C_i(p^*)} U_i(x_i)$.

(2) The allocation, x^*, is feasible. That is, $\sum_i x_i^* \leq \sum_i e_i$.

Notice that this can be seen as an alternative definition that does not use preferences over lists, $\tilde{U}_i$, nor prices over lists, $\tilde{p}$. This means that we are ready to compare an equilibrium of the economy with uncertain delivery with the equilibrium with public state verification (Arrow-Debreu-McKenzie under uncertainty) and the equilibrium with differential information proposed by Radner (1968). Everything boils down to the choice sets. In Arrow-Debreu-McKenzie: $X_i^{AD} = \mathbb{R}_+^{SL}$; in Radner (1968): $X_i^R = \mathbb{R}_+^{SL} \cap P_i$; here: $X_i(p) = \mathbb{R}_+^{SL} \cap C_i(p)$. It should be clear that, for all prices p:

$$X_i^{AD} \subseteq X_i(p) \subseteq X_i^R.$$

7 Existence of equilibrium

If the correspondence from prices to the deliverable budget set were continuous, existence of equilibrium would be guaranteed (we could apply Berge's Maximum Theorem, and then Kakutani's Fixed Point Theorem). But, as we illustrate in Appendix 3, $C_i(p)$ is not lower hemicontinuous (this property fails when prices in some state are null, or when prices in two undistinguished states are collinear).

7.1 A sequence of economies

In order to establish existence of equilibrium, we construct a sequence of economies. In these economies, the choice set is not constrained to satisfy the endogenous deliverability restrictions. But violating these restrictions implies an utility penalty. The penalty is a function of the greatest of the differences between the cheapest bundles and the bundles that are delivered.

These economies have no relation with reality. They are an artifice to establish existence of equilibrium.

In the economy $\mathcal{E}^j$, if state s occurs, the utility penalty imposed on agent i is:

$$Z_i^{js}(x_i, p) = j \max_{t \in P_i(s)} \{p^s \cdot x_i^s - p^s \cdot x_i^t\}.$$

Since $s \in P_i(s)$, the maximum is at least zero, thus penalties are never negative. Penalties increase along the sequence of economies, and this is actually the only difference between the economies in the sequence.

In the economy $\mathcal{E}^j$, the utility functions of the agents are:

$$U_i^j(x_i,p) = U_i(x_i) - j\sum_{s=1}^{S}\mu_i^s \max_{t\in P_i(s)}\{p^s\cdot x_i^s - p^s\cdot x_i^t\}.$$

For any $j \in \mathbb{N}$, the utility functions are continuous in prices and bundles, $(x_i,p) \in \mathbb{R}_+^{SL}\times\Delta^{SL}$. The maximum of linear functions is a convex function, and multiplying a convex function by a negative constant, $-j$, yields a concave function. Hence, the objective function, $U_i^j(x_i,p)$, is concave in the first variable. Observe also that the utility penalty preserves the property of *no satiation*. The plan $x_i+\epsilon\bar{1}$ is always preferred to x_i (observe that the utility penalty remains constant). The fact that the utility functions depend (continuously) on prices does not interfere with existence of equilibrium.[19]

Lemma 7.1.

Let $\mathcal{E}^j$ be an Arrow-Debreu-McKenzie economy such that, for each agent i:

- initial endowments are strictly positive, $e_i \gg 0$;

- the utility functions are $U_i^j(x_i,p) = U_i(x_i) - j\sum_{s=1}^{S}\mu_i^s \max_{t\in P_i(s)}\{p^s\cdot x_i^s - p^s\cdot x_i^t\}$, with $U_i(x_i)$ continuous, concave and weakly increasing.

Then, there exists an Arrow-Debreu-McKenzie equilibrium.

The sequence of economies has a sequence of equilibria, $\{(x^j,p^j)\}_{j\in\mathbb{N}}$, in the compact set that contains the total endowments of the economy, $[0,e_T]^n\times\Delta^{SL}$, where $e_T=\sum_i e_i$. There exists a subsequence that converges. For the limit, (x^*,p^*), to be an essential equilibrium of the original economy, the following conditions must be satisfied:

(1) Feasibility: $\sum_i x_i^* \le \sum_i e_i = e_T$;

(2) Budget restriction: $\forall i:\ p^*\cdot x_i^* \le p^*\cdot e_i$;

(3) Deliverability: $\forall i:\ x_i^* \in C_i(p^*)$;

(4) Optimality: $\forall i:\ x_i \in B_i(p^*)\cap C_i(p^*) \Rightarrow U_i(x_i^*) \ge U_i(x_i)$.

7.2 The first three conditions

It is straightforward to show that the first three conditions are satisfied.

[19] With price dependent preferences, it is known that equilibrium exists (Arrow and Hahn, 1971). In the context of economies with uncertain delivery, see our previous paper (2008).

Lemma 7.2.

Consider a sequence of economies $\{\mathcal{E}^j\}_{j=1}^{+\infty}$ defined as in Lemma 7.1, and a corresponding sequence of equilibria, $\{x^j, p^j\}_{j=1}^{+\infty}$.

Then, the sequence of equilibria has an accumulation point, (x^*, p^*), that satisfies:

(1) Feasibility: $\sum_i x_i^* \leq \sum_i e_i = e_T$;

(2) Budget restriction: $p^* \cdot x_i^* \leq p^* \cdot e_i, \ \forall i$;

(3) Deliverability: $x_i^* \in C_i(p^*), \ \forall i$.

The difficult part of the proof is to verify condition (4): that the limit, (x^*, p^*), maximizes the utility of the agents in the deliverable budget set, $B_i(p^*) \cap C_i(p^*)$. The fact that C_i is not lower hemicontinuous (as shown in Appendix 3) could prevent (x^*, p^*) from being optimal. There could be a deliverable consumption plan $y_i \in B_i(p^*) \cap C_i(p^*)$ that is not even nearly deliverable in the economies in the sequence. In spite of having a low utility level for high j (because of the penalty), this bundle could be optimal in the original economy, and, in this case, (x^*, p^*) would not be an equilibrium (an example of non-existence of equilibrium is given in Appendix 2).

For this fourth condition to hold, we need extra assumptions. One is the existence of an agent that prevents prices from being null. We now introduce this agent.

7.3 The ϵ-agent

The ϵ-agent can have an arbitrarily small endowment (this is why it is called an ϵ-agent), but is perfectly informed and has a utility function with bounded marginal utilities. Below a certain price level, the demand of such an agent exceeds the aggregate endowment.

The effect of introducing this agent is to impose a strictly positive lower bound on equilibrium prices (in the sequence of economies and in the limit).

Lemma 7.3.

Let the agent ϵ be such that: (i) $P_\epsilon(s) = \{s\}, \ \forall s \in \Omega$; (ii) U_ϵ is continuously differentiable, strictly increasing and concave; and (iii) $e_\epsilon \gg 0$.

Consider a sequence of prices, $\{p_n\}_{n \in \mathbb{N}}$, that converges to the boundary of the simplex ($\exists s, l: \lim_{n \to +\infty} p_n^{sl} = 0$).

Then, for sufficiently large n, the demand of agent ϵ for at least one of the goods is greater than the aggregate endowment of this good.

7.4 The fourth condition

To prove the fourth condition, and establish existence of equilibrium, we make two additional assumptions: that there is an ϵ-agent in the economy; and that agents have equal preferences in states that they do not distinguish.

Theorem 1.

Consider an economy with uncertain delivery, $\mathcal{E} \equiv (e_i, u_i, \mu_i, P_i)_{i=1}^n$, such that:

- *Preferences are represented by a vector of Von Neumann-Morgenstern (1944) utility functions $u_i^s : \mathbb{R}_+^L \to \mathbb{R}$, which are continuous, concave and weakly increasing.*
- *Preferences are the same in undistinguished states: $t \in P_i(s) \Rightarrow u_i^t(\cdot) = u_i^s(\cdot)$.*
- *Initial endowments are constant across undistinguished states, $e_i \in \mathbb{R}_+ \cap P_i$.*[20]
- *One of the agents is an ϵ-agent: (i) $P_\epsilon(s) = \{s\},\ \forall s \in \Omega$; (ii) U_ϵ is continuously differentiable, strictly increasing and concave; and (iii) $e_\epsilon \gg 0$.*

Then, there exists an equilibrium of the economy with uncertain delivery.

The strategy of the proof is to assume (by way of contradiction) that there exists a x_i' in $B_i(p^*) \cap C_i(p^*)$ that is preferred to x_i^*, and then find that there exists a similar x_i which belongs to $B_i(p^j) \cap C_i(p^j)$, for large j. This contradicts that (x^j, p^j) is an equilibrium of $\mathcal{E}^j$, because x_i would also be preferred to x_i^j in the economy $\mathcal{E}^j$.

[20] It is not necessary to consider that endowments are strictly positive because the ϵ-agent guarantees irreducibility (McKenzie, 1981).

8 Concluding remarks

We have introduced a new general equilibrium model of trade *ex ante* with differential information. The agents make state-contingent trade contracts before receiving their private information, and then use their private information to enforce contracts.

It is crucial that the information provided by the agents to enforce contracts is not aggregated by any institution. Otherwise, such an institution could announce the true state of nature, and the economy would be as described by Debreu (1959, chapter 7).

Agents find it useful to trade lists, which are a sort of incomplete contracts (an agent that buys a list has to accept any possible outcome compatible with the list). We have shown that these contracts are enforceable if and only if agents select P_i-measurable plans of lists.

Using a no-arbitrage argument, we found a fundamental value property of prices: the price of a plan of lists is equal to the price of the cheapest consumption plan that is compatible with the plan of lists. Furthermore, we found that an agent that buys a plan of lists should expect to receive this cheapest consumption plan.

Endowments and preferences were assumed to be constant across states that the agent does not distinguish (P_i-measurable). While the assumption of P_i-measurable endowments is used to analyze arbitrage and short selling, preferences are only assumed to be P_i-measurable for the existence result.

For the existence result, we also needed to introduce in the economy the ϵ-agent, who guarantees that if some price converges to zero, then aggregate demand becomes greater than the aggregate endowment. Without such an assumption, equilibrium may not exist, as shown by the counter-example in Appendix 2.

The ϵ-agent could be removed if we: (i) assume that, for any state of nature, there exists an agent that can prove that it is the true state ($\forall s, \exists i : \ P_i(s) = \{s\}$); and (ii) impose adequate bounds on marginal utility to guarantee that aggregate demand grows beyond the aggregate endowment, when some price approaches zero.

Finally, we remark that a model in which all trade is *ex ante* does not cover the cases in which agents arrive at the market with different information (Akerlof, 1970), a setting in which trade is at the *interim* stage. The contributions of Radner (1979) and Allen (1981) suggest that, in this setting, prices reveal all the private information of the agents.

In a model of trade *ex ante*, prices cannot reveal the information of the agents, because agents haven't received their information yet.

The consequences of allowing trade after agents receive their information (*ex post*) remain to be examined. This is left for future research.

Appendix 1: The proofs

Proposition 5.1:

Absence of arbitrage opportunities implies that:

$$\forall \tilde{x}_i, \tilde{y}_i \in \mathbb{F}(\mathbb{R}_+^{SL}) : \quad \tilde{p}(\tilde{x}_i \oplus \tilde{y}_i) = \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i).$$

Proof of Proposition 5.1:

Let $\tilde{z}_i = \tilde{x}_i \oplus \tilde{y}_i = \{z_i \in \mathbb{R}_+^{SL} : \ \exists x_i \in \tilde{x}_i, \ y_i \in \tilde{y}_i, \ z_i = x_i + y_i\}$.

If $\tilde{p}(\tilde{z}_i) < \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i)$, then an agent can buy $\tilde{z}_i$ and sell both lists $\tilde{x}_i$ and $\tilde{y}_i$. By construction of $\tilde{z}_i$, for each $z_i^s \in \tilde{z}_i^s$, there exist $x_i^s \in \tilde{x}_i^s$ and $y_i^s \in \tilde{y}_i^s$ such that $x_i^s + y_i^s = z_i^s$. When receiving z_i^s, the agent has enough resources to deliver x_i^s and y_i^s, in order to keep the contracts for delivery of $\tilde{x}_i$ and $\tilde{y}_i$. In the process, the agent retains some rent.

If $\tilde{p}(\tilde{z}_i) > \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i)$, then an agent can sell $\tilde{z}_i$ and buy both lists $\tilde{x}_i$ and $\tilde{y}_i$. Receiving $x_i^s \in \tilde{x}_i^s$ and $y_i^s \in \tilde{y}_i^s$, the agent delivers $z_i^s = x_i^s + y_i^s$, keeping the contract for delivery of $\tilde{z}_i$. Again, the agent retains some rent. **QED**

Proposition 5.2:

Absence of arbitrage opportunities implies that:

$$\tilde{x} \subseteq \tilde{y} \ \Rightarrow \ \tilde{p}(\tilde{y}) \leq \tilde{p}(\tilde{x}).$$

Proof of Proposition 5.2:

If $\tilde{p}(\tilde{x}) < \tilde{p}(\tilde{y})$, an agent that buys $\tilde{x}$ and sells $\tilde{y}$ retains some rent.

In each state of nature, s, the agent can use exactly what is received, $x^s \in \tilde{x}^s$, to keep the contract for delivery of $\tilde{y}^s$, because $x^s \in \tilde{y}^s$. **QED**

Proposition 6.1:

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Recall that the following are true:

(i) [enforceability] $x_i^* \in M_i(\tilde{x}_i^*), \ \forall i$;

(ii) [pricing of lists] $\tilde{p}^*(\tilde{x}_i^*) = \tilde{p}^*(M_i(\tilde{x}_i^*)) \leq p^* \cdot x_i^*, \ \forall i$;

(iii) [no satiation] $\tilde{p}^*(\tilde{x}_i^*) = p^* \cdot e_i, \ \forall i$;

(iv) [feasibility] $\sum_i x_i^* \leq \sum_i e_i$.

Then, for each $i = 1, ..., n$:

(1) $\tilde{p}^*(\tilde{x}_i^*) = \min_{x \in M_i(\tilde{x}_i^*)} \{p^* \cdot x\}$;

(2) $x_i^* \in arg \min_{x \in M_i(\tilde{x}_i^*)} \{p^* \cdot x\}$.

Proof of Proposition 6.1:

Suppose that the lists $\tilde{x}_i^*$ is strictly cheaper than the corresponding delivery, x_i^*:

$\exists i: \ \tilde{p}^*(\tilde{x}_i^*) < p^* \cdot x_i^*$.

Summing across agents, using (ii): $\sum_i \tilde{p}^*(\tilde{x}_i^*) < \sum_i p^* \cdot x_i^*$.

By *no satiation* (iii): $\sum_i \tilde{p}^*(\tilde{x}_i^*) = \sum_i p^* \cdot e_i$.

This implies that the equilibrium allocation is not feasible.

$$\sum_i p^* \cdot e_i < \sum_i p^* \cdot x_i^* \ \Rightarrow \ \exists (s,l): \ \sum_i e_i^{sl} < \sum_i x_i^{sl*}.$$

Contradiction that, together with (ii), proves that: $\tilde{p}^*(\tilde{x}_i^*) = \tilde{p}^*(M_i(\tilde{x}_i^*)) = p^* \cdot x_i^*$.

Using Corollary 5.1, we finish the proof.

$$\left\{ \begin{array}{l} \tilde{p}^*(\tilde{x}_i^*) \leq \min_{z \in M_i(\tilde{x}_i^*)} \{p^* \cdot z\} \\ \tilde{p}^*(\tilde{x}_i^*) = p^* \cdot x_i^* \geq \min_{z \in M_i(\tilde{x}_i^*)} \{p^* \cdot z\} \end{array} \right. \Rightarrow \tilde{p}^*(\tilde{x}_i^*) = p^* \cdot x_i^* = \min_{z \in M_i(\tilde{x}_i^*)} \{p^* \cdot z\}.$$

QED

Proposition 6.2:

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Then:

$(M(\tilde{x}^*), x^*, \tilde{p}^*)$ is also an equilibrium of the economy with uncertain delivery.

Proof of Proposition 6.2:

The price of $M(\tilde{x}^*)$ is not higher (Corollary 5.2) and the utility is the same (Assumption 4.2).

If $\tilde{x}_i$ solves the problem of the agent, then $M_i(\tilde{x}_i)$ also does. **QED**

Proposition 6.3:

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ be an equilibrium of the economy with uncertain delivery. Recall that prices of singleton lists are denoted by p^*. Let $\tilde{q}^*(\tilde{z}) = \min_{z \in \tilde{z}} \{p^* \cdot z\}$. Then:

i) $\forall \tilde{y}^*$ s.t. $x^* \subseteq \tilde{y} \subseteq M(\tilde{x}^*)$: $(\tilde{y}^*, x^*, \tilde{q}^*)$ is also an equilibrium.

ii) (x^*, p^*) is an essential equilibrium.

Proof of Proposition 6.3:

By Proposition 6.2, $(M(\tilde{x}^*), x^*, \tilde{p}^*)$ is an equilibrium of the economy with uncertain delivery.

If $M_i(\tilde{x}^*)$ solves the problem of agent i under prices $\tilde{p}^*$, then it also solves the problem of the agent under prices $\tilde{q}^*$.

This is so because: (i) the price of $M_i(\tilde{x}^*)$ remains the same, $\tilde{q}^*(M_i(\tilde{x}^*)) = \tilde{p}^*(M_i(\tilde{x}^*))$; (ii) prices of other lists do not decrease, $\forall z \in (\mathbb{F}(\mathbb{R}_+^L))^S$: $\tilde{q}^*(\tilde{z}) \geq \tilde{p}^*(\tilde{z})$; and (iii) preferences remain the same, $q^* = p^* \Rightarrow \tilde{U}_i(\cdot, \tilde{q}^*) = \tilde{U}_i(\cdot, \tilde{p}^*)$.

The price and the utility of $M_i(x_i^*)$ and $M_i(\tilde{x}_i^*)$ are the same, thus $(M(x^*), x^*, \tilde{q}^*)$ is also an equilibrium of the economy with uncertain delivery.

The same applies to any $\tilde{y}^*$ s.t. $x^* \subseteq \tilde{y} \subseteq M(\tilde{x}^*)$ **QED**

Proposition 6.4:

Let $\tilde{p}$ be an essential price system, $\tilde{x}_i \in arg \max_{\tilde{x}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{x}_i, \tilde{p})$, and $x_i = Y_i(\tilde{x}_i, p)$. Then:

$$x_i \in arg \max_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i).$$

Proof of Proposition 6.4:

Suppose that there exists $y_i \in B_i(p) \cap C_i(p)$ that is preferred to x_i:

$$U_i(y_i) > U_i(x_i).$$

Since $y_i \in C_i(p)$:

$$\tilde{U}_i(M_i(y_i), \tilde{p}) \geq U_i(y_i) > U_i(x_i) = \tilde{U}_i(\tilde{x}_i, \tilde{p}).$$

Since $\tilde{p}$ is an essential price system:

$\tilde{p}(\tilde{x}_i) = p \cdot x_i$ and $\tilde{p}(M_i(y_i)) = p \cdot y_i$.

If $y_i \in B_i(p)$, then $M_i(y_i) \in B_i(\tilde{p})$. Contradiction. **QED**

Proposition 6.5:

Let $\tilde{p}$ be an essential price system, and $x_i \in arg \max_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i)$. Then:

$$M_i(x_i) \in arg \max_{\tilde{x}_i \in B_i(\tilde{p})} \tilde{U}_i(\tilde{x}_i, \tilde{p}).$$

Proof of Proposition 6.5:

We know that $\tilde{U}_i(M_i(x_i), \tilde{p}) = U_i(x_i)$

Suppose that there exists $\tilde{y}_i \in B_i(\tilde{p})$ that is preferred to $M_i(x_i)$:

$$\tilde{U}_i(\tilde{y}_i, \tilde{p}) > \tilde{U}_i(M_i(x_i), \tilde{p})$$

Let $y_i = Y_i(\tilde{y}_i, p)$. Then:

$$U_i(y_i) = \tilde{U}_i(\tilde{y}_i, \tilde{p}) > \tilde{U}_i(M_i(x_i), \tilde{p}) = U_i(x_i).$$

Since $\tilde{p}$ is an essential price system:

$\tilde{y}_i \in B_i(\tilde{p}) \Rightarrow y_i \in B_i(p)$. Contradiction. **QED**

Proposition 6.6:

The pair (x^*, p^*) is an essential equilibrium of the economy with uncertain delivery if and only if:

(1) Each agent's choice is optimal, $x_i^* \in arg \max_{x_i \in B_i(p^*) \cap C_i(p^*)} U_i(x_i)$.

(2) The allocation, x^*, is feasible. That is, $\sum_i x_i^* \leq \sum_i e_i$.

Proof of Proposition 6.6:

The proof follows from Definitions 6.1 and 6.3 and from Propositions 6.4 and 6.5. **QED**

Lemma 7.1:

Let $\mathcal{E}^j$ be an Arrow-Debreu-McKenzie economy such that, for each agent i:

- initial endowments are strictly positive, $e_i \gg 0$;

- the utility functions are $U_i^j(x_i, p) = U_i(x_i) - j \sum_{s=1}^{S} \mu_i^s \max_{t \in P_i(s)} \{p^s \cdot x_i^s - p^s \cdot x_i^t\}$, with $U_i(x_i)$ continuous, concave and weakly increasing.

Then, there exists an Arrow-Debreu-McKenzie equilibrium.

Proof of Lemma 7.1:

Restrict the choice set to the compact $[0, T]$, with $T = 2 \sum_i e_i$.

Consider correspondences, ψ_i, which assign to given prices, p, bundles, x_i', that maximize $U_i^j(x_i, p)$ in the budget set, $B_i(p)$.

$\psi_i : [0, T]^n \times \Delta^{SL} \longrightarrow [0, T]$;

$x_i' \in \psi_i(x, p) \Leftrightarrow x_i' = arg \max_{x_i \in B_i(e_i, p)} U_i^j(x_i, p)\ ,\ \forall i.$

Consider also a correspondence, ψ_p, that assigns to the total demand, $\sum_i x_i$, the prices, p', which maximize the value of excess demand:

$\psi_p : [0, T]^n \times \Delta^{SL} \longrightarrow \Delta^{SL}$;

$p' \in \psi_p(x, p) \Leftrightarrow p' = arg \max_{p \in \Delta^{SL}} \{p \cdot \sum_i (x_i - e_i)\}.$

The objective functions, U_i^j and $V_p(x, p) = p \cdot \sum_i (x_i - e_i)$, are continuous, and $B_i(p)$ is a continuous correspondence. We can, therefore, use Berge's Maximum Theorem to show that each of the correspondences ψ_i and ψ_p is upper hemicontinuous with non-empty and compact values. They also have convex values because the objective functions are concave. The product correspondence retains these properties and maps a compact set into itself:

$\psi \equiv \prod_{i=1}^{n} \psi_i \times \psi_p$;

$\psi : [0, T]^n \times \Delta^{SL} \longrightarrow [0, T]^n \times \Delta^{SL}$;

$(x', p') \in \psi(x, p) \Leftrightarrow x_i' \in \psi_i(x, p),\ \forall i$ and $p' \in \psi_p(x, p)$.

Existence of a fixed-point, (x^*, p^*), follows from Kakutani's Theorem.

It is clear that x_i^* solves the problem of agent i.

The fact that p^* maximizes the value of excess demand implies that:

$p' \cdot \sum_i (x_i^* - e_i) \leq p^* \cdot \sum_i (x_i^* - e_i) \leq 0$, for all $p' \in \Delta^{SL}$.

Making $p' = e^j = (0, ..., 1, ..., 0)$, for each j, shows that x^* is feasible: $\sum_i (x_i^* - e_i) \leq 0$.

The usual extension from $[0, T]^n$ to $\mathbb{R}_+^{nSL}$ applies. **QED**

Lemma 7.2:

Consider a sequence of economies $\{\mathcal{E}^j\}_{j=1}^{+\infty}$ defined as in Lemma 7.1, and a corresponding sequence of equilibria, $\{x^j, p^j\}_{j=1}^{+\infty}$.

Then, the sequence of equilibria has an accumulation point, (x^*, p^*), that satisfies:

(1) Feasibility: $\sum_i x_i^* \leq \sum_i e_i = e_T$;

(2) Budget restriction: $p^* \cdot x_i^* \leq p^* \cdot e_i,\ \forall i$;

(3) Deliverability: $x_i^* \in C_i(p^*),\ \forall i$.

Proof of Lemma 7.2:

Conditions (1) and (2) follow from the fact that (x^*, p^*) is the limit of a sequence of equilibria.

(1) The set of feasible allocations is closed, and the limit allocation, x_i^*, is the limit of a sequence of feasible allocations, therefore it is feasible.

(2) The limit allocation, x_i^*, is the limit of a sequence of allocations in the sequence of budget sets. Therefore, it also belongs to the limit budget set.

Suppose that x_i^* does not satisfy the budget restriction of agent i. Let $\alpha = 3\|e_T\| + 1$, and select $\epsilon > 0$ such that $p^* \cdot x_i^* - p^* \cdot e_i = \alpha\epsilon$. Choosing a sufficiently high j, we can guarantee that $\|x^* - x^j\| < \epsilon$ and $\|p^* - p^j\| < \epsilon$. With $p^j = p^* + dp$, $x^j = x_i^* + dx_i$, and manipulating:

$$(p^* + dp) \cdot (x_i^* + dx_i) - (p^* + dp) \cdot e_i = p^* \cdot x_i^* - p^* \cdot e_i + p^* \cdot dx_i + dp \cdot x_i^* + dp \cdot dx_i - dp \cdot e_i =$$

$$= \alpha\epsilon + (p^* + dp) \cdot dx_i + dp \cdot (x_i^* - e_i) > \alpha\epsilon - \epsilon - \epsilon \cdot 3\|e_T\| = 0.$$

This means that x^j would not satisfy the budget restriction of $\mathcal{E}^j$. Contradiction.

(3) The limit allocation, x^*, satisfies the deliverability restrictions in the limit economy. To see this, suppose that x^* violated one of the restrictions by more than $\delta > 0$, then, for a sufficiently high j, x^j would also violate the same restriction by more than δ. That is, for $t \in P_i^s$, $\exists j_0 \in \mathbb{N}$:

$$p^{s*} \cdot x^{s*} > p^{s*} \cdot x^{t*} + \delta \Rightarrow p^{sj} \cdot x^{sj} > p^{sj} \cdot x^{tj} + \delta, \text{ for all } j > j_0.$$

Utility among feasible allocations is bounded by $U_i(e_T)$, so we can consider a j that is sufficiently high for $j\delta > U_i(e_T) - U_i(e_i)$. It would follow that $U_i^j(x^j) < U_i(x^j) - j\delta < U_i(x^j) - U_i(e_T) + U_i(e_i) < U_i(e_i) = U_i^j(e_i)$, which is a contradiction. **QED**

Lemma 7.3:

Let the agent ϵ be such that: (i) $P_\epsilon(s) = \{s\},\ \forall s \in \Omega$; (ii) U_ϵ is continuously differentiable, strictly increasing and concave; and (iii) $e_\epsilon \gg 0$.

Consider a sequence of prices, $\{p_n\}$, that converges to the boundary of the simplex $(\exists s, l : \lim_{n \to +\infty} p_n^{sl} = 0)$.

Then, for sufficiently large n, the demand of agent ϵ for at least one of the goods is greater than the aggregate endowment of this good.

Proof of Lemma 7.3:

Define the set $E = \{x \in \mathbb{R}_+^{SL} :\ U_\epsilon(x) \geq U_\epsilon(e_\epsilon)\ \wedge\ x \leq 2\sum_i e_i\}$.

Find the minimum and the maximum of the partial derivatives of the utility function in the compact set E. Let $a^{sl} = \min_{x \in E} \frac{\partial U_\epsilon}{\partial x^{sl}}$ and $b^{sl} = \max_{x \in E} \frac{\partial U_\epsilon}{\partial x^{sl}}$. Let $a = \min_{sl \in SL} a^{sl}$ and $b = \max_{sl \in SL} b^{sl}$.

To satisfy the equalities between marginal rates of substitution and price ratios, for the demand to be an element of E, we must have: $\frac{a}{b} \leq \frac{p^{sl}}{p^{s'l'}} \leq \frac{b}{a}$, $\forall s, l, s', l'$.

For sufficiently large n, this is not true. For at least one commodity, demand becomes greater than the aggregate endowment. **QED**

Theorem 1:

Consider an economy with uncertain delivery, $\mathcal{E} \equiv (e_i, u_i, \mu_i, P_i)_{i=1}^n$, such that:

- Preferences are represented by a vector of Von Neumann-Morgenstern (1944) utility functions $u_i^s : \mathbb{R}_+^L \to \mathbb{R}$, which are continuous, concave and weakly increasing.
- Preferences are the same in undistinguished states: $t \in P_i(s) \Rightarrow u_i^t(\cdot) = u_i^s(\cdot)$.
- Initial endowments are constant across undistinguished states, $e_i \in \mathbb{R}_+ \cap P_i$.
- One of the agents is an ϵ-agent: (i) $P_\epsilon(s) = \{s\}$, $\forall s \in \Omega$; (ii) U_ϵ is continuously differentiable, strictly increasing and concave; and (iii) $e_\epsilon \gg 0$.

Then, there exists an equilibrium of the economy with uncertain delivery.

Proof of Theorem 1:

It is not necessary to consider that endowments are strictly positive because the ϵ-agent guarantees irreducibility (McKenzie, 1981).[21]

In the presence of the ϵ-agent, we are sure that the limit of the sequence of equilibrium prices is in the interior of the simplex.

Given Lemma 7.1 and Lemma 7.2, all that is left to prove is (4), which states that the limit of the sequence of equilibria (x^*, p^*) is composed by optimal choices in the original economy with uncertain delivery, that is:

$$\forall i: \; x_i \in B_i(p^*) \cap C_i(p^*) \Rightarrow U_i(x_i^*) \geq U_i(x_i).$$

Assume, by way of contradiction, that there exists a $y_i \in B_i(p^*) \cap C_i(p^*)$ such that $U_i(y_i) > U_i(x_i^*)$.

We need to show that this would imply that, for high j, (x^j, p^j) is not an equilibrium of $\mathcal{E}^j$.

<u>A preliminary remark</u>

[21] Notice that, dividing the agents in two groups: the group that has the ϵ-agent strictly desires the endowments of the other group; and has endowments that are strictly desired by the other group.

Suppose that prices for delivery in s and in $t \in P_i(s)$ are parallel: $p^{*s} = ap^{*t}$. The two deliverability conditions that involve prices p^{*s} and p^{*t} yield equalities.

$$\begin{cases} p^{*s} \cdot y_i^s \leq p^{*s} \cdot y_i^t \\ p^{*t} \cdot y_i^t \leq p^{*t} \cdot y_i^s \end{cases} \Rightarrow \begin{cases} ap^{*t} \cdot y_i^s \leq ap^{*t} \cdot y_i^t \\ p^{*t} \cdot y_i^t \leq p^{*t} \cdot y_i^s \end{cases} \Rightarrow \begin{cases} p^{*s} \cdot y_i^s = p^{*s} \cdot y_i^t \\ p^{*t} \cdot y_i^t = p^{*t} \cdot y_i^s \end{cases}$$

The two consumption bundles, y_i^s and y_i^t, must cost the same in both states. The utility functions u_i^s and u_i^t are also equal, because s and t belong to the same element of the agent's partition of information. If $u_i^s(y_i^s) > u_i^s(y_i^t)$, then the agent would be better off selecting y_i^s for consumption in both states. Thus, we must have $u_i^s(y_i^s) = u_i^s(y_i^t)$. Since the utility functions are concave, the agent is not worse off consuming the average bundle in both states. Notice that if the original vector satisfies the deliverability conditions, then this average vector also does. Define the consumption vector x_i'' by modifying y_i, considering this average bundle whenever there are two parallel prices. Therefore, we will have $x_i''^s = x_i''^t$ whenever $p^{*s} = ap^{*t}$.

Main argument of the proof

Reformulating, we assume that there exists a $x_i'' \in B_i(p^*) \cap C_i(p^*)$ such that $U_i(x_i'') > U_i(x_i^*)$, with $x_i''^s = x_i''^t$ whenever $p^{*s} = ap^{*t}$.

The neighbor $x_i' = (1-\delta)x_i''$ is still preferred to x_i^* (for small $\delta > 0$), also belongs to $C_i(p^*)$, and belongs to the interior of the budget sets in $\mathcal{E}$ and $\mathcal{E}^j$ (for high j):

$$U_i(x_i') > U_i(x_i^*); \quad p^* \cdot x_i' < p^* \cdot e_i; \quad p^j \cdot x_i' < p^j \cdot e_i.$$

Since the utility functions are continuous, there exists a radius $\epsilon > 0$ such that the neighbors of x_i' are still preferred to x_i^* and, therefore, to x_i^j, for high j (according to U_i, which does not include the possible utility penalty):

$$d(x_i, x_i') < \epsilon \Rightarrow U_i(x_i) > U_i(x_i^j), \text{ for sufficiently high } j.$$

We are assuming that the bundle x_i' satisfies the deliverability conditions for the equilibrium prices p^*. Consider, without loss of generality, the following element of the agent's information partition: $P_i(s) = \{1, ..., s\}$. It should be clear that this reasoning extends to all the elements of P_i. The conditions for delivery in these states are written below, with all $k^{ij} \geq 0$.

$$\begin{cases} p^{*1} \cdot x_i'^1 \leq p^{*1} \cdot x_i'^2; \\ ... \\ p^{*1} \cdot x_i'^1 \leq p^{*1} \cdot x_i'^s; \\ p^{*2} \cdot x_i'^2 \leq p^{*2} \cdot x_i'^1; \\ ... \\ p^{*2} \cdot x_i'^2 \leq p^{*2} \cdot x_i'^s; \\ ... \\ ... \\ p^{*s} \cdot x_i'^s \leq p^{*s} \cdot x_i'^1; \\ ... \\ p^{*s} \cdot x_i'^s \leq p^{*s} \cdot x_i'^{s-1}. \end{cases} \Leftrightarrow \begin{cases} p^{*1} \cdot x_i'^2 - p^{*1} \cdot x_i'^1 = k^{12} \geq 0; \\ ... \\ p^{*1} \cdot x_i'^s - p^{*1} \cdot x_i'^1 = k^{1s} \geq 0; \\ p^{*2} \cdot x_i'^1 - p^{*2} \cdot x_i'^2 = k^{21} \geq 0; \\ ... \\ p^{*2} \cdot x_i'^s - p^{*2} \cdot x_i'^2 = k^{2s} \geq 0; \\ ... \\ ... \\ p^{*s} \cdot x_i'^1 - p^{*s} \cdot x_i'^s = k^{s1} \geq 0; \\ ... \\ p^{*s} \cdot x_i'^{s-1} - p^{*s} \cdot x_i'^s = k^{s,s-1} \geq 0. \end{cases}$$

We will find an x_i that is a neighbor of x_i' and belongs to $C_i(p^j)$ (which contradicts the fact that the allocation x^j is an equilibrium of $\mathcal{E}^j$). This would prove (4) by contradiction.

Let $d(x_i, x_i') < \epsilon$. We already know that $U(x_i) > U(x_i^*)$. Consider a sufficiently high j for $U_i(x_i) > U_i(x_i^j)$ and also for $d(p^j, p^*) < \epsilon$.

<u>Case 1</u>: All inequalities are such that $k^{st} > 0$.

Denote $dx_i = x_i - x_i'$ and $dp^j = p^j - p^*$. Pick the lowest k^{st} among those that are strictly positive and denote it by k^{min}. Manipulating the condition which guarantees that in state s, the bundle $x_i'^s$ is not more expensive than $x_i'^t$:

$$p^{*s} \cdot x_i'^t - p^{*s} \cdot x_i'^s = (p^{js} - dp^{js}) \cdot (x_i^t - dx_i^t) - (p^{js} - dp^{js}) \cdot (x_i^s - dx_i^s) = k^{st} \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s = k^{st} + p^{js} \cdot dx_i^t + dp^s \cdot (x_i^t - dx_i^t) - p^{js} \cdot dx_i^s - dp^s \cdot (x_i^s - dx_i^s) \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s > k^{st} - \epsilon - \epsilon(\|e_T\| + \epsilon) - \epsilon - \epsilon(\|e_T\| + \epsilon) \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s > k^{st} - 2\epsilon - 2\epsilon(\|e_T\| + \epsilon) = k^{st} - 2\epsilon(\|e_T\| + 1 + \epsilon).$$

Let $\epsilon_2 = 2\epsilon(\|e_T\| + 1 + \epsilon) > 0$. We have:

$$p^{js} \cdot x_i^t - p^{js} \cdot x_i^s > k^{st} - \epsilon_2.$$

Choosing an $\epsilon > 0$ small enough to make $\epsilon_2 < k^{min}$ guarantees that the strict inequalities for x_i' and p^* remain strict for any $x_i \in B(x_i', \epsilon)$ and p^j (with j large enough).

There is no utility penalty, therefore, $U_i^j(x_i) > U_i^j(x_i^j)$. We have a contradiction. The consumption bundle in the equilibrium sequence, x_i^j, is not a maximizer of U_i^j.

<u>Case 2</u>: For every $t \in P_i(s)$, prices p^{*s} and p^{*t} are not parallel.

The difference relative to *Case 1* lies in checking that the inequalities which are not strict at

(x_i', p^*) are still satisfied at (x_i, p^j) (with j large enough). The inequalities that are not strict are those for which $k^{st} = 0$.

Let $\gamma^{st} = \left(1 - \dfrac{p^{*s} \cdot p^{*t}}{\|p^{*s}\| \|p^{*t}\|}\right) \|p^{*s}\|$. Let γ^{min} be the lowest of all strictly positive γ^{st}, with $t \in P_i(s)$. Since we have a lower bound on equilibrium prices, γ^{st} is only zero when prices p^{*s} and p^{*t} are parallel (we are excluding this case, for now).

Keep x_i sufficiently close to x_i' in order to preserve the strict inequalities (pick $\epsilon > 0$ such that $\epsilon_2 < k^{min}$), and select displacements parallel to prices: $dx_i^s = -\frac{\epsilon}{2} \frac{p^{*s}}{\|p^{*s}\|}$.

Let $\epsilon_3 = \frac{\epsilon \gamma^{min}}{8\|e_T\|}$, and consider a j that is high enough for: $d(p^j, p^*) < \min\{\epsilon_3, \epsilon\}$.

Consider an inequality that is not strict, for example: $p^{*a} \cdot x_i'^b = p^{*a} \cdot x_i'^a$, implying that $k^{ab} = 0$. Let's verify that this generic deliverability condition still holds in $\mathcal{E}^j$.

$$p^{ja} \cdot x_i^b - p^{ja} \cdot x_i^a = (p^{*a} + dp^{ja}) \cdot (x_i'^b + dx_i^b) - (p^{*a} + dp^{ja}) \cdot (x_i'^a + dx_i^a) =$$
$$= p^{*a} \cdot (x_i'^b + dx_i^b) + dp^{ja} \cdot (x_i'^b + dx_i^b) - p^{*a} \cdot (x_i'^a + dx_i^a) - dp^{ja} \cdot (x_i'^a + dx_i^a) =$$
$$= p^{*a} \cdot dx_i^b + dp^{ja} \cdot (x_i'^b + dx_i^b) - p^{*a} \cdot dx_i^a - dp^{ja} \cdot (x_i'^a + dx_i^a) >$$
$$> p^{*a} \cdot dx_i^b - \epsilon_3(\|e_T\| + \epsilon) - p^{*a} \cdot dx_i^a - \epsilon_3(\|e_T\| + \epsilon) =$$
$$= p^{*a} \cdot dx_i^b - p^{*a} \cdot dx_i^a - 2\epsilon_3(\|e_T\| + \epsilon) >$$
$$> -p^{*a} \cdot \tfrac{\epsilon}{2} \tfrac{p^{*b}}{\|p^{*b}\|} + p^{*a} \cdot \tfrac{\epsilon}{2} \tfrac{p^{*a}}{\|p^{*a}\|} - 4\epsilon_3 \|e_T\| =$$
$$= -\tfrac{\epsilon}{2} \tfrac{p^{*a} \cdot p^{*b}}{\|p^{*a}\| \|p^{*b}\|} \|p^{*a}\| + \tfrac{\epsilon}{2} \tfrac{p^{*a} \cdot p^{*a}}{\|p^{*a}\| \|p^{*a}\|} \|p^{*a}\| - 4\epsilon_3 \|e_T\| =$$
$$= \tfrac{\epsilon}{2} \tfrac{p^{*a} \cdot p^{*a}}{\|p^{*a}\| \|p^{*a}\|} \|p^{*a}\| - \tfrac{\epsilon}{2} \tfrac{p^{*a} \cdot p^{*b}}{\|p^{*a}\| \|p^{*b}\|} \|p^{*a}\| - \tfrac{\epsilon}{2} \gamma^{min} =$$
$$= \tfrac{\epsilon}{2} (1 - \tfrac{p^{*a} \cdot p^{*b}}{\|p^{*a}\| \|p^{*b}\|}) \|p^{*a}\| - \tfrac{\epsilon}{2} \gamma^{min} \geq 0$$

In sum, this displacement dx_i implies that:

$$p^{ja} \cdot x_i^b - p^{ja} \cdot x_i^a > 0.$$

The deliverability condition is verified, and thus $U_i^j(x_i) > U_i^j(x_i^j)$. Contradiction.

<u>Case 3</u>: Prices p^{*s} and p^{*t} are parallel.

The same displacement as in case 2, $dx_i^s = -\frac{\epsilon}{2} \frac{p^{*s}}{\|p^{*s}\|}$, is good for the case in which prices p^{*a} and p^{*b} are parallel. In this case: $x_i'^a = x_i'^b$ and also $dx_i^a = dx_i^b$. Hence, $x_i^a = x_i^b$ and the conditions remain satisfied in equality.

All deliverability conditions are satisfied, therefore: $U_i^j(x_i) = U_i(x_i) > U_i(x_i^j) \geq U_i^j(x_i^j)$. The consumption bundle in the equilibrium sequence, x_i^j, does not maximize U_i^j, because x_i is preferred. This contradiction proves (4). QED

Appendix 2: Example of non-existence of equilibrium

Consider an economy in which two agents trade a single good under uncertainty. There are three states of nature, and their future endowments depend on the state of nature that occurs:

$e_A = (100, 100, 1)$ and $e_B = (1, 100, 100)$.

Agents only observe their endowments.

$P_A = \{\{1, 2\}; \{3\}\}$ and $P_B = \{\{1\}; \{2, 3\}\}$.

The different states occur with objective and publicly known probabilities:

$\mu = (\mu^1, \mu^2, \mu^3) = (0.45, 0.1, 0.45)$.

A significant level of risk aversion induces agents to trade *ex ante*, in order to maximize expected utility:

$$U_i(x_i) = \sum_{s=1}^{S} \mu^s \sqrt{x_i^s}.$$

Prices in states 1 and 3 must be strictly positive, or else the demands of agent B and A would be infinite for the corresponding contingent goods.

With strictly positive prices for all the contingent goods, if agents selected different consumption levels in states that they did not distinguish, then they would end up receiving the cheapest of the alternatives, which would be the one with the lowest consumption level. In this case, we must have:

$x_A = (x_A^{12}, x_A^{12}, x_A^3)$ and $x_B = (x_B^1, x_B^{23}, x_B^{23})$.

Since agents are at the frontier of their budget sets:

$$\begin{cases} (p^1 + p^2)x_A^{12} + p^3 x_A^3 = 100(p^1 + p^2) + p^3; \\ p^1 x_B^1 + (p^2 + p^3)x_B^{23} = p^1 + 100(p^2 + p^3). \end{cases}$$

Adding the two:

$$p^1(x_A^{12} + x_B^1) + p^2(x_A^{12} + x_B^{23}) + p^3(x_A^3 + x_B^{23}) = 101p^1 + 200p^2 + 101p^3.$$

For this to be an equilibrium, the allocation must be feasible:

$$\begin{cases} x_A^{12} + x_B^1 \leq 101; \\ x_A^{12} + x_B^{23} \leq 200; \\ x_A^3 + x_B^{23} \leq 101. \end{cases}$$

With strictly positive prices, the conditions are verified in equality. This implies that the allocation is of the form:

$$\begin{cases} x_A = (x_A^{12}, x_A^{12}, x_A^3) = (x_A^3 + 99, x_A^3 + 99, x_A^3); \\ x_B = (x_B^1, x_B^{23}, x_B^{23}) = (x_B^1, x_B^1 + 99, x_B^1 + 99). \end{cases}$$

The only individually rational allocation of this form corresponds to the initial endowments. There is no trade. But are agents maximizing their utility levels?

$$\begin{cases} x_A = (100, 100, 1); \\ x_B = (1, 100, 100). \end{cases} \Rightarrow \begin{cases} U(x_A) = 0.45 * 10 + 0.1 * 10 + 0.45 * 1 = 5.95; \\ U(x_B) = 0.45 * 1 + 0.1 * 10 + 0.45 * 10 = 5.95. \end{cases}$$

Suppose that $p^1 = p^3$. Agent A can trade consumption in s_1 for consumption in s_3. But consuming less in s_1 implies that delivery in s_2 will also be of this lower quantity. In any case, the agent can select:

$$x_1' = (x_1'^{12}, x_1'^{12}, x_1'^{3}) = (81, 81, 20).$$

The corresponding utility level is:

$$U(x_1') = 0.45 * 9 + 0.1 * 9 + 0.45 * 4.47 = 6.96.$$

In the case with asymmetric prices ($p^1 \neq p^3$), the same trade is even more favorable for one of the agents. We reached a contradiction, implying that there is no equilibrium with strictly positive prices.

With $p^2 = 0$, an alternative bundle can be big enough to violate feasibility and still be deliverable. The deliverability restriction is not relevant because it is of the form $0 \cdot x^2 \leq 0 \cdot x^s$. Agents can choose a consumption level for state 2 that is big enough to violate feasibility and still desire to increase it. There cannot be a rational expectations equilibrium with $p^2 = 0$.

Appendix 3: The deliverability correspondence

The set of bundles that satisfy the deliverability restrictions depends on the prevailing prices. Consider the correspondence from prices to the set of deliverable bundles:

$$C_i : \Delta^{SL} \longrightarrow \mathbb{R}_+^{SL};$$

$$C_i(p) = \left\{ x \in \mathbb{R}_+^{SL} : \forall s \in \Omega,\ p^s \cdot x^s = \min_{t \in P_i(s)} \{p^s \cdot x^t\} \right\}.$$

If the correspondence $B_i(p) \cap C_i(p)$ were continuous, we could apply Berge's maximum theorem and Kakutani's fixed point theorem to establish existence of equilibrium in economies with uncertain delivery.

Upper hemicontinuity of C_i at p_0 means that, given an arbitrary open set, V, containing $C_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, we have $C_i(p) \subseteq V$.

The correspondence is closed since all the restrictions are inequalities which are not strict. With a compact range, a closed-valued correspondence is upper hemicontinuous if and only if it is closed. Therefore, when restricted to a bounded economy (for example, by the total initial endowments in the economy), C_i is upper hemicontinuous.

Lower hemicontinuity of C_i at p_0 means that given an arbitrary open set, V, intersecting $C_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, the image $C_i(p)$ also intersects V.

The correspondence under study, C_i, is not lower hemicontinuous. Lower hemicontinuity fails when prices in are null ($p^s = 0$) or collinear ($p^s = ap^t$).

When prices are null, the deliverability restrictions disappear. It is always true that $0 \cdot x^s \leq 0 \cdot x^t$. But with a small perturbation, the restrictions appear. This is why l.h.c. fails.

When prices are collinear, the failure of l.h.c. is more subtle.

Consider an economy with two goods, A and B, and two states of nature, s and t. Let $p_0 = (p_0^s, p_0^t) = (p_0^{As}, p_0^{Bs}; p_0^{At}, p_0^{Bt}) = (\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4})$. The bundle $x_0 = (1, 0; 0, 1)$ belongs to the deliverable set, since:

$p_0^s \cdot x_0^s \leq p_0^s \cdot x_0^t \Leftrightarrow \frac{1}{4} \leq \frac{1}{4}$, and

$p_0^t \cdot x_0^t \leq p_0^t \cdot x_0^s \Leftrightarrow \frac{1}{4} \leq \frac{1}{4}$.

Delivering $(1, 0)$ in state s and $(0, 1)$ in state t does not violate deliverability because both bundles have the same price in both states.

A small perturbation in prices can make $(0, 1)$ cheaper in state s and $(1, 0)$ cheaper in state t. Consider an open ball around x_0 with radius $0 < \epsilon < \frac{1}{10}$. After a perturbation in prices to $p = (\frac{1}{4} + \delta, \frac{1}{4} - \delta, \frac{1}{4} - \delta, \frac{1}{4} + \delta)$, this ball does not intersect the deliverable set.

Suppose that there existed a vector $dx = (\epsilon^{As}, \epsilon^{Bs}, \epsilon^{At}, \epsilon^{Bt})$ such that $x = (1 + \epsilon^{As}, \epsilon^{Bs}; \epsilon^{At}, 1 + \epsilon^{Bt})$ is inside that open ball and belongs to the deliverable set:

(1) $(\frac{1}{4} + \delta, \frac{1}{4} - \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \leq (\frac{1}{4} + \delta, \frac{1}{4} - \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \Leftrightarrow$

$\Leftrightarrow (\frac{1}{4} + \delta)(1 + \epsilon^{As}) + (\frac{1}{4} - \delta)\epsilon^{Bs} \leq (\frac{1}{4} + \delta)\epsilon^{At} + (\frac{1}{4} - \delta,)(1 + \epsilon^{Bt}) \Leftrightarrow$

$\Leftrightarrow \frac{1}{4} + \frac{1}{4}\epsilon^{As} + \delta + \delta\epsilon^{As} + \frac{1}{4}\epsilon^{Bs} - \delta\epsilon^{Bs} \leq \frac{1}{4}\epsilon^{At} + \delta\epsilon^{At} + \frac{1}{4} + \frac{1}{4}\epsilon^{Bt} - \delta - \delta\epsilon^{Bt} \Leftrightarrow$

$\Leftrightarrow \frac{1}{4}(\epsilon^{As} + \epsilon^{Bs} - \epsilon^{At} - \epsilon^{Bt}) + \delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta;$

(2) $(\frac{1}{4} - \delta, \frac{1}{4} + \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \leq (\frac{1}{4} - \delta, \frac{1}{4} + \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \Leftrightarrow$

$\Leftrightarrow (\frac{1}{4} - \delta)\epsilon^{At} + (\frac{1}{4} + \delta)(1 + \epsilon^{Bt}) \leq (\frac{1}{4} - \delta)(1 + \epsilon^{As} + (\frac{1}{4} + \delta)\epsilon^{Bs}) \Leftrightarrow$

$\Leftrightarrow \frac{1}{4}(\epsilon^{At} + 1 + \epsilon^{Bt} - 1 - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + 1 + 1 + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs} \leq 0 \Leftrightarrow$

$\Leftrightarrow \frac{1}{4}(\epsilon^{At} + \epsilon^{Bt} - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs}) \leq -2\delta.$

Adding the two inequalities, we obtain:

(1 + 2) $\delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta \Leftrightarrow \epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \leq -2.$

Which is impossible, because $\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \geq -4\epsilon > -\frac{4}{10}$.

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