Incomplete Markets, Transitory Shocks, and Welfare*

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Abstract

Although equilibrium allocations in models with incomplete markets are generally not Pareto-efficient, it is often argued that quantitative welfare losses from missing assets are small when time horizons are long and shocks are transitory. In this paper we use a computational analysis to show that even in the simplest infinite horizon model without aggregate uncertainty welfare losses can be substantial.

Furthermore we show that in this model – contrary to previous results in the literature – welfare losses from incomplete markets do not necessarily disappear when one considers calibrations of the model in which agents become very patient. We identify two scenarios under which welfare losses remain substantial. First, when the economic model is calibrated to higher frequency data, the period persistence of negative income shocks must increase as well. In this case the welfare loss of incomplete markets remains constant even as agents' rate of time preference tends to one. Secondly, for a fixed specification of endowment processes, an exogenous decrease of agents' rate of discounting should not affect their abilities to borrow. With exogenous borrowing constraints, the incomplete markets welfare does not converge to the complete markets welfare.

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1 Introduction

Although competitive equilibria are generally not Pareto-efficient when financial markets are incomplete, in the applied literature it is often argued that incomplete markets "do not matter" and that the welfare losses due to missing financial securities are quantitatively small. This argument comes in two parts. First, following Lucas' (1987) observation on the welfare costs of business cycles, it is argued that the overall welfare gains from risk sharing are quantitatively small. A second argument states that in a model with transitory shocks and patient agents a single bond often suffices to realize most of the potential welfare gains from risk sharing and that the welfare gains from additional assets are very small (see e.g. Levine and Zame (2000)).

Comparing the welfare agents achieve in autarky to the complete-markets welfare in a realistically calibrated model where agents have von-Neumann-Morgenstern utility with relatively low risk aversion, one readily notices that the differences are often small in terms of wealth equivalences. However, this observation crucially depends on the specification of preferences and endowment shocks (see e.g. van Wincoop (1999) for examples where there are substantial gains from risk sharing). In models where even the welfare gains from perfect risk sharing are small, it is obvious that market incompleteness cannot have large effects on welfare, because the autarky welfare provides a lower bound on any equilibrium welfare. In determining the quantitative welfare effects of incomplete markets one therefore must view welfare losses from missing assets relative to the welfare achieved in autarky. A more interesting question is then to determine what percentage of the total welfare gains from perfect risk sharing can be realized with a limited number of assets (see Magill and Quinzii (2000) for a similar argument).

We consider a simple infinite horizon model with 2 types of agents and with a single bond. Using Heaton and Lucas' (1996) calibration of idiosyncratic shocks to yearly US data we show that with a single bond there are likely to be substantial gains from additional financial assets. Using the algorithm developed in Judd et al. (2000) we compute (approximate) incomplete-markets equilibria. We consider the effect of agents' risk aversion and of the magnitude of the shock on welfares. Somewhat surprisingly we find that the relative (to autarky) welfare losses from incomplete markets generally decrease as agents' risk aversion increases or as the magnitude of the shock increases. This finding shows that it is impossible to project results that are found in models with low welfare gains of risk sharing to models with high welfare gains.

A different and more interesting argument against the importance of market incompleteness is that in models with transitory shocks, patient agents, and long time horizons a single bond suffices to smooth out negative endowment shocks. In a recent theoretical paper Levine and Zame (2000) show that in a Lucas (1978) style exchange economy with Markovian endowment shocks and no aggregate uncertainty agents' welfare converges to the complete markets welfare as the discount factor β converges to one. They also show that their result is somewhat fragile to the assumption of aggregate uncertainty and they point out that they are unable to provide estimates for the actual welfare losses.

In this paper we provide numerical examples for the convergence and we verify that welfares do not converge if there is aggregate uncertainty and if Levine and Zame's conditions for this case are not satisfied.

The main contribution of this paper is to show that even without aggregate uncertainty, the result of Levine and Zame (2000) does not imply that in economies which are calibrated to high frequency data welfare losses are small. The analytical approach of Levine and Zame (2000) focuses on the interpretation of discount factors as just describing agents' attitudes to patience. But in macroeconomics discount factors are often interpreted as being related to the length of time periods in the model. Realistic calibration of an economic model must then mean that the discount factor depends on the length of a period. While one can argue that daily trading in financial assets is possible and that a realistic β should therefore be close to one, one must then also calibrate endowment shocks appropriately. In particular, the persistence of shocks must increase as the length of a period decreases. When considering a sequence of economies in which the persistence of shocks increases with β in a way which ensures that the complete-markets sharing rule remains the same (i.e. the length of a period shortens without changing the complete markets allocation) the welfare losses from incomplete markets remain approximately constant and, contrary to the result of Levine and Zame (2000), do not converge to zero. A crude approximation of agents' value function explains intuitively why one cannot possibly expect convergence in this case.

A second point, which is worth pointing out, is that Levine and Zame (2000) impose an implicit debt constraint to rule out Ponzi schemes. Many applied papers (see e.g. den Haan (1999)) assume explicit and much tighter debt constraints – these obviously lead to larger welfare losses. However, these debt constraints constitute an additional market imperfection.

An important issue for welfare losses in incomplete markets is the number of assets, their dividends and the specification of agents' endowments. It is an important but nearly unmanageable empirical task to correctly specify the stochastic process of existing assets' dividends and individual endowments. Since we focus on the long-time-horizon aspect of the problem and ask how much borrowing is needed for agents to be able to smooth out transitory shocks we consider an incomplete markets economy with a single bond. It is currently computationally too burdensome to consider models with more than 1 asset and very patient agents.

The paper is organized as follows. Section 2 describes the standard model of an infinite-horizon pure exchange economy. In Section 3 we outline our basic computational strategy and show that for the calibration of idiosyncratic shocks used in Heaton and Lucas (1996), welfare losses due to incomplete markets are substantial when compared to autarky. In Section 4 we show that the incomplete markets welfare does generally not converge to the complete markets welfare if the length of a period decreases. In this case both the persistence of the endowment shock as well as the discount factor increase. In Section 5 we provide examples of how explicit debt constraints can lead to substantial welfare losses. Section 6 concludes the paper.

2 The Economic Model

We examine an infinite horizon pure exchange economy with heterogeneous agents and incomplete asset markets. Time is indexed by $t = 0, 1, 2, ...$ A time-homogeneous Markov process of exogenous states (y_t) takes values in a discrete set $Y = \{1, 2, \ldots, S\}$. The Markov transition matrix is denoted by Π. Let Σ denote the set of all possible histories σ of the exogenous states. A date-event σ_t is the history of states along a history σ up to time t, i.e. $\sigma_t = (y_0 y_1 \cdots y_t)$. There are S successors of any node σ_t, namely $\sigma_t s = (y_0 y_1 \cdots y_t s)$ for each $s \in Y$. Each node $\sigma_t, t \geq 1$, has a unique predecessor $\sigma_t^* = (y_0 y_1 \cdots y_{t-1})$. To simplify notation the event tree includes the root nodes' predecessor σ_0^*. In each date-event $\sigma \in \Sigma$ there is a single perishable consumption good.

We assume that there are finitely many types of infinitely-lived agents $h \in \mathcal{H} = \{1, 2, .., H\}$. Agent h's individual endowment at time event σ_t is a function $e^h : Y \to I\!R_{++}$ depending on the current shock y_t alone. The aggregate endowment of the economy in state y_t is $e(y_t) = \sum_{h=1}^{H} e^h(y_t)$. Occasionally it will be more convenient to write $e^h(\sigma)$. It will then always be understood that $e^h(\sigma) = e^h(y)$ where $\sigma = (\sigma^* y)$. Each agent h has a time-separable von-Neumann-Morgenstern utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\}.$$

We assume that the Bernoulli functions $u_h(.) : I\!R_{++} \to I\!R$ are strictly monotone, C^2, strictly concave, and satisfy the Inada property, that is, $\lim_{x\to 0} u'(x) = \infty$. We also assume that discount factor $\beta \in (0, 1)$ is the same for all agents and that expectations are taken under the true Markov-probabilities.

Let the matrix

$$e = \begin{pmatrix} e^1(1) & \cdots & e^1(S) \\ \vdots & & \vdots \\ e^H(1) & \cdots & e^H(S) \end{pmatrix}$$

represent possible individual endowments. The vector of utility functions is $u = (u^1, \ldots, u^H)$. We collect the primitives of the economy as $\mathbf{E} = (e, u, \Pi, \beta)$.

2.1 Equilibrium Concepts

In order to evaluate the welfare effects of incomplete markets we define an Arrow-Debreu equilibrium for complete markets.

Definition 1 *An Arrow-Debreu equilibrium for an economy* $\mathbf{E}$ *is a collection of prices* $(p(\sigma))_{\sigma\in\Sigma}$ *and a consumption allocation* $(c^h(\sigma))_{\sigma\in\Sigma}^{h\in\mathcal{H}}$ *such that markets clear and agents maximize, i.e.*

- $\sum_{h\in\mathcal{H}}(c^h(\sigma) - e^h(\sigma)) = 0$ *for all* $\sigma \in \Sigma$.
- $(c^h(\sigma))_{\sigma\in\Sigma} \in \arg\max u^h(c)$ *s.t.* $\sum_{\sigma\in\Sigma} p(\sigma)(c(\sigma) - e^h(\sigma)) = 0$

In contrast to the Arrow-Debreu equilibrium we want to examine economies where there is a single one-period bond at each node $\sigma \in \Sigma$ and where agents have to trade this bond in order to transfer wealth across time and states. We define the notion of a financial market equilibrium for an economy where agents face an implicit debt constraints as in Levine and Zame (1996) or Magill and Quinzii (1994).

Definition 2 *A financial market equilibrium for an economy* $\mathbf{E}$ *with a single bond is a process of portfolio holdings and consumptions* $(\theta^h(\sigma), c^h(\sigma))_{\sigma\in\Sigma}^{h\in\mathcal{H}}$ *as well as asset prices* $(q(\sigma))_{\sigma\in\Sigma}$ *satisfying the following conditions:*

(1) $\sum_{h=1}^{H} \theta^h(\sigma) = 0$ *for all* $\sigma \in \Sigma$.

(2) For each agent h :

$$\begin{aligned} (\theta^h, c^h) \in \arg \quad & \max_{\theta,c} U_h(c) \ \textit{s.t.} \\ & c(\sigma) = e^h(\sigma) + \theta(\sigma^*) - \theta(\sigma)q(\sigma) \\ & \sup_{\sigma\in\Sigma} |q(\sigma)\theta(\sigma)| < \infty \end{aligned}$$

3 A Yearly Calibration and Convergence to Arrow-Debreu

In this section we present a first example. We show that for a model which is calibrated to yearly data the welfare loss in incomplete markets is substantial when compared to the autarky outcome. We then demonstrate that in this example the welfare converges to the complete markets welfare as $\beta \to 1$. Levine and Zame (2000) prove this result analytically and our example is meant to illustrate that convergence is very fast for realistic values of β. Finally we show that with aggregate uncertainty, welfare losses initially decrease but that they do not converge to zero.

3.1 The Example Economy

Heaton and Lucas (1996) use the income series from the Panel Studies of Income Dynamics to calibrate processes for idiosyncratic income shocks. In the resulting model the shock can take 2 different values, $(e^h(1), e^h(2)) = (3.77, 6.23)$. The transition matrix is given by

$$\Pi = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}.$$

We assume that there are $H = 2$ agents having identical CRRA Bernoulli utilities $u^h(c) = \frac{c^{1-\gamma}}{1-\gamma}$ where the coefficient of relative risk aversion is given by γ. We vary both risk aversion and the magnitude of the shock and compute equilibria for $\gamma \in \{0.5, 1.5, 2.5, 3.5\}$ and for $e^1(1) \in \{2, 3, 3.77, 4.5\}$. For the cases without aggregate uncertainty $e^2(y) = 10 - e^1(y)$ for all $y \in Y$.

3.2 Computational Procedure

In all examples below we assume that there exist a recursive equilibrium where the interest rate and the agents' portfolio choice are functions of the last-period portfolio and the current shock alone. Although with finitely many agents recursive equilibria of this type do not always exist (see Kubler and Schmedders (2000)), with a single bond and only two states they usually do exist. We use the computational procedure developed in Judd et al. (2000) to approximate these equilibria numerically. Unfortunately there is no formal procedure that assures that the computed welfares are close to the actual equilibrium welfares. We choose the number of spline-nodes in such a way that the maximum relative error in the agents' Euler equations lies consistently below 10^{-8}. For the results in Table 1 below we then recompute the welfare loss for an economy with a continuum of i.i.d ex ante identical agents and the results turn out to be almost identical. It follows from Huggett (1993) and Aiyagari (1994) that in these economies a recursive equilibrium always exists and the price of the bond is constant across states and time. While there are no formal techniques which can be used to evaluate how close the computed equilibrium price is to the true equilibrium price of the bond, given an equilibrium price, the dynamic programming techniques in Santos (1998) can be applied to find error bounds on the true welfares agents achieve. Moreover, for our simple two-state problem upper bounds on the true equilibrium price can be established. All this should give us some confidence in our later results.

In order to approximate equilibria for models with an implicit debt constraint we follow the procedure in Zhang (1997) to determine the theoretical borrowing limits. An implicit debt constraint implies that at all nodes agents must be able to pay off their debt in finite time. Therefore, it must be impossible that the interest payment on debt exceeds an agent's endowment. For a model without aggregate uncertainty, we approximate this debt limit by $-\frac{1}{1-\beta}$, since we need a compact region of admissible portfolios for computations. The equilibrium price will always be above β and so the true limit might be larger, but the additional welfare gained is negligible and we can focus on this approximation. In models with aggregate uncertainty $q(s) < \beta$ in all states s with a bad aggregate shock. In these cases we start with a conservative estimate and increase the set of admissible portfolio holdings when necessary.

3.2.1 Welfare Losses

In order to evaluate the welfare losses from incomplete financial markets we compute the wealth equivalent of the welfare loss from incomplete markets and put this in relation to the welfare loss from autarky (i.e. from a situation where $c^h(\sigma) = e^h(\sigma)$ for all $\sigma \in \Sigma$). In this we follow Magill and Quinzii (2000) and their definition of unexploited gains from trade.

Given our specification of preferences (which we will use throughout the paper), we can derive an analytic solution for the complete markets outcome. We then compute the welfare loss λ as follows. Let W^h_{CM}, W^h_A, W^h_B denote the wealth equivalents of agent h's utility for complete markets, autarky and incomplete markets with a single bond respectively, i.e. $W^h = ((1-\gamma)U^h)^{1/(1-\gamma)}$. Then

$$\lambda^h = \frac{(W^h_{CM} - W^h_B)/W^h_{CM}}{(W^h_{CM} - W^h_A)/W^h_{CM}} = \frac{W^h_{CM} - W^h_B}{W^h_{CM} - W^h_A}.$$

λ^h measures how much much consumption the agent would be willing to give up at time $t = 0$ in order to avoid the incomplete markets allocation relative to how much the agent would be willing to give up in order to avoid autarky.

For economies with no aggregate uncertainty and homothetic utility this is equivalent to measuring how much of his consumption at each node an agent would be willing to give up to avoid the incomplete markets economy and put this in relation to how much the agent would be willing to give up to avoid autarky.

3.3 Results

Table 1 displays the welfare losses (in percent) due to a missing second asset for different specifications of the shock and the preferences. The economy starts in state $y_0 = 1$. In each entry of the table, the first number is the welfare loss for agent 1 who starts with a bad idiosyncratic shock, the second number is the welfare loss for agent 2 who starts with a good idiosyncratic shock.

shock	$\gamma = 0.5$		$\gamma = 1.5$		$\gamma = 2.5$		$\gamma = 3.5$	
(2,8)	12.0037	16.4969	8.0027	14.3480	5.9018	11.5214	4.4070	11.2448
(3,7)	12.0753	13.4989	11.0493	12.8368	10.9580	12.8607	10.6737	13.1865
(3.77,6.23)	12.7238	13.1545	12.4901	13.1238	12.5803	13.3058	12.5821	13.5719
(4.5,5.5)	12.9695	13.0886	12.9233	13.1212	12.8937	13.1718	12.8789	13.2411

Table 1: Welfare gains from complete markets.

While for almost all cases the welfare loss from incomplete markets is substantial, the changes of the welfare loss as the magnitude of the shock changes or as risk aversion changes seem counterintuitive at first. In order to understand these changes, note that an increase of the magnitude of the shock has two effects. If an agent starts in his good shock, due to discounting, an increase of first period endowments tend to increase his utility. On the other hand, with imperfect risk-sharing opportunities, the increase of the magnitude of the shock tends to decrease welfare. With complete financial markets only the first effect is relevant – as the magnitude of the shock increases the first agent is better off while the second agent is worse off. In the autarky allocation the second effect is always much stronger than the first – both agents lose as the magnitude of the shock increases. With incomplete financial markets, these two effects tend to offset each other. This explains that the first agent's welfare loss generally decreases as the shock increases while this effect is much less significant for the second agent. The first agents complete markets welfare decreases while the second agent's complete market welfare increases.

Finally there is a third effect which explains why welfare losses for the first agent (and for large shocks also for the second agent) decrease as risk aversion increases. A higher risk aversion leads to a lower equilibrium interest rate. An agent can borrow more easily to self-insure against the endowment shock. For example, the fact that the first agent's welfare loss is so small for the case of high risk aversion and the large idiosyncratic shock can be explained by the fact that the larger shock does lead to more income risk for the agent – but since there is a single bond and the interest rate is relatively low the large income shock can be mostly smoothed out by borrowing.

3.3.1 Patience

The economies considered above are calibrated to yearly data and in most assets (certainly in bonds) the frequency of trade is much higher. Following Levine and Zame's (1998, 2000) theoretical analysis we increase the agents discount factors. We focus on the calibration of preferences from Heaton and Lucas (1996) and fix $\gamma = 1.5$. We consider two specifications of the shock, $e^h = (3.77, 6.23)$ and $e^h = (2, 8)$ and we vary β to roughly match the interest rate for two-year, yearly, quarterly, monthly and weekly data, i.e. we choose $\beta \in \{0.9, 0.95, 0.99, 0.996, 0.999\}$. Table 2 shows the welfare losses as agents become more and more patient for both endowment specifications.

β	small shock		large shock	
0.9	21.1735	23.6049	14.4033	26.9125
0.95	12.4901	13.1238	8.0027	14.3480
0.99	2.8714	2.8751	2.2266	2.7610
0.996	1.1718	1.1756	1.0065	1.1560
0.999	0.0469	0.0471	0.2753	0.2786

Table 2: Welfare convergence as $\beta \to 1$.

Without aggregate uncertainty there is fast convergence to the complete-markets welfares for both specifications of the shock. The analytical approach of Levine and Zame (1998, 2000) reveals the reason why welfares converge in an economy without aggregate uncertainty. In an economy with no aggregate uncertainty and a single bond, the price of the bond is always at least as large as the discount factor. Thus, when the discount factor $\beta \to 1$, the bond price converges to 1 and the interest rate approaches 0. Consequently the implicit debt constraint does no longer keep agents from borrowing at each bad shock and they are able to smooth consumption just as in complete markets.

3.3.2 Aggregate Uncertainty

As Levine and Zame point out, their result does not necessarily hold when there is aggregate uncertainty that is not traded.

However, it turns out that with little aggregate uncertainty, the welfare loss from incomplete markets decreases substantially as agents become more patient. In order to quantify the behavior of economies with aggregate uncertainty, we introduce an aggregate shock of approximately 6 percent of aggregate endowments, which is independent of the idiosyncratic shock, i.e. we have 4 states and individual endowments are given by

$$e^1 = (3.77 \cdot 0.97, 3.77 \cdot 1.03, 6.23 \cdot 0.97, 6.23 \cdot 1.03) \text{ and } e^2 = (6.23 \cdot 0.97, 6.23 \cdot 1.03, 3.77 \cdot 0.97, 3.77 \cdot 1.03)$$

The transition matrix is given by

$$\Pi = \begin{pmatrix} 0.375 & 0.375 & 0.125 & 0.125 \\ 0.375 & 0.375 & 0.125 & 0.125 \\ 0.125 & 0.125 & 0.375 & 0.375 \\ 0.125 & 0.125 & 0.375 & 0.375 \end{pmatrix}.$$

Table 3A shows how the welfare loss decreases with β.

β	welfare gains	
0.9	20.1940	24.3118
0.95	11.4386	14.0907
0.99	3.2420	3.8148
0.996	1.9591	2.2411
0.999	1.0648	1.1125
0.9995	0.7978	0.8110

Table 3a: Welfare gains with small aggregate uncertainty.

While for low β the increase in incomplete markets welfare is substantial it appears to converge to a welfare bounded away from the complete markets welfare. Although this claim cannot be verified computationally we repeat the same experiment for a much larger idiosyncratic shock of 20 percent. Preferences and probabilities are held constant. The endowments are now given by

$e^1 = (3.77 \cdot 0.9, 3.77 \cdot 1.1, 6.23 \cdot 0.9, 6.23 \cdot 1.1)$ and $e^2 = (6.23 \cdot 0.9, 6.23 \cdot 1.1, 3.77 \cdot 0.9, 3.77 \cdot 1.1)$

β	welfare gains	
0.9	21.5106	29.2299
0.95	14.6742	19.0034
0.99	7.9387	8.8004
0.996	6.5683	6.8471
0.999	5.9068	5.9744
0.9995	5.8063	5.8416

Table 3b: Welfare gains with large aggregate uncertainty.

Table 3B shows that with substantial aggregate uncertainty there is no convergence in welfares. The welfare loss decreases substantial up to $\beta = 0.996$ and remains almost constant when β increases further. One possible explanation for the initial decrease is that with higher β the persistence of the negative shock plays a less important role (we will make this argument precise in Section 4 below). In order to isolate the role of persistence we now assume that all shocks are iid.

β	welfare gains	
0.9	9.7598	11.1444
0.95	6.0533	6.7915
0.99	2.9724	3.2302
0.996	2.4902	2.5853
0.999	2.1861	2.2057
0.9995	2.1433	2.1530

Table 4: Welfare gains with i.i.d. shocks.

Table 4 shows that even with iid shocks agents' incomplete market welfare initially increases (compared to autarky and complete markets) as β increases from 0.9 to 0.99. However, after that initial increase, the welfare loss seems to converge to around 2 percent.

The initial increase in welfare as well as the fact that welfare remains bounded away from the complete markets welfare can be explained as follows. As Levine and Zame point out, the reason why welfares converge in an economy without aggregate uncertainty is that as $\beta \to 1$, the bond price converges to 1 and the interest rate approaches 0. In this case the implicit debt constraint does no longer keep agents from borrowing at each bad shock. With aggregate uncertainty, however, the interest rate in the bad aggregate state usually remains bounded away from zero even as $\beta \to 1$. This fact follows from the agents' Euler equations: at least one agent has to have more consumption in one of the future good aggregate states than today. With only one bond, by concavity, this implies that the price of the bond has to be bounded away from one, even if β is arbitrarily close to one. However, initially, as β increases from 0.9 to 0.99 the interest rate decreases substantially, even with aggregate uncertainty. This fact implies that the implicit debt constraint moves further out and agents have more opportunities to borrow and insure against the negative idiosyncratic shock. When the aggregate shock is big, additional welfare losses become small right after $\beta = 0.99$. A further increase in β obviously has insignificant effects on the interest rate and the resulting increase in welfare is insignificant as well. However, with a small aggregate shock, the interest rate is very close to one, and increasing β from 0.99 to 0.996 increases agents' ability to borrow substantially. Thereafter, however, even small aggregate uncertainty ensures that the incomplete markets welfares do not converge to the complete markets welfare.

4 Persistence of Shocks

For economies without aggregate uncertainty Levine and Zame (1998, 2000) show that the incomplete markets welfare converges to the complete markets welfare as agents become more and more patient. The simple examples of the previous section demonstrate that even with aggregate uncertainty the welfare losses due to incomplete financial markets tend to decrease substantially when agents discount factors increase. Even if we disregard the (fairly widely accepted) assumption of positive discounting in agents' utilities, these results are only valid for a very narrow interpretation of agents' discount factor. The results have no implications for a sequence of economies where assets can be traded more and more often. In this case, the shocks as well as the agents' discount factors must change. We now show that these results do not generalize to sequences of economies where the probabilities in the transition matrix change along with the discount factors.

We consider a sequence of economies where the probability of the negative shock at time $t+1$ given a negative shock at time t increases as β converges to one. In order to stress that the increased persistence of shocks is simultaneous with a shorter length of the time period we call this notion of persistence π-persistence. We are going to determine the change of the π-persistence of the shock such that the complete-markets sharing rule remains constant. If calibration is taken seriously, the complete-markets allocation of a given economy should be independent of the choice of the length of a period. The following proposition shows that we can determine (unambiguously) how the π-persistence must change to leave the complete markets allocation constant. Note that we use homothetic preferences and so the actual size of endowments is irrelevant. Also, we do not need to change any parameters of the utility functions since we only consider time-separable utility functions.

Proposition 1 *Let $(c^h)_{h\in H}$ be a consumption allocation in an Arrow-Debreu equilibrium of the economy $\mathbf{E} = (e, u, \Pi_0, \beta_0)$. Then $(c^h)_{h\in H}$ is also an equilibrium allocation for the economy $\tilde{\mathbf{E}} = (e, u, \Pi_1, \beta_1)$ with $\beta_1 \geq \beta_0$ if Π_1 satisfies*

$$\pi_1(y|s) = \frac{\beta_0}{\beta_1}\frac{1-\beta_1}{1-\beta_0}\pi_0(y|s) \tag{1}$$

for all $y, s \in Y$ with $y \neq s$.

Note that the relative probabilities of leaving state s and jumping to state y remain constant for all $y \neq s$. The following lemma is needed to prove the proposition.

Lemma 1 *Let $0 < \beta_0 \leq \beta_1 < 1$ and Π_0 be a transition matrix. Then*

$$[I - \beta_0\Pi_0] = \frac{1-\beta_1}{1-\beta_0}\left[I - \beta_1\Pi_1\right]$$

for the transition matrix Π_1 *with*

$$\pi_1(y|s) = \frac{\beta_0}{\beta_1}\frac{1-\beta_1}{1-\beta_0}\pi_0(y|s)$$

for all $y, s \in Y$ *with* $y \neq s$.

Proof.

Direct computation proves the lemma: The off-diagonal elements of $[I - \beta_1\Pi_1]$ equal

$$\begin{aligned} -\beta_1\pi_1(y|s) &= -\beta_1\frac{\beta_0}{\beta_1}\frac{1-\beta_1}{1-\beta_0}\pi_0(y|s) \\ &= \frac{1-\beta_1}{1-\beta_0}(-\beta_0\pi_0(y|s)) \end{aligned}$$

and the diagonal elements equal

$$\begin{aligned} 1-\beta_1(1-\sum_{y\neq s}\pi_1(y|s)) &= 1-\beta_1(1-\sum_{y\neq s}\frac{\beta_0}{\beta_1}\frac{1-\beta_1}{1-\beta_0}\pi_0(y|s)) \\ &= 1-\beta_1+\frac{1-\beta_1}{1-\beta_0}\beta_0\sum_{y\neq s}\pi_0(y|s) \\ &= \frac{1-\beta_1}{1-\beta_0}(1-\beta_0+\beta_0\sum_{y\neq s}\pi_0(y|s)). \quad \square \end{aligned}$$

Proof of the Proposition.

By the first welfare theorem individual consumptions in an Arrow-Debreu equilibrium solely depend on the exogenous shock y. Therefore, the (necessary) first-order conditions for agents' optimality imply that Arrow-Debreu price for consumption at node σ_t only depend on the exogenous shock as well as the probability of σ and on t. Hence, there exist S prices $p_1, ..., p_S$ collinear to equilibrium marginal utilities $u'(c_1), ..., u'(c_S)$ such that an agent's budget constraint for all states $s = 1, ..., S$ can be written as

$$[I-\beta\Pi]^{-1}\begin{pmatrix} p_1(c_1-e_1) \\ \vdots \\ p_S(c_S-e_S) \end{pmatrix} = 0$$

If p_s remain constant agents' budget constraints do not change if $[I-\beta\Pi]^{-1}$ is multiplied by a positive number. Thus, the old equilibrium allocation remains feasible at the old equilibrium prices. But it is clear from the agents' first-order conditions that it also remains optimal. $\square$

In order to illustrate the proposition, consider the specification from Section 3 above. With only two states, the π-persistence of the income shock has to increase with β in the following way.

$$\pi_{11}(0.95) = 0.75,\ \pi_{11}(0.99) = 0.9520,\ \pi_{11}(0.996) = 0.9809 \text{ and } \pi_{11}(0.999) = 0.9952.$$

Note that Lemma 1 implies that for β converging to one $\pi(y|y)$ also converges to one for all shocks y, and $\pi(y|s)$ converges to zero for all $y \neq s$. The resulting economies then have the property that the complete-markets allocation remains constant. With such an increase of π-persistence one would expect the convergence of incomplete-markets welfare to the complete markets welfare to be at least slower. As it turns out, (even when there is no aggregate uncertainty) there will be no convergence and λ^h remains almost constant.

An increase in β with the associated increase in π-persistence changes each agent h's utility in the Arrow Debreu equilibrium by $\frac{1-\beta_0}{1-\beta_1}$. It is easy to verify that the ratio of autarky welfare to complete markets welfare does not change by such a simultaneous change in β and Π. By the recursive structure of the economy the autarky welfare for the S possible shocks is given by

$$\begin{pmatrix} U_A^h(1) \\ \vdots \\ U_A^h(S) \end{pmatrix} = [I - \beta\Pi]^{-1} \begin{pmatrix} u_h(e^h(1)) \\ \vdots \\ u_h(e^h(S)) \end{pmatrix}.$$

By Lemma 1, when β increases and Π changes accordingly, $(I - \beta\Pi)$ is multiplied by $(1-\beta_0)/(1-\beta_1)$ and the ratio of complete markets utility to autarky utility does not change.

In order to determine the change in the incomplete-markets utility and in λ we compute an equilibrium for several example economies. We fix the idiosyncratic shock to the Heaton and Lucas specification (i.e. $e^h \in \{3.77, 6.23\}$) and the coefficient of relative risk aversion to 1.5. We consider the case without aggregate uncertainty as well as the two examples with aggregate uncertainty from Section 3 above. The small aggregate shock is approximately 6 percent, the large is approximately 20 percent. For $\beta = 0.95$ we fix Π as in Section 3 above. Table 5 shows the changes of welfare for the three cases we consider.

β	no aggregate uncertainty		small		large	
0.95	12.4901	13.1238	12.7457	13.5967	14.6742	19.0034
0.99	12.4884	13.1231	16.3286	17.6131	22.5528	25.8788
0.996	12.4723	13.1235	16.4566	18.1582	22.9010	27.1338
0.999	12.4911	13.1375	16.5692	18.4513	23.9440	27.5862

Table 5: No convergence with increased persistence.

In all three examples, there is no convergence to the complete markets welfares. On the contrary, for the case without any aggregate uncertainty, the welfare losses λ remain almost constant. We do not know whether the welfare losses stay exactly constant since the small variations of the numbers could be due the approximation of the implicit debt constraint by the term $1/(1-\beta)$. In the presence of aggregate uncertainty, the welfare loss increases as β increases.

4.1 The Impact of Persistence

It seems quite surprising that without aggregate uncertainty, the welfare losses remain almost constant. This result is robust with respect to preferences and shocks: we considered larger idiosyncratic shocks and different risk aversions as in Section 3 and in all cases the changes in λ turn out to be insignificant.

In order to obtain an economic intuition for this result it is helpful to first analyze the impact of increased persistence in the presence of a constant discount factor. In such a case the agents change their savings behavior when the probability of repeated negative shocks increases. Under such a scenario the agents realize that without a change in their savings policy the probability of hitting their borrowing constraint is also increasing. Therefore, as soon as they are hit by a bad shock, they reduce their consumption in order to reduce the amount they have to borrow. As a consequence the agents' welfare in the incomplete markets economy is decreasing.

Now, if in addition to the increase in persistence also the discount factor increases, the agents face two opposing effects. First, the increase in the discount factor leads to a smaller interest rate which in turn leads to a loosening of the implicit debt constraint. Secondly, as described in the previous paragraph, the increased persistence of negative shocks leads the agents to more cautious borrowing. These two effects more or less cancel each other out resulting in the observed constant welfare losses in Table 5.

4.1.1 A partial equilibrium intuition

This interpretation becomes plausible if we consider an individual agent's maximization problem in an economy without aggregate uncertainty and with $q = \beta$ and investigate how his welfare changes as Π and β change. It is easy to see that in this case the implicit debt constraint implies that $\theta > \frac{1}{1-q}$, because if the agent borrows more than $\frac{1}{1-q}$ it is impossible to maintain positive consumption and pay back his debt in finite time.

Consider now two optimization problems, one with β_0, Π_0 and price $q_0 = \beta_0$ and one with $\beta_1 > \beta_0$, price $q_1 = \beta_1$ and Π_1 calculated according to Equation (1).

Consider the agent's problem[1]

$$\max_{\theta,c} U(c) \text{ s.t. } c(\sigma) = e(\sigma) + \theta(\sigma^*) - \theta(\sigma)q_1 \text{ and } \inf_{\sigma\in\Sigma} q(\sigma)\theta(\sigma) > -\infty$$

Under the Markovian structure, with time-separable utility, it is well known how to formulate this problem as a dynamic programming problem. In particular, the optimal choice at node σ will be a function of the current shock y and last periods bond holding $\theta(\sigma^*)$. There exists a differentiable value function V_1 such that the Bellman equation

$$V_1(\theta_-, y) = \max_{\theta\in I\!R} u(e + \theta_- - q_1\theta) + \beta_1 \sum_{s\in Y} \pi_1(s|y)V_1(\theta, s) \tag{2}$$

is satisfied for all $\theta > \frac{1}{1-\beta_1}$. Let $\theta_1(\theta_-, y)$ denote the optimal policy function for β_1, Π_1 and let $c_1(\theta_-, y)$ denote the associated optimal consumption. Define $\tilde{\theta}(\theta_-, y) = \frac{1-\beta_1}{1-\beta_0}\theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)$. This is certainly a feasible trading strategy (i.e. does not violate the implicit debt constraint) under q_0 if θ_1 is feasible under q_1. Let $\tilde{c}(\theta_-, y)$ be the consumption induced by $\tilde{\theta}(\theta_-, y)$ given price q_0. Let $D(\theta_-, y)$ denote the difference between the optimal consumption under β_1 given portfolio $\frac{1-\beta_0}{1-\beta_1}\theta_-$ and $\tilde{c}(\theta_-, y)$.

Substituting the budget constraints and using $\beta_i = q_i$ we obtain

$$D(\theta_-, y) = \theta_- \cdot (\frac{1-\beta_0}{1-\beta_1} - \frac{1-\beta_1}{1-\beta_1}) - \beta_1\theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y) + \beta_0\frac{1-\beta_1}{1-\beta_0}\theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)$$

Simplifying this expression we obtain

$$D(\theta_-, y) = \frac{\beta_1 - \beta_0}{1-\beta_0}(\frac{1-\beta_0}{1-\beta_1}\theta_- - \theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)).$$

Define a function $\tilde{V}(\theta_-, y)$ by

$$\tilde{V}(\theta_-, y) = u(\tilde{c}(\theta_-, y)) + \beta_0 \sum_{s\in Y} \pi_0(s|y)\tilde{V}(\tilde{\theta}(\theta_-, y), s) \tag{3}$$

Clearly the agents utility under π_0, β_0 is not less than $\tilde{V}(0, y)$. However, we will show that

$$\epsilon(\theta_-, y) := \tilde{V}(\theta_-, y) - \frac{1-\beta_1}{1-\beta_0}V_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y) \approx 0$$

which implies that the agents utility under β_0, Π_0 cannot be substantially smaller than under β_1, Π_1.

Since

$$\epsilon(\theta_-, y) = \tilde{V}(\theta_-, y) - V_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y) + \frac{\beta_1 - \beta_0}{1-\beta_0}V_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)$$

[1]In this subsection we drop the agent specific superscripts to simplify notation.

we can use Equation (2), substitute for π_1 and obtain

$$\begin{aligned}\epsilon(\theta_-, y) &= u(\tilde{c}(\theta_-, y)) - u(c_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)) + \frac{\beta_1 - \beta_0}{1-\beta_0}(V_1(\theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)) \\ &\quad - V_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y)) + \beta_0 \sum_s \pi_0(s|y)\epsilon(\theta, s)\end{aligned}$$

Since, by the envelope theorem $V_1'(., y) = u'(.)$, a first-order Taylor expansion implies that for small changes in portfolios

$$\epsilon(\theta_-, y) \approx u'(c_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y))(D(\theta_-, y) - \frac{\beta_1 - \beta_0}{1-\beta_0}(\frac{1-\beta_0}{1-\beta_1}\theta_- - \theta_1(\frac{1-\beta_0}{1-\beta_1}\theta_-, y))) + \beta_0 \sum_s \pi_0(s|y)\epsilon(\theta, y)$$

By the definition of $D(., y)$ this implies that $\epsilon(\theta_-, y) \approx 0$.

The second order Taylor-terms will generally not cancel and depending of the curvature of the utility function they could be non-negligible. Moreover, in general equilibrium, the price q is not constant across all nodes and will always lie above β. However, the computational examples show that the changes in welfare are very small for many realistically calibrated examples.

4.1.2 Aggregate Uncertainty

From the calculations in Section 3 above, one would expect that in a model with aggregate uncertainty the welfare losses should increase with β when π-persistence increases. However, the decrease in the incomplete market's welfare is small because the volatility of exchange rates decreases as the π-persistence increases. While for $\beta = 0.95$ the probability of a good aggregate shock, given a bad aggregate shock is 0.5 it drops to 0.096 for $\beta = 0.99$ and to 0.0382 for $\beta = 0.996$.

Therefore the lower bound on the bond price will converge to β as $\beta \to 1$ and agents can borrow more and more in order to self-insure against bad aggregate or individual shocks. For β close to one, an economy with aggregate uncertainty is similar to an economy without and agents' welfare does not steadily decrease as β and the associated π-persistence increase.

5 An Explicit Debt Constraint

Although we have argued so far that a discount factor close to one must mean that the economy is calibrated to high-frequency data and that therefore the π-persistence of negative income shocks has to be very high as well, it is of independent interest to investigate under which conditions the incomplete markets welfares converge to the complete markets welfare for a sequence of economies where only the discount factor changes. For example, one could argue that the real yearly interest-rate is not much higher than 1 percent and that therefore even for a model which is calibrated to yearly data, β should be more around 0.99 than 0.95; it is then important to understand the welfare consequences of such an argument.

Recall our explanation why even a little aggregate uncertainty destroys the convergence result. As Levine and Zame (1998, 2000) show the equilibrium bond price will lie above β whenever the Bernoulli function exhibits a convex first derivative (as it is the case for CRRA utility). As β increases, the bond price converges to one. The implicit debt constraint then implies that agents can take on more and more debt (for $q(\sigma) \geq 1$ for all $\sigma \in \Sigma$, the implicit debt constraint is meaningless since agents' debt can become arbitrarily large and with zero interest rate they can still repay their debt in finite time). However, with aggregate uncertainty, the interest rate in a bad aggregate state will remain bounded away from 0 and the implicit debt constraint remains to have bite.

Without aggregate uncertainty the same phenomenon can be achieved by introducing an explicit debt constraint which does not explode as β converges to one. As mentioned in the previous section, one must impose a restriction on portfolio strategies in order to rule out Ponzi schemes. Choosing an implicit as opposed to an tight explicit debt constraint has important effects on welfare.

We examine an explicit debt constraint

$$q(\sigma)\theta^h(\sigma) \geq -B \quad \text{for all } \sigma \in \Sigma, \tag{4}$$

for some positive number B. While this borrowing constraint forbids agents to enter into a Ponzi scheme it clearly introduces a market imperfection. We cannot eliminate the possibility that in equilibrium the debt constraint for agent h is binding for some $\sigma \in \Sigma$ and thereby altering the nature of the equilibrium.

For $\beta = 0.99$ the resulting interest rate lies around 1 percent. For the case of an implicit debt constraint, an agent whose worst endowment is $\underline{e}^h$ is allowed to borrow up to $100 \cdot \underline{e}^h$ each period. If the length of a period is taken to be one year (i.e. the Heaton and Lucas calibration for the idiosyncratic shock is assumed to be realistic) this implies that agents borrow up to 100 times their yearly income without any collateral. This assumption appears to be very extreme. In Table 3 we document how the welfare loss of incomplete markets increases as the borrowing constraint becomes more realistic for the small shock $e^h = (3.77, 6.23)$ as well as for the large shock $e^h = (2, 8)$. An explicit debt constraint x is taken to imply that the agent is allowed to borrow up to x times his worst state endowments. For example, for $x = 50$ and the small shock, the agent is allowed to borrow $50 \cdot 3.77$. The transition matrix is taken from Section 3 above, $\gamma^h = 1.5$ and $\beta = 0.99$.

x	small shock		large shock	
IDC ($\approx$ 100)	2.8714	2.8751	2.2266	2.7610
50	2.9207	3.2375	2.4799	3.5703
10	3.0999	3.3167	5.6444	7.3962
5	3.8048	5.1845	11.8619	13.7480
4	4.6584	6.2742	14.7715	16.6311
3	6.2742	8.1511	19.1300	20.9157
2	9.7645	11.8943	26.3647	27.9866
1	19.6430	21.8663	40.3742	41.6489

Table 6: Welfare impact of an explicit debt constraint.

Table 6 shows how welfare losses increase as the debt constraint becomes tighter. Surprisingly the increase is insignificant when the amount the agent is allowed to borrow decreases by 50 percent from the initial implicit debt constraint. For the case of the small shock agents can smooth out most of their bad shock even if they are only allowed to borrow up to 10 times their worst endowments. Only when the borrowing constraint becomes very tight does the welfare loss increase significantly.

6 Conclusion: Incomplete Markets Matter for Welfares

In this paper we have shown in the context of simple infinite-horizon models that the welfare of economic agents can be severely affected by the presence of market incompleteness. The differences between agents' welfare in incomplete and complete markets can be substantial.

We have also shown that welfare losses from incomplete markets do not always disappear when agents become extremely patient. First, when an economic model is calibrated to higher frequency data, the period persistence of shocks must increase as well. In the infinite-horizon model under discussion such a calibration results in almost constant welfare losses of incomplete markets as agents' rate of time preference converges to 1. Secondly, for a fixed specification of endowment processes, an exogenous decrease of agents' rate of discounting should not affect their abilities to borrow. With exogenous borrowing constraints, the incomplete markets welfare does not converge to the complete markets welfare.

References

[1] Aiyagari, S.R., Uninsured Idiosyncratic Risk and Aggregate Saving, *Quarterly Journal of Economics* 109 (1994), 659–684.

[2] den Haan, W., The importance of the number of different agents in a heterogeneous asset-pricing model, mimeo UCSD (1999).

[3] Heaton, J. and D.J. Lucas, Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing, *Journal of Political Economy* 104 (1996), 443–487.

[4] Huggett, M., The risk-free rate in heterogeneous agent incomplete-insurance economies, *Journal of Economic Dynamics and Control* 17 (1993), 953-969.

[5] Judd, K.L., F. Kubler, and K. Schmedders, Computing Equilibria in Infinite Horizon Finance Economies: The Case of One Asset, *Journal of Economic Dynamics and Control* 24 (2000), 1047–1078.

[6] Kubler, F. and K. Schmedders, Recursive equilibria with incomplete markets, forthcoming *Macroeconomic Dynamics*(2000).

[7] Levine, D.K. and W.R. Zame, Debt constraint and equilibrium in infinite horizon economies with incomplete markets, *Journal of Mathematical Economics* 26 (1996), 103–131.

[8] Levine, D.K. and W.R Zame, Risk Sharing and Market Incompleteness, in *The Theory of Markets*, P.J.J. Herings, G. van der Laan, A.J.J. Talman (eds.), 217–222, North-Holland, Amsterdam (1998).

[9] Levine, D.K. and W.R. Zame, Does Market Incompleteness Matter?, working paper, UCLA (2000).

[10] Lucas, R.E.Jr., Asset Prices in an Exchange Economy, *Econometrica* 46 (1978), 1429–1445.

[11] Lucas, R.E.Jr., *Models of Business Cycles*, Basil Blackwell, (1987).

[12] Magill, M. and M. Quinzii, Infinite Horizon Incomplete Markets, *Econometrica* 62 (1994), 853–880.

[13] Magill, M. and M. Quinzii, Infinite Horizon CAPM Equilibrium, *Economic Theory* 15 (2000), 103–138.

[14] Santos, M., Numerical Solutions of Dynamic Economic Models, ITAM discussion paper 9804 (1998).

[15] van Wincoop, E., How big are potential welfare gains from international risk sharing? *Journal of International Economics* 47, (1999), 109–134.

[16] Zhang, H.H., Endogeneous Borrowing Constraints with Incomplete Markets, *Journal of Finance* 52 (1997), 2187–2209.