Serial Cost Sharing in Multidimensional Contexts*

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First version: May 2001

This version: May 2002

Accepted for publication in
Mathematical Social Sciences
as of April 26, 2002

*We thank Patrick González, Philippe Mahenc, Michel Moreaux, François Salanié, Yves Sprumont, Yuntong Wang, and specially three anonymous referees for their helpful comments. We also thank the Institut d'économie industrielle de l'Université des Sciences Sociales de Toulouse for its hospitality during the Fall 1999.

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1 Introduction

The Serial Cost Sharing Rule has received much attention since its introduction by Shenker (1990) and its extensive analysis by Moulin and Shenker (1992, 1994). It was originally conceived for problems where n agents request different quantities of a private good, the sum of which is produced by a single facility. This rule can be constructed from two ethical axioms: Equal Treatment of Equals (in terms of demands) and Independence of Larger Demands (a protection of small demanders against larger ones). It satisfies other interesting properties and has other characterizations as well. It is therefore natural to investigate whether this rule can be extended to more general problems while keeping similar properties.

Sprumont (1998) brings a partial answer to this question by proposing the Axial Serial Rule for the case where each agent requests a single specific good. Koster et al. (1998) propose a similar extension, the Radial Serial Rule, for the context where agents request several but homogeneous private goods. Both rules use stand alone costs as a basis to compare demands. As its name implies, the Radial Serial Rule uses intermediate demands that are constructed by changing the original demands in a proportional way.

None of the two problems considered by these authors is more general than the other. Our paper considers a more general context where each agent requests a list of goods that may be private, public, or specific to some agents and where aggregate demand is not necessarily the sum of individual demands. By allowing agents to have vectors of personalized demands, we generalize both the models of Sprumont (1998) and of Koster et al. (1998). Moreover, we admit more general paths along which demands may be scaled down to construct intermediate demands. This yields a rule that we call the Path Serial Rule.

The paper presents an analysis of this rule in the light of properties found in the literature on cost sharing rules. These properties are transposed or extended to the general context whenever possible. Otherwise, they are weakened by requiring that the predicate holds only for changes in the demands taking place along the specified paths. This is the case of the Serial Principle defined by Sprumont (1998) since it is shown that this principle is incompatible with the basic Equal Treatment of Equals. The weaker Path Serial Principle characterizes the Path Serial Rule together with the Equal Treatment of Equivalent Demands in terms of stand alone costs, a stronger form of Equal Treatment of Equals. The Path Serial Rule also satisfies properties similar in spirit to the ones that hold for the original Serial Rule. However, we show that a characterization in terms of other properties, as in Moulin and Shenker (1994), Sprumont (1998), and Koster et al. (1998) depends crucially on the fact that the Axial, the Radial and the Path Serial Rule use stand alone costs to compare demands.

The paper is organized as follows. In Section 2, we review some of the known results on serial cost sharing and its generalization, in particular those of Kolpin (1996), Sprumont (1998) and Koster et al. (1998). We also give an overview of the paper. The formulation of the problem and the main definitions are given in Section 3. The section ends with an example illustrating the importance of the more general form of the problems to be considered. The Path Serial Cost Sharing Rule is defined in Section 4. Three sets of properties that can be imposed on a cost sharing rule are presented and discussed in Section 5. The incompatibility result and the characterization of the Path Serial Rule in the general context are the object of Section 6. A brief conclusion follows as Section 7. Two more technical proofs appear in the last section.

2 Overview of the paper

With the original Serial Cost Sharing Rule, agents are ordered according to their demands. Then, the cost of producing n times the demand of agent 1, which is called an intermediate demand, is shared equally among all agents. In addition, agents 2 to n must bear equally the incremental cost of another intermediate demand in which the demand of agents 2 to n is increased to the level of the demand of agent 2. Next, the incremental cost of an intermediate demand, in which the demand of agents 3 to n is increased to the level of the demand of agent 3, is shared equally among agents 3 to n and so on until total demand is satisfied.

The two rules of Sprumont (1998) and Koster et al. (1998) consist in first ordering individual demands according to their stand alone costs. Next, a first intermediate demand is constructed by reducing demands of agents 2 to n along a ray or an axis down to the point where their stand alone costs is the same as for agent 1. A second intermediate demand is constructed by reducing demands of agents 3 to n down to the point where their stand alone costs is the same as for agent 2, etc. Finally, the serial formula is applied to the costs of these intermediate demands. The rules bear the names Axial or Radial because of the way in which individual demands are reduced to construct intermediate demands.

Rays are particular cases of increasing paths. In some circumstances, it may be natural to adjust all components of the demand of an agent along the ray to which it belongs. In others, this may not be appropriate. For technological considerations or simply as an expression of preferences, they may have to be adjusted in a non linear way. As pointed out by Koster et al. (1998), one can envisage other extensions of the Serial Rule using more general paths to scale the demands. This idea leads to the definition of the Path Serial Rule, which may be seen as an application of the original Serial Rule to a path reduction of the problem.

The original Serial Cost Sharing Rule has a simple characterization: it is the only cost sharing rule to satisfy Equal Treatment of Equals (ETE) and Independence of Larger Demands (ILD). Condition (ETE) requires that equal demands be treated identically while (ILD) requires that the cost share of an agent be independent of the size of the demands that are larger than his or hers. In more general contexts, these conditions do not have much bite since demands are not directly comparable. Sprumont (1998) proposes stronger forms of these conditions, which he calls respectively Symmetry (S) and the Serial Principle (SP). The idea behind Symmetry is that if the demands of two agents can be interchanged without altering total cost, then the two goods should be deemed comparable. Then, if the two agents request the same quantity of these two goods, they should be charged the same amount. A stronger property introduced by Koster et al. (1998) is Equal Treatment of Equivalent Demands (ETV), where demands are equivalent when they have the same stand alone cost.

(ILD) implies the ordering of agents according to their demands, which may be impossible in more general contexts. Sprumont's answer is to order agents according to the cost shares produced by the rule. Then, the Serial Principle says that an agent who pays less than another agent should not see her cost share change if this other agent increases his demand. Koster et al. (1998) define a weaker form of this property called the Radial Serial Principle (RSP), which says that an agent who pays less than another agent should not see her cost share change if this other agent increases his demand along the ray to which it belongs. In the more general context considered here, this property becomes the Path Serial Principle (PSP). Thus, the Axial, the Radial, and the Path Serial Rules are characterized by (ETV) together with (SP), (RSP), and (PSP) respectively.

Moulin and Shenker (1994) show that the original Serial Cost Sharing Rule enjoys other remarkable ethical and coherency properties. Among other results, it is characterized by the combination of Additivity, Separability (for separable cost functions), Free Lunch, and Fair Ranking. Additivity requires that a rule yields the same results, whether it is applied separately to different cost elements or to their sums. Separability says that if cost is a linear function of total demand, then it should be allocated proportionally to the demands. Free Lunch says that if the cost of an n-fold replication of an agent's demand is zero, so should be the cost share of this agent. Fair Ranking, also called No-Domination, says that the cost shares of agents should be ordered as their demands. It implies (ETE). While Separability, Free Lunch, and Fair Ranking can be transposed to the Path Serial Rule, this is not the case of Additivity. Indeed, Kolpin (1996) shows that there is no extension of the Serial Rule to a multidimensional context satisfying Scale Invariance and Additivity. Thus, none of the Axial, Radial, and Path Serial Rules satisfies Additivity.

The original Serial Rule is immune to arbitrary changes in the way output is measured. It satisfies a property introduced by Sprumont (1998) and called Ordinality, which says that arbitrary changes in the units in which output is measured should not affect cost shares. A weaker property is Scale Invariance, which prescribes the invariance of cost shares with respect to linear changes in the units. We extend Ordinality to the general context considered in this paper by requiring that the paths that belong to the specification of a problem be transformed with the demands and the cost function to give an equivalent problem. We also impose that demands that enter symmetrically in the cost function be transformed with the same function in order to preserve symmetry. The Path Serial Rule satisfies Ordinality thus defined.

Coming back to other characterizations, Sprumont (1998) shows that the Axial Serial Rule is the only cost sharing rule that satisfies Symmetry (S), Rank Independence of Irrelevant Agents (RIIA), Independence of Null Agents (INA), Ordinality (O), and the Serial Principle (SP). Condition (RIIA) imposes that the ranking of two agents' cost shares depends on their demands only. Put differently, a change in an agent's demand must not affect the interpersonal ranking of others' cost shares. (INA) states that agents with null demands can be entirely removed from a problem without altering the outcome for the others. This implies that agents with null demands pay zero.

Koster et al. (1998) assert that Sprumont's characterization of the Axial Serial Rule does not extend to the Radial Serial Rule in their homogeneous context. We reinforce their result by showing that (SP) and (ETE) are incompatible in this context. Since (ETE) is hardly a disputable requirement, (SP) must be weakened in some way. Therefore, Koster et al. define the Radial Serial Principle (RSP), by restricting (SP) to rays. We define the Path Serial Principle (PSP) in a similar way, by restricting (SP) to paths.

Koster et al. also have a characterization theorem, asserting that the Radial Serial Rule is the only cost sharing rule that satisfies (RSP), (INA), (RIIA), (ETE), and Radial Ordinality (RO), a weaker form or ordinality. We argue that there is an implicit assumption behind the theorems of Sprumont and of Koster et al.: stand alone cost is the proper basis for the comparison of heterogeneous demands. We show that, by choosing other basis, we get other rules that also satisfy all the conditions of their theorems. In the general context of this paper, these other rules and the Path serial rule satisfy (S), (RIIA), (INA), and (O) in addition to (PSP).

3 The Cost Sharing Problem

A cost sharing problem starts with a profile of demands, to which a cost function is applied. In some cases, as with serial cost sharing, demands may have to be scaled down to meet certain conditions. The cost sharing problem must thus be completed by a description of how this should be made. We address each of these elements in the next three subsections. Then, we examine special cases of this problem found in the literature and we present an example to illustrate the generality of our approach.

3.1 The demands

Throughout this paper, there is a fixed set of divisible *commodities* $K = \{1, \ldots, k\}$ and a fixed set of *agents* $N = \{1, \ldots, n\}$. The commodities may be goods, characteristics serving to describe needs, or specifications of a certain facility. A commodity may be specific to a particular agent or a subset of agents. This means that these agents are the only ones to be able to consume, use, or enjoy the commodity in question. Hence, they will be the only ones to demand positive quantities of this commodity. As for non specific commodities, they may be private or public or anything in between.

For each agent $i \in N$, let there be a positive integer $m_i \leq k$ and a one-to-one function $\ell_i : \{1, \ldots, m_i\} \rightarrow K$, specifying the list of commodities that may be requested by this agent. Next, let M_i be the range of ℓ_i, i.e. $M_i = \{\ell_i(1), \ldots, \ell_i(m_i)\}$. In plain words, M_i is the subset of commodities for which agent i may request a positive quantity. We assume that $K = \cup_{i=1}^n M_i$. Thus, for each commodity, there is at least one agent concerned by this commodity.

The demand of agent i is described by a vector $q_i \in \mathbb{R}_+^{m_i}$. The scalar q_{ih} is the demand of commodity $\ell_i(h) \in M_i$ by agent i. Let $M = \{M_1, \ldots, M_n\}$ with cardinality $m = \sum_{i=1}^n m_i \leq nk$. A *profile of demands* is an element $Q \in \mathbb{R}_+^m = \prod_{i=1}^n \mathbb{R}_+^{m_i}$. Given a subset $S \subset N$ and $Q \in \mathbb{R}_+^m$, let $Q^S \in \mathbb{R}_+^m$ be the vector obtained from Q by changing all components $q_j,\ j \in N \backslash S$, for components of 0.

3.2 The cost function

To continue the description of the problem, we assume that the agents jointly own a facility to jointly produce any list of commodities that are requested. The cost of producing a bundle $Y \in \mathbb{R}_+^m$ is $C(Y)$. A *cost function* $C : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ also induces n *stand alone cost functions* $c_i : \mathbb{R}_+^{m_i} \rightarrow \mathbb{R}_+$ defined by:

$$c_i(y_i) = C\left(Y^{\{i\}}\right) \ \forall i \in N$$

Given $y_i, y'_i \in \mathbb{R}_+^{m_i}$, let $y_i \ll y'_i$ mean $y_{ih} < y'_{ih},\ h = 1, \ldots, m_i$. We shall say that $c_i : \mathbb{R}_+^{m_i} \to \mathbb{R}_+$ is increasing if $y_i \ll y'_i$ implies $c_i(y_i) < c_i(y'_i)$. Thus, c_i is increasing if an increase in all components of y_i yields a cost increase.

Let $\mathbb{C}(m)$ be the class of continuous and non-decreasing functions $C : \mathbb{R}_+^m \to \mathbb{R}_+$ satisfying $C(0) = 0$ and whose induced functions $c_i,\ i = 1, \ldots, n$, are increasing. We shall work with this class of functions throughout the paper. Whereas we need the mild assumption that each c_i be increasing, we do not want to impose and we do not need that C be increasing. In other words, $Y \leq Y' \in \mathbb{R}_+^m$ and $y_i \ll y'_i$ for some i do not necessarily imply $C(Y) < C(Y')$. Indeed, C may be the result of a more or less complex aggregation and optimization procedure. Thus, it is not necessarily increasing in all its components as, for example, when some public goods are involved.

A function $C \in \mathbb{C}(m)$ is *symmetric in the components* i and j if $C(Y) = C(Y_{ij})\ \forall Y \in \mathbb{R}_+^m$ where Y_{ij} is the vector Y with the components i and j interchanged. This requires that $m_i = m_j$ but not necessarily $M_i = M_j$. A function $C \in \mathbb{C}(nk)$ is *symmetric* if it is symmetric in the components i and j $\forall i, j \in N$. For a symmetric function, we let $m_i = k\ \forall i$. Thus, $m = nk$.

Note that if $C \in \mathbb{C}(m)$ is symmetric in the components i and j, then $c_i(x) = c_j(x)\ \forall x \in \mathbb{R}_+^{m_i}$. Indeed, $c_i(x) = C\left(Y^{\{i\}}\right) = C\left(Y^{\{j\}}\right) = c_j(x)$ for any $Y \in \mathbb{R}_+^m$ such that $y_i = y_j = x$. The middle equality follows from the fact that the difference between $Y^{\{i\}}$ and $Y^{\{j\}}$ amounts to an interchange of the components i and j.

3.3 The paths

Serial cost sharing requires that larger demands be initially scaled down to a level equivalent to smaller ones. In some circumstances, it may be natural to adjust all components of the demand of an agent along the ray to which it belongs, i.e. proportionally. This is the method used in the Radial Serial Rule. In other circumstances, this may not be appropriate. As pointed out by Koster et al. (1998) in their Remark 3.7, one can envisage other extensions of the serial rule using more general paths to scale the demands. This is the idea developed in this paper. This approach requires that we add the rules according to which demands must be scaled, to Q and C in the definition of a cost sharing problem.

For each $i \in N$, we consider functions $h_i : \mathbb{R}_+^{m_i+1} \to \mathbb{R}_+^{m_i}$, which map each $y \in \mathbb{R}_+^{m_i}$ and $\tau \in \mathbb{R}_+$ onto a vector $h_i(y, \tau) \in \mathbb{R}_+^{m_i}$. Assume that $h_i(y, \cdot)$ is non-decreasing, increasing without bound in at least one component, and that for each $y \in \mathbb{R}_+^{m_i}$, there exists a $\tau' \in \mathbb{R}_+$ (necessarily unique) such that $h_i(y, \tau') = y$. Let $\mathcal{H}_i$ be the class of these functions. Then, $h_i(y, \mathbb{R}_+)$ is the path through y defined by $h_i(y, \cdot)$. Clearly, the class $\{h_i(y, \mathbb{R}_+) : y \in \mathbb{R}_+^{m_i}\}$

scans $\mathbb{R}_+^{m_i}$ since h_i is defined for each $y \in \mathbb{R}_+^{m_i}$. Finally, let $h_i^R : \mathbb{R}_+^{m_i} \backslash \{0\} \times \mathbb{R}_+ \to \mathbb{R}_+^{m_i}$ be defined by $h_i^R(y, \tau) = \tau y$. This function defines the ray through a $y \neq 0$ in $\mathbb{R}_+^{m_i}$. Index i may be dropped in the homogeneous case.

We do not impose that $h_i(y, 0) = 0$ and that $h_i(y, \cdot)$ be continuous and increasing in all components. However, given a function $C \in \mathbb{C}(m)$, we restrict ourselves to the class of functions $\mathcal{H}_i(c_i) \subset \mathcal{H}_i$ for which $c_i(h_i(y, \cdot))$ is continuous and increasing, with $c_i(h_i(y, 0)) = 0$. Since $c_i(0) = 0$ and since c_i is increasing, this implies that there is at least one null component in $h_i(y, 0)$. In words, a path starts on an axis but not necessarily at the origin. The cost of the bundle at the starting point is null and increasing thereafter. This definition of $\mathcal{H}_i(c_i)$ insures that for any $\alpha \in \mathbb{R}_+$, there is a unique τ_α such that $c_i(h_i(y, \tau_\alpha)) = \alpha$.

Let $\mathcal{H}(C) = \mathcal{H}_1(c_1) \times \cdots \times \mathcal{H}_n(c_n)$, $H(Y, \tau) = (h_1(y_1, \tau_1), \ldots, h_n(y_n, \tau_n))$, and $\mathbb{C}(m) \times \mathcal{H} = \{(C, H) : C \in \mathbb{C}(m) \text{ and } H \in \mathcal{H}(C)\}$. A *cost sharing problem* is a triple $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$. Accordingly, a *cost sharing rule* is a mapping $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfying the budget balance condition:

$$\sum_{i=1}^{n} \xi_i(Q, C, H) = C(Q)$$

The vector $\xi(Q, C, H)$ is the list of cost shares for the problem (Q, C, H).

We assume that H is exogenous as is the case of Q. The choice of h_i may come from agent i, be imposed by the planner or be negotiated between all those concerned. The criteria leading to the adoption of a particular h_i may include technological considerations or preferences. For example, the different components of q_i may pertain to different technical characteristics of a facility and for technological reasons that only agent i knows, any change in q_i should be done according to a function h_i (not necessarily linear) supplied by the agent. Alternatively, h_i may be the expression of a preference by the agent. In the example given below, each agent has a two-component demand, gas in summer and gas in winter. If these demands are to be reduced, some agents may prefer a reduction of gas available in summer rather than a proportional reduction of both. Others may have different desiderata.

3.4 Special cases

With some abuse of notation, we write $M_i = \{i\}$ $\forall i$ to describe the case where each agent requests a quantity of a single *specific* or *personalized* good as in Sprumont (1998). This implies $M = K = N$. In this case, a problem may be written as a pair $(Q, C) \in \mathbb{R}_+^n \times \mathbb{C}(n)$.

At the other end of the spectrum, when $M_i = K$ $\forall i$ and $C(Y) = c\left(\sum_i y_i\right)$ with $c : \mathbb{R}_+^k \to \mathbb{R}_+$, all commodities are homogeneous and private. Following Moulin and Shenker (1994) and Sprumont (1998), we call these functions and the resulting problems *homogeneous.* If in addition, $h_i = h^R \ \forall i$ as with the Radial Serial Rule, then a problem may be written as a pair $(Q, C) \in \mathbb{R}_+^{nk} \times \mathbb{C}(nk)$.

Moulin-Shenker	$k = 1$	$M_i = K \ \forall i$	$C(Y) = c(\sum_i y_i)$
Koster et al. (1998)		$M_i = K \ \forall i$	$C(Y) = c(\sum_i y_i)$
Sprumont (1998)	$k = n$	$M_i = \{i\} \ \forall i$	

Table 1: Special cases

Table 1 summarizes the different features of problems considered in the literature thus far. One can see that none of the two problems considered by Sprumont (1998) and Koster et al. (1998) is more general than the other. Koster et al. study homogeneous problems while Sprumont suppose only one commodity for each agent. By comparison, we allow $m_i > 1$ and $M_i \neq M_j \ \forall i, j$. This means that agents may have vectors of demands for goods that are specific to them. Thus, we generalize both the model of Sprumont and the one of Koster et al. Actually, we allow for any mix of private, public, and specific commodities. In addition, the h_i define paths that are not necessarily rays.

3.5 An example

We conclude this section with an example that illustrates the kind of problem that can fit in this general framework. Suppose there are three cities A, B and C that must be supplied with natural gas from point S. Thus, a pipeline must be built to link the three cities to the supply S. The possible links are represented in Figure 1. Thus, B could be fed directly from S or through A.

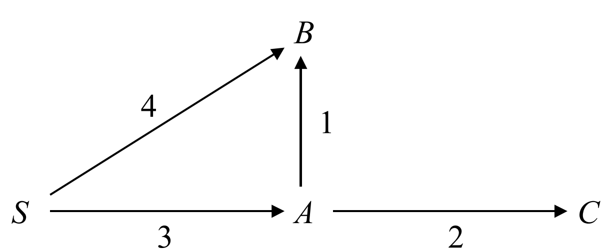

Figure 1: A Network of pipelines

Let there be two periods, summer and winter. Each city has a demand in each period and this demand is expected to remain the same forever. Thus, the profile of demands is a sextuple:

$$Q = ((q_{as}, q_{aw}), (q_{bs}, q_{bw}), (q_{cs}, q_{cw}))$$

Not only is gas in winter different from gas in summer but gas available in one city is different from gas available in a different city. Gas is a specific good. Indeed, supplying additional gas to A has an impact on costs that is different from the impact on costs of supplying the same quantity to B or C.

A network of pipelines may be represented by a $\gamma \in \mathbb{R}_+^4$ specifying the capacity of each of the four segments labelled $1, 2, 3, 4$. If we assume that the marginal cost of a segment is decreasing with its capacity and if cost is to be minimized, only one of segments 1 and 4 will be built. In other words, only two networks are possible: γ_1 with 0 capacity on the last segment and γ_2 with 0 capacity on the first segment. Capacity on each segment of each network depends on the profile of demands. In other words, γ_1 and γ_2 are functions of Q. More precisely, $\gamma_1 : \mathbb{R}_+^6 \to \mathbb{R}_+^4$ is defined by:

$$\gamma_1(Q) = (\max\{q_{bs},\ q_{bw}\},\ \max\{q_{cs},\ q_{cw}\},\ \max\{q_{as} + q_{bs} + q_{cs},\ q_{aw} + q_{bw} + q_{cw}\},\ 0)$$

Indeed, the capacities on segments 1 and 2 must be sufficient to carry the largest quantities required by B and C respectively over the two periods. Moreover, the capacity on segment 3 must be sufficient to carry the largest of the total quantity required by the three cities over the two periods. $\gamma_2 : \mathbb{R}_+^6 \to \mathbb{R}_+^4$ is defined in a similar way.

$$\gamma_2(Q) = (0,\ \max\{q_{cs},\ q_{cw}\},\ \max\{q_{as} + q_{cs},\ q_{aw} + q_{cw}\},\ \max\{q_{bs},\ q_{bw}\})$$

Suppose that the cost of building a network γ is given by a function $c : \mathbb{R}_+^4 \to \mathbb{R}_+$. Then, C would be given by:

$$C(Q) = \min\{c(\gamma_1(Q)), c(\gamma_2(Q))\}$$

Thus far, this problem fits neither the framework considered by Koster et al. (1998), since the goods (gas in A, B, or C) are specific to agents, neither the one of Sprumont (1998) since there is a vector of specific goods for each agent. To make the problem still more different, suppose that each city requires that scaling up, if needed, be done in a proportional way but that scaling down be done in a non proportional way. More precisely, in the latter case, the largest demand should first be reduced until it reaches the size of the smallest demand and then both demands should be reduced proportionally. The prescribed path is described by the following function $h_i : \mathbb{R}^3_+ \to \mathbb{R}^2_+$:

$$h_i(y,\tau) = \begin{cases} (\min\{\max\{\tau y_s, y_s\}, \tau y_w\},\ \tau y_w) & \text{if} \quad y_s \le y_w \\ (\tau y_s,\ \min\{\max\{\tau y_w, y_w\}, \tau y_s\}) & \text{if} \quad y_s > y_w \end{cases}$$

This function belongs to $\mathcal{H}_i$. One can insure that it belongs to $\mathcal{H}_i(c_i)$ by imposing that c_i be increasing, be it slightly, with respect to both y_{is} and y_{iw}.

4 The Path Serial Cost Sharing Rule

The original version of the Serial Cost Sharing Rule has been introduced by Shenker (1990) and characterized by Moulin and Shenker (1992, 1994) in the context where the individuals request a single private good, i.e. $k = 1$ and $C(Q) = c(\sum_i q_i)$. Before presenting an extension of this rule to the general context considered here, we give the definition of the Direct Serial Rule introduced by Sprumont (1998). This is simply the original serial rule applied to a problem $(Q,C) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$ that is not necessarily homogeneous. This direct rule will prove useful in assessing the properties of the Path Serial Rule.

Definition 1 (The Direct Serial Rule) Consider a problem $(Q,C) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$ where Q is naturally ordered, i.e. $q_1 \le \ldots \le q_n$. Then, consider the *intermediate demands* $Q^i = (q^i_1, \ldots, q^i_n) \in \mathbb{R}^n_+,\ i = 1, \ldots, n$, defined by $q^i_j = \min\{q_i, q_j\}\ \forall j \in N$. The *Direct Serial Rule* $\xi^{DS} : \mathbb{R}^n_+ \times \mathbb{C}(n) \to \mathbb{R}^n_+$ is defined by:

$$\xi^{DS}_i(Q,C) = \sum_{j=1}^{i} \frac{C(Q^j) - C(Q^{j-1})}{n+1-j},\ i = 1, \ldots, n.$$

In the context of Moulin and Shenker, $C(Q^1) = c(nq_1)$, $C(Q^2) = c(q_1 + (n-1)q_2)$, $C(Q^3) = c(q_1 + q_2 + (n-2)q_3)$, and so on. Thus all agents share equally the cost $c(nq_1)$ of a list of identical demands $(q_1, \ldots, q_1)$. Then, agents $2, \ldots, n$ shares equally $c(q_1 + (n-1)q_2) - c(nq_1)$, i.e. the cost increase when the demand is changed from $(q_1, \ldots, q_1)$ to $(q_1, q_2, \ldots, q_2)$, and so on. Note that in this context, $q_1 \le \ldots \le q_n$ is equivalent to $c_1(q_1) \le c_2(q_2) \le \ldots \le c_n(q_n)$. This is not so for a more general problem $(Q,C) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$, since different agents may request different commodities. Actually, the order between agents may depend on the units in which these demands are expressed and thus, the cost shares may depend on this choice. This is certainly something that we want to avoid. In addition, with heterogeneous commodities, $(q_1, \ldots, q_1)$ is not necessarily a vector of identical demands. Thus, there is no point in defining intermediate demands in this way. Moreover, this would not work in the general context where the number of commodities may be different from one M_i to the other.

The Path Serial Rule that we now define takes care of these particularities. In essence, it consists in first ordering individual demands according to their stand alone costs. Next, a first intermediate demand is constructed by reducing demands of agents 2 to n along the respective paths specified by the h_i, down to the points where their stand alone costs are the same as for agent 1. A second intermediate demand is constructed by reducing demands of agents 3 to n, along the same paths, down to the point where their stand alone costs are the same as for agent 2, etc. Finally, the formula of the Direct Serial Rule is applied to the costs of these intermediate demands.

Definition 2 (The Path Serial Rule) Given a problem $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$, suppose, without loss of generality, that agents are ranked according to their $c_i(q_i)$:

$$c_1(q_1) \leq c_2(q_2) \leq \ldots \leq c_n(q_n).$$

Then, for each i, consider the *intermediate demand* $Q^i = (q_1^i, \ldots, q_n^i) \in \mathbb{R}_+^m$ defined by:

$$\begin{cases} q_j^i = q_j & \text{if} \quad c_j(q_j) \leq c_i(q_i) \\ q_j^i \in h_j(q_j, \mathbb{R}_+) \text{ and } c_j\left(q_j^i\right) = c_i(q_i) & \text{if} \quad c_j(q_j) > c_i(q_i) \end{cases}$$

By definition of $\mathcal{H}(C)$, these intermediate demands are uniquely defined. Finally, the cost allocation of the *Path Serial Rule* is given by the following formula:

$$\xi_i^{PS}(Q, C, H) = \sum_{j=1}^{i} \frac{C(Q^j) - C(Q^{j-1})}{n+1-j}, \ i = 1, \ldots, n.$$

Remark 1 The Radial Serial Rule ξ^{RS} of Koster et al. (1998) is the Path Serial Rule ξ^{PS} with the use of h_i^R as the scaling function for any i and any pair $(Q, C) \in \mathbb{R}_+^m \times \mathbb{C}(m)$. In short, $\xi^{RS}(Q, C) = \xi^{PS}\left(Q, C, H^R\right)$ where $H^R = \left(h_1^R, \ldots, h_n^R\right)$. Both ξ^{PS} and ξ^{RS} reduce to the Axial Rule ξ^A of Sprumont (1998) when $M_i = \{i\}$ $\forall i$ and all three reduce to the original serial rule in the context of the single private good. We say that they are Serial Extensions of the original Serial Rule.

Remark 2 One might question the important role assigned to the stand alone costs in the Path Serial Rule. From the point of view of agents, this may be the best criterion, at least under increasing returns, since any subset of agents is then guaranteed not to pay more on total than its stand alone cost. Yet, this is not the only way to order demands and to define equivalent demands. Koster et al. (1998) indicate that a more general binary relation could be used to do so. In line with their formulation, c_i could be replaced by any other function, say g_i, having similar properties. More precisely, let $G = (g_1, \ldots, g_n)$ be the vector of such functions. Then, a problem would be defined by a four-tuple (Q, C, H, G) where each h_i would now be requested to belong to the class of functions $\mathcal{H}_i(g_i) \subset \mathcal{H}_i$ for which $g_i(h_i(y, \cdot))$ is continuous and increasing, with $g_i(h_i(y, 0)) = 0$. The definition of the Path Serial Rule would be modified accordingly. Examples of modifications of the Path Serial Rule along this line are given in Section 6.

Moulin and Shenker (1992, 1994) show that the Serial Rule, i.e. ξ^{DS}, has interesting ethical and consistency properties in the context of the single private good. What can be said of the Path Serial Rule ξ^{PS}? More generally, does there exist a serial extension that possesses the same or similar interesting properties?

Before addressing this question, we associate with any problem $(Q, C, H) \in \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H}(C)$, a reduced problem of a particular interest in the following way. Let $c^{-1}_{h_i y} : \mathbb{R}_+ \to \mathbb{R}^{m_i}_+$, $i = 1, \ldots, n$, be defined by

$$c^{-1}_{h_i y}(\alpha) = h_i(y, \tau) : c_i(h_i(y, \tau)) = \alpha \tag{1}$$

and $c^H_Q : \mathbb{R}^n_+ \to \mathbb{R}_+$ by:

$$c^H_Q(x) = C\left(c^{-1}_{h_1 q_1}(x_1), \ldots, c^{-1}_{h_n q_n}(x_n)\right) \tag{2}$$

Finally, let $\breve{c}(Q) = (c_1(q_1), \ldots, c_n(q_n))$. The pair $\left(\breve{c}(Q), c^H_Q\right) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$ is a new cost sharing problem, which is normalized in the following sense.

Definition 3 A problem $(Q, C) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$ is said *normalized* if $\left(\breve{c}(Q), c^H_Q\right) = (Q, C)$. In particular, the problem $\left(\breve{c}(Q), c^H_Q\right) \in \mathbb{R}^n_+ \times \mathbb{C}(n)$ defined by (1-2) is normalized. We call it the *normalized path reduction* of (Q, C, H).

When $M_i = \{i\}$ $\forall i$, the function $c^{-1}_{h_i y}$ is simply the inverse c^{-1}_i and $\left(\breve{c}(Q), c^H_Q\right)$ is the normalized problem of Sprumont (1998). We now state, without proof, a very important lemma.

Lemma 1 *The Path Serial Rule ξ^{PS} is given by:*

$$\xi^{PS}(Q, C, H) = \xi^{DS}\left(\breve{c}(Q), c^H_Q\right)$$

i.e. by applying the Direct Serial Rule to the normalized path reduction of the problem (Q, C, H).

The last lemma says that ξ^{PS} consists in applying ξ^{DS} to a problem in which each demand is replaced by its stand alone cost as with ξ^{A}. However, there is more to it than just a transformation of vectors of quantities into a scalar. The definition of the cost function that applies to the reduced demands involves a projection of each demand onto a manifold of dimension one, i.e. a path.

Not surprisingly, there is a cost associated with this reduction, even in the homogeneous case. As we shall see, ξ^{PS} does not satisfy all of the properties that ξ^{DS} meets in the single specific good context of Sprumont or the single private good context of Moulin-Shenker. However, since ξ^{PS} is tantamount to applying the Axial Rule to the normalized path reduction of the problem, it satisfies a restriction of some of these properties to the paths along which the rules operates.

5 Properties of the Path Serial Rule

This section is devoted to the presentation and discussion of three sets of properties. The first one includes a strong form of scale invariance called Ordinality, which is satisfied by the Path Serial Rule. In the other two, one finds the Equal Treatment of Equivalent demands and the Path Serial Principle, which together characterize the Path Serial Rule.

5.1 Ordinality

Almost everybody would agree with the requirement that final cost shares should not depend on the units in which demands are measured. In the literature on cost sharing and game theory, one often finds a condition called Scale Invariance, which says that linear but otherwise arbitrary and independent rescalings of the units should not affect cost shares. Sprumont (1998) argues that no rescaling of the units should affect cost shares, not only linear ones, a requirement which he calls ordinality. We adopt the same point of view and we transpose his definition to the general context of this paper. Since a problem is now defined as a triple, which include scaling functions h_i defining paths, we add the natural requirement that the latter be transformed by the same function that is applied to the units and the cost function to transform a problem into an equivalent one. We also add another natural requirement: the same transformation must be used for the demands of two agents if they enter symmetrically in the cost function. This is essential to preserve the symmetry of the cost function, without which some complications may arise.

Let $\mathcal{F}(m)$ be the class of separable, increasing and one-to-one correspondences $f : \mathbb{R}_+^m \to \mathbb{R}_+^m$. More precisely, each $f \in \mathcal{F}(m)$ is a list of m increasing one-to-one correspondences $f_{i\ell} : \mathbb{R}_+ \to \mathbb{R}_+,\ \ell = \ell_i(1), \ldots, \ell_i(m_i)\,;\ i = 1, \ldots, n$. Let $f(Y) = (f_1(y_1), \ldots, f_n(y_n))\ \forall Y \in \mathbb{R}_+^m$. We define two problems $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ and $(Q', C', H') \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C')$ as *ordinally equivalent* if there exists a transformation $f \in \mathcal{F}(m)$ such that:

- $\forall i, j \in N : f_i = f_j$ if C is symmetric in the components i and j,
- $Q = f(Q')$,
- $\forall i, \in N : h_i(q_i, \mathbb{R}_+) = f_i(h'_i(q'_i, \mathbb{R}_+))$ or equivalently $h_i(f_i(q'_i), \mathbb{R}_+) = f_i(h'_i(q'_i, \mathbb{R}_+))$
- $C'(Y) = C(f(Y))\ \forall Y \in \mathbb{R}_+^m$.

Under this definition, the demand of each commodity by each agent may be transformed by any increasing function. This function may be different from one commodity to the other and from one agent to the other, except when the cost function is symmetric for these two agents. However, separability requires that the transformation of each unit be done independently from the others. The path $h'_i(q'_i, \mathbb{R}_+)$ is also transformed into the path $h_i(q_i, \mathbb{R}_+)$ and the cost function C into C' in parallel with the transformation of q'_i into q_i.

Definition 4 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Ordinality* (O) if for any pair of ordinally equivalent problems $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ and $(Q', C', H') \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C')$, we have $\xi(Q, C, H) = \xi(Q', C', H')$.

We now give an example of two ordinally equivalent problems. Consider a problem $(Q, C, H) \in \mathbb{R}_+^4 \times \mathbb{C}(4) \times \mathcal{H}(C)$ with $q_1 = (1, 1)\,,\ q_2 = (1, 4)\,,$

$$C(Y) = 2y_{11} + y_{12} + y_{21} + y_{22}$$

and $h_i = h_i^R$ for $i = 1, 2$. Next, let $f_1 : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ be defined by $f_1(y_1, y_2) = (y_1, y_2)$ and $f_2 : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by $f_2(y_1, y_2) = (y_1, y_2^2)$. We obtain an equivalent problem $(Q', C', H') \in \mathbb{R}_+^4 \times \mathbb{C}(4) \times \mathcal{H}(C)$ by defining:

$$q'_1 = f_1^{-1}(q_1) = (1, 1)\,; \qquad q'_2 = f_2^{-1}(q_2) = (1, 2)\,;$$

$$C'(Y) = C(f(Y)) = 2y_{11} + y_{12} + y_{21} + y_{22}^2$$

H' must meet the condition $h_i(f_i(q'_i), \mathbb{R}_+) = f_i(h'_i(q'_i, \mathbb{R}_+))$ or equivalently $h'_i(q'_i, \mathbb{R}_+) = f_i^{-1}(h_i(f_i(q'_i), \mathbb{R}_+))\,,\ i = 1, 2$. Accordingly, we set:

$$h'_i(y, \tau) = f_i^{-1}(h_i(f_i(y), \tau))\ \ \forall \tau \in \mathbb{R}_+,\ i = 1, 2$$

Substituting the definitions of f_i and h_i in the preceding identity, we get $h'_1 = h_1$ and:

$$h'_2(y,\tau) = h_2\left(\tau y_1, \sqrt{\tau} y_2\right) \quad \forall \tau \in \mathbb{R}_+$$

While h_2 defines rays, this is not the case of h'_2.

Remark 3 If f is linear, then (O) reduces to the standard Scale Invariance. In particular, f transforms rays into rays. The above example shows that this is not necessarily the case with an arbitrary $f \in \mathcal{F}(m)$. Consequently, the Radial Serial Rule does not satisfy (O) since this rule operates along rays. Put differently, the requirement that rays be transformed into rays places some restriction on the class of admissible transformation functions. This restriction led Koster et al. (1998) to define a weaker invariance condition that they name Radial Ordinality.

More precisely, two problems (Q, C), $(Q', C') \in \mathbb{R}^m_+ \times \mathbb{C}(m)$ are said to be *ray-to-ray equivalent* if there exists a transformation $f \in \mathcal{F}(m)$ such that the prescriptions for ordinal equivalence hold with the third one replaced by:

- $\forall i, \in N : h_i^R(q_i, \mathbb{R}_+) = f_i\left(h_i^R(q'_i, \mathbb{R}_+)\right)$ or equivalently $h_i^R(f_i(q'_i), \mathbb{R}_+) = f_i\left(h_i^R(q'_i, \mathbb{R}_+)\right)$

Then, *Radial Ordinality* (RO) requires that the cost shares be the same for two ray-to-ray equivalent problems.

Interestingly, two ordinally equivalent problems have the same normalized path reduction. Thus, a cost sharing rule defined on the normalized path reduction of a problem, satisfies (O).

Lemma 2 *Given a pair of ordinally equivalent problems* $(Q, C, H) \in \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H}(C)$ *and* $(Q', C', H') \in \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H}(C')$, *we have* $\left(\check{c}(Q), c_Q^H\right) = \left(\check{c}'(Q'), c'^{H'}_{Q'}\right)$.

The proof is given in subsection 8.1. Combining Lemmas 1 and 2, we obtain the following corollary.

Corollary 1 *The Path Serial Rule* ξ^{PS} *satisfies (O).*

5.2 Equal Treatment of Equivalents Demands

The two essential features of the Serial Cost Sharing Rule introduced by Moulin and Shenker (1992) is the equal treatment of equal demands and the protection it offers to agents with small demands against larger ones. In the general context of this paper, demands are not necessarily comparable in terms of quantities. Sprumont (1998) and Koster et al. (1998) address this problem and propose reinforcements of the properties just mentioned. In this subsection, we review the definitions pertaining to the equal treatment of equivalent demands. The protection against larger demands follows in the next subsection. We start with a property called Fair Ranking or No-Domination, which implies equal treatment of equal demands for homogeneous problems.

Definition 5 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Fair Ranking* (FR) *if for any homogeneous problem* $(Q, C, H) \in \mathbb{R}_+^{nk} \times \mathbb{C}(nk) \times \mathcal{H}(C)$ *and* $i, j \in N$, the following holds:

$$q_i \leq q_j \Rightarrow \xi_i(Q, C, H) \leq \xi_j(Q, C, H)$$

Definition 6 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Equal Treatment of Equals* (ETE) if for any homogeneous problem $(Q, C, H) \in R_+^{nk} \times C(nk) \times \mathcal{H}(C)$ and $i, j \in N$, the following holds:

$$q_i = q_j \Rightarrow \xi_i(Q, C, H) = \xi_j(Q, C, H)$$

Koster et al. (1998) introduce a stronger condition, based on the notion of equivalent demands. A very natural criterion to order two individual demands is their stand alone cost. Thus, q_i and q_j are equivalent if $c_i(q_i) = c_j(q_j)$. This criterion yields the following reinforcement of the above two properties.

Definition 7 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Fair Ranking with respect to stand alone cost* (FRV) if for all $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ *and* $i, j \in N$, the following holds:

$$c_i(q_i) \leq c_j(q_j) \Rightarrow \xi_i(Q, C, H) \leq \xi_j(Q, C, H)$$

Definition 8 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Equal Treatment of Equivalents demands* (ETV) if for all $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ and $i, j \in N$, the following holds:

$$c_i(q_i) = c_j(q_j) \Rightarrow \xi_i(Q, C, H) = \xi_j(Q, C, H)$$

Sprumont (1998) uses the property of Symmetry instead, which can also be extended to the general context. Comparing the demands of two different agents does not in general make sense. It does make sense however if the two lists of commodities requested by the two agents are sufficiently similar. One circumstance in which the commodities requested by agent i can be declared similar to those requested by agent j is when the cost function is symmetric in the components i and j, i.e. $C(Y) = C(Y_{ij})\ \forall Y \in \mathbb{R}_+^m$.

Definition 9 A cost sharing rule $\xi : \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}^n_+$ satisfies *Symmetry* (S) if for all $(Q, C, H) \in \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H}(C)$ and $i, j \in N$ such that C is symmetric in the components i and j, the following holds:

$$q_i = q_j \Rightarrow \xi_i(Q, C, H) = \xi_j(Q, C, H)$$

Remark 4 (ETV) implies (S), which implies (ETE). The converse is not true in general. (ETE) does not imply (S) since a cost function may be symmetric without being homogeneous. Only for homogeneous problems are (S) and (ETE) identical. (S) does not imply (ETV) since we can have $c_i(q_i) = c_i(q_j)$ without having $q_i = q_j$.

Remark 5 The fact that ξ^{DS} satisfies (FR) on the class of normalized problems implies that ξ^{PS} satisfies (FRV). More generally, let a rule $\xi : \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}^n_+$ be defined from a rule $\xi^N : \mathbb{R}^n_+ \times \mathbb{C}(n) \to \mathbb{R}^n_+$ by $\xi(Q, C, H) = \xi^N\left(\check{c}(Q), c^H_Q\right)$. Then, ξ satisfies (FRV) if and only ξ^N satisfies (FR) on the class of normalized problems. As a corollary, ξ satisfies (ETV) if and only if ξ^N satisfies (ETE) on the class of normalized problems.

5.3 The Serial Principle

This subsection is devoted to the independence of the contributions of agents with small demands with respect to the size of larger demands. The original condition has been defined for the single private good case. We first extend the definition to the general context. However, this condition does not have much bite in this context since demands are not necessarily comparable. Sprumont (1998) proposes a more powerful condition called the Serial Principle. The latter being incompatible with Equal Treatment of Equals in the general context, it must be weakened. We propose the Path Serial Principle. The subsection ends with two related conditions called Rank Independence of Irrelevant Agents and Independence of Null Agents.

Definition 10 A cost sharing rule $\xi : \mathbb{R}^m_+ \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}^n_+$ satisfies *Independence of Larger Demands* (ILD) if for two cost sharing problems (Q, C, H) and $(Q', C, H) \in \mathbb{R}^{nk}_+ \times \mathbb{C}(nk) \times \mathcal{H}(C)$ such that $M_i = K\ \forall i$ and any $i \in N$ such that $q'_i = q_i$ and

$$\begin{aligned} q'_j &= q_j \quad \forall j \in N \backslash \{i\} : q_j < q_i \\ q'_j &\geq q_j \quad \forall j \in N \backslash \{i\} : q_i \leq q_j \end{aligned}$$

the following holds:

$$\xi_i(Q, C, H) = \xi_i(Q', C, H)$$

In the general context, (ILD) has no real content since demands cannot be compared. In the homogeneous case, things are better but the condition remains weak since the relation $\leq$ on $\mathbb{R}^k$ is not complete. To obviate this problem, Sprumont (1998) proposes that demands be ordered according to the cost shares produced by the cost sharing rule itself. This yields the Serial Principle, which requires that an agent's cost share be unaffected by increases in the demands of those who initially pay more than him.

Definition 11 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies the *Serial Principle* (SP) if for two cost sharing problems (Q, C, H) and $(Q', C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$, and any $i \in N$ such that $q_i' = q_i$ and

$$\begin{array}{rcll} q_j' & = & q_j & \forall j \in N \backslash \{i\} : \xi_j(Q, C, H) < \xi_i(Q, C, H) \\ q_j' & \geq & q_j & \forall j \in N \backslash \{i\} : \xi_i(Q, C, H) \leq \xi_j(Q, C, H) \end{array}$$

the following holds:

$$\xi_i(Q, C, H) = \xi_i(Q', C, H)$$

In the general context of this paper, (SP) is a very demanding condition. It is not necessarily satisfied by a serial extension. Actually, as shown in the next section, it is incompatible with (ETE), hence with (ETV), which is a basic property of the Path Serial Rule. Since (ETE) is a hardly disputable equity condition, the only avenue left is to weaken the Serial Principle (SP) into a less demanding condition. We consider a restriction of (SP) to paths.

Definition 12 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies the *Path Serial Principle* (PSP) if given two cost sharing problems (Q, C, H) and $(Q', C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$, any $i \in N$ such that $q_i' = q_i$ and

$$\begin{array}{ll} q_j' = q_j & \forall j \in N \backslash \{i\} : \xi_j(Q, C, H) < \xi_i(Q, C, H) \\ q_j \in h_j\left(q_j', \mathbb{R}_+\right) \text{ and } q_j' \geq q_j & \forall j \in N \backslash \{i\} : \xi_i(Q, C, H) \leq \xi_j(Q, C, H) \end{array}$$

the following holds:

$$\xi_i(Q, C, H) = \xi_i(Q', C, H)$$

Remark 6 From Lemma 1 and the fact that ξ^{DS} satisfies (SP) in $\mathbb{R}_+^n \times \mathbb{C}(n)$, we obtain that ξ^{PS} satisfies (PSP). Actually, the fact that ξ^{DS} satisfies (ILD) on the class of normalized problems in $\mathbb{R}_+^n \times \mathbb{C}(n)$ yields the same conclusion.

We conclude this section by transposing to the general context, two properties introduced by Sprumont (1998). The first imposes on a cost sharing rule that the ranking of two agents' cost shares depends on their demands alone. Thus, a change in an agent's demand must not affect the interpersonal ranking of the other cost shares. The second says that an agent with a null stand alone cost can be entirely removed from any problem without altering the outcome for the other agents. This implies of course that agents with null demands pay zero. Both are satisfied by the Path Serial Rule.

Definition 13 A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ satisfies *Rank Independence of Irrelevant Agents* (RIIA) if for two cost sharing problems (Q, C, H) and $(Q', C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ such that $q_i = q'_i$ and $q_j = q'_j$ for some $i, j \in N$, then:

$$\xi_i(Q, C, H) \leq \xi_j(Q, C, H) \Leftrightarrow \xi_i(Q', C, H) \leq \xi_j(Q', C, H)$$

Definition 14 Given a profile $Q \in \mathbb{R}_+^m$, let $\xi^{N\setminus\{i\}}$ be the restriction of ξ to the reduced profile $Q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$ and C_{-i} and H_{-i} be the restrictions of C and H respectively to Q_{-i}. A cost sharing rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n \times \mathcal{H}$ satisfies *Independence of Null Agents* (INA) if for any $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$ and any $i \in N$, the following holds:

$$c_i(q_i) = 0 \Rightarrow \xi_j^{N\setminus\{i\}}(Q_{-i}, C_{-i}, H_{-i}) = \xi_j(Q, C, H) \ \forall j \in N\setminus\{i\}$$

Remark 7 Note that the premise of the condition as defined by Sprumont, reads $q_i = 0$. Thus, our condition is slightly stronger than his in the general context of this paper.

Remark 8 (INA) implies that $\xi_i(Q, C, H) = 0 \ \forall i : q_i = 0$, a property called *no exploitation* by some authors. However (INA) says more. If an agent with a null stand alone cost is removed from the problem, this must not change the contributions of the remaining agents. This is a form of consistency. The Path Serial Principle implies the first part of (INA) but not the latter. Also note that (INA) implies Free Lunch, which Moulin and Shenker (1994) use to characterize the Serial Rule in the single private good context. It also implies another condition called Dummy in cooperative game theory, which says that if an agent does not affect the cost of any coalition that she might join, then her cost share must be zero.

6 The Main Results

Not surprisingly, paralleling the characterization of the original Serial Rule, the Path Serial Rule is the only cost sharing rule that satisfies Equal Treatment of Equivalents (ETV) and the Path Serial Principle (PSP). To justify the weakening of (SP) into (PSP), we first show that (ETE) and (SP) are incompatible in the general context. This means that (S) and (SP) are also incompatible. So are (ETV) and (SP), hence the use of (PSP) in the characterization theorem. To conclude the section, we discuss other characterizations of the Axial and the Radial Serial Rule given by Sprumont (1998) and Koster et al. (1998) respectively.

Theorem 1 *If $m_i \geq 2$ for at least one i, there does not exist a cost sharing rule that satisfies (ETE) and (SP).*

This result follows from the enlargement of the demand space alone. The proof is given in subsection 8.2.

Theorem 2 *ξ^{PS} is the only cost sharing rule that satisfies (ETV) and (PSP).*

It has been pointed out in Remarks 5 and 6 that ξ^{PS} satisfies (ETV) and (PSP). The proof of the converse follows from the definition of the rule and the axioms. Since it is now standard, it is omitted.

Remark 9 Combining Remarks 5 and 6, we get the following characterization. If a rule $\xi : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ is defined from a rule $\xi^N : \mathbb{R}_+^n \times \mathbb{C}(n) \to \mathbb{R}_+^n$ by $\xi(Q, C, H) = \xi^N \left(\breve{c}(Q), c_Q^H\right)$ and if ξ^N satisfies Independence of Larger Demands (ILD) and Equal Treatment of Equals (ETE) on the class of normalized problems, then $\xi = \xi^{PS}$.

Are there other characterizations of the Path Serial Rule? Sprumont (1998) shows that the Axial Rule $\xi^A : \mathbb{R}_+^n \times \mathbb{C}(n) \to \mathbb{R}_+^n$ is the only cost sharing rule that satisfies Ordinality (O), (SP), Independence of Null Agents (INA), Rank Independence of Irrelevant Agents (RIIA), and Symmetry (S) in the context where $M_i = \{i\}\ \forall i$. However, there is an implicit assumption behind this result: stand alone cost is the proper basis for the comparison of heterogeneous demands. This is an implication of choosing the normalized problem as the proper form to which to apply a given rule. But, why choose the normalized form? We show below that, by choosing to apply the Direct Serial Rule to a different member of the equivalence class of the problem, and there may be good reasons to do so, we get a different rule that satisfies all the conditions of Sprumont's Theorem.

Koster et al. (1998) have a similar theorem, which asserts that the Radial Serial Rule ξ^{RS} is the only cost sharing rule that satisfies the Radial Serial Principle (RSP), (INA), (RIIA), (ETE), and Radial Ordinality (RO). Clearly, Koster et al. assume that the stand alone cost of the demand of an agent is the proper aggregate of this demand. Thus, the Radial Serial Rule is defined by applying the Direct Serial Rule to the normalized reduced form of the problem. Again, why choose the normalized reduced form? For a different choice of the reduction, we get a different rule having the same properties.

To illustrate these points, we now present three rules along the lines that have just been suggested. The first is designed for the general context of this paper and is different from ξ^{PS}, and thus from ξ^{RS} and ξ^A, except for homogeneous problems. The second rule is defined specifically for homogeneous problems and the third for general problems, including homogeneous ones.

One can imagine the first rule, which we call ξ^{T_1}, in the following manner. Given a problem, the planner decides to apply multiplicative factors θ_i to the stand alone cost functions before using them. In all fairness, if the cost function is symmetric in the components i and j, she chooses $\theta_i = \theta_j$.

Given a problem $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$, let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}_{++}^n$ be such that $\theta_i = \theta_j$ if c_Q^H is symmetric in the components i and j. Then, let the functions $\hat{c}_i : \mathbb{R}_+^{m_i} \to \mathbb{R}_+,\ i = 1, \ldots, n$, be defined by $\hat{c}_i(y) = \theta_i c_i(y)$. Finally, ξ^{T_1} is defined in the same manner as ξ^{PS} except that agents are ordered according to their $\hat{c}_i(q_i)$:

$$\hat{c}_1(q_1) \leq \hat{c}_2(q_2) \leq \ldots \leq \hat{c}_n(q_n)$$

and the intermediate demands $\hat{Q}^1, \ldots, \hat{Q}^n$, are constructed using the functions $\hat{c}_i$ instead of c_i. Clearly, ξ^{T_1} is different from ξ^{PS} whenever $\exists i, j : \theta_i \neq \theta_j$.

Remark 10 Consider the problem $\left(\hat{c}(Q), \hat{c}_Q^H\right) \in \mathbb{R}_+^n \times \mathbb{C}(n)$ where $\hat{c}(Q) = (\hat{c}_1(q_1), \ldots, \hat{c}_n(q_n))$ and $\hat{c}_Q^H$ is obtained by replacing c with $\hat{c}$ in (1) and (2). It can be checked that $\left(\hat{c}(Q), \hat{c}_Q^H\right)$ is ordinally equivalent to $\left(\breve{c}(Q), c_Q^H\right)$, but it is not normalized. The rule ξ^{T_1} is actually defined by applying the Direct Serial Rule to this different path reduction of the problem:

$$\xi^{T_1}(Q, C, H) = \xi^{DS}\left(\hat{c}(Q), \hat{c}_Q^H\right)$$

Theorem 3 *The rule $\xi^{T_1} : \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H} \to \mathbb{R}_+^n$ is a serial extension that satisfies (O), (PSP), (INA), (RIIA), and (S).*

Proof. ξ^{T_1} is a serial extension since $\theta_i = \theta_j\ \forall i, j$ in the single private good case. It satisfies (S) by construction and (O) since all ordinally equivalent problems have the same normalized path reduced form $\left(\breve{c}(Q), c_Q^H\right)$ and hence the same path reduced form $\left(\hat{c}(Q), \hat{c}_Q^H\right)$. Since removing an agent from the problem does not change θ_j for the others, ξ^{T_1} inherits (INA) from ξ^{DS} as well as (RIIA) and (PSP). Recall that (PSP) is the restriction of (SP) to the paths $h_i(y, \mathbb{R}_+)$. ■

Corollary 2 *The Path Serial Rule ξ^{PS} satisfies (O), (PSP), (INA), (RIIA), and (S).*

Proof. ξ^{PS} is ξ^{T_1} with $\theta_i = 1\ \forall i \in N$. ■

Remark 11 ξ^{T_1} applies to the context of Sprumont (1998), where $M_i = \{i\}\ \forall i$. In this context, ξ^{T_1} satisfies (SP) and it is different from ξ^A.

We now define a rule, which we call ξ^{T_2}, for homogeneous problems. With this rule, the planner wishes to penalize in some way an abusive use of good 1 relative to good 2. Given a homogeneous problem $(Q, C, H) \in \mathbb{R}_+^{nk} \times \mathbb{C}(nk) \times \mathcal{H}(C)$, she selects a function $g \in \mathcal{F}(k)$, which presumably depends on the units in which demands are expressed, and she defines $N_1(Q) = \{i \in N : g_1(q_{i1}) > g_2(q_{i2})\}$ and $N_2(Q) = N \backslash N_1(Q)$, with $N_1(Q) = \emptyset$ if $k = 1$. Moreover, this partition is invariant over the class of ordinally equivalent problems: for two ordinally equivalent problems (Q, C, H) and (Q', C', H') with $q_i = f(q_i')$ $\forall i$ and $f \in \mathcal{F}(k)$, $N_1(Q') = N_1(Q)$. Put differently, if the formulation were to be changed for (Q', C', H'), then the planner would replace g by $g \circ f$. Then, let the functions $\hat{c}_i : \mathbb{R}_+^{m_i} \to \mathbb{R}_+,\ i = 1, \ldots, n$, be defined by:

$$\hat{c}_i(y) = \begin{cases} \gamma c_i(y) & \text{if } i \in N_1(Q) \\ c_i(y) & \text{if } i \in N_2(Q) \end{cases}$$

where γ is any positive real number. Otherwise, ξ^{T_2} is defined exactly as is ξ^{T_1}.

Theorem 4 *The rule $\xi^{T_2} : \mathbb{R}_+^{nk} \times \mathbb{C}(nk) \to \mathbb{R}_+^n$ is a serial extension that satisfies (O), (PSP), (INA), (RIIA), and (ETE) on homogeneous problems.*

Proof. ξ^{T_2} is a serial extension since $N_1(Q) = \emptyset$ in the single private good case. It satisfies (O) since the partition of N is preserved under an ordinal transformation and since all ordinally equivalent problems have the same path reduced form $\left(\hat{c}(Q), \hat{c}_Q^H\right)$. As ξ^{T_1}, it inherits (PSP), (INA), (RIIA), and (ETE) from ξ^{DS}. ∎

Remark 12 Since rays are particular cases of path, ξ^{T_2} satisfies (RO), (RSP), (INA), (RIIA), and (ETE) on homogeneous problems, showing that the Radial Serial Rule is not the only one to possess these properties.

The third rule, called ξ^{T_3}, applies to general problems. Given a problem $(Q, C, H) \in \mathbb{R}_+^m \times \mathbb{C}(m) \times \mathcal{H}(C)$, we now partition N into two subsets $N_1(Q, C, H)$ and $N_2(Q, C, H)$ according to the following rule:

$$\begin{aligned} N_1(Q, C, H) &= \{i \in N : c_i(h_i(q_i, \mathbb{R}_+)) = [0, \infty)\} \\ N_2(Q, C, H) &= N \setminus N_1(Q, C) \end{aligned}$$

Finally, we define the functions $\hat{c}_i : \mathbb{R}_+^{m_i} \to \mathbb{R}_+,\ i = 1, \ldots, n$, by

$$\hat{c}_i(y) = \begin{cases} \gamma c(y) & \text{if } i \in N_1(Q, C, H) \\ c(y) & \text{if } i \in N_2(Q, C, H) \end{cases}$$

where γ is any positive real number. From now on, the new rule ξ^{T_3} is defined exactly as ξ^T.

Clearly, ξ^{T_3} satisfies (O), (PSP), (INA), (RIIA), and (S). That ξ^{T_3} is different from ξ^{PS}, and thus from ξ^{RS} and ξ^{A}, can be seen by considering the following homogeneous problem $\left(Q, C, H^R\right)$, where $Q = ((3,3),(0,2))$, $H^R = \left(h^R, h^R\right)$, and $C(Q) = c(q_1 + q_2)$ with $c(y) = y_1 + a\left(1 - b^{-y_2}\right)$ and where a and b are finite numbers satisfying $a > 0$ and $b > 1$. Thus, recalling that $h^R(q_i, \mathbb{R}_+)$ is the ray through q_i, we have $c\left(h^R(q_1, \mathbb{R}_+)\right) = [0, \infty)$ and $c\left(h^R(q_2, \mathbb{R}_+)\right) = [0, a)$, from which $N_1\left(Q, C, H^R\right) = \{1\}$ and $N_2\left(Q, C, H^R\right) = \{2\}$.

7 Conclusion

We have shown that serial cost sharing can be extended to the general context where agents request several commodities that can be public, private, or specific to some of them and where aggregation may be very general. Actually, aggregation may be so general as to involve optimization. We have defined the Path Serial Rule to meet this kind of problem. As it names implies, it consists in scaling down the demands along paths that belong to the specification of the problem, in order to construct the intermediate demands that are at the root of serial cost sharing. Put differently, the Path Serial Rule consists in applying the original Serial Cost Sharing Rule to a projection of each demand onto the specified path.

We have been able to characterize the Path Serial Rule only by the Equal Treatment of Equivalents (demands) and the Path Serial Principle. Yet, the Path Serial Principle is considerably weaker than the corresponding property in the single good context. Indeed, this principle says something about how cost shares should behave when demands change along the specified paths. Anything can happen for other types of changes in the demands.

The Path Serial Rule satisfies other properties such as Independence of Null Agents, Rank Independence of Irrelevant Agents, and Ordinality. However, we have exhibited other rules that have the same properties. We have also pointed out that the characterizations of the Axial Serial Rule by Sprumont (1998) and of the Radial Serial Rule by Koster et al. (1998), in terms of similar properties, rely on the implicit assumption that stand alone costs are the proper numbers to compare heterogeneous demands, or the proper aggregates of multi-commodity demands.

In a companion paper (Téjédo and Truchon, 2002), we address two additional issues: monotonicity and bounds of cost shares. Moulin and Shenker (1994) prove that, under appropriate assumptions on the cost function, the original Serial Rule produces cost shares that are monotone with respect to own and others' demands and that lay between reasonable bounds. Moulin (1996) shows that it satisfies the Stand Alone Test under increasing returns, i.e. no subset of agents pay more than their stand alone cost. We transpose these results to the Path Serial Rule under an assumption of diminishing incremental cost. However, Monotonicity and Cross Monotonicity is restricted to hold along paths.

8 Proofs

8.1 Proof of Lemma 2

Let $f:\mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ be the ordinal transformation. Thus, we have $Q = f(Q')$, $h_i(q_i,\mathbb{R}_+) = f_i(h'(q_i',\mathbb{R}_+))\ \forall i$, and $C'(Y) = C(f(Y))\ \forall Y \in \mathbb{R}_+^m$. The latter implies

$$c_i'(y) = c_i(f_i(y)) \quad \forall y \in \mathbb{R}_+^{m_i}\ \forall i \in N$$

and:

$$c_i'(q_i') = c_i(q_i) \quad \forall i \in N$$

In short, $\breve{c}'(Q') = \breve{c}(Q)$. Next, we show that $c_{Q'}'^{H'} = c_Q^H$. Given a $x \in \mathbb{R}_+^n$ and a $i \in N$, let τ and τ' be two real numbers such that:

$$c_i(h_i(q_i,\tau)) = x_i = c_i'(h_i'(q_i',\tau')) = c_i(f_i(h_i'(q_i',\tau')))$$

Since the two problems are ordinally equivalent, $h_i(q_i,\mathbb{R}_+) = f_i(h_i'(q_i',\mathbb{R}_+))$, i.e. $h_i(q_i,\tau)$ and $f_i(h_i'(q_i',\tau'))$ are both on the path $h_i(q_i,\mathbb{R}_+)$. Since $c_i(h_i(q_i,\cdot))$ is an increasing function, $c_i(h_i(q_i,\tau)) = c_i(f_i(h_i'(q_i',\tau')))$ implies $h_i(q_i,\tau) = f_i(h_i'(q_i',\tau'))$. Using the latter with $c_{h_iq_i}^{-1}(x_i) = h_i(y,\tau)$, and $c_{h_i'q_i'}'^{-1}(x_i) = h_i'(q_i',\tau')$, we get:

$$\begin{aligned} c_{Q'}'^{H'}(x) &= C'\left(c_{h_1'q_1'}'^{-1}(x_1),\ldots,c_{h_n'q_n'}'^{-1}(x_n)\right) = C\left(f\left(c_{h_1'q_1'}'^{-1}(x_1),\ldots,c_{h_n'q_n'}'^{-1}(x_n)\right)\right) \\ &= C\left(c_{h_1q_1}^{-1}(x_1),\ldots,c_{h_nq_n}^{-1}(x_n)\right) = c_Q^H(x) \end{aligned}$$

8.2 Proof of Theorem 1

Consider the two homogeneous problems (Q,C,H) and $\left(\tilde{Q},C,H\right) \in \mathbb{R}_+^4 \times \mathbb{C}(4) \times \mathcal{H}(C)$ where $Q = ((2,1),(1,2))$, $\tilde{Q} = ((2,1),(2,3))$ and $C(Y) = (y_{11}+y_{21})^3 + (y_{12}+y_{22})^3$ and let ξ be a cost sharing rule that satisfying (ETE) and (SP). Suppose that $\xi_1(Q,C,H) \leq \xi_2(Q,C,H)$. The latter implies:

$$\xi_1(Q,C,H) \leq \frac{C(Q)}{2} = 27 \tag{3}$$

Next, consider the profile of demands $Q^1 = ((2,1),(2,1))$. By (ETE), we have $\xi_1(Q^1,C,H) = \dfrac{C(Q^1)}{2} = 36$. Thus, by (SP):

$$\xi_1(Q,C,H) = \xi_1\left(\tilde{Q},C,H\right) = \xi_1\left(Q^1,C,H\right) = 36 \tag{4}$$

We have a contradiction between (3) and (4).

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