TOWARD SUSTAINABILITY: TECHNOLOGY TRANSITION AND ENDOGENOUS POPULATION GROWTH

By

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In order to reach the state of economic sustainability, the problem of technology transition emphasizes the possibility of substituting for the exhaustible resource with an everlasting source of energy input. This paper aims at providing an analysis of this problem in an overlapping-generation model where the population is not a datum, but endogenous in the sense that it results from fertility decisions made by economic agents. First, we provide a new proof of the existence of competitive equilibrium under infinite time horizon. Here the difficulty lies in the fact that the market size is itself endogenous, because fertility – hence the population – is an individual decision at every point in time. Second, and perhaps most interestingly, the oil stock might not be entirely depleted, and the unused part *in situ* may serve the role of storing value for wealth transmission over time, just as money. But in contrast with paper money, which has no intrinsic value, leaving productive oil *in situ* as a bubble certainly adds another dimension to the inefficiency of overlapping-generation model. In this case, there are infinitely many equilibria as well as many steady states, depending on the data that characterize the initial state of the economy. Moreover, the convergence to some steady state, far from being simply monotone, might exhibit cyclical behavior, such as damped oscillation, limit cycles, etc.

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1. INTRODUCTION

WHILE THE ROLE OF EXHAUSTIBLE RESOURCES in economic growth has almost been thoroughly explored (see the well-known Review of Economic Studies 1974 Symposium), the question of endogenous population and fertility decision has only attracted attention – once again – quite recently (see Becker and Barro (1988), Barro and Becker (1989), Becker, Murphy, and Tamura (1990), Galor and Weil (1998), among others). To our knowledge, a synthesis of these two strands of literature has not been undertaken, despite the fact that it certainly contributes to our understanding of the real world. This is the task that we propose to accomplish in this paper.

In the first strand of literature, the size of population – assumed to be exogenously given – has seldom been a concern. The basic questions addressed by this strand of the literature are (i) how does the market allocate an exhaustible resource stock over time? and (ii) what is the time path of the resource price? This strand of the literature also attempted to provide an answer to the following two questions: (iii) is the market efficient in allocating the exhaustible resource over time? and (iv) what are the implications of resource exhaustibility in the context of economic growth? Some answers to these questions can be found in Dasgupta and Heal (1974, 1979). The answer to question (ii) is that the resource price rises at the rate of return for holding assets – the so-called Hotelling rule – and this would warrant allocation efficiency along the resource extraction path, with the resource being depleted asymptotically. As to the third question, which is raised in the dynamic framework, similar results are obtained (see Stiglitz (1974)), and the existence and characterization of the optimal solution have been fully analyzed – in the setting of an optimal growth model – by Mitra (1980). When the resource is essential, the economy would sink in the long run to the trivial steady state in which both the resource and the capital stock vanish, and so does the consumption in the limit. This unpleasant outcome could only be avoided when the exhaustible resource can be easily substituted for by a reproducible capital.

With respect to the second strand of the literature on economic growth in which the size of the population is endogenous, only reproducible capital has been considered as a factor of production besides labor. In the class of models that follow the Ramsey-Solow tradition in which all economic decisions are conferred to a single infinitely lived agent (a planner, or the head of a dynasty), the population size tends to a stationary level; see, for example Razin and U Ben-Zion (1975) or Nerlove, Razin, and Sadka (1987). When capital is human – as in Barro and Becker, op. cit. – rather than physical, the appropriate model is of the Uzawa-Lucas variety (see Lucas (1988)). Using also the dynastic-utility formulation, these authors showed that the economy exhibits exponential growth at a rate equal to a positive endogenous fertility rate. Works in this direction have been carefully surveyed in Tamura (2000). On the other hand, in the overlapping-generation framework, Samuelson (1975) investigated the optimal size of population in the long run and pointed out that the incentive to maintain an increasing fertility rate would ultimately lead to an inefficient allocation outcome. Erhlich and Lui (1991) further discussed this question, and provided a concise literature survey in Erhlich and Lui (1997).

Surprisingly, there are not many studies bringing together natural resources and population in a comprehensive synthetic model of economic growth. Exceptions are Nerlove, Razin, and Sadka (1986) and Eckstein, Stern, and Wolpin (1988). These papers focused on indestructible land as a production factor. Nerlove et al. relied upon the dynastic-utility approach to show the efficiency of the competitive market outcome with endogenous population, while Eckstein et al. used the overlapping-generation framework and demonstrated that as long as the fertility decision is taken into account, the population growth will not be excessive; the market outcome will be efficient; and the economy will reach a stationary long run consumption level above the Malthusian subsistence level. The value of land depends on the time path of land per capita and, since land is fixed in quantity, the problem of over-accumulation of capital is simply ruled out. On the other hand, Nerlove (1993) is, to our knowledge, the only study that links the use of a renewable resource to the fertility decision. Studies that link exhaustible resources with fertility decisions are simply non-existent; see Rault and Nerlove (1997) and Robinson and Srinivasan (1997).

The overlapping-generation framework was seldom used in the study of the extraction of exhaustible resources. Kemp and Long (1979) and, recently, Olson and Knapp (1997) are exceptions, maybe because studying the extraction of exhaustible resources in this framework is too involved. Kemp and Long assumed that the resource is not essential in the production process, and showed that the resource could be partially depleted, inducing, therefore, a form of inefficiency in this case. On the other hand, Olson and Knapp considered the exhaustible resource as an essential factor of production. They established the existence of an equilibrium, and provided a characterization of the market outcome. Market efficiency in this study is warranted; however, the economy would ultimately collapse into the trivial steady state of zero output in the limit. The convergence to this degenerate state need not be monotone, but may happen in damped oscillations, and the pattern of resource extraction as well as the time path of the resource price could possibly exhibit non-classical behavior.

As we have seen, if there is no possibility of substitution for an essential exhaustible resource, say oil, used as an input in the production process, the whole economy might altogether glide to the trivial steady state of zero consumption in the long run, a regretful doomsday. In order to reach the state of economic sustainability, the problem of technology transition emphasizes the possibility of substituting for the exhaustible resource with an everlasting source of energy input, say solar energy, which could be made available through investments in the so-called backstop technology (see, for example, Hung and Quyen (1993, 1994)). In models of economic growth, oil is used first, with solar energy gradually being brought in to substitute for oil. By the time the oil stock is entirely depleted, the backstop capital will have reached the Golden Rule stationary level, and its marginal productivity is equal to the interest rate. Can these results be carried over into a dynamic general-equilibrium framework, especially when the population is not a datum, but endogenous in the sense that it results from fertility decisions made by economic agents? This paper aims at providing some answers to these questions in an overlapping-generation model. We consider the fertility decision problem in an economy where production requires, besides labor, an energy input. The energy input comes from oil, an exhaustible resource, and could be substituted for by solar energy. The adoption of the overlapping-generation model as a modeling strategy could be justified on the following grounds. First, unlike the dynastic-utility formulation, which assumes perfect foresight on the part of the head of the dynasty and to whom the task of inter-temporal resource planning is assigned, the overlapping-generation model provides a decentralized setting. Individual decisions about resource allocation for each generation are all made explicit, and the market mechanism which consistently links these decentralized decisions through time worked out. Second, some results obtained under the overlapping-generation framework are in sharp contrast to those obtained from the single infinitely-lived dynastic head framework. Oil and solar energy might be used simultaneously in the production process. The oil stock might be incompletely depleted, and under this scenario, the unexploited part of the oil stock would then become a means of storing and transferring wealth through time. Although intrinsically productive, oil plays in this case the role of a bubble which persists throughout the whole time horizon. It is remarkable that introducing oil into the overlapping-generation framework gives rise to multiple equilibria with complex dynamics, including the possibility of convergence to some steady state through damped oscillation, limit cycles, etc.

A major task of our paper is to provide a proof of the existence of a competitive equilibrium for an economy – with capital, oil, and fertility choice – formulated under the overlapping-generation framework. Our existence poof proceeds in three stages.

In the first stage, we show that when the economy is truncated at the end of a finite number of periods, the truncated economy thus obtained always has a competitive equilibrium. The proof involves a modification of the technique developed by Debreu, Gale, and Nikaido[3]. In the proof technique developed by these researchers, the number of consumers is exogenously given. Hence in their model the market demand for each commodity can be obtained by aggregating the individual demands of a known number of consumers. In our model, because fertility decisions are endogenous, the number of young and old individuals – and hence the sum of their individual demands for each commodity – is endogenous. Thus we cannot invoke directly the results of these researchers to assert the existence of a competitive equilibrium, and must make some adjustments to account for the endogenous temporal variations in the number of consumers in each market in each period. The proof of the existence of a competitive equilibrium for a truncated economy is long and involved and is relegated to Appendix B. Most of the technical arguments in the proof are deployed to prevent the economy from collapsing by showing that the birth rate in each period is positive.

In the second stage, we consider the sequence of truncated economies, with each element of the sequence indexed by the period at the end of which the original economy is truncated, and – through a series of technical arguments[4] – establish some bounds on the

[3] See Nikaido (1970, Chapter 10).

[4] The technical arguments involve the limiting values of the various endogenous variables – the birthrates, the oil and backstop capital investments by a young individual, the price of oil, and the price of renewable energy – either when energy resources are abundant or when the price of energy is near its critical value $\rho^{\max}$. These technical arguments are needed to ensure that the birthrates are bounded below and away

endogenous variables that apply uniformly to all the truncated economies. In the third stage, we use these bounds to show that a sequence of competitive equilibria – one competitive equilibrium for each truncated economy – has a subsequence that converges in the product topology of a denumerable family of finite-dimensional Euclidean spaces. The limit of the subsequence is a competitive equilibrium of the infinite time horizon economy. The technique we employ in establishing convergence is Cantor's famous diagonal trick used in the proof of the following version of the Tychonoff theorem: "*The product of a denumerable family of compact metrizable spaces is compact and metrizable.*" The interested reader can consult Dieudonné (1976, (12.5.9)). We would like to point out that Balasko and Shell (1980) and Balsko, Cass, and Shell (1980) were the first researchers who applied the Tychonoff theorem or Cantor's diagonal trick to prove the existence of a competitive equilibrium for an overlapping-generation model of a pure exchange economy. Our overlapping-generation model has both capital and an exhaustible resource. It also has an endogenous population structure because fertility decisions are determined by the maximizing behavior of the successive young generations. Compared to the models of Balasko and Shell, op cit., and Balasko, Cass, and Shell, op cit., our model is much more complex, and it is not possible to invoke the results of these researchers to assert that it has a competitive equilibrium. Furthermore, unlike the proof of these researchers, our existence is accessible to readers who only possesse a rudimentary knowledge of real analysis. Also, we believe that our proof of the existence of a competitive equilibrium for an overlapping-generation model with endogenous fertility, oil, and capital has its own merits, and thus have presented it in great detail. We would also like to point out the advantage of the existence proof technique based on the Tychonoff theorem over that based on the monotone mapping theorem (see, for example, Stokey, Lucas, and Prescott (1989) or Olson and Knapp, op cit.) that is often used to establish the existence of a competitive equilibrium for simple macroeconomics models with one state variable and formulated under the overlapping-generation framework. In the latter technique, it is necessary to establish first that the operator a fixed point of which constitutes a competitive equilibrium of the overlapping-generation model has some desired monotonicity property, and this is hard to show, especially when the model has several state variables, as is the case of our model.

The paper is organized as follows. In Section 2, the overlapping-generation model is presented. In the model, four classes of economic agents exist in each period: a young generation, an old generation, competitive firms producing the consumption good, and competitive firms producing solar energy. The consumption good is produced from labor and energy, with the energy input coming either from oil or the backstop or from both. We shall assume that the consumption good can also be used as investment goods to augment the backstop capital. As the oil stock dwindles, accumulating backstop capital is the only way to prevent eventually a drastic reduction in consumption. In Section 3, we study preliminarily the competitive equilibrium of an economy which does not have any oil left and which is endowed only with backstop capital. We demonstrate the existence of a unique forward-looking temporary equilibrium as well as the existence of at least a steady state under infinite time horizon. We then show the possibility of oscillation and

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from 0. They are the price we pay for by our specification of a more realistic sub-utility function of offspring.

of a 2-cycle in the dynamic convergence to a steady state. To support all these findings, we provide a numerical example for each case. In Section 4, we focus on an economy endowed with both oil and backstop capital. The hard task in Section 4 is to establish the existence of a competitive equilibrium under infinite horizon. In Section 5, we characterize oil extraction and provide the conditions under which the oil stock will be exhausted either in finite time. We then turn to Section 6 where incomplete oil depletion occurs and present some numerical examples. One key feature is that when both sources of energy are used in production, although the total energy required is well determined, its composition is not. Henceforth, many time paths of resource usage might prevail in equilibrium, and therefore there exist accordingly many time paths of resource and backstop capital all of which satisfy market clearing conditions. From the moment oil will not be brought into use as a production factor, the remaining oil stock serves as a means of storing value which earns the equilibrium rate of return for asset holding. Depending on how much oil will be left under the ground unexploited, there will be an equilibrium capital price path, as studied in Section 3, and, therefore, a resulting capital accumulation trajectory. Again, steady states are multiple and the convergence to a steady state exhibits complex dynamics that might be range from monotone to cyclical convergence. We also provide a set of numerical examples to highlight these properties. In Section 7, we bring together all disparate elements into a synthetic characterization of the competitive equilibria that emerge from our model. Section 8 contains some concluding remarks.

2. THE MODEL

In this section, we present the overlapping-generation model. First, we describe the technology. Second, we describe the economic agents and the problems they face. Finally, we define the competitive equilibrium.

2.1. *The Technology*

The perfectly competitive firms produce a consumption good from two inputs – labor and energy – according to a neoclassical production function, say $Y = F(E, L)$, where Y denotes the output; E the energy input; and L the labor input. We impose the following conditions on F:

ASSUMPTION 1: *The production function F is assumed to be linear homogenous, concave, and continuously differentiable. Furthermore, $F(0,E) = F(L,0) = 0$ for all $E \geq 0, L \geq 0,$ and $\partial F(E,L)/\partial E > 0,\ \partial F(E,L)/\partial L > 0$ for all $(E,L) >> 0$. Also, F satisfies the following Inada* conditions:

(i) *For any $L > 0$, we have*

$$\ell im_{E\to 0}\partial F(E,L)/\partial E = \infty \text{ and } \ell im_{E\to\infty}\partial F(E,L)/\partial E = 0.$$

(ii) *For any $E > 0$, we have*

$$\ell im_{L\to 0}\partial F(E,L)/\partial L = \infty \text{ and } \ell im_{L\to\infty}\partial F(E,L)/\partial L = 0.$$

In what follows, we shall let $e = E/L$ denote the energy input per worker and $f(e) = F(e,1)$ denote the output of the consumption good – as a function of e – produced by a worker.

In our economy, energy inputs come from two sources: oil and a backstop, say solar energy. While oil can be extracted at negligible cost, its ultimate stock is limited. The backstop, on the other hand, can provide a perpetual flow of energy. However, harnessing the Sun's energy requires investments in backstop capital, say solar collectors. In any period, the amount of solar energy harnessed is assumed to be proportional to the stock of backstop capital K, and, to simplify the exposition, we shall assume that the proportionality constant is equal to unity, i.e., one unit of backstop capital produces one Btu. Also, we shall assume that backstop capital depreciates at rate $\delta, 0 \le \delta \le 1$.

If Q_t is the amount of oil – also measured in Btu's – extracted for use as part of the energy input in period t and K_t is the stock of backstop capital in that period, then the total energy input used in period t is $E_t = Q_t + K_t$. Furthermore, if L_t is the labor input used in period t, then the output of the consumption good in that period is $Y_t = F(Q_t + K_t, L_t)$. We assume that the consumption good can also be used as investment goods to augment the stock of backstop capital. As time goes on and the oil resources dwindle, it is imperative that investments in the backstop be made to prevent a drastic reduction in consumption. The accumulation of backstop capital only influences the output of the consumption good indirectly through the amount of solar energy delivered by the backstop sector to the economy. Because there is only one kind of capital in the model, namely backstop capital, we shall from now on refer to backstop capital simply as capital.

2.2. *Economic Agents*

In this economy, four classes of economic agents coexist in each period: a young generation, an old generation, competitive firms producing the consumption good, and competitive firms producing solar energy. These economic agents interact on five markets – oil, solar energy, labor, backstop capital, and the consumption good. An individual works when she is young. She has to allocate her wages among current consumption, raising children, and saving for her old-age consumption. The two real assets in the economy are oil and capital, which represent the only possible forms of saving.

At the beginning of each period $t, t = 0,1,...,$ the state of the economy is represented by a list (X_t, K_t, N_t^0, N_t^1), where X_t, K_t, N_t^0, and N_t^1 represent, respectively, the remaining oil stock, the stock of capital, the number of young individuals, and the number of old individuals. The initial state of the economy, i.e., (X_0, K_0, N_0^0, N_0^1), is assumed to be known.

For each $t = 0,1,...,$ let ϕ_t, φ_t, ω_t, ρ_t, and p_t denote, respectively, the price of oil, the price of solar energy, the wage rate, the rental rate of backstop capital, and the price of the consumption good – all in period t. Also, we shall choose the consumption good as the numéraire in each period and set $p_t = 1, t = 0,1,...$ The list $(\phi_t, \varphi_t, \omega_t, \rho_t)$ is called the price system in period t. By a price system we mean an infinite sequence $(\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$.

2.2.1. *The Old Generation*

The real assets in each period are owned by the old generation of that period. An old individual in period t owns X_t / N_t^1 units of oil and K_t / N_t^1 units of backstop capital. The total amount of funds at the disposal of such an individual at the end of the period is $\phi_t \frac{X_t}{N_t^1} + (1 - \delta + \rho_t)\frac{K_t}{N_t^1}$. Because the individual dies at the end of the period, she will leave nothing behind. Her consumption is thus given by

$$c_t^1 = \phi_t \frac{X_t}{N_t^1} + (1 - \delta + \rho_t)\frac{K_t}{N_t^1}.$$

2.2.2. *The Young Generation*

A young individual owns nothing except for 1 unit of labor that she supplies in-elastically on the labor market. She has to allocate her labor income among current consumption, raising children, and saving for old-age consumption. A lifetime plan for a young individual of period t is a list $\left(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1}\right)$, where c_t^0, c_{t+1}^1, b_t, x_{t+1}, and k_{t+1} denote, respectively, her current consumption, her old-age consumption, the number of offspring she raises – at the constant cost h in terms of real resources per child – the amount of oil she buys as investment, and the amount of capital she buys – also for investment purposes. A lifetime plan is feasible if it satisfies the following two temporal budget constraints:

(1) $\quad \omega_t - c_t^0 - hb_t - \phi_t x_{t+1} - k_{t+1} = 0,$

(2) $\quad c_{t+1}^1 - \phi_{t+1} x_{t+1} - (1 - \delta + \rho_{t+1})k_{t+1} = 0.$

The lifetime utility associated with such a lifetime plan is assumed to be given by

(3) $\quad u(c_t^0) + \gamma u(c_{t+1}^1) + v(b_t),$

where $u(c)$ is the single-period sub-utility function associated with consumption, and $v(b)$ is the sub-utility function of offspring. Also, $\gamma, 0 < \gamma < 1,$ is a parameter representing the factor she uses to discount future utilities. It should be emphasized that for parents children have intrinsic value, and the number of offspring is here considered as a consumption good from their viewpoint.[5]

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[5] One may think of b_t as a measure of quality of a child – in period t – for an economy in an economy endowed with a constant population, say N_0. Then the economy's human capital at the beginning is $N_0^0 b_0$, and the economy's human capital in the following periods are given by $N_0 b_t, t = 1,2,...$ We think

We impose the following assumption on the single-period sub-utility function of consumption and the sub-utility function of offspring:

ASSUMPTION 2:

(i) *The sub-utility function* $u(c)$ *is defined for all* $c>0$. *It is continuously differentiable, strictly concave, and strictly increasing. Furthermore, it satisfies the following Inada conditions:* $\ell im_{c\to 0}u'(c)=+\infty$ *and* $\ell im_{c\to +\infty}u'(c)=0$.

(ii) *The sub-utility function of offspring* $v(b)$ *is defined for all* $b\geq 0$. *It is continuously differentiable and concave. Furthermore, there exists a number* $b^{\max}>1$ *such that* $v'(b)>0, 0\leq b<b^{\max}$, and $v'(b)\leq 0, b\geq b^{\max}$.

Note that $b^{\max}$ represents the saturation number of offspring. Because the single-period sub-utility function of consumption and the sub-utility function[6] of offspring are both assumed to be concave and increasing, current consumption, old-age consumption, and offspring are all normal goods. Furthermore, because it is costly to raise children, the optimal number of children is strictly less than $b^{\max}$.

The problem of a young individual in period t is to find a feasible lifetime plan that maximizes (3). Let

(4) $$r_t=\max\left\{\frac{\phi_t}{\phi_{t-1}},1-\delta+\rho_t\right\}$$

denote the rate of return to savings for a young individual of period t. Her lifetime utility maximization problem can be restated under the following form:

(5) $$\max_{\left(c_t^0,c_{t+1}^1 b_t\right)}u(c_t^0)+\gamma u(c_{t+1}^1)+v(b_t)$$

subject to the following single budget constraint:

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that when one talks about human capital, the investments involved should encompass both the efforts made with regard to the quantity of children and the efforts made with regard to the quality of each child. Here, the quantity-quality trade-off is relevant and merits a further separate study. In this paper, we try not to be abusive by engaging in a lax interpretation of human capital. Therefore, we choose to consider b as the number of offspring, and thus $N_t^0=N_0^0 b_0 \ldots b_{t-1}$ is the size of the young generation in period t. The size of the young generation in each period is endogenous precisely because b_t is a decision variable for a young individual in period t.

[6] Observe that part (ii) of Assumption 2 rules out homothetic preferences. If preferences are homothetic, the Engel curves of current consumption, future consumption, and offspring are all straight lines. In particular, when labor income rises, the demand for offspring rises in the same proportion as the rise in labor income, which seems to be untenable. Furthermore, for an economy that is sustained only by renewable energy resources, homothetic preferences imply that from any initial condition the economy enters a steady state after one period: the transition to its steady state level of the birth rate lasts exactly one period, and this also seems unreasonable (see also footnote 3). On the other hand, it can be shown that homothetic preferences allow for a much simpler proof of the existence of a competitive equilibrium for the case the economy begins with a positive stock of fossil fuels.

(6) $$c_t^0 + \frac{1}{r_{t+1}} c_{t+1}^1 + hb_t - \omega_t = 0.$$

Note that the budget constraint (6) asserts that the present value of consumption over two periods plus the current cost of raising children are equal to the current labor income, where the discounted price of future consumption is given by $1/r_{t+1}$.Because $u(c)$ is strictly concave for all $c > 0$, and $v(b)$ is strictly concave in $0 \le b \le b^{\max}$, the lifetime utility maximization stated under the form represented by (5) and (6) has a unique solution (c_t^0, c_{t+1}^1, b_t). The first-order conditions $u'(c_t^1)/u'(c_t^0) = r_{t+1}/\gamma$ and $v'(b_t) = hu'(c_t^0)$ are standard and have the usual interpretations. Given the parameters γ and h, one these first-order conditions and the budget constraint (6) can be solved to obtain the demand functions $c_t^0(\omega_t, r_{t+1}), c_{t+1}^1(\omega_t, r_{t+1})$, and $b_t(\omega_t, r_{t+1})$. Note that the Inada condition imposed on the sub-utility function associated with consumption implies that $c_t^0 > 0$ and $c_{t+1}^1 > 0$; that is, current consumption and future consumption are both positive. However, the number of offspring raised by a young individual of period t might be zero if the current wage rate is low enough.

The saving of the individual is

(7) $$s_t = \omega_t - c_t^0 - hb_t,$$

and the division of saving between oil and capital depends on their relative rates of return. If $\phi_{t+1}/\phi_t > 1 - \delta + \rho_{t+1}$, i.e., if investment in oil yields a higher rate of return than investment in capital, then all the savings will be put into oil: the optimal investment mix is $x_{t+1} = s_t/\phi_t$ and $k_{t+1} = 0$. On the other hand, if $\phi_{t+1}/\phi_t < 1 - \delta + \rho_{t+1}$, then all the savings will be put into capital: the optimal investment mix is $x_{t+1} = 0$ and $k_{t+1} = s_t$. When $\phi_{t+1}/\phi_t = 1 - \delta + \rho_{t+1}$, the two real assets yield the same rate of return. The individual is indifferent between oil and capital, and k_{t+1} can assume any value between 0 and s_t.

What happens to the optimal lifetime plan (c_t^0, c_{t+1}^1, b_t) when the rate of return to saving in the next period rises? To answer this question, recall that $1/r_{t+1}$ represents the price – in terms of current consumption – of 1 unit of old-age consumption. Because a rise in r_{t+1} makes the price of 1 unit of old-age consumption cheaper, we expect that the substitution effect will cause old-age consumption to rise at the expense of current consumption and the number of offspring. Furthermore, as r_{t+1} rises, real lifetime income also rises with r_{t+1}. The income effect will raise current consumption, old-age consumption, and the number of offspring. The income effect reinforces the substitution effect and causes old-age consumption to rise even more, and thus, all in all, we get $\partial c_{t+1}^1 / \partial r_{t+1} > 0$. However, for current consumption and the number of offspring, the net impact is ambiguous because the substitution effect and the income effect operate in opposite directions. The net impact on s_t is thus ambiguous although c_{t+1}^1 is increasing in r_{t+1}. To obtain sharper results, we shall make the following assumption:

ASSUMPTION 3: *For a young individual, current consumption, old-age consumption, and offspring are gross substitutes*

Assumption 3 is often made in overlapping-generation models and looks quite innocuous at the macroeconomic level; see, for example Azariadis (1993, Section 7.4) and Azariadis and Drazen (1990). Thanks to this assumption, we obtain the result that young-age consumption declines when the discounted price of future consumption declines, i,e., $\partial c_t^0 / \partial r_{t+1} \leq 0$. Similarly, for the number of offspring, we have $\partial b_t / \partial r_{t+1} \leq 0$. It follows immediately from (7) that $\partial s_t / \partial r_{t+1} \geq 0$. On the other hand, as already discussed immediately after Assumption 2, for a young individual, current consumption, old-age consumption, and offspring are all normal goods. Thus we expect c_t^0, c_{t+1}^1, and b_t to rise with the current wage rate ω_t. Furthermore, because $c_{t+1}^1 = r_{t+1} s_t$, saving also rises with ω_t.

Now according to Assumption 2, the Inada condition is imposed on the sub-utility function of consumption, but not on the sub-utility function of offspring. Thus we can expect that when the labor income of a young individual is too low, she will choose not to raise children. To determine the critical level of labor income that triggers the extinction of the population at the end of the following period, suppose that $r_{t+1} = (1-\delta)$, the minimum rate of return to saving that is possible. Next, let

(8) $\omega^{\min}(\delta) = Inf\{\omega_t | b_t > 0, \text{ given that } r_{t+1} = (1-\delta)\}$.

As defined, $\omega^{\min}(\delta)$ is the critical wage rate at or below which a young individual will choose not to raise children, given that the rate of return to saving is equal to its minimum possible level. Using Assumption 3, we can assert that $\omega^{\min}(\delta)$ declines when δ rises. In what follows, we shall suppress the rate of capital depreciation in the notation for this critical wage rate and write it simply as $\omega^{\min}$, except when we want to stress its dependence on δ. Furthermore, if $\omega^{\min}$ is the labor income of a young individual, then using Assumption 3, we can assert that for any rate of return above the minimum level $1-\delta$, the individual still chooses not to raise children. Let $e^{\min}$ denote the energy input per worker that gives rise to the critical wage rate $\omega^{\min}$. These two critical variables are linked by the following relation:

(9) $\omega^{\min} = f(e^{\min}) - e^{\min} f'(e^{\min})$.

Recall that as $\omega_t \downarrow \omega^{\min}$, the number of offspring she raises will tend to 0 while the saving for old-age consumption is bounded below and away from 0, which implies that $s(\omega_t, r_{t+1}) / b(\omega_t, r_{t+1})$, the saving/offspring ratio will tend to infinity. It is this property that prevents the population from becoming extinct in finite time. When the labor income declines to the critical level $\omega^{\min}$, the birthrate approaches 0, but the saving for old-age consumption – although low – is still bounded below and away from 0, allowing for a high saving/offspring ratio. The high saving/offspring ratio means a high level of energy input per worker in the next period, with an ensuing high wage rate in that period. A high wage rate in the next period leads to a high birthrate in that period, which gives the population a chance to bounce back. The saving/offspring ratio also tends to infinity when labor income tends to infinity. The reason is that the birthrate, although rises with income, remains bounded above by the saturation level $b^{\max}$ while saving increases without bound. Thus when the wage rate is high, the cost of raising children becomes a negligible fraction of labor income, and most of the labor income is spent on current consumption and on investments to provide for old-age consumption. Due to these reasons, we can expect the curve $\omega_t \to s(\omega_t, r_{t+1})/b(\omega_t, r_{t+1})$, $\omega_t > \omega^{\min}$, to have a U-shape. The shape of this curve for the numerical example presented in Section 3.4 is depicted in Figure 1.

2.2.3. *Solar Energy Producers*

Solar energy is produced by competitive firms from capital. Given that the price of solar energy in period t is φ_t and that the rental rate of capital – also in period t – is ρ_t, the profit maximization problem of the competitive firms is particularly simple. The representative solar energy producer solves the following profit maximization problem:

(10) $\quad \max_{(K^{\#}, S^{\#})}[\varphi_t S^{\#} - \rho_t K^{\#}]$

subject to the following technological constraint:

(11) $\quad S^{\#} - K^{\#} \leq 0,$

where $K^{\#}$ and $S^{\#}$ represent, respectively, this firm's demand for capital and its output of solar energy. The ordered pair $(K^{\#}, S^{\#})$ is called a production plan of the representative firm in the backstop sector, and a production plan is feasible if it satisfies the technological constraint (11). If $\varphi_t > \rho_t$, then because of the assumption that one unit of capital produces one unit of energy, the backstop sector will demand an unbounded amount of capital to produce an unbounded amount of solar energy and make infinite profits. Because of limited capital, this situation obviously cannot arise in equilibrium. On the other hand, if $\varphi_t < \rho_t$, then the backstop sector will shut down. When $\varphi_t = \rho_t$, the profits of the producers in this sector are zero, and the output of this sector is indeterminate.

2.2.4. *Producers of the Consumption Good*

In each period t, the representative firm in the consumption good sector solves the following profit maximization problem:

(12) $\quad \max_{(Q,S,L,Y)}[Y - \phi_t Q - \varphi_t S - \omega_t L]$

subject to the technological constraint

(13) $\quad Y - F(Q + S, L) \leq 0,$

where Q, S, L, and Y represent, respectively, the oil input, the solar energy input, the labor input, and the output of the consumption good. The list (Q, S, L, Y) is called a production plan of the representative firm, and a production plan is technological feasible if it satisfies the constraint (13).

Let (Q_t, S_t, L_t, Y_t) be a solution of the problem constituted by (12) and (13). As usual, the price of a factor is equal to its marginal productivity; that is, $\partial F / \partial L = \omega$ for labor, $\partial F / \partial E = \phi_t$ for oil, and $\partial F / \partial E = \varphi_t$ for solar energy when these inputs are used. In particular, we have (i) $Q_t = 0$ if $\phi_t > \varphi_t$, (ii) $S_t = 0$ if $\phi_t < \varphi_t$. When $\phi_t = \varphi_t$, the mix (Q_t, S_t) is indeterminate, although the sum $E_t = Q_t + S_t$ is uniquely determined.

2.3. *Definition of Competitive Equilibrium*

Let $\mathsf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ be *a price system. An allocation induced by* P is a list of infinite sequences

$$\mathsf{A} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty}\right)$$

with the following properties:

(i) $c_0^1 = [\phi_0 X_0 + (1 - \delta + \rho_0) K_0] / N_0^1$;

(ii) $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ is the optimal lifetime plan for a young individual of period t when the price system P prevails.

(iii) $(K_t^{\#}, S_t^{\#})$ is an optimal production plan of the representative firm in the backstop sector in period t when the price system P prevails.

(iv) (Q_t, S_t, L_t, Y_t) is an optimal production plan of the representative firm in the consumption good sector in period t when the price system P prevails.

(v) $(X_t, K_t, N_t^0, N_t^1) = N_{t-1}^0 (x_t, k_t, b_{t-1}, 1),\ t = 1,2,...$

Observe that (i) represents the consumption of an old individual in period 0, and (ii) describes the dynamics of the system driven by the lifetime utility maximization behavior of the successive generations. The pair (P,A) is said to constitute *a competitive equilibrium* if the following market-clearing conditions are satisfied for each $t = 0,1,...,$

(vi) $X_{t+1} + Q_t = X_t$,

(vii) $S_t = S_t^{\#}$,

(viii) $L_t = N_t^0$,

(ix) $K_t^{\#} = K_t$,

(x) $N_t^1 c_t^1 + N_t^0 (c_t^0 + h b_t + k_{t+1}) = Y_t + (1 - \delta) K_t$.

Observe that (vi) represents the equilibrium condition on the oil market; (vii) the equilibrium on the solar energy market; (viii) the equilibrium condition on the labor market; (ix) the equilibrium condition on the capital market; and (x) the equilibrium condition on the consumption good market.

3. COMPETITIVE EQUILIBRIUM FOR AN ECONOMY WITHOUT OIL RESOURCES

In this section, we analyze an economy without oil resources. The analysis can also be used to characterize the behavior of our economy in its final phase, after the oil stock has been exhausted and is now completely sustained by renewable energy.

Before beginning the analysis, it is worth glancing at the other extreme case of an economy endowed only with oil, but no capital. This is the case studied in detail by Olson and Knapp (1997) in a model where there is no consideration of fertility decision. Our model only needs to be slightly modified to accommodate this new feature. First, the problem of a young individual is still represented by (5) and (6). Next, note that since oil is the only input in the production of the consumption good, factor pricing in a competitive market requires $\phi_t = f'(q_t)$ and $\omega_t = f(q_t) - q_t f'(q_t)$, where $q_t = x_t - x_{t+1}$ is the oil input per worker. To warrant positive consumption in period t requires $q_t > 0$; and to warrant asset holding, one must have $r_{t+1} = \phi_{t+1} / \phi_t = f'(q_{t+1}) / f'(q_t)$, which is exactly the Hotelling rule. Now note that because oil is an exhaustible resource, its stock must decline inexorably over time, and the implication of this fact is that the population will be extinct in the long run. To see why, suppose that the birthrate tends to a limit that is greater than or equal to 1 in the long run. Because the cost of raising one child is $h > 0$, the oil input per worker in the long run must be bounded away from 0, which obviously cannot be maintained given the fixed stock of oil. Thus in the long run the remaining oil stock and the number of young individuals will both vanish. Furthermore, because the output of the economy tends to 0 through time, the total labor income of workers must also tend to 0 through time. This result implies that the oil asset bought by the successive young generations will tend to 0 in the long run, i.e., the oil stock will be asymptotically depleted. Sine all the oil stock will be depleted, inefficiency in term of over accumulation of assets is ruled out. The convergence to the trivial steady state may be monotone

Let us now return to our main purpose of this section: analyzing an economy without oil resources. Suppose that the economy begins in state $\left(K_0, N_0^0, N_0^1\right)$ in period 0, with $K_0 > 0, N_0^0 > 0,$ and $N_0^1 > 0;$ that is, the economy begins without any oil resources, but with a positive stock of capital and a positive population. In each period, economic agents thus interact only on four markets: the market for labor, the market for backstop capital, the market for solar energy, and the market for the consumption good. Because one unit of capital produces one Btu, and because in equilibrium the profit in the backstop sector is 0, the price of renewable energy is equal to the rental rate of capital. Thus, we shall conduct our analysis in terms of the rental rate of capital without explicitly mentioning the price of renewable energy.

3.1. *The Capital/Labor Ratio*

When there are no oil resources, all the energy needs of the economy are provided by the backstop. To prevent the population from becoming extinct in period 1, we shall assume the initial capital/labor ratio is higher than the critical energy input per worker, i.e., $\kappa_0 > e^{\min}$, where we have let $\kappa_0 = K_0 / N_0^0$.

In period 0, the equilibrium rental rate of capital and the equilibrium wage rate are given, respectively, by $\overline{\rho}_0 = f'(\kappa_0)$ and $\overline{\omega}_0 = f(\kappa_0) - \kappa_0 f'(\kappa_0)$. The consumption of an old individual in period 0 is given by $\overline{c}_0^1 = (1-\delta+\overline{\rho}_0)K_0 / N_0^1$. As for a young individual of period 0, she solves the lifetime utility maximization problem obtained by setting $t = 0$, $\omega_0 = \overline{\omega}_0$, and $r_1 = 1-\delta+\rho_1$ in (6), with ρ_1 representing the rental rate of capital in period 1 that is yet to be determined. The current consumption, the old-age consumption, the number of offspring, and the saving under the form of capital that constitute the solution of the preceding lifetime utility maximization problem are denoted by $c^0(\overline{\omega}_0, 1-\delta+\rho_1)$, $c^1(\overline{\omega}_0, 1-\delta+\rho_1)$, $b(\overline{\omega}_0, 1-\delta+\rho_1)$, and $k(\overline{\omega}_0, 1-\delta+\rho_1)$, respectively. The capital/labor ratio in period 1 that is generated by this optimal lifetime plan is

$$\kappa(\overline{\omega}_0, 1-\delta+\rho_1) = \frac{k(\overline{\omega}_0, 1-\delta+\rho_1)}{b(\overline{\omega}_0, 1-\delta+\rho_1)} = \frac{s(\overline{\omega}_0, 1-\delta+\rho_1)}{b(\overline{\omega}_0, 1-\delta+\rho_1)}.$$

3.2. *Existence and Uniqueness of Competitive Equilibrium for an Economy without Oil Resources*

For any wage rate $\omega_0 > \omega^{\min}$ in period 0 and any rental rate of capital $\rho_1 \geq 0$ in period 1, let

(14) $\quad \zeta(\omega_0, \rho_1) = f'\big(\kappa(\omega_0, 1-\delta+\rho_1)\big).$

As defined, $\zeta(\omega_0, \rho_1)$ represents the rental rate of capital in period 1 generated by the maximizing behavior of a young individual of period 0, given that ω_0 is her labor income and ρ_1 is the rental rate of backstop capital that this individual expects to prevail in period 1. Figure 2 depicts the curve $\zeta(\omega_0,.) : \rho_1 \to \zeta(\omega_0, \rho_1), \rho_1 \geq 0$, for the numerical example presented in Section 3.4.

It is clear that the curve $\zeta(\omega_0,.)$ is continuous. Because a young individual must save for her old-age consumption, her capital investment is always positive even if its rate of return is 0; that is, $0 < k(\omega_0, 1-\delta) < \omega_0$. Furthermore, because $\omega_0 > \omega^{\min}$, we must also have $b(\omega_0, 1-\delta) > 0$. Hence $\kappa(\omega_0, 1-\delta) > 0$, which implies $0 < \zeta(\omega_0, 0) < +\infty$. Also, according to Assumption 3, the capital/labor ratio $\kappa(\omega_0, 1-\delta+\rho_1)$ is increasing in ρ_1, which implies that $\zeta(\omega_0,.)$ is downward sloping. Now as ρ_1 continues to rise, if $b(\omega_0, 1-\delta+\rho_1) = 0$ for some value $\rho_1 = \widetilde{\rho}_1$, then $\kappa(\omega_0, 1-\delta+\rho_1) \to +\infty$ when $\rho_1 \uparrow \widetilde{\rho}_1$, which means that $\zeta(\omega_0, \rho_1) \downarrow 0$ as $\rho_1 \uparrow \widetilde{\rho}_1$, and $\zeta(\omega_0,.)$ must have crossed the 45-degree line before ρ_1 reaches $\widetilde{\rho}_1$. Point B in Figure 2 represents such a value of $\widetilde{\rho}_1$. On the other hand, if $b(\omega_0, \rho_1) > 0$ for all $\rho_1 \geq 0$, then $\zeta(\omega_0,.)$ must also cross the 45-degree line at a single point. In either case, $\zeta(\omega_0,.)$ crosses the 45-degree line at a single point, labeled point A in Figure 2. Point A represents the equilibrium rental rate of capital in period 1. Note that at point A the birthrate is positive. We have just established the following lemma:

LEMMA 1: *Suppose that* ω_0, *the wage rate prevailing in period* 0, *is strictly above the critical level* $\omega^{\min}$. *Then there exists a unique value for the rental rate of capital in period* 1 *that satisfies the following condition*:

(15) $\quad f'\big(\kappa(\omega_0, 1-\delta+\rho_1)\big) = \rho_1.$

The unique value of ρ_1, *say* $\rho_1 = g(\omega_0)$, *that solves* (15) *is the equilibrium rental rate of capital in period* 1, *given that* ω_0 *is the wage rate prevailing in period* 0. *Furthermore, the equilibrium birthrate in period* 0 *is strictly positive*.

Now let $\overline{\rho}_1 = g(\overline{\omega}_0)$ denote the equilibrium rental rate of capital in period 1. The equilibrium capital/labor ratio in period 1 is then given by $\overline{\kappa}_1 = \kappa(\omega_0, 1-\delta+\overline{\rho}_1) = [f']^{-1}(\overline{\rho}_1)$. The equilibrium price of solar energy in period 1 is $\overline{\varphi}_1 = \overline{\rho}_1$, and the equilibrium wage rate in period 1 is $\overline{\omega}_1 = f(\overline{\kappa}_1) - \overline{\kappa}_1 f'(\overline{\kappa}_1) > \omega^{\min}$. Also, the state of the system in period 1 is

$$\left(\overline{K}_1, \overline{N}_1^0, \overline{N}_1^1\right) = \left(\overline{N}_0^0 k(\overline{\omega}_0, 1-\delta+\overline{\rho}_1), \overline{N}_0^0 b(\overline{\omega}_0, 1-\delta+\overline{\rho}_1), \overline{N}_0^0\right),$$

where we have let $\left(\overline{K}_0, \overline{N}_0^0, \overline{N}_0^1\right) = \left(K_0, N_0^0, N_0^1\right)$.

The procedure used to obtain

$$\left(\overline{\varphi}_1, \overline{\omega}_1, \overline{\rho}_1\right), \left(c^0(\overline{\omega}_0, 1-\delta+\overline{\rho}_1), c^1(\overline{\omega}_0, 1-\delta+\overline{\rho}_1), b(\overline{\omega}_0, 1-\delta+\overline{\rho}_1), k(\overline{\omega}_0, 1-\delta+\overline{\rho}_1)\right)$$

and $\left(\overline{K}_1, \overline{N}_1^0, \overline{N}_1^1\right)$ can be repeated ad infinitum to obtain a price system[7] $\overline{\mathsf{P}} = \left(\overline{\varphi}_t, \overline{\omega}_t, \overline{\rho}_t\right)_{t=0}^{\infty}$ and an allocation induced by $\overline{\mathsf{P}}$, say

$$\overline{\mathsf{A}} = \left(\overline{c}_0^1, (\overline{c}_t^0, \overline{c}_{t+1}^1, \overline{b}_t, \overline{k}_{t+1})_{t=0}^{\infty}, (\overline{S}_t, \overline{L}_t, \overline{Y}_t)_{t=0}^{\infty}, (\overline{K}_t^{\#}, \overline{S}_t^{\#})_{t=0}^{\infty}, (\overline{K}_t, \overline{N}_t^0, \overline{N}_t^1)_{t=0}^{\infty}\right),$$

where

$$\begin{aligned}&\left(\overline{c}_t^0, \overline{c}_{t+1}^1, \overline{b}_t, \overline{k}_{t+1}\right)\\ &\quad = \left(c^0(\overline{\omega}_t, 1-\delta+\overline{\rho}_{t+1}), c^1(\overline{\omega}_t, 1-\delta+\overline{\rho}_{t+1}), b(\overline{\omega}_t, 1-\delta+\overline{\rho}_{t+1}), k(\overline{\omega}_t, 1-\delta+\overline{\rho}_{t+1})\right),\end{aligned}$$

and

$$\overline{S}_t = \overline{S}_t^{\#} = \overline{K}_t^{\#} = \overline{K}_t,$$
$$\overline{L}_t = \overline{N}_t^0,$$
$$\overline{Y}_t = F(\overline{S}_t, \overline{L}_t).$$

The pair $\left(\overline{\mathsf{P}}, \overline{\mathsf{A}}\right)$, thus constructed, constitutes a competitive equilibrium for an economy without oil resources. Furthermore, it is clear there is no other competitive equilibrium. We summarize the result just obtained in the following proposition:

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[7] Note that if preferences are homothetic, then the equilibrium prices of solar energy remain the same after one period; that is $\overline{\rho}_t = \overline{\rho}_1, t = 2,3,\ldots$ That is, the economy reaches a steady state in period 1.

PROPOSITION 1: *For an economy that has no oil resources, but that is sustained by a source of renewable energy – provided by a backstop technology – there exists a unique competitive equilibrium.*

We would like to mention that this proposition is similar to Theorem 13.1 found in Azariadis, op.cit, page 198. The sufficient condition for the uniqueness of the forward-looking equilibrium asserted by Theorem 13.1 is also obtained in our model, thanks to our Assumption 3 regarding the gross substitute relationship among all the consumption goods.

3.3. *Steady States for an Economy without Oil Resources: Existence and Uniqueness*

Let

(16) $\rho^{\max} = f'(e^{\min})$

denote the the critical price of energy at or above which a young individual will choose not to raise children. Now for any initial rental rate of capital ρ_0 that satisfies the condition $0 < \rho_0 < \rho^{\max}$, define

(17) $G(\rho_0) = g(\omega(\rho_0))$,

where we have let $\omega(\rho_0)$ denote the prevailing equilibrium wage rate when ρ_0 is the equilibrium rental rate of capital; that is, $\omega(\rho_0) = f(\kappa_0) - \rho_0\kappa_0$, with $\kappa_0 = [f']^{-1}(\rho_0)$.

The map $G: \rho_0 \to G(\rho_0), 0 < \rho_0 < \rho^{\max}$, plays a fundamental role in our analysis. It describes the transition of the equilibrium rental rate of capital from one period to another for an economy without oil or for an economy that has exhausted its oil resources. A fixed point of G represents a steady-state level for the rental rate of capital. The following lemma, the proof of which is given in Appendix A, presents some of the limiting behavior of G.

LEMMA 2: *We have* (i) $\ell im_{\rho_0 \uparrow \rho^{\max}} G(\rho_0) = 0$ *and* (ii) $\ell im_{\rho_0 \downarrow 0} G(\rho_0) = 0$. *Also, for all* ρ_0 *in a right neighborhood of* 0*, we have:* $G'(\rho_0) > 1$.

We shall extend the curve $G: \rho_0 \to G(\rho_0), 0 < \rho_0 < \rho^{\max}$, to all of the closed interval $[0, \rho^{\max}]$ by setting $G(0) = 0$ and $G(\rho^{\max}) = 0$. Figure 3 depicts this curve for the numerical example in Section 3.4.

Let

(18) $G^{\max} = \max_{0 \le \rho_0 \le \rho^{\max}} G(\rho_0)$.

Because and $G(0) = G(\rho^{\max}) = 0$ and $G(\rho_0) > 0$ for all $\rho_0 \in (0, \rho^{\max})$, we must have $G^{\max} > 0$. To preclude the possibility that the population becomes extinct in finite time, we shall make the following assumption:

ASSUMPTION 4: *We have* $G^{\max} < \rho^{\max}$.

Assumption 4 implies that for an economy that is sustained completely by renewable energy, if the rental rate of capital is currently below the critical level $\rho^{\max}$, it will remain below $\rho^{\max}$ in the next period. In the literature of one-dimensional discrete dynamical systems, the interval $[0, G^{\max}]$ is known as a *confining set.*[8] If $f'(\kappa_0) \le G^{\max}$, then the equilibrium rental rate of capital will evolve inside the confining interval $[0, G^{\max}]$. If $f'(\kappa_0) > G^{\max}$, then the equilibrium rental rate of capital will enter the interval $[0, G^{\max}]$ in period 1, and will never leave the confining interval after that.

Now as ρ_0 rises from 0 to $\rho^{\max}$, the curve *G* rises from the origin and stays above the 45-degree line initially. It reaches the maximum value $G^{\max}$ at some point inside the open interval $(0, \rho^{\max})$, then descends to the point $\rho^{\max}$ on the horizontal axis when ρ_0 reaches $\rho^{\max}$. Hence it must cross the 45-degree line at least once, and the rental rate of capital at such a crossing represents the rental rate of capital in a steady state. We have just established the following proposition:

PROPOSITION 2: *For an economy that has no oil resources, but that is sustained by a backstop technology, there exists at least a steady state.*

The shape of the curve $G: \rho_0 \to G(\rho_0), 0 \le \rho_0 \le \rho^{\max}$ – first rising from 0, then returning to 0 – suggests a possible rich dynamics. Depending on the preferences, the technology, and the values of their parameters, convergence to a steady state might be monotone or in damped oscillation. There might even be cycles. To derive some of the conditions that lead to these possibilities, let us first recall that according to the definition of *G,* we have

(19) $\quad G(\rho_0) = f'\big(\kappa\big(\omega(\rho_0),\ 1-\delta + G(\rho_0)\big)\big).$

Differentiating (19) with respect to ρ_0, then manipulating the result, we obtain

$$(20) \qquad G'(\rho_0) = f''\big(\kappa\big(\omega(\rho_0),\ 1-\delta+G(\rho_0)\big)\big)\begin{bmatrix} D_1\kappa\big(\omega(\rho_0),\ 1-\delta+G(\rho_0)\big)\omega'(\rho_0) \\ + D_2\kappa\big(\omega(\rho_0),\ 1-\delta+G(\rho_0)\big)G'(\rho_0) \end{bmatrix}$$

$$= \eta_{f',e}(\rho_0)\left[\eta_{\kappa,\omega}(\rho_0)\eta_{\omega,\rho}(\rho_0)\frac{G(\rho_0)}{\rho_0} + \eta_{\kappa,r_1}(\rho_0)\frac{G'(\rho_0)G(\rho_0)}{1-\delta+G(\rho_0)}\right],$$

where $\eta_{f',e}(\rho_0), \eta_{\kappa,\omega}(\rho_0), \eta_{\omega,\rho}(\rho_0)$, and $\eta_{\kappa,r_1}(\rho_0)$ denote, respectively, the elasticity of the marginal productivity of energy with respect to the energy input, the elasticity – with respect to current labor income – of the capital/labor in the next period that is generated by the maximizing behavior of a young individual of the current period, the cross price elasticity of the wage rate with respect to the rental rate of capital in the same period, and

[8] See, for example, Easton (1998, p. 20).

the elasticity of the capital/labor ratio in the next period – with respect to the net rate of return to capital investment in that period. More precisely,

$$(21)\qquad \eta_{f',e}(\rho_0) = \left.\frac{ef''(e)}{f'(e)}\right|_{e=\kappa(\omega(\rho_0),1-\delta+G(\rho_0))},$$

$$(22)\qquad \eta_{\kappa,\omega}(\rho_0) = \frac{D_1\kappa\big(\omega(\rho_0),1-\delta+G(\rho_0)\big)\omega(\rho_0)}{\kappa\big(\omega(\rho_0),1-\delta+G(\rho_0)\big)},$$

$$(23)\qquad \eta_{\omega,\rho}(\rho_0) = \frac{\rho_0\omega'(\rho_0)}{\omega(\rho_0)},$$

and

$$(24)\qquad \eta_{\kappa,r_1}(\rho_0) = \frac{D_2\kappa\big(\omega(\rho_0),1-\delta+G(\rho_0)\big)\big(1-\delta+G(\rho_0)\big)}{\kappa\big(\omega(\rho_0),1-\delta+G(\rho_0)\big)}.$$

At a fixed point of $G,$ we have $G(\rho_0)=\rho_0,$ and we shall denote by $\overline{\eta}_{f',e}(\rho_0),$ $\overline{\eta}_{\kappa,\omega}(\rho_0),$ $\overline{\eta}_{\omega,\rho}(\rho_0),$ and $\overline{\eta}_{\kappa,r_1}(\rho_0)$ the values assumed by the elasticities in (21), (22), (23), and (24), respectively, at the fixed point by replacing $G(\rho_0)$ with ρ_0 in these expressions. The derivative represented by (20) now becomes

$$(25)\qquad G'(\rho_0) = \overline{\eta}_{f',e}(\rho_0)\left[\overline{\eta}_{\kappa,\omega}(\rho_0)\overline{\eta}_{\omega,\rho}(\rho_0) + \overline{\eta}_{\kappa,r_1}(\rho_0)\frac{G'(\rho_0)\rho_0}{1-\delta+\rho_0}\right],$$

Solving (25) for $G'(\rho_0)$, we obtain

$$(26)\qquad G'(\rho_0) = \frac{\overline{\eta}_{f',e}(\rho_0)\overline{\eta}_{\kappa,\omega}(\rho_0)\overline{\eta}_{\omega,\rho}(\rho_0)[1-\delta+\rho_0]}{1-\delta+\rho_0[1+\overline{\eta}_{f',e}(\rho_0)\overline{\eta}_{\kappa,r_1}(\rho_0)]}.$$

Expression (26) represents the slope of $G:\rho_0\to G(\rho_0), 0\le\rho_0\le\rho^{\max},$ at a fixed point of this curve. We have the following result:

PROPOSITION 3: *Consider an economy that has no oil resources and that is sustained by renewable energy provided by a backstop. Let* ρ_0 *be the rental rate of capital at a steady state of this economy. If*

(27) $\quad 0 < G'(\rho_0) < 1$

then the convergence to this steady state is monotone. On the other hand, if

(28) $\quad -1 < G'(\rho_0) < 0,$

then the economy approaches this steady state in damped oscillations. Finally, if

(29) $\quad G'(\rho_0) < -1,$

then the fixed point is an unstable steady state, and there exists a 2- cycle (ρ^*,ρ^{**}) *that satisfies* $\rho^* < \rho_0 < \rho^{**}.$

PROOF: The conditions (27) and (28) characterize the convergence to a fixed point of a one-dimensional map in discrete time, while condition (29) characterizes the existence of 2-cycles. See, for example, Azariadis (1993, Chapters 7 and 8).

3.4. *Numerical Example*

Suppose that the lifetime utility function is

$$(30)\qquad \frac{[c^0]^{1-\sigma}}{1-\sigma}+\gamma\frac{[c^1]^{1-\sigma}}{1-\sigma}-\frac{1}{2\beta}\left(b-b^{\max}\right)^2.$$

In (30), c^0, c^1, and b denote, respectively, the current consumption, the future consumption, and the number of offspring raised by a young individual. Also, $\sigma, 0<\sigma<1,$ is the parameter characterizing the sub-utility function of consumption, $b^{\max}>1$ is the saturation number of offspring, and $\beta>0$ is the parameter characterizing the sub-utility function of offspring. The output produced by one worker – as a function of the energy input – is assumed to be given by $f(e)=ae^{\alpha}$, where $a>0$ represents the technological level of the economy and $\alpha, 0<\alpha<1,$ is a parameter.

Given the current wage rate ω_0, the part of the income that remains after paying for the cost of raising b children is $\omega_0-hb,$ which must be divided between current consumption and backstop capital investment. That is, the individual has to solve the following utility maximization problem

$$(31)\qquad \max_{(c^0,c^1)}\left(\frac{[c^0]^{1-\sigma}}{1-\sigma}+\gamma\frac{[c^1]^{1-\sigma}}{1-\sigma}\right)$$

subject to

$$(32)\qquad c^0+\frac{1}{1-\delta+\rho_1}c^1=m,$$

where we have let

$$(33)\qquad m=\omega_0-hb.$$

It is simple to show that the solution of the maximization problem constituted by (31), (32), and (33) is given by

$$(34)\qquad c^0=\frac{m}{1+\gamma^{\frac{1}{\sigma}}\left(1-\delta+\rho_1\right)^{\frac{1-\sigma}{\sigma}}}$$

and

$$(35)\qquad c^1=\frac{\gamma^{\frac{1}{\sigma}}[1-\delta+\rho_1]^{\frac{1}{\sigma}}m}{1+\gamma^{\frac{1}{\sigma}}\left(1-\delta+\rho_1\right)^{\frac{1-\sigma}{\sigma}}}.$$

Substituting (34) and (35) into the objective function in (31), we obtain the following expression for the indirect utility function that represents the discounted utilities of consumption over the two periods:

$$U(m,\rho_1) = \frac{m^{1-\sigma}}{1-\sigma}\left(\left[\frac{1}{1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}}\right]^{1-\sigma} + \gamma\left[\frac{\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1}{\sigma}}}{1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}}\right]^{1-\sigma}\right)$$

(36)

$$= \frac{m^{1-\sigma}}{1-\sigma}\left(1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\mu}.$$

It is clear that $U(m,\rho_1)$ is strictly increasing and strictly concave in m. Furthermore, $\ell im_{m\to 0}\frac{\partial U}{\partial m}(m) = +\infty$ and $\ell im_{m\to +\infty}\frac{\partial U}{\partial m}(m,\rho_1) = 0.$ Also, note that the curve $m \to \frac{\partial U}{\partial m}(m,\rho_1)$ shifts upward when ρ_1 rises. Given the choice of $b,$ the lifetime utility obtained is then given by

$$\chi(b) = U(\omega_0 - hb, \rho_1) - \frac{1}{2\beta}\left(b - b^{\max}\right)^2$$

(37)

$$= \frac{(\omega_0 - hb)^{1-\sigma}}{1-\sigma}\left(1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\sigma} - \frac{1}{2\beta}\left(b - b^{\max}\right)^2.$$

Differentiating (37) with respect to $b,$ we obtain:

(38) $$\chi'(b) = -\left[\frac{h}{(\omega_0 - hb)^{\sigma}}\left(1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\sigma} + \frac{b}{\beta}\right] + \frac{b^{\max}}{\beta}.$$

Observe that in (38) the expression inside the square brackets is positive and strictly increasing with $b, 0 \le b < \omega_0 / h.$ Furthermore, it tends to infinity when $b \to \omega_0 / h.$ Hence the individual will choose to raise children if and only if

(39) $$\chi'(0) = -\frac{h}{(\omega_0)^{\sigma}}\left(1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\sigma} + \frac{b^{\max}}{\beta} > 0.$$

The inequality in (39) can be simplified to

(40) $$\omega_0 > \left(\frac{\beta h}{b^{\max}}\right)^{\frac{1}{\sigma}}\left(1+\gamma^{\frac{1}{\sigma}}(1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right).$$

Given ρ_1, the rental rate of backstop capital in period 1, the expression on the right side of (40) represents the *critical level of labor income* at or below which the young individual will choose not to raise children. When we set $\rho_1 = 0$ in the expression on the right side of inequality (40), we obtain

$$(41) \qquad \omega^{\min} = \left(\frac{\beta h}{b^{\max}}\right)^{\frac{1}{\sigma}} \left(1 + \gamma^{\frac{1}{\sigma}} (1-\delta)^{\frac{1-\sigma}{\sigma}}\right),$$

the current version of the critical wage rate defined by (8). For a wage rate above $\omega^{\min}$, there is the possibility that a young individual might choose not to raise children if the rental rate of capital in the next period is high enough, and we have already indicated that the critical wage rate in this case is given by the right side of (40). However, according to Lemma 1, this will not happen in equilibrium.

When her labor income exceeds the critical wage level, which is given by the right side of (40), the optimal number of children that a young individual chooses to raise is the unique value of b that satisfies following the first-order condition:

$$(42) \qquad \frac{h\beta}{(\omega_0 - hb)^{\sigma}} \left(1 + \gamma^{\frac{1}{\sigma}} (1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\sigma} = b^{\max} - b.$$

It follows directly from (42) that a rise in the current wage rate ω_0 will induce a rise in the number of offspring b that the young individual chooses to raise. Furthermore, the part of labor income left after the cost of raising children has been paid, namely $\omega_0 - hb$, will also rise with ω_0. Not surprisingly, current consumption, future consumption, and the number of offspring are all normal goods. It also follows directly from (42) that a rise in ρ_1, the rental rate of capital in the next period, will induce a decline in b. The decline in b implies a rise in $\omega_0 - hb$, which in turn will trigger a rise in $\frac{1}{(\omega_0 - hb)^{\mu}} \left(1 + \gamma^{\frac{1}{\sigma}} (1-\delta+\rho_1)^{\frac{1-\sigma}{\sigma}}\right)^{\sigma}$, i.e., a decline in c^0, the current consumption. The decline in current consumption, given the rise in $\omega_0 - hb$, implies a rise in savings, which in turn implies a rise in future consumption. These results indicate that the preferences represented by (30) satisfy Assumption 3, namely future consumption is a gross substitute of both offspring and current consumption.

For the simulation exercise, the numerical values chosen for the parameters are: $a = 4, \alpha = 0.5, \sigma = 0.5, \beta = 25, \gamma = 0.65, h = 0.25, b^{\max} = 9, \delta = 0.75.$ Also, the initial backstop capital labor ratio is taken to be $\kappa_0 = 0.09$.

The critical energy input per worker is $e^{\min} = 0.07$, which yields the following values for the critical wage rate and the critical rental rate of backstop capital: $\omega^{\min} = 0.53$ and

$\rho^{\max} = 7.50$. Also, $G^{\max} = 2.89 < \rho^{\max}$, and Assumption 4 is satisfied. The largest confining interval in which the rental rate of backstop capital, i.e., the price of renewable energy, evolves is $[0, G^{\max}] = [0, 2.89]$.

As explained at the end of Section 2.2.2, we expect the curve $\omega_t \to s(\omega_t, r_{t+1}) / b(\omega_t, r_{t+1})$, $\omega_t > \omega^{\min}$, to have a U-shape. Figure 1 below depicts the shape of this curve for the present numerical example.

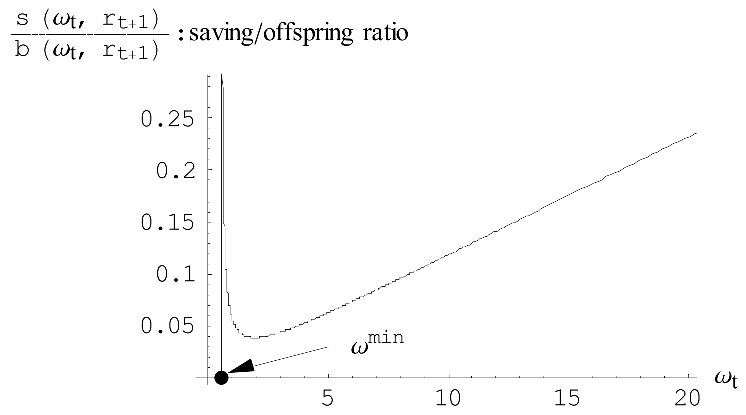

Figure 1.— The saving/offspring ratio as a function of labor income, given the rate of return to saving

Also, recall from Section 3.2 that the curve $\zeta(\omega_0,.) : \rho_1 \to \zeta(\omega_0, \rho_1), \rho_1 \geq 0$, represents the rental rate of capital in period 1 generated by the maximizing behavior of a young individual of period 0, given that ω_0 is her labor income and ρ_1 is the rental rate of backstop capital that this individual expects to prevail in period 1. Figure 2 depicts this curve for the present numerical example.

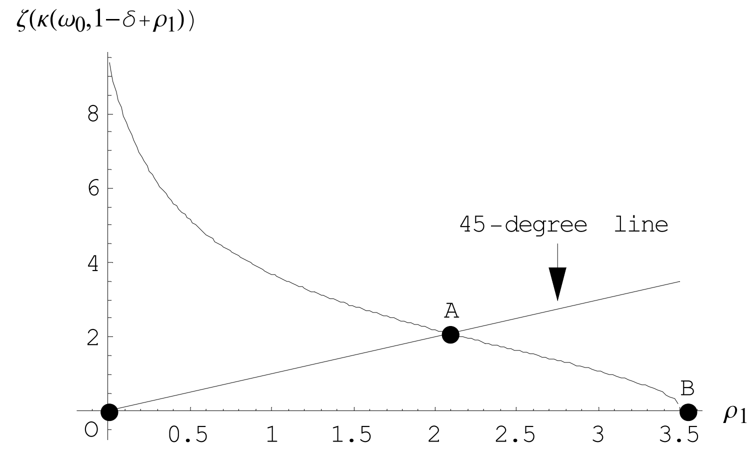

FIGURE 2.— The equilibrium rental rate of backstop capital in the next period, given the current wage rate.

The results of the simulation exercise are presented in the following table:

TABLE I
THE DYNAMIC COMPETITIVE EQUILIBRIUM: CONVERGENCE TO STEADY STATE IN DAMPED OSCILLATION
$(a=4, \alpha=0.5, \sigma=0.5, \beta=25, \gamma=0.65, h=0.25, b^{\max}=9, \delta=0.75)$

Period	κ_t (capital/labor ratio)	ρ_t (rental rate of capital)	ω_t (wage rate)	b_t (birth rate)
0	0.09	6.667	0.6	0.003
1	38.53	0.322	12.41	6.306
2	0.45	2.993	1.34	0.714
3	0.83	2.201	1.82	1.363
4	0.59	2.600	1.54	0.988
5	0.69	2.406	1.66	1.157
6	0.64	2.503	1.60	1.069
7	0.66	2.455	1.63	1.112
8	0.65	2.479	1.61	1.090
9	0.66	2.467	1.62	1.100
10	0.65	2.473	1.62	1.096
11	0.66	2.467	1.62	1.098
12	0.66	2.471	1.62	1.097

Clearly, Table I shows a steady state the existence of which is assured by Proposition 1. Our computations allow us to depict in Figure 3 the curve $G:\rho_0 \to G(\rho_0), 0 \le \rho_0 \le \rho^{\max}$. The steady state in this example is unique.

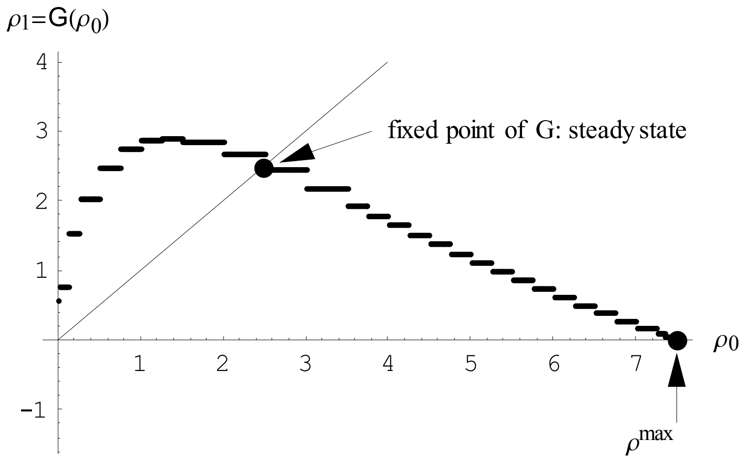

FIGURE 3.— The transition of the rental rate of capital from one period to another: the curve $G:\rho_0 \to G(\rho_0), 0 \le \rho_0 \le \rho^{\max}$.

A perusal of Table I reveals several interesting results. First, the economy converges to a steady state in about 12 periods. Second, the convergence is in damped oscillation. As can be seen from the second column of the table, the capital/labor ratio enters a small neighborhood of its steady state value in about 5 periods. The rapid convergence is due to the assumption on the sub-utility function of offspring. When the capital/labor ratio is low, the wage is also low. A low labor income in turn induces a young individual to give more weight to future consumption at the expense of the number of offspring she raises, resulting in a higher capital/labor ratio in the next period. In the simulation exercise, the initial capital/labor ratio has been chosen to be rather low. As can be seen from Table I, the equilibrium wage in period 0 is only 0.6, which induces an initial optimal birth rate of 0.003. The young generation of period 0 almost chooses to stop producing children. The capital/labor ratio in period 1 is 38.53, which is high, and the resulting high wage in that period induces the young generation of that period to raise more children. The number of offspring raised by a young individual of period 1 is 6.306, which helps to drive down the capital labor ratio in period 2. The special features of the sub-utility function of offspring thus have a stabilizing influence on the economy, and prevent the population from an abrupt collapse.

For the values chosen for the parameters, the population grows at the rate of 9.743% per period in steady state. In steady state, the lifetime plan of a young individual is given by

(43) $\bar{c}^0 = 0.63, \bar{c}^1 = 1.96, \bar{b} = 1.097, \bar{k} = 0.72.$

The lifetime plan (43) involves a capital/labor ratio of $\bar{\kappa} = 0.66,$ and yields a lifetime utility of 2.15.

If the cost of raising children is high, the marginal utility offspring is low, or the productivity of backstop capital is low, the steady state might involve a contracting population, i.e., a steady birth rate strictly less than 1. In this case, the economy will become extinct in the long run. Indeed, if the saturation number of offspring is $b^{\max} = 8$ instead of $b^{\max} = 9,$ the steady-state birth rate will be 0.96. A lower value of the saturation number of offspring implies that parents have less love for children, which leads to a birthrate below the replacement rate. If the saturation number of offspring assumes the value of $b^{\max} = 8.28234,$ then the steady-state birthrate is $\bar{b} = 1,$ i.e., the population becomes stable in the long run.

Because of the shape of the curve $G: \rho_0 \to G(\rho_0), 0 \le \rho_0 \le \rho^{\max},$ rising from 0 at $\rho_0 = 0,$ then declining to 0 when ρ_0 reaches the critical value $\rho^{\max}$, it is not surprising that the economy being analyzed has cycles. Indeed, if the parameters assume the following values:

$$a = 7.45, \alpha = 0.75, \sigma = 0.5, \beta = 15, \gamma = 0.75, h = 0.45, b^{\max} = 10, \delta = 1.0,$$

then the economy has a stable two-cycle $(\rho^*, \rho^{**}) = (4.610, 5.330),$ with the rental rate of backstop capital alternating between 4.610 and 5.330. In terms of birthrates, the two-cycle is $(b^*, b^{**}) = (1.612, 0.623),$ which indicates that the population will grow by 0.4% every two periods.

The traditional demographic explanation for fertility decisions rests on the interplay between a variable stock of humans and a relatively fixed factor – land – along the line of Malthus (1798). The numerical example just presented suggests that our model can generate fluctuations in fertility rates without appealing to a fixed factor, such as land. Convergence to a steady state in damped oscillation or stable cycles can be generated by varying the values of the parameters of the model. The source of the fluctuations in fertility can be found in the properties exhibited by the sub-utility function of offspring: a saturation level and a very small number – almost zero – of children produced when wages are low.

4. EXISTENCE OF COMPETITIVE EQUILIBRIUM FOR AN ECONOMY WITH OIL RESOURCES

In this section we give a proof of the existence of a dynamic competitive equilibrium for an economy with oil resources. The proof consists of three stages. In the first stage, we show that when the economy is truncated at the end of a finite number of periods, the truncated economy thus obtained always has a competitive equilibrium. In the second stage, we consider the sequence of truncated economies, with each element of the sequence indexed by the period at the end of which the original economy is truncated, and – through a series of technical arguments – establish some bounds on the endogenous variables that apply uniformly to all the truncated economies. In the third stage, we use these bounds to show that a sequence of competitive equilibria – one competitive equilibrium for each truncated economy – has a subsequence that converges in the product topology of a denumerable family of finite-dimensional Euclidean spaces. The limit of the subsequence is a dynamic competitive equilibrium of the infinite time horizon economy. The convergence is established by employing Cantor's diagonal trick used in proving the Tychonoff theorem; see, for example, Dieudonné (1976, (12.5.9)).

Suppose that the economy begins at time 0 in state (X_0, K_0, N_0^0, N_0^1), with $X_0 > 0, K_0 \geq 0$. Let $\xi_0 = X_0 / N_0^0$ and $\kappa_0 = K_0 / N_0^0$ denote the initial oil endowment/labor ratio and the initial capital/labor ratio, respectively. The initial energy endowments/labor ratio is thus equal to $\xi_0 + \kappa_0$. To keep the problem from becoming degenerate, we shall assume that the initial energy endowments per worker is sufficient to prevent the population from becoming extinct in the next period. Thus we make the following assumption:

ASSUMPTION 4: *The energy endowment per worker in period* 0 *is higher than the critical level* $e^{\min}$; *that is,* $\xi_0 + \kappa_0 > e^{\min}$.

Now let T be a non-negative integer. If we truncate our economy at the end of period T, then we obtain an economy with a finite time horizon that we call the truncated economy with time horizon T. A price system for the truncated economy with time horizon T is a finite sequence $\mathsf{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$. An *allocation induced by* P^T is a list of finite sequences

$$\mathsf{A}^T = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (K_t^\#, S_t^\#)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0\right)$$

with the following properties:

(i) $c_0^1 = [\phi_0 X_0 + (1-\delta+\rho_0)K_0]/N_0^1.$

(ii) $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ is the optimal lifetime plan for a young individual of period t, when the price system P^T prevails.

(iii) (Q_t, S_t, L_t, Y_t) is an optimal production plan of the representative firm in the consumption good sector in period t, when the price system P^T prevails.

(iv) $(K_t^\#, S_t^\#)$ is an optimal production plan for the representative producer of solar energy in period t, when the price system P^T prevails.

(v) $(X_t, K_t, N_t^0, N_t^1) = N_{t-1}^0(x_t, k_t, b_{t-1}, 1),\ t = 1,2,...,T.$

(vi) $c_T^0 = \omega_T.$

Observe that (vi) represents the consumption of a young individual in period T. Because the problem ends at the end of period T, a young individual of this period has no future to plan for and thus will neither save nor raise children; she will consume all the wages she earns. The pair $\left(\mathsf{P}^T, \mathsf{A}^T\right)$ is said to constitute *a competitive equilibrium for the truncated economy with time horizon T* if the following market-clearing conditions are satisfied:

(vii) $X_{t+1} + Q_t = X_t,\ 0 < t < T,$ and $X_T = Q_T.$

(vii) $S_t = S_t^\#,\ 0 \le t \le T,$

(viii) $L_t = N_t^0,\ 0 \le t \le T,$

(ix) $K_t^\# = K_t,\ 0 \le t \le T,$

(x)
$$[\phi_t X_t + (1-\delta+\rho_t)K_t] + [N_t^0(c_t^0 + b_t + k_{t+1})] = Y_t + (1-\delta)K_t,\ 0 \le t < T,$$
$$[\phi_T X_T + (1-\delta+\rho_T)K_T] + [N_T^0 c_T^0] = Y_T + (1-\delta)K_T.$$

PROPOSITION 4: *Consider an economy with a positive stock of oil and possibly a positive stock of backstop capital. For any integer $T \ge 0$, the truncated economy with time horizon T has a competitive equilibrium, say* $\left(\mathsf{P}^T, \mathsf{A}^T\right)$, with $\mathsf{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$ and

$$\mathsf{A}^T = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (K_t^\#, S_t^\#)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0\right).$$

Under such a competitive equilibrium, the birthrate in each period before the last period is positive. If we let $\xi_t = X_t / N_t^0$, $\kappa_t = K_t / N_t^0$, and $q_t = Q_t / N_t^0$ denote, respectively, the equilibrium oil endowment/labor ratio, the equilibrium capital/labor ratio, and the equilibrium oil input per worker – all in period t – then the following relationship holds between the price of oil and the price of renewable energy:

(44) $\phi_t \ge \rho_t,$ $(t = 0,...,T),$

with equality holding if $q_t > 0$. *Furthermore, the equilibrium price of energy in period t, namely* $\min\{\phi_t, \rho_t\}$, *satisfies the following condition*:

(45) $\quad 0 < \min\{\phi_t, \rho_t\} = f'(q_t + \kappa_t) = \rho_t < \rho^{\max}, \qquad (t = 0,\ldots,T).$

In particular, *the equilibrium price of oil and the equilibrium price of renewable energy in period* 0 *are equal, i.e.,*

(46) $\quad \phi_0 = \rho_0 = f'(q_0 + \kappa_0).$

The proof of Proposition 4 is based on the well-known technique developed by Debreu, Gale, and Nikaido.[9] Because fertility choice is endogenous, the number of consumers – old and young individuals – in each period is also endogenous. The proof of this proposition requires a modification of the technique developed by these researchers to accommodate the endogenous number of consumers in each period and to ensure that the population does not become extinct before the end of the time horizon. Because it is long and involves numerous technical arguments, the proof of Proposition 4 is relegated to Appendix B.

Now for each integer $T = 1,2,\ldots,$ let $(\mathsf{P}^T, \mathsf{A}^T)$, where $\mathsf{P}^T = (\phi_t^T, \varphi_t^T, \omega_t^T, \rho_t^T)_{t=0}^T$ and

$$\mathsf{A}^T = \begin{pmatrix} c_0^{1,T}, (c_t^{0,T}, c_{t+1}^{1,T}, b_t^T, x_{t+1}^T, k_{t+1}^T)_{t=0}^{T-1}, (Q_t^T, S_t^T, L_t^T, Y_t^T)_{t=0}^T, \\ (K_t^{\#,T}, S_t^{\#,T})_{t=0}^T, (X_t^T, K_t^T, N_t^{0,T}, N_t^{1,T})_{t=0}^T, c_T^{0,T} \end{pmatrix},$$

be a competitive equilibrium[10] for the truncated economy with time horizon *T*. We shall show that the sequence $(\mathsf{P}^T, \mathsf{A}^T)_{T=1}^\infty$ has a subsequence that converges in the product topology of a denumerable family of Euclidean spaces and that the limit of this subsequence is a dynamic competitive equilibrium for the original economy.

To begin the existence proof, we claim that the equilibrium prices of oil in period 1 are bounded above uniformly for all truncated economies; that is, $\sup_{T\geq 1} \phi_1^T < +\infty$. Indeed, if this is not true, then for each positive integer *n* we can find a positive integer $T(n)$ such that for the truncated economy with time horizon $T(n)$, the equilibrium price of oil in period 1 satisfies the inequality $\phi_1^{T(n)} \geq n$. The rate of return to oil investment for a young individual of period 0 in this truncated economy is then given by $\phi_1^{T(n)} / \phi_0^{T(n)} \geq n / \phi_0^{T(n)} >$ $n / \rho^{\max}$. The rate of return to capital for such a young individual is given by $1 - \delta + \rho_1^{T(n)}$ $< 1 - \delta + \rho^{\max}$, where the strict inequality has been obtained with the help of (45). Thus when *n* is large enough the rate of return to oil investment will be strictly higher than the rate of return to capital investment for a young individual of period 0 in the truncated

[9] See Nikaido (1970, Chapter 10).

[10] Observe that we have used the superscript *T* to indicate the time horizon of the truncated economy in question. The superscript *T* is needed to distinguish one truncated economy from another in the sequence of truncated economies used in the proof of Proposition 5. Such a superscript is not needed in the proof – given in the appendix – of Proposition 4 because in that proof we consider only one truncated economy and thus no possibility for confusion might arise.

economy with time horizon $T(n)$, and she will only invest in oil. The price of energy in period 1 for that truncated economy is then given by

$$\phi_1^{T(n)} = f'\left(q_1^{T(n)} + \kappa_1^{T(n)}\right)$$
$$= f'\left(q_1^{T(n)}\right) < \rho^{\max},$$

where the strict inequality has been obtained with the help of (45). The strict inequality contradicts the reductio ad absurdum hypothesis, and the claim is proved.

To continue, let us now consider the sequence of four-dimensional vectors $\left(\phi_0^T, \rho_0^T, \phi_1^T, \rho_1^T\right)_{T=1}^{\infty}$. According to (45) and (46), the sequence $\left(\phi_0^T, \rho_0^T, \rho_1^T\right)_{T=1}^{\infty}$ lies in a bounded set of the three-dimensional Euclidean space. We have also just shown that the sequence $\left(\phi_1^T\right)_{T=1}^{\infty}$ is bounded. Thus the sequence $\left(\phi_0^T, \rho_0^T, \phi_1^T, \rho_1^T\right)_{T=1}^{\infty}$ has a convergent subsequence that we denote by $\left(\phi_0^{\tau_0(n)}, \rho_0^{\tau_0(n)}, \phi_1^{\tau_0(n)}, \rho_1^{\tau_0(n)}\right)_{n=1}^{\infty}$, where $\tau_0 : n \to \tau_0(n)$ is an increasing map from the set of positive integers into itself. Let

$$\left(\phi_0, \rho_0, \phi_1, \rho_1\right) = \ell im_{n\to+\infty}\left(\phi_0^{\tau_0(n)}, \rho_0^{\tau_0(n)}, \phi_1^{\tau_0(n)}, \rho_1^{\tau_0(n)}\right).$$

The convergence of the subsequence $\left(\phi_0^{\tau_0(n)}, \rho_0^{\tau_0(n)}\right)_{n=1}^{\infty}$ implies the convergence of the subsequence of wage rates $\left(\omega_0^{\tau(n)}\right)_{n=1}^{\infty}$, the limit of which we denote by ω_0. For a young individual of period 0 in the subsequence of truncated economies with time horizons $\tau_0(n), n = 1,2,...,$ the convergence of her labor income, the current and future prices of oil, as well as the rental rate of capital in her old age, implies the convergence of her current consumption, her old-age consumption, and the number of children she raises. Furthermore, because the capital investment by a young individual of period 0 is bounded above by her labor income, the sequence $\left(k_1^{\tau_0(n)}\right)_{n=1}^{\infty}$ has a convergence subsequence that by abuse of notation we still denote by $\left(k_1^{\tau_0(n)}\right)_{n=1}^{\infty}$, and denote its limit by k_1. The convergence of $\left(k_1^{\tau_0(n)}\right)_{n=1}^{\infty}$ then implies the convergence of $\left(x_1^{\tau_0(n)}\right)_{n=1}^{\infty}$ the limit of which will be denoted by x_1. Thus

$$\ell im_{n\to+\infty}(c_0^{0,\tau_0(n)}, c_1^{1,\tau_0(n)}, b_0^{\tau_0(n)}, x_1^{\tau_0(n)}, k_1^{\tau_0(n)}) = (c_0^0, c_1^1, b_0, x_1, k_1)$$

exists and is an optimal lifetime plan for a young individual of period 0, given that ω_0 is her labor income; ϕ_0 is the current price of oil; ρ_0 is the current price of renewable energy; ϕ_1 is the price of oil in period 1; and ρ_1 is the rental rate capital in period 1. Next, note that the argument used to prove Claim 3 in Section B.7 of Appendix B can also be used here to assert that $b_0 > 0.$ Also, in the limit, the oil endowment/worker ratio and the capital/labor ratio in period 1 are given by $(\xi_1, \kappa_1) = \left(x_1 / b_0, k_1 / b_0\right).$ The argument used to establish Claim 5 in Section B.7 of Appendix B can also be used here to show that $\xi_1 + \kappa_1 > e^{\min}$.

As for an old individual of period 0, her consumption in the limit is given by $c_0^1 = \ell im_{n\to+\infty} c_0^{1,\tau_0(n)}$.

The argument used to prove $\sup_{T\geq 1}\phi_1^T < +\infty$ can be repeated to show that $\sup_{T\geq 1}\phi_2^T < +\infty$. Hence the sequence $\left(\phi_0^{\tau_0(n)}, \rho_0^{\tau_0(n)}, \phi_1^{\tau_0(n)}, \rho_1^{\tau_0(n)}, \phi_2^{\tau_0(n)}, \rho_2^{\tau_0(n)}\right)_{n=1}^{\infty}$ has a convergent subsequence that we denote by $\left(\phi_0^{\tau_1(\tau_0(n))}, \rho_0^{\tau_1(\tau_0(n))}, \phi_1^{\tau_1(\tau_0(n))}, \rho_1^{\tau_1(\tau_0(n))}, \phi_2^{\tau_1(\tau_0(n))}, \rho_2^{\tau_1(\tau_0(n))}\right)_{n=1}^{\infty}$, where τ_1 is an increasing map whose domain and range are equal to the image of the map τ_0. More precisely, the following limits exist

$$\ell im_{n\to+\infty}\left(\phi_1^{\tau_1(\tau_0(n))}, \rho_1^{\tau_1(\tau_0(n))}, \phi_2^{\tau_1(\tau_0(n))}, \rho_2^{\tau_1(\tau_0(n))}\right) = (\phi_1, \rho_1, \phi_2, \rho_2),$$

$$\ell im_{n\to+\infty}\omega_1^{\tau_1(\tau_0(n))} = \omega_1,$$

and

$$\ell im_{n\to+\infty}(c_1^{0,\tau_1(\tau_0(n))}, c_2^{1,\tau_1(\tau_0(n))}, b_1^{\tau_1(\tau_0(n))}, x_2^{\tau_1(\tau_0(n))}, k_2^{\tau_1(\tau_0(n))}) = (c_1^0, c_2^1, b_1, x_2, k_2).$$

Furthermore, $(c_1^0, c_2^1, b_1, x_2, k_2)$ is an optimal lifetime plan for a young individual of period 1 whose labor income is ω_1 and who faces the prices $(\phi_1, \rho_1, \phi_2, \rho_2)$. As in period 0, the birthrate in period 1 is also positive, and the energy endowment/labor ratio in period 2 is also above the critical level $e^{\min}$. The process just described can be repeated ad infinitum to obtain

(i) a sequence of increasing maps $\tau_0, \tau_1, \tau_2, ...$ with the domain and the range of one map – except τ_0 – being the image of the preceding map;

(ii) a price system $\mathsf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$; and

(iii) an allocation induced by P, say

$$\mathsf{A} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty}\right),$$

with the following properties:

(iv) for each $t = 0,1,...,$ we have

$$\ell im_{n\to+\infty}\left(\phi_{t'}^{\tau_t(...\tau_1(\tau_0(n)))}, \varphi_{t'}^{\tau_t(...\tau_1(\tau_0(n)))}, \omega_{t'}^{\tau_t(...\tau_1(\tau_0(n)))}, \rho_{t'}^{\tau_t(...\tau_1(\tau_0(n)))}\right)_{t'=0}^{t} = \left(\phi_{t'}, \varphi_{t'}, \omega_{t'}, \rho_{t'}\right)_{t'=0}^{t};$$

(v) for each $t = 0,1,...,$ the lifetime plan $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}$ is the optimal choice for a young individual of period *t* when she faces the price system P;

(vi) the pair (P, A) constitutes a competitive equilibrium for the economy with infinite time horizon.

We summarize the results just obtained in the following proposition:

PROPOSITION 5: *Consider an economy with a positive stock of oil and possibly a positive stock of backstop capital. This economy has a competitive equilibrium, say* (P, A), *with* $\mathsf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ *and*

$$\mathsf{A} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty}\right).$$

Under such a competitive equilibrium, the birthrate in each period is positive. If we let $\xi_t = X_t / N_t^0$, $\kappa_t = K_t / N_t^0$, *and* $q_t = Q_t / N_t^0$ *denote, respectively, the equilibrium oil endowment/labor ratio, the equilibrium capital/labor ratio, and the equilibrium oil input per worker – all in period t – then the following relationship holds between the price of oil and the price of renewable energy*:

(47) $\phi_t \geq \rho_t$, $(t = 0,1,...)$,

with equality holding if $q_t > 0$. *Furthermore, the equilibrium price of energy in period t, namely* $\min\{\phi_t, \rho_t\}$, *satisfies the following condition*:

(48) $0 < \min\{\phi_t, \rho_t\} = f'(q_t + \kappa_t) = \rho_t < \rho^{\max}$, $(t = 0,1,...)$.

In particular, *the equilibrium price of oil and the equilibrium price of renewable energy in period* 0 *are equal, i.e.,* $\phi_0 = \rho_0 = f'(q_0 + \kappa_0)$.

5. OIL EXTRACTION UNDER COMPETITIVE EQUILIBRIUM

In this section, we study the pattern of oil extraction under competitive equilibrium. To this end, let (P,A) be a dynamic competitive equilibrium, where $\mathsf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ and

$$\mathsf{A} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty}\right).$$

Recall that we have let $\xi_t = X_t / N_t^0$, $\kappa_t = K_t / N_t^0$, and $q_t = Q_t / N_t^0$ denote, respectively, the equilibrium oil endowment/labor ratio, the equilibrium capital/labor ratio, and the equilibrium oil input per worker.

Intuitively, we expect that in any period if the oil endowment per worker is large, but the capital/labor ratio is not, then the oil input per worker will be high. To see why, let us look at the identity that represents the division of the output of the consumption good between the two factors of production – energy and labor – in period *t*:

$$\begin{aligned} f(q_t + \kappa_t) &= \omega_t + (q_t + \kappa_t) f'(q_t + \kappa_t) \\ &= c_t^0 + hb_t + \phi_t(\xi_t - q_t) + k_{t+1} + (q_t + \kappa_t) f'(q_t + \kappa_t). \end{aligned} \tag{49}$$

Now when ξ_t is large, but q_t remains bounded above, say $q_t < M$, where M is a positive number, then the output of the consumption good produced by a worker will be bounded above by $f(M + \kappa_t)$ and the price of energy will be bounded below by $f'(M + \kappa_t)$. Furthermore, when ξ_t is large, $\xi_t - q_t$ is also large, which implies that $\phi_t(\xi_t - q_t)$ will be large. Thus when ξ_t is large, the second line of (49) will be large, while $f(q_t + \kappa_t)$ remains bounded above, and this is not possible. We have just established the following result:

LEMMA 3: *If* ξ_t *is large, but* κ_t *is not, then* q_t *will be large.*

To alleviate some of the technical arguments concerning the limiting behavior of the economy when the energy endowments/worker ratio is extremely high or close to the critical level $e^{\min}$, we shall assume that the earnings of the factor labor relative to the earnings of the factor energy are not very high. We state this assumption more precisely as follows:

ASSUMPTION 5: *For any positive value of e, we have* $f(e) - ef'(e) \le \overset{o}{b} ef'(e),$ *where* $\overset{o}{b}$ *is a constant satisfying* $1 < \overset{o}{b} < b^{\max}$.

Now when the energy endowments/worker ratio in a period is high, we expect the energy input per worker to be high; the number of offspring produced to be high; and the energy endowments/worker ratio in the next period to be high, but lower than the energy input per worker in the current period. To see why, suppose that $\xi_t + \kappa_t$ is large. If κ_t is large, then obviously $q_t + \kappa_t$ is large. If κ_t is not large, then ξ_t must be large, and $q_t + \kappa_t$ is also large in this case. A high energy input per worker in period *t* means a high wage rate in this period, which in turn implies a high number of offspring and a high level of saving for a young individual of period *t.* The saving/offspring ratio for such an individual then satisfies the following chain of inequalities:

$$\frac{s_t}{b_t} = \frac{x_{t+1} f'(q_t + \kappa_t) + k_{t+1}}{b_t}$$

$$= f'(q_t + \kappa_t)\left[\xi_{t+1} + \frac{\kappa_{t+1}}{f'(q_t + \kappa_t)}\right] < \frac{\omega_t}{b_t} < \frac{\overset{o}{b}}{b_t}(q_t + \kappa_t) f'(q_t + \kappa_t),$$

where that the last inequality has been obtained by invoking Assumption 5. Thus we have

$$\xi_{t+1} + \frac{\kappa_{t+1}}{f'(q_t + \kappa_t)} < \frac{\overset{o}{b}}{b_t}(q_t + \kappa_t).$$

Now when ξ_t is large, $f'(q_t + \kappa_t)$ will be small and $b_t > \overset{o}{b}$. In this case, we have the following chain of inequalities:

$$\xi_{t+1} + \kappa_{t+1} < \xi_{t+1} + \frac{\kappa_{t+1}}{f'(q_t + \kappa_t)} < \frac{\overset{o}{b}}{b_t}(q_t + \kappa_t) < q_t + \kappa_t.$$

We have just proved the following lemma:

LEMMA 4: *If* $\xi_t + \kappa_t$ *is large, then* $q_t + \kappa_t$ *is large and* b_t *is close to* $b^{\max}$. *Furthermore,* $\xi_{t+1} + \kappa_{t+1}$ *is also large, but* $\xi_{t+1} + \kappa_{t+1} < q_t + \kappa_t \le \xi_t + \kappa_t$.

The following lemma asserts that when oil resources are abundant capital will not be accumulated.

LEMMA 5: *If* ξ_t *is large, but* κ_t *is not, then a young individual of period t will put all her saving in oil.*

PROOF: If ξ_t is large, but κ_t is not, then q_t is large according to Lemma 3. According to Lemma 4, $\xi_{t+1}+\kappa_{t+1}$ is also large. Furthermore, the price of energy in period $t+1$ is higher than the price of energy in period *t,* i.e.,

(50) $\phi_{t+1} \geq f'(q_{t+1}+\kappa_{t+1}) \geq f'(\xi_{t+1}+\kappa_{t+1}) > f'(q_t+\kappa_t) = \phi_t.$

To prove the lemma, suppose $\kappa_{t+1} > 0$. There are then two cases to consider: (i) κ_{t+1} is large when ξ_t is large and (ii) κ_{t+1} remains bounded when ξ_t becomes indefinitely large. In case (i), the rental rate of capital in period $t+1$, namely $\rho_{t+1} = f'(q_{t+1}+\kappa_{t+1})$, will be close to 0, which implies that the rate of return to capital investment will be close to $1-\delta \leq 1$. However, according to (50), we have $\phi_{t+1}/\phi_t > 1$, i.e., for a young individual of period *t,* the rate of return to oil investment is greater than 1, and it will not be optimal for her to invest in oil. Case (i) thus cannot arise in equilibrium. In case (ii), ξ_{t+1} will be large, which, according to Lemma 3, implies that the price of energy in period $t+1$ will be low, and investing in capital will yield a rate of return close to $1-\delta$, which, again according to (50), is also strictly lower than the rate of return to oil investment. ■

LEMMA 6: *There exist two values, say ρ^- and ρ^+, which satisfy $0<\rho^-<\rho^+<\rho^{\max}$ and which do not depend on the rate of capital depreciation, such that $\rho^- < f'(q_t+\kappa_t) < \rho^+$, for all $t = 0,1,...$*

The proof of Lemma 6 is given in Appendix C. This lemma asserts the existence of a lower bound greater than 0 and an upper bound less than $\rho^{\max}$ for the equilibrium prices of energy through time, and that these bounds are independent of the rate of capital depreciation. The interval $[\rho^-,\rho^+]$ thus constitutes a confining interval for all the competitive equilibria, regardless of the rate of capital depreciation.

Now from the perspective of a young individual the decision on whether to invest in oil or capital depends on the rates of return of these assets. For capital investment, the rate of depreciation has a particularly important role to play. A high rate of depreciation discourages capital investment, while a low rate of depreciation, ceteris paribus, makes this asset relatively more attractive than oil. Thus when the rate of depreciation is low, we expect capital investment to be favored over oil investment; the successive generations prefer to invest only in capital, and we can expect that the oil stock will be exhausted in finite time. The following proposition confirms this intuition.

PROPOSITIN 6: *If the rate of capital depreciation is not too high, say $\delta < \rho^-$, then*

(i) *there exists a competitive equilibrium under which the oil stock is exhausted in finite time; and*

(ii) *there exists no competitive equilibrium under which the oil stock is exhausted asymptotically.*

PROOF: To prove the proposition, we first claim that if the rate of capital depreciation is not too high, say $\delta < \rho^-$, then there exists an integer $\overline{T}$ such that for any integer $T > \overline{T}$ and any competitive equilibrium of the truncated economy with time horizon T, the oil stock is exhausted in or before the penultimate period. To prove the claim, suppose that it is not true. Then for any positive integer n, there exists a positive integer $T, T > n$, and a competitive equilibrium for he truncated economy with time horizon T, say $\left(\mathsf{P}^T, \mathsf{A}^T\right)$, with $\mathsf{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$ and

$\mathsf{A}^T = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (\hat{K}_t, \hat{S}_t)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0\right)$,

such that $X_T > 0$. Because $X_T > 0$, the oil investment of every young generation before the last period must be positive, which implies that the price of oil must rise through time at a rate greater than or equal to the rate of capital investment, i.e.,

$$\frac{\phi_{t+1}}{\phi_t} \geq 1 - \delta + \rho_{t+1}, \qquad (t = 0, ..., T-1).$$

In particular, for $t = T-1$, we have

$$\frac{\phi_T}{\phi_{T-1}} \geq 1 - \delta + \rho_T.$$

Because oil exhaustion always occurs in a truncated economy, all of the remaining oil resources at the beginning of period *T* must be extracted for use in the consumption good sector, and this will constrain the price of oil in period *T* not to exceed the rental rate of capital in that period, i.e., $\phi_T \leq \rho_T$. Using this inequality in the preceding inequality, we obtain

$$\frac{\phi_T}{\phi_{T-1}}(1 - \phi_{T-1}) \geq 1 - \delta,$$

which constrains the price of oil in the penultimate period to be bounded above by 1, i.e., $\phi_{T-1} < 1$. However, we know that the price of oil must rise through time from the initial level $\phi_0 = f'(q_0 + \kappa_0 \geq f'(\xi_0 + \kappa_0) > 0$ at or above the rate of return to capital investment. Furthermore, according to Lemma 6, we must have $\rho_t > \rho^-$, for $t = 0, ..., T$. Hence, using the hypothesis $\delta < \rho^-$, we obtain $1 - \delta + \rho_t > 1 - \delta + \rho^- > 1$, for $t = 0, ..., T$. This last result implies that the price of oil in period $T-1$ will be arbitrarily large when T is large, which contradicts the hypothesis of the reductio ad absurdum argument. The claim is now established.

We are now ready to prove Proposition 6. To this end, note that in the sequence of truncated economies used in the proof of Proposition 5, oil resources are depleted by period $\overline{T}$ in all the truncated economies with time horizon greater than or equal to $\overline{T}$. Hence in the economy that is the limit of a subsequence of the sequence of truncated economies oil exhaustion also occurs by period $\overline{T}$. This proves part (i) of Proposition 6. To prove (ii) of Proposition 6, note that if the oil stock is exhausted asymptotically, then the price of oil must rises indefinitely through time at or above the rate of return to capital investment. The price of oil thus will tend to infinity when *t* tends to infinity. However, we have already argued in the proof of the claim that the price of oil in the period preceding a period in which oil is extracted for use in the production of the consumption good is bounded above by 1. Thus, part (ii) of Proposition 6 is established. ■

6. INCOMPLETE OIL EXHAUSTION UNDER COMPETITIVE EQUILIBRIUM

In the preceding section, we show that when the rate of capital depreciation is not too high, there exists a competitive equilibrium under which oil exhaustion occurs in finite time. The following question immediately arises. Are there equilibria under which oil resources are only exhausted asymptotically or under which part of the oil stock is left forever under the ground unexploited. The answer to the first part of the question is negative if the rate of capital depreciation is not too high, according to Proposition 6, while the answer to the second part is maybe, as illustrated by the numerical example given in Section 6.2 below.

6.1. *Steady State under Incomplete Oil Exhaustion*

Consider a competitive equilibrium (P,A), where $\mathsf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ and

$$\mathsf{A} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty}\right),$$

under which the oil stock is partially depleted. Let T be the last period the oil stock is exploited. Then we have $0 < Q_T < X_T$ and $Q_t = 0, t > T$.

Because all the young generations of period T or after put their savings in both oil and capital, we must have

$$\frac{\phi_{t+1}}{\phi_t} = 1 - \delta + \rho_{t+1}, \qquad (t \geq T).$$

Furthermore, the current budget constraint for a young individual of period $t \geq T$ can be expresses under the following form:

$$\text{(51)} \qquad \phi_t \frac{X_T}{N_t^0} = \omega_t - c_t^0 - hb_t - k_{t+1}.$$

Forwarding (51) by one period, we obtain

$$\text{(52)} \qquad \phi_{t+1} \frac{X_T}{N_{t+1}^0} = \omega_{t+1} - c_{t+1}^0 - hb_{t+1} - k_{t+2}.$$

Dividing (52) by (51), we obtain

$$\text{(53)} \qquad \begin{aligned} \frac{\phi_{t+1}}{\phi_t}\frac{N_t^0}{N_{t+1}^0} &= \frac{1-\delta+\rho_{t+1}}{b_t} \\ &= \frac{\omega_{t+1} - c_{t+1}^0 - hb_{t+1} - \kappa_{t+2}b_{t+2}}{\omega_t - c_t^0 - hb_t - \kappa_{t+1}b_{t+1}}, \end{aligned}$$

where we have let $\kappa_t = k_t / b_t$ denote the capital/labor ratio in period $t, t > T$. Now note that $\rho_t = f'(\kappa_t)$, $\omega_t = f(\kappa_t) - \kappa_t f'(\kappa_t)$, and $b_t = b(\omega_t, 1 - \delta + f'(\kappa_{t+1}))$, $t \geq T$. Hence (53) is a second-order nonlinear difference equation in the capital/labor ratio $\kappa_t, t \geq T$. If

(54) $\quad \overline{\overline{\kappa}} = \lim_{t\to+\infty} \kappa_t$

exists, then in the limit, the second equality in (53) becomes

(55) $\quad \dfrac{1-\delta+\overline{\overline{\rho}}}{\overline{\overline{b}}} = 1,$

where we have let $\overline{\overline{\rho}} = \lim_{t\to+\infty} \rho_t$ and $\overline{\overline{b}} = \lim_{t\to+\infty} b_t$.

If the rate of capital depreciation is not too high, then $\overline{\overline{\rho}} > \delta$ according to Lemma 6, and we must have $\overline{\overline{b}} = 1-\delta+\overline{\overline{\rho}} > 1,$ which means that in steady state the population and the price of oil all grow at a rate equal to the rate of return to capital investment. Furthermore, a young individual of any period owns only a fraction of the oil owned by her parent, with the fraction being the inverse of the number of children raised by the parent: the same oil stock is owned by each of the successive young generations, and due to population growth each young individual in later periods owns a smaller and smaller part of the economy's oil stock.

When is incomplete oil exhaustion a likely outcome under competitive equilibrium? To answer this question, let us look at the following more detailed representation of the division of output among the various uses in a steady state under incomplete oil exhaustion:

(56) $\quad (\overline{\overline{s}} - \overline{\overline{b}}\overline{\overline{\kappa}}) = f(\overline{\overline{\kappa}}) - \overline{\overline{\kappa}} f'(\overline{\overline{\kappa}}) - \overline{\overline{c}}^{0} - h\overline{\overline{b}} - \overline{\overline{b}}\overline{\overline{\kappa}}.$

In (56), we have let $\overline{\overline{s}}$ represent the saving of a young individual. The left side of (56) thus represents the funds allocated to oil investment. The right side of (56) represents what remains of the output of the consumption good produced per worker after (i) the factor capital has received its remuneration; (ii) the young individual has paid for her current consumption; (iii) the young individual has paid the costs of raising children; and (iv) the young individual has paid for the cost of capital investment required to sustain the steady state of the economy. If the earning of capital relative to output is high, there will be little left for wages. Furthermore, out of the low wages, the young individual must pay for her current consumption, the cost of raising children, and capital investment. Because the birthrate is higher than 1, the cost of raising children will be substantial if the cost of raising a child is high. There might not exist any value of $\overline{\overline{\kappa}}$ such that the right side of (56) is positive, a necessary condition for incomplete oil exhaustion. When such a value exists, one can always construct a competitive equilibrium under which the oil stock is only partially exploited, as asserted by the following proposition:

PROPOSITION 7: *For any value of* $\kappa > e^{\min}$, *let* $\rho(\kappa) = f'(\kappa)$ *and* $\omega(\kappa) = f(\kappa) - \kappa f'(\kappa)$ *denote, respectively, the rental rate of capital and the wage rate that prevail when only renewable energy is used in the production of the consumption good and when* κ *is the capital/labor ratio. Also, recall that* $c^0(\omega_t, r_{t+1})$ *and* $b(\omega_t, r_{t+1})$ *denote, respectively, the current consumption of a young individual of period t and the number of children she* *raises, given that ω_t is the prevailing wage rate in period t and r_{t+1} is the rate of return to her saving. If there exists a value of κ such that*

(57) $\quad f(\kappa)-\kappa\rho(\kappa)-c^0\big(\omega(\kappa),1-\delta+\rho(\kappa)\big)-\big(h+b\big(\omega(\kappa),1-\delta+\rho(\kappa)\big)\big)\kappa>0,$

then there exist infinitely many steady states in which part of the oil stock is left under the ground unexploited forever. Furthermore, the capital/labor ratio and the birthrate are lower in a steady state with incomplete oil exhaustion than in the steady state with complete oil exhaustion.

PROOF: Before proving Proposition 7, note that if the variable κ in (57) is interpreted as the steady-state capital/labor ratio under incomplete oil exhaustion, then the expression on the left side of inequality (57) represents the part of the saving put into oil by a young individual. Inequality (57) thus represents a necessary condition of incomplete oil exhaustion.

Now let κ_0 be a value of κ that satisfies (57) and

(58) $\quad \xi_0=\dfrac{1}{\phi_0}\Big[f(\kappa_0)-\kappa_0\rho(\kappa_0)-c^0\big(\omega(\kappa_0),1-\delta+\rho(\kappa_0)\big)-\big(h+b\big(\omega(\kappa_0),1-\delta+\rho(\kappa_0)\big)\big)\kappa_0\Big],$

where we have let $\phi_0=\rho(\kappa_0)$. Here we shall interpret κ_0 as the initial capital/labor ratio and ξ_0 – defined by (58) – as the initial oil endowment per worker. Next, let $b_0=b\big(\omega(\kappa_0),1-\delta+\rho(\kappa_0)\big)$ and suppose that in period $t, t=0,1,...,$ the price of oil, the rental rate of capital, and the wage rate are given, respectively, by $\phi_t=\phi_0 b_0^t$, $\rho_t=\rho(\kappa_0)$, $\omega_t=\omega(\kappa_0)$. It is straightforward to verify that when the price system $\mathsf{P}=\big(\phi_t,\rho_t,\omega_t\big)_{t=0}^{\infty}$ prevails, a young individual of each period $t=0,1,...,$ will have the same labor income, will have the same current and old-age consumption, will raise the same number of children, will invest in the same quantity of capital per child, and will spend the same amount of real resources to buy oil. The price system constructed and the lifetime plans induced by this price system thus constitute a competitive equilibrium. Under this competitive equilibrium, the oil stock is never exploited; the capital/labor ratio is constant; the birthrate is constant; the rate of return to capital is equal to the birthrate; and the price of oil rises through time geometrically at a rate equal to the birthrate. The competitive equilibrium thus constructed is thus a steady state for an economy with exhaustible resources.

Now note that if there exists a value of κ that satisfies (57), then by continuity all the capital/labor ratios in a small neighborhood of κ also satisfy (57), which implies that if there exists one steady state, then there exist infinitely many steady states.

Finally, note that when κ is high, the left side of (57) will be negative due to the Inada condition $\lim_{\kappa\to+\infty} f'(\kappa)=0$. Let $\bar{\kappa}$ be the smallest value of κ such that the left side of (57) is less than or equal to 0. Then any value of κ that satisfies (57) will be strictly less than $\bar{\kappa}$; that is, the capital/labor ratio in a steady state with incomplete oil exhaustion is less than that in the steady state with complete oil exhaustion. The lower capital/labor ratio in a steady state with incomplete oil exhaustion means a lower wage rate and a higher rate of return to saving. These last results imply – according to Assumptions 2 and 3 – a lower birthrate in a steady state with incomplete oil exhaustion than in the steady state with complete oil exhaustion. ■

Incomplete oil exhaustion is likely to exist if wages account for a proportion that is much higher than capital remuneration and if the cost of raising a child is low. If oil resources are abundant, the oil input per worker will be high according to Lemma 3. Thus the amount of oil left under the ground unexploited in the case of incomplete oil exhaustion will be relatively small so that in equilibrium successive young generations can afford to pay for the investment in this asset out of their wages.

6.2. *Numerical Example*

Suppose that preferences are represented by the following lifetime utility function: $Logc^0 + \gamma Logc^1 + v(b)$, with $v(b) = \beta Log(b+1), 0 \le b \le \hat{b}$, where β is a positive parameter and $\hat{b}$ is a constant greater than 1 but less than the saturation number of offspring $b^{\max}$. As specified, the single-period sub-utility function is logarithmic and the sub-utility function of offspring is also logarithmic in the relevant range $[0, \hat{b}]$. With logarithmic preferences, current consumption, the number of offspring, and saving depend only on labor income, not on the rate of return to saving. These features of logarithmic preferences allow for a closed-form solution of the lifetime utility maximization problem, and make the computations of competitive equilibria less burdensome. Given these preferences, it is simple to show that the optimal lifetime plan for a young individual of period t is given by

$$c_t^0 = \frac{h + \omega_t}{1 + \beta + \gamma}, \quad c_{t+1}^1 = \frac{\gamma(h + \omega_t)}{1 + \beta + \gamma} r_{t+1}, \quad b_t = \frac{\beta\omega_t - h(1+\gamma)}{h(1 + \beta + \gamma)}, \quad s_t = \frac{\gamma(h + \omega_t)}{1 + \beta + \gamma}.$$

Here, we recall, ω_t is her labor income; r_{t+1} is the rate of return to her saving; and s_t is the value of her saving. As for the output of the consumption good produced by a worker, we assume that it is given by $f(e) = e^{\alpha}$, $0 < \alpha < 1$. The following values for the parameters are assumed: $\alpha = 0.10$, $\beta = 0.73$, $\gamma = 0.77$, $h = 0.13$, and $\delta = 0.15$. The value of $\alpha = 0.10$ means that most of the output goes to labor as its remuneration. The cost of raising a child is $h = 0.13$, which is sufficiently low so that there are some funds left after a young individual has paid for her current consumption, for the cost of raising children, and for the cost of capital investment. If oil resources are abundant, the oil input per worker will be high, and the amount of oil left under the ground unexploited will be relatively small so that young generations can afford to pay for the investment in this asset out of their wages. In the numerical example, the oil stock is only about 18% of the stock of capital at the time extraction activities are terminated. Also, the initial oil endowment per worker and the initial capital/labor ratio are assumed to be given by $\xi_0 = 2.011$ and $\kappa_0 = 0$, respectively. The competitive equilibrium with incomplete oil exhaustion is presented in Table II.

TABLE II

COMPETITIVE EQUILIBRIUM WITH INCOMPLETE OIL EXHAUSTION

($\alpha = 0.10, \beta = 0.73, \gamma = 0.77,\ h = 0.13,\ \delta = 0.15,\ \xi_0 = 2.011,\ \kappa_0 = 0$)

Period	ξ_t	κ_t	q_t	ϕ_t	ρ_t	ω_t	b_t
0	2.011	0	0.480	0.194	0.194	0.836	1.179
1	1.898	0	0.453	0.204	0.204	0.831	1.168
2	1.253	0	0.421	0.218	0.218	0.825	1.155
3	0.721	0.097	0.287	0.237	0.237	0.818	1.138
4	0.381	0.165	0.175	0.264	0.264	0.808	1.116
5	0.185	0.209	0.081	0.305	0.305	0.795	1.087
6	0.095	0.232	0	0.372	0.372	0.778	1.047
…	…	…	…	…	…	…	…
t	$\frac{0.095}{b_6^{t-6}}$	0.232	0	$0.372 b_6^{t-6}$	0.372	0.778	1.047

As can be seen from Table II, the economy begins with a large oil endowment per worker, but no backstop capital. For the first three periods, all the energy requirements of the economy are met by drawing down the oil resources. Capital begins to be accumulated at the end of the third period, and energy produced by the backstop technology provides part of the energy inputs in the fourth period. The two technologies – fossil fuels and the backstop – are both exploited during three periods, with the backstop gradually replacing oil. From period 6 on, the oil stock is not exploited anymore. The amount of oil that remains at the beginning of period 6 is left forever under the ground, unexploited, and all the energy requirements of the economy are met by the backstop technology. The economy enters a steady state at the beginning of period 6, and in this steady state, the population and the aggregate capital stock both grow geometrically at the same rate of 1.047, which is also the rate of return to capital investment. Because the oil stock is no longer exploited, it passes from one generation to another, and each young individual owns a fraction – which is equal to the inverse of the birthrate in this steady state – of the oil belonging to a young individual of the previous period.

A competitive equilibrium with incomplete oil exhaustion is obviously not Pareto efficient because the oil stock is not exhausted. Although oil has an intrinsic value as an input in the production of the consumption good, the part of the oil stock left unexploited serves no production purposes. Its only use is a store of value, a means through which successive young generations transfer their incomes made during their working days to days of retirement. In this manner, the part of the oil stock left unexploited serves a basic function of money: a store of value. In contrast with paper money, which has no intrinsic value and might have a zero price in equilibrium,[11] oil left under the ground unexploited always has a positive value, which, according to our version of Hotelling rule in general equilibrium setting, must appreciate at the rate of interest, namely the real value of the

[11] See McCandless and Wallace (1991, Chapter 10).

solar energy harnessed from the marginal unit of capital. This surprising feature – a sort of oil bubble so to speak – is first encountered here.

7. A CHARACTERIZATION OF COMPETITIVE EQUILIBRIUM

It has been observed that in their efforts to obtain energy, humans first used firewood, then coal, fossil fuels, electricity, nuclear power... and energy from the Sun that will come some day. Energy resources that are more abundant and easy to harvest are first exploited, to be followed by energy resources that are less accessible and more costly to produce. The objective of this section is to present a characterization of this process of energy substitution under a competitive equilibrium. To concentrate on the influence of exhaustible resources on fertility decisions and on the process of technology substitution, we shall mimic those observations by only considering the case in which fossil fuels are abundant – but backstop capital is not – at the beginning, i.e., ξ_0 is large, but κ_0 is negligible. Also, we shall assume that the population is either stable or growing in the long run.

When fossil fuels are abundant at the beginning, a competitive equilibrium consists of three phases. In the first phase, the energy inputs used in the production of the consumption good come solely from oil, according to Lemma 3. Because the population does not become extinct in the long run, oil alone cannot sustain the economy indefinitely, and the backstop must begin to provide part of the energy requirements of the economy in finite time. The time interval that encompasses the introduction of the backstop and the end of extraction activities constitutes the second phase of a competitive equilibrium: the phase of technology substitution. The third phase of a competitive equilibrium begins after all extraction activities have been terminated, either due to oil exhaustion or because the competitive equilibrium in question involves incomplete oil depletion.

According to Lemma 3, the oil input per worker in period 0, namely q_0, will be high, which means a high wage rate in this period. The high labor income will induce a young individual of period 0 to raise a number of children close to the saturation level $b^{\max}$, which a fortiori implies $b_0 > \overset{o}{b}$, where $\overset{o}{b}$, we recall, is the parameter described earlier in Assumption 5. The division of the output of the consumption good produced by a worker between the two factors of production – labor and oil – is represented by the identity $f(q_0) = \omega_0 + \phi_0 q_0$. Furthermore, using Assumption 5 and the fact that $b_0 > \overset{o}{b}$, we can assert that $\omega_0 < b_0 \phi_0 q_0$. Also, according to Lemma 5, a young individual of period 0 will invest all her saving in oil. Thus we have $\kappa_1 = 0$ and

$$(59) \qquad \xi_1 = \frac{\xi_0 - q_0}{b_0} = \frac{s_0}{b_0 \phi_0} < \frac{\omega_0}{b_0 \phi_0} < q_0 < \xi_0.$$

The chain of inequalities in (59) indicate that the oil endowment per worker in period 1 is strictly less than the oil input per worker in period 0, which is in turn less than the oil endowment per worker in period 0. Because $q_1 \le \xi_1$, we must have

(60) $\phi_1 = f'(q_1) \geq f'(\xi_1) > f'(q_0) = \phi_0,$

i.e., the price of oil in period 1 will be higher than the price of oil in period 0. If the oil endowment per worker in period 1 is still large, the preceding argument can be repeated to assert that the oil input per worker – although lower than that in period 1 – is still high, and a version of (59) as well as a version of (60) also hold for period 1.

During the early phase of the competitive equilibrium, the oil endowment per worker falls rapidly. There are two reasons behind this fast decline. First, the price of oil must be low to clear the oil market. More precisely, the low price of oil induces the firms producing the consumption good to use more of this input. The high oil input per worker also means a high wage rate, allowing the young generation to save more, which, coupled with a low oil price, make it possible for the young generation to buy the rest of the oil stock as investment. Second, the initial high birthrates mean less oil is available for each worker in the following periods. One implication of the fast decline in the oil endowment per worker is a slow-down in the population growth. As the price of oil rises, the wage rate declines. The income effect on fertility decision is a decrease in the birthrate. If the rate of return to oil investment rises through time, the substitution effect – according to Assumption 3 – will reinforce the income effect and cause the birthrate to fall even further. This result bears out in the numerical example presented in Section 6.2. Thus we can expect the birthrate – which is high at the beginning – to decline steadily through time as the oil stock is being exploited.

In the second phase, technology substitution – backstop for fossil fuels – takes place. When both technologies are exploited in a period, say t, of the second phase, we must have

(61) $$\begin{aligned}\frac{\phi_t}{\phi_{t-1}} &= 1-\delta+\rho_t \\ &= 1-\delta+\phi_t.\end{aligned}$$

Observe that when (61) holds, we must have $\phi_{t-1} < 1;$ that is, when the two technologies are being exploited at the same time, the price of energy cannot be too high. If both technologies are exploited in each of the periods of the second phase, (61) will hold during the time interval in which the technology substitution takes place. Furthermore, according to Lemma 6, if the rate of capital depreciation is not too high, then the second line of (61) will be greater than 1, which implies that the price of oil as well as the rate of return to oil and capital both rise during the phase of technology substitution. Again as our discussion of the first phase, the rise in energy prices means a fall in the wage rate. Furthermore, a rise in the return to saving will induce a young individual to save more at the expense of children, according to Assumption 3. Thus, the birthrate continues to decline in the second phase.

How long does the second phase last? To answer this question, one must determine precisely the time ϕ_{t-1} exceeds 1, which requires many more technical arguments that we have not carried out. Needless to say, the length of the technology substitution phase depends critically on the rate of capital depreciation. In the particular case of $\delta = 1,$ (61)

is reduced to $\phi_t / \phi_{t-1} = \phi_t$, which leads to $\phi_t = \phi_{t-1} = 1$; that is when capital depreciates completely at the end of each period, a necessary condition for the two technologies to co-exist for a period is that the price of oil in that and in the previous period to be equal to 1, a result that cannot possibly arise in equilibrium. Thus, when capital depreciates completely, technology substitution occurs abruptly, with the backstop being brought into use only after the fossil fuels have been exhausted. The second phase does not exist in this case. This result is not hard to understand. When capital depreciates completely, it is not different from oil – an exhaustible resource – from the perspective of an investor: both fetch the same price on the energy markets and both are used up at the end of the production process.

During the third phase of a competitive equilibrium all the energy needs of the economy are met by the backstop. There are two possible scenarios to consider: complete oil exhaustion and incomplete oil exhaustion.

If the oil resources have been completely depleted when the third phase begins, a situation that might arise according to Proposition 6, then the evolution of the economy from this time on is completely determined by the backstop technology and the preferences, as described in Section 3. Proposition 3 in this section describes how the economy behaves in the long run. Depending on the values of the various elasticities, the economy might converge to a steady state in a monotone manner, in damped oscillation, or it might converge to a stable cycle. The existence of an exhaustible resource has only a fleeting impact in the short run, especially at the beginning when the resource makes it possible for the population to grow rapidly and for capital to accumulate in a less painful manner. One can visualize through time the process of transforming oil into the consumption good and into backstop capital. In the long run oil does not influence the birthrate; its only impact is to allow for a population with a larger absolute size.

In Section 6.2, we presented a numerical example for which oil exhaustion is incomplete. The competitive equilibrium computed and exhibited in that section is not the only equilibrium of this numerical model. There is another competitive equilibrium under which the oil stock is exhausted after a finite number of periods have elapsed. This competitive equilibrium is presented in Table III. Under this competitive equilibrium, the oil stock is depleted rapidly, and is completely exhausted at the end of period 5. After that the economy is completely sustained by the backstop. Compared to the competitive equilibrium with incomplete oil exhaustion, capital accumulation begins one period sooner under the competitive equilibrium with complete oil exhaustion. The two technologies co-exist during four periods, and the economy enters a steady state – without any oil left – in period 6, with a higher capital/labor ratio and a higher birthrate.

TABLE III

COMPETITIVE EQUILIBRIUM WITH COMPLETE OIL EXHAUSTION

$(\alpha = 0.10, \beta = 0.73, \gamma = 0.77,\ h = 0.13,\ \delta = 0.15,\ \xi_0 = 2.011,\ \kappa_0 = 0)$

Period	ξ_t	κ_t	q_t	ϕ_t	ρ_t	ω_t	b_t
0	2.011	0	0.480	0.194	0.194	0.836	1.179
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2	0.826	0.084	0.337	0.218	0.218	0.825	1.155
3	0.424	0.161	0.222	0.237	0.237	0.818	1.138
4	0.177	0.214	0.127	0.264	0.264	0.808	1.116
5	0.044	0.246	0.044	0.305	0.305	0.795	1.087
6	0	0.261	0	0.335	0.335	0.787	1.068
7	0	0.263	0	0.332	0.332	0.788	1.068
8	0	0.263	0	0.332	0.332	0.788	1.068
…	…	…	…	…	…	…	…
t	0	0.263	0	0.332	0.332	0.788	1.068

Note that under both competitive equilibria, the convergence to steady state is monotone, due to logarithmic preferences.

In the case the competitive equilibrium involves incomplete oil exhaustion, there are infinitely many possible steady states according to Proposition 7. For the numerical example of Section 6.2, the competitive equilibrium presented in that sub-section is one among the infinitely many possible equilibria with incomplete oil exhaustion. With a deliberate choice of the same set of parameters, Table III provides the outcome of an equilibrium with complete oil exhaustion. The possibility of multiple equilibria arises from the indeterminate mix of solar energy and oil use that we have pointed out in solving the profit maximization problem constituted by (12) and (13) in Section 2.2.4. Starting from the same initial oil stock and the same initial capital stock, the economy may evolve through time along a different equilibrium trajectory, reaching the point at which solar energy is substituted for oil at a time different from the time technology substitution occurs under incomplete oil exhaustion. Furthermore, at the time technology substitution takes place, the oil stock and the capital stock under complete oil exhaustion may assume values that are different from those under incomplete oil exhaustion (including the special case the oil stock is completely exhausted when the backstop is first brought into use). Which steady state the economy will converge to in the long run depends on the state of the system at the time the backstop completely replaces oil in the production of the consumption good. In each of these steady states, the capital/labor ratio and the birthrate are both lower than those in the state with complete oil exhaustion.

8. CONCLUSION

The new feature in this paper is the analysis of the transition to sustainability for an economy in which the labor force – or population – is endogenous and in which the production technology allows for the substitution of an exhaustible, says oil, by an everlasting source of energy, says energy from the Sun. As modeling strategy, we chose the overlapping-generations framework, and adopt the simplest specification possible. In the model, an individual agent has a two-period life; treats offspring as a consumption good; and transfers his wealth over time through either oil or capital asset, or both.

Before discussing the issue of transition, let us first return to Section 3, which deals with the case the economy has no oil and must accumulate capital to harness the energy from the Sun. As mentioned in our survey of the literature on endogenous fertility decision, the basic advances – in our view – have been accomplished in Becker and Barro (1988), and Barro and Becker (1989), with the useful dynastic formulation: each one-period lived parent takes into consideration not only its own consumption, but also the number as well the well-being of their children. Offspring are treated as a complex composite good, with weights put on children as a consumption and on their utility as a ''quality'' measure of that consumption. Thanks to this assumption, recurrent consideration allows us to subsume the utility of all future generations in the dynastic head's objective, transforming the dynamic problem into one of a social planner with infinite life and perfect foresight in the Ramsey-Solow tradition. Barro and Becker, op cit., pointed out but not elucidated – the possibility of multiple steady states and the complex dynamics of convergence. In contrast with these researchers, we have adopted a truncated foresight approach, with two-period lived individual agents who care for his young and old age consumption, and consider children as a consumption good which provides some pleasure of its own. Our view is thus somewhat more ''market oriented'' than the central planning view, and the existence of dynamic competitive equilibrium, the possibility of multiple steady states, as well as the complex dynamic convergence, are all properly worked out.

Section 3 is, however, a prelude to the analysis of the transition to economic sustainability, which consists of replacing exhaustible oil by solar energy as the source of energy needed to produce the consumption goods. The matters involved, which are analyzed in the rest of our paper, are rather intriguingly complicated. First, the existence of competitive equilibrium is problematic in that the market size is itself endogenous, because fertility – hence the population – is an individual decision at every point in time. Second, and perhaps most interestingly, the oil stock might not be entirely depleted, and the unused part *in situ* may serve the role of storing value for wealth transmission over time, just as a money. In this event, there are infinitely many equilibria as well as many steady states, depending on the data that characterize the initial state of our economy. This raises the problem of indeterminacy of equilibrium encountered in the literature. Again, the dynamic convergence remains, as before, complex. The path toward an equilibrium, far from being simply monotone, might exhibit cyclical behavior. Last but not least, in the event of incomplete resource exhaustion, oil serves the role of storing value as money. In contrast with paper money, which has no intrinsic value, oil is a factor of production. Leaving valuable oil *in situ* certainly adds another dimension to the inefficiency of overlapping-generation model in this case. In the terminology of Tirole (1985), oil becomes a financial bubble despite its productivity in contributing to real production.

One shortcoming of this paper is that it does not get into normative considerations. It is well known that inefficiency of many kinds usually arise in overlapping-generations models. In addition to the eventual ''under-accumulation'' of capital, we now face the possibility of oil bubble, an ''over-accumulation'' of a resource asset. Beside the prescription of affecting the pattern of capital accumulation through taxation imposed on bequests, gifts, etc. in order to restore economic efficiency, the obvious policy implication of our work is how to induce complete resource exhaustion by some public intervention which should, at some point in time, discourage asset holding under the form of *in situ* resource. The task is not that simple, because of the general equilibrium repercussion of any policy on the rest of the economy. This would lead us too far afield, so we keep it on our research agenda for some time to come.

APPENDIX A: THE PROOF OF LEMMA 2

First, note that $\ell im_{\rho_0 \uparrow \rho^{\max}} \kappa(\omega(\rho_0),1-\delta) = +\infty.$ Furthermore, according to Assumption 3, we have $\kappa(\omega(\rho_0),1-\delta) < \kappa(\omega(\rho_0),1-\delta+g(\omega(\rho_0))),$ which implies that

$$f'(\kappa[\omega(\rho_0),1-\delta]) > f'(\omega(\rho_0),1-\delta+g(\omega(\rho_0))) = G(\rho_0).$$

Hence

$$\begin{aligned}\ell im_{\rho_0 \uparrow \rho^{\max}} G(\rho_0) &= \ell im_{\rho_0 \uparrow \rho^{\max}} f'(\omega(\rho_0),1-\delta+g(\omega(\rho_0))) \\ &\leq \ell im_{\rho_0 \uparrow \rho^{\max}} f'(\kappa[\omega(\rho_0),1-\delta]) = 0,\end{aligned}$$

which is (i) of the Lemma 2. Next, note that as $\rho_0 \downarrow 0,$ the capital/labor ratio and the wage rate associated with $\rho_0,$ namely $\kappa_0,$ and $\omega(\rho_0),$ both tend to infinity. Because the current wage rate tends to infinity, the current consumption and the future consumption of a young individual both tend to infinity – even when the rental rate of capital in the next period is 0. Also, the number of offspring raised by a young individual will rise to the saturation level $b^{\max}$. Hence the capital/labor ratio generated by the maximizing behavior of a young individual of period 0 will tend to infinity, which implies that the rental rate capital in period 1, namely $G(\rho_0),$ will tend to 0, establishing (ii).

As for the last statement of Lemma 2, note that as $\rho_0 \downarrow 0,$ the capital/labor ratio generated by the maximizing behavior of a young individual, say $\kappa_1,$ satisfies the following inequality:

$$\kappa_1 < \frac{f(\kappa_0)}{b^{\max}} = \frac{1}{b^{\max}}\left[\frac{f(\kappa_0)}{\kappa_0}\right]\kappa_0.$$

Due to the Inada condition imposed on f, the expression inside the square brackets tends to 0 as $\kappa_0 \to +\infty$. Hence κ_1 / κ_0 is arbitrarily small when ρ_0 is sufficiently small, which means that $G(\rho_0) = \rho_1 = f'(\kappa_1) > f'(\kappa_0) = \rho_0$ for all ρ_0 in a right neighborhood of 0. ■

APPENDIX B: THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM FOR A TRUNCATED ECONOMY

B.1. *The Price System*

Let T be a nonnegative integer and $\mathsf{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t, p_t)_{t=0}^T$ be a price system for the truncated economy with time horizon T. Here $\phi_t, \varphi_t, \omega_t, r_t,$ and p_t denote, respectively, the price of oil, the price of solar energy, the wage rate, the rental rate of capital, and the price of the consumption good – all in period $t, t = 0,...,T.$ For mathematical convenience, we normalize the price system P^T so that it belongs to the $(5T+4)-$dimensional unit simplex $\Delta^{(5T+4)}$, i.e., $\sum_{t=0}^T (\phi_t + \varphi_t + \omega_t + \rho_t + p_t) = 1.$

B.2. *A Bound for the Economy*

In the Debreu-Gale-Nikaido technique, a bound, sufficiently large, is imposed upon the economy to obtain the solutions of the various maximization problems, even when some of the prices are zero. To find a bound for the economy under consideration, let $(\widehat{X}_t, \widehat{K}_t, \widehat{N}_t^0, \widehat{N}_t^1, \widehat{Y}_t)_{t=0}^T$ be the finite sequence defined recursively in the following manner.

For $t = 0,$ set

(B.2.1) $$\left(\widehat{X}_0, \widehat{K}_0, \widehat{N}_0^0, \widehat{N}_0^1, \widehat{Y}_0\right) = \left(X_0, K_0, N_0^0, N_0^1, F(X_0 + K_0, N_0^0)\right).$$

For $0 < t \le T,$ set

(B.2.2) $$\begin{aligned}&\left(\widehat{X}_t, \widehat{K}_t, \widehat{N}_t^0, \widehat{N}_t^1, \widehat{Y}_t\right)\\ &= \left(\widehat{X}_{t-1}, \widehat{Y}_{t-1} + (1-\delta)\widehat{K}_{t-1}, \frac{\widehat{Y}_{t-1} + (1-\delta)\widehat{K}_{t-1}}{h}, \widehat{N}_{t-1}^0, F(\widehat{X}_t + \widehat{K}_t, \widehat{N}_t^0)\right).\end{aligned}$$

As defined, the five elements of the list $\left(\widehat{X}_t, \widehat{K}_t, \widehat{N}_t^0, \widehat{N}_t^1, \widehat{Y}_t\right)$ represent, respectively, upper bounds on the oil stock, the stock of backstop capital, the size of the young generation, the size of the old generation, and the output of the consumption good – all in period $t, t = 0,...,T.$ A bound, say $M,$ on the economy can then be obtained by choosing M such that

(B.2.3) $$M > \widehat{X}_t + \widehat{K}_t + \widehat{N}_t^0 + \widehat{N}_t^1 + \widehat{Y}_t, \qquad (t = 0,...,T).$$

The value of M thus chosen ensures that value of any state variable at any time and under any possible evolution of the economy is bounded above by M.

B.3. *The Dynamics Generated by Lifetime Utility Maximization*

Let $\Gamma(\mathsf{P}^T)$ be the set of all possible realizations of the system generated by the maximization behavior of the successive young generations when the price system P^T

prevails. More precisely, an element of $\Gamma(\mathrm{P}^T)$ is a finite sequence, say $(X_t, K_t, N_t^0, N_t^1)_{t=0}^T$, defined recursively as follows.

For $t=0$, set $(X_t, K_t, N_t^0, N_t^1) = (X_0, K_0, N_0^0, N_0^1)$, the initial state of the economy. Next, suppose that (X_t, K_t, N_t^0, N_t^1) has been defined for $0 \le tT-1$. Let $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ be the lifetime plan that solves the following "more restrained" lifetime utility maximization problem:

(B.3.1) $\quad \max_{(c^0, c^1, b, x, k)} u(c^0, c^1, b)$

subject to the following current and future budget constraints

(B.3.2) $\quad p_t(c^0 + hb + k) + \phi_t x - \omega_t \le 0,$

(B.3.3) $\quad p_{t+1}c^1 - \phi_{t+1}x - p_{t+1}(1-\delta+\rho_{t+1})k \le 0,$

and the following three "*bounding constraints*"

(B.3.4) $\quad N_t^0(c^0 + hb + k) - M \le 0,$

(B.3.5) $\quad N_t^0 x - M \le 0,$

(B.3.6) $\quad N_t^0 c^1 - M \le 0.$

The bounding constraints are typical of the Debreu-Gale-Nikaido proof technique. Here they are temporarily imposed to ensure that the above lifetime utility maximization problem has a solution, even when some of the prices are zero. At the end of the existence proof, it will be shown that these bounding constraints are not binding.

We have already explained that the first three components of $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ are always unique, while the last two components, namely x_{t+1} and k_{t+1}, are only unique when the rate of return on oil investment and the rate of return on backstop capital investment are different. When these two rates of return are equal, the investment mix (x_{t+1}, k_{t+1}) are indeterminate and is only required to satisfy the constraint $p_t(c_t^0 + hb_t + k_{t+1}) + \phi_t x_{t+1} - \omega_t \le 0$. Thus for any two elements of $\Gamma(\mathrm{P}^T)$, say (X_t, K_t, N_t^0, N_t^1) and $(\hat{X}_t, \hat{K}_t, \hat{N}_t^0, \hat{N}_t^1)$, we always have $(N_t^0, N_t^1) = (\hat{N}_t^0, \hat{N}_t^1)$, but it might happen that $(X_t, K_t) \ne (\hat{X}_t, \hat{K}_t)$.

It is clear that $\Gamma(\mathrm{P}^T)$ is a nonempty convex compact subset of the $4(T+1)-$dimensional Euclidean space. Furthermore, it is simple to show that the map $\Gamma : \mathrm{P}^T \to \Gamma(\mathrm{P}^T)$ is a closed point-to-set map.

B.4. *The Dynamics Generated by Profit Maximization*

B.4.1. *Profit Maximization in the Backstop Sector*

For each $t = 0, ..., T$, let

(B.4.1.1) $\Psi_t^{\#}(\mathsf{P}^T) = \arg\max_{(K^{\#},S^{\#})}[\varphi_t S^{\#} - \rho_t K^{\#}]$

subject to the technological constraint

(B.4.1.2) $S^{\#} - K^{\#} \leq 0,$

and the following additional bounding constraint

(B.4.1.3) $S^{\#} + K^{\#} - M \leq 0.$

As defined, $\Psi_t^{\#}(\mathsf{P}^T)$ is the set of feasible production plans that maximize the profits, subject to the bounding constraint (B.4.1.3), in period t of the energy producers in the backstop sector when the price system P^T prevails. The set $\Psi_t^{\#}(\mathsf{P}^T)$ is clearly nonempty, convex, and compact. Furthermore, it is simple to show that the point-to-set map $\Psi_t^{\#} : \mathsf{P}^T \to \Psi_t^{\#}(\mathsf{P}^T)$ is closed.

B.4.2. *Profit Maximization in the Consumption Good Sector*

For each $t = 0,...,T,$ let

(B.4.2.1) $\Psi_t(\mathsf{P}^T) = \arg\max_{(Q,S,L,Y)}[p_t Y - \phi_t Q - \varphi_t S - \omega_t L]$

subject to the technological constraint

(B.4.2.2) $Y - F(Q+S, L) \leq 0$

and the following additional bounding constraint

(B.4.2.3) $Q + S + L + Y - M \leq 0.$

As defined, $\Psi_t(\mathsf{P}^T)$ is the set of feasible production plans that maximize the profits – subject to the bounding constraint (B.4.2.3) – in period t of the representative firm in the consumption good sector when the price system P^T prevails. The set $\Psi_t(\mathsf{P}^T)$ is clearly nonempty, convex, and compact. Furthermore, it is simple to show that the point-to-set map $\Psi_t : \mathsf{P}^T \to \Psi_t(\mathsf{P}^T)$ is closed.

B.5. *Excess Demand and Walras' Law*

Let $\Psi^{\#}(\mathsf{P}^T) = \prod_{t=0}^{T} \Psi_t^{\#}(\mathsf{P}^T)$ and $\Psi(\mathsf{P}^T) = \prod_{t=0}^{T} \Psi_t(\mathsf{P}^T).$ Now for each sequence (X_t, K_t, N_t^0, N_t^1) in $\Gamma(\mathsf{P}^T),$ each sequence $(K_t^{\#}, S_t^{\#})_{t=0}^T$ in $\Psi^{\#}(\mathsf{P}^T),$ and each sequence $(Q_t, S_t, L_t, Y_t)_{t=0}^T$ in $\Psi(\mathsf{P}^T),$ let us associate a finite sequence of five-dimensional vectors, say $(Z_t)_{t=0}^T$, defined as follows. For $t = 0,...,T-1,$ set

$$\text{(B.5.1)} \qquad Z_t = \begin{pmatrix} Q_t + X_{t+1} - X_t, \\ S_t - S_t^{\#}, \\ L_t - N_t^0, \\ K_t^{\#} - K_t, \\ N_t^0(c_t^0 + hb_t + k_{t+1}) + N_t^1 c_t^1 - Y_t - (1-\delta)K_t \end{pmatrix}.$$

Observe that the elements in the list on the right side of (B.5.1) represent, respectively, the excess demands for oil, solar energy, backstop capital, labor, and the consumption good – all in period t. Furthermore, $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}$ is the sequence of lifetime plans that give rise to the sequence $(X_t, K_t, N_t^0, N_t^1)_{t=1}^T$. Also, in period 0, the total consumption of the old generation is

(B.5.2) $$N_0^1 c_0^1 = \min\{M, \frac{\phi_0 X_0 + [(1-\delta)p_0 + \rho_0]K_0}{p_0}\}.$$

For $t = T$, set

(B.5.3) $$Z_T = \begin{pmatrix} Q_T - X_T, \\ S_T - S_T^{\#}, \\ L_T - N_T^0, \\ K_T^{\#} - K_T, \\ N_T^0 c_T^0 + N_T^1 c_T^1 - Y_T - (1-\delta)K_T \end{pmatrix},$$

where the total consumption of the young generation in the last period is

(B.5.4) $$N_T^0 c_T^0 = \min\{M, \frac{N_T^0 \omega_T}{p_T}\}.$$

Let $\mathsf{P}_t^T = (\phi_t, \varphi_t, \omega_t, \rho_t, p_t)$ denote the price system in period $t, t = 0,...,T$. We claim that $\mathsf{P}_t^T.Z_t =\leq 0, t = 0,...,T$, i.e., Walras' law holds in each period. Indeed, for $t = 0,...,T-1$, the value of the list of excess demands in period t is

(B.5.5) $$\begin{aligned} \mathsf{P}_t^T.Z_t = {} & \phi_t(Q_t + X_{t+1} - X_t) + \varphi_t(S_t - S_t^{\#}) + \omega_t(L_t - N_t^0) + \rho_t(K_t^{\#} - K_t) \\ & + p_t\left(N_t^0(c_t^0 + hb_t + k_{t+1}) + N_t^1 c_t^1 - Y_t - (1-\delta)K_t\right) \\ = {} & [\phi_t Q_t + \varphi_t S_t + \omega_t L_t - Y_t] + [\rho_t K_t^{\#} - \varphi_t S_t^{\#}] \\ & + [N_t^0(p_t(c_t^0 + hb_t + k_{t+1}) - \omega_t))] \\ & + [p_t N_t^1 c_t^1 - \phi_t X_t - (p_t(1-\delta) + \rho_t)K_t] \leq 0. \end{aligned}$$

On the right side of the second equality in (B.5.5), the expression inside the first pair of square brackets is less than or equal to zero because (Q_t, S_t, L_t, Y_t) is an optimal production plan for the representative firm in the production good sector; the expression inside the second pair of square brackets is less than or equal to zero because $(K_t^{\#}, S_t^{\#})$ is an optimal production for the representative producer of solar energy; the expression inside the third pair of square brackets is less than or equal to zero because it represents the current budget constraint of the young generation; and the expression in the last pair of square brackets is less than or equal to zero because it represents the budget constraint of the old generation.

As for the value of excess demand in period T, we have

$$
\begin{aligned}
\text{(B.5.6)}\qquad \mathsf{P}_T^T.Z_T &= \phi_T(Q_T - X_t) + \varphi_T(S_T - S_T^{\#}) + \omega_T(L_T - N_T^0) + \rho_T(K_T^{\#} - K_T) \\
&\quad + p_T\left(N_T^0 c_T^0 + N_t^1 c_t^1 - Y_t - (1-\delta)K_t\right) \\
&= [\phi_T Q_T + \varphi_T S_T + \omega_T L_T - Y_T] + [\rho_T K_t^{\#} - \varphi_T S_T^{\#}] \\
&\quad + [N_t^0 (p_T c_T^0 - \omega_T)] \\
&\quad + [p_T N_T^1 c_T^1 - \phi_T X_T - (p_T(1-\delta) + \rho_T)K_T] \le 0.
\end{aligned}
$$

B.6. *Application of the Gale-Nikaido Theorem*

Now let $\mathsf{Z}(\mathsf{P})$ denote the set of all the finite sequences $(Z_t)_{t=0}^T$ thus defined, when $(X_t, K_t, N_t^0, N_t^1)_{t=0}^T$, $(K_t^{\#}, S_t^{\#})_{t=0}^T$, and $(Q_t, S_t, L_t, Y_t)_{t=0}^T$ range over $\Gamma(\mathsf{P}^T)$, $\Psi^{\#}(\mathsf{P}^T)$, and $(Q_t, S_t, L_t, Y_t)_{t=0}^T$, respectively. It is clear that $\mathsf{Z}(\mathsf{P})$ is a nonempty, convex, and compact set. Furthermore, $\mathsf{Z}: \mathsf{P} \to \mathsf{Z}(\mathsf{P})$ is a closed point-to-set map from the simplex $\Delta^{(4T+3)}$ into the $4(T+1)-$dimensional Euclidean space. Next, using (B.5.5) and (B.5.6) we can write

$$
\text{(B.6.1)}\qquad \sum_{t=0}^{T} \mathsf{P}_t^T.Z_t \le 0, \quad \text{for all } (Z_t)_{t=0}^T \text{ in } \mathsf{E}(\mathsf{P}) \text{ and all } \mathsf{P}^T \text{ in } \Delta^{(4T+3)};
$$

that is, Walras' law holds for the truncated economy.

According to the Gale-Nikaido theorem,[12] there exists a price system $\overline{\mathsf{P}}^T = \left(\overline{\phi}_t, \overline{\varphi}_t, \overline{\omega}_t, \overline{\rho}_t, \overline{p}_t\right)_{t=0}^T$ and a sequence of excess demand vectors $\left(\overline{Z}_t\right)_{t=0}^T$ in $\mathsf{Z}(\overline{\mathsf{P}}^T)$ such that $\overline{Z}_t \le 0, t = 0,...,T$. Let $(\overline{X}_t, \overline{K}_t, \overline{N}_t^0, \overline{N}_t^1)_{t=0}^T$, $(\overline{K}_t^{\#}, \overline{S}_t^{\#})_{t=0}^T$, and $(\overline{Q}_t, \overline{S}_t, \overline{L}_t, \overline{Y}_t)_{t=0}^T$ be the three sequences that give rise to $\left(\overline{Z}_t\right)_{t=0}^T$. Also, let $(\overline{c}_t^0, \overline{c}_{t+1}^1, \overline{b}_t, \overline{x}_{t+1}, \overline{k}_{t+1})_{t=0}^{T-1}$ be the sequence of lifetime plans that gives rise to $(\overline{X}_t, \overline{K}_t, \overline{N}_t^0, \overline{N}_t^1)_{t=0}^T$. It follows from the definition of $\left(\overline{Z}_t\right)_{t=0}^T$ and from its properties that

$$
\text{(B.6.2)}\qquad
\begin{aligned}
&\overline{Q}_t + \overline{X}_{t+1} - \overline{X}_t \le 0, \\
&\overline{S}_t - \overline{S}_t^{\#} \le 0, \\
&\overline{L}_t - \overline{N}_t^0 \le 0, \\
&\overline{K}_t^{\#} - \overline{K}_t \le 0, \\
&\overline{N}_t^0(\overline{c}_t^0 + h\overline{b}_t) + \overline{K}_{t+1} + \overline{N}_t^1 \overline{c}_t^1 - \overline{Y}_t - (1-\delta)\overline{K}_t \le 0, \qquad (t = 0,...,T),
\end{aligned}
$$

and

[12] See Nikaido (1970, Theorem 45.1, p.320).

$$\begin{aligned}
&\overline{Q}_T - \overline{X}_T \le 0,\\
&\overline{S}_T - \overline{S}_T^{\#} \le 0,\\
\text{(B.6.3)}\qquad &\overline{L}_T - \overline{N}_T^0 \le 0,\\
&\overline{K}_T^{\#} - \overline{K}_T \le 0,\\
&\overline{N}_T^0\overline{c}_T^0 + \overline{N}_T^1\overline{c}_T^1 - \overline{Y}_T - (1-\delta)\overline{K}_T \le 0.
\end{aligned}$$

For $t = 0,$ (B.6.2) assumes the following form:

$$\begin{aligned}
&\overline{Q}_0 + \overline{X}_1 - \overline{X}_0 \le 0,\\
&\overline{S}_0 - \overline{S}_0^{\#} \le 0,\\
\text{(B.6.4)}\qquad &\overline{L}_0 - \overline{N}_0^0 \le 0,\\
&\overline{K}_0^{\#} - \overline{K}_0 \le 0,\\
&\overline{N}_0^0(\overline{c}_0^0 + h\overline{b}_0) + \overline{K}_1 + \overline{N}_0^1\overline{c}_0^1 - \overline{Y}_0 - (1-\delta)\overline{K}_0 \le 0.
\end{aligned}$$

It follows from the first inequality in (B.6.4) that

$$\text{(B.6.5)}\qquad \overline{Q}_0 \le \overline{X}_0 = X_0 = \widehat{X}_0$$

and

$$\text{(B.6.6)}\qquad \overline{X}_1 \le \overline{X}_0 = X_0 = \widehat{X}_0.$$

Using (B.6.5), the second inequality, the fourth inequalities in (B.6.4), and the fact that the production plans $(\overline{K}_0^{\#}, \overline{S}_0^{\#})$ and $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0)$ satisfy the technological constraints, we can write

$$\begin{aligned}
&\overline{Q}_0 + \overline{S}_0 + \overline{L}_0 + \overline{Y}_0\\
&\quad \le \overline{X}_0 + \overline{K}_0 + \overline{N}_0^0 + F(\overline{X}_0 + \overline{K}_0, \overline{N}_0^0)\\
\text{(B.6.7)}\qquad &\qquad = X_0 + K_0 + N_0^0 + F(X_0 + K_0, N_0^0)\\
&\qquad\quad = \widehat{X}_0 + \widehat{K}_0 + \widehat{N}_0^0 + F(\widehat{X}_0 + \widehat{K}_0, \widehat{N}_0^0)\\
&\qquad\qquad = \widehat{X}_0 + \widehat{K}_0 + \widehat{N}_0^0 + \widehat{Y}_0 < M.
\end{aligned}$$

The strict inequality between the expression on the left of the first inequality in (B.6.7) and M means that the bounding constraint (B.4.3) is not binding on the production plan $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0).$ We claim that the production plan $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0)$ actually maximizes the profits in period 0 – subject only to the technological constraint – of the representative firm in the consumption good sector when the price system $\overline{\mathsf{P}}^T$ prevails. Indeed, if this were not true, then we can find a feasible production plan, say $(Q, S, L, Y),$ which violates the bounding constraint (B.4.2.3), but yields a profit higher than that of $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0).$

Next, let λ be a number strictly between 0 and 1, then consider the following production plan

(B.6.8) $\lambda(\overline{Q}_0,\overline{S}_0,\overline{L}_0,\overline{Y}_0)+(1-\lambda)(Q,S,L,,Y).$

Using the assumed strict concavity of the production function of the consumption good, we can assert that the production plan defined by (B.6.8) is feasible and yields a higher profit than that of the production plan $(\overline{Q}_0,\overline{S}_0,\overline{L}_0,\overline{Y}_0)$. If λ is chosen close to 1, the production plan defined by (B.6.8) will also satisfy the bounding constraint (B.4.2.3), contradicting the optimality of $(\overline{Q}_0,\overline{S}_0,\overline{L}_0,\overline{Y}_0)$.

Similarly, the production plan $(\overline{K}_0^{\#},\overline{S}_0^{\#})$ maximizes the profits in period 0 – subject only to the technological constraint – of the representative firm in the backstop sector when the price system $\overline{\mathsf{P}}^T$ prevails.

Next, we claim that when the price system $\overline{\mathsf{P}}^T$ prevails, the lifetime plan $(\overline{c}_0^0,\overline{c}_1^1,\overline{b}_0,\overline{x}_1,\overline{k}_1)$ maximizes the lifetime utility of a young individual in period 0, subject only to the current and future budget constraints. Indeed, it follows from the last inequality in (B.6.4) that

$$
\begin{aligned}
&\overline{N}_0^0(\overline{c}_0^0+h\overline{b}_0)+\overline{K}_1\\
&\quad\le \overline{Y}_0+(1-\delta)\overline{K}_0=F(\overline{Q}_0+\overline{K}_0,\overline{L}_0)+(1-\delta)\overline{K}_0\\
&\quad\le F(\overline{X}_0+\overline{K}_0,\overline{N}_0^0)+(1-\delta)\overline{K}_0\\
&\quad=F(X_0+K_0,N_0^0)+(1-\delta)K_0\\
&\quad=F(\widehat{X}_0+\widehat{K}_0,\widehat{N}_0^0)+(1-\delta)K_0\\
&\quad=\widehat{Y}_0+\widehat{K}_0<M.
\end{aligned}
\tag{B.6.9}
$$

The strict inequality between the expression on the first line of (B.6.9) and M means that the bounding constraint (B.3.4) is not binding on $(\overline{c}_0^0,\overline{c}_1^1,\overline{b}_0,\overline{x}_1,\overline{k}_1)$. Also, inequality (B.6.6) means that the bounding constraint (B.3.5) is not binding on $(\overline{c}_0^0,\overline{c}_1^1,\overline{b}_0,\overline{x}_1,\overline{k}_1)$, either. To continue, observe that from (B.6.9) we can draw

(B.6.10) $\overline{K}_1\le\widehat{Y}_0+\widehat{K}_0=\widehat{K}_1<M.$

Furthermore, note that when $t=1$, (B.6.2) assumes the following form

$$
\begin{aligned}
&\overline{Q}_1+\overline{X}_2-\overline{X}_1\le 0,\\
&\overline{S}_1-\overline{S}_1^{\#}\le 0,\\
&\overline{L}_1-\overline{N}_1^0\le 0,\\
&\overline{K}_1^{\#}-\overline{K}_1\le 0,\\
&\overline{N}_1^0(\overline{c}_1^0+h\overline{b}_1)+\overline{K}_2+\overline{N}_1^1\overline{c}_1^1-\overline{Y}_1-(1-\delta)\overline{K}_1\le 0.
\end{aligned}
\tag{B.6.11}
$$

From the last inequality in (B.6.11), we draw the following inequality $\overline{N}_1^1\overline{c}_1^1 \le \overline{Y}_1 + (1-\delta)\overline{K}_1$. Also, because $\overline{N}_1^1 = \overline{N}_0^0$, we can write

$$
\begin{aligned}
\overline{N}_0^0\overline{c}_1^1 = \overline{N}_1^1\overline{c}_1^1 &\le \overline{Y}_1 + (1-\delta)\overline{K}_1 \\
&\le F(\overline{Q}_1 + \overline{S}_1, \overline{L}_1) + (1-\delta)\overline{K}_1 \\
&\le F(\overline{X}_1 + \overline{K}_1, \overline{N}_1^0) + (1-\delta)\overline{K}_1 \qquad \text{(B.6.12)} \\
&\le F(\widehat{X}_1 + \widehat{K}_1, \widehat{N}_1^0) + (1-\delta)\widehat{K}_1 \\
&\le \widehat{Y}_1 + (1-\delta)\widehat{K}_1 < M.
\end{aligned}
$$

On the second line in (B.6.12), the inequality follows from the fact that the production plan $(\overline{Q}_1, \overline{S}_1, \overline{L}_1, \overline{Y}_1)$ is technologically feasible. The inequality on the third line is obtained by using the first two inequalities in (B.6.11), the fact that $(\overline{K}_1^{\#}, \overline{S}_1^{\#})$ is a feasible production plan for the backstop sector, and the fourth inequality in (B.6.11). The inequality on the fourth line of (B.6.12) follows from (A6.6), the fact that $\widehat{X}_0 = \widehat{X}_1$, and (B.6.10). Thus the bounding constraint (B.3.6) is not binding on $(\overline{c}_0^0, \overline{c}_1^1, \overline{b}_0, \overline{x}_1, \overline{k}_1)$, either.

The manner used to show that in period 0 no technologically feasible production plan yields a profit higher than that of the production plan $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0)$ can also be used here to show that the lifetime plan $(\overline{c}_0^0, \overline{c}_1^1, \overline{b}_0, \overline{x}_1, \overline{k}_1)$ solves the lifetime utility maximization problem of a young individual in period 0, subject only to the current and future budget constraints, when the price system $\overline{\mathsf{P}}^T$ prevails.

Furthermore, it follows directly from (A6.9) that $\overline{N}_0^0 h\overline{b}_0 < \widehat{Y}_0 + (1-\delta)\widehat{K}_0$, which in turn implies

$$\overline{N}_1^0 = \overline{N}_0^0\overline{b}_0 \le \frac{\widehat{Y}_0 + (1-\delta)\widehat{K}_0}{h} = \widehat{N}_1^0. \qquad \text{(B.6.13)}$$

Also, because $\overline{N}_1^1 = \overline{N}_0^0 = N_0^0 = \widehat{N}_0^0 = \widehat{N}_1^1$, we have

$$\left(\overline{X}_1, \overline{K}_1, \overline{N}_1^0, \overline{N}_1^1, \overline{Y}_1\right) \le \left(\widehat{X}_1, \widehat{K}_1, \widehat{N}_1^0, \widehat{N}_1^1, \widehat{Y}_1\right). \qquad \text{(B.6.14)}$$

Using (B.6.14) and the results that when the price system $\overline{\mathsf{P}}^T$ prevails,

(i) the production plan $(\overline{Q}_0, \overline{S}_0, \overline{L}_0, \overline{Y}_0)$ maximizes the profit of the representative firm in the consumption good sector in period 0, and

(ii) the lifetime plan $(\overline{c}_0^0, \overline{c}_1^1, \overline{b}_0, \overline{x}_1, \overline{k}_1)$ solves the lifetime utility maximization problem of a young individual in period 0, subject only to the current and future budget constraints,

we can start an induction on t to show that under $\overline{\mathsf{P}}^T$,

(i) the production plan $(\overline{Q}_t, \overline{S}_t, \overline{L}_t, \overline{Y}_t)$ maximizes the profit of the representative firm in the consumption good sector in period $t, t = 0,...,T$, and

(ii) the lifetime plan $(\overline{c}_t^0, \overline{c}_{t+1}^1, \overline{b}_t, \overline{x}_{t+1}, \overline{k}_{t+1})$ solves the lifetime utility maximization problem of a young individual in period $t, t = 0,...,T$, subject only to the current and future budget constraints.

B.7. *Positive Prices and Positive Birthrates*

The results obtained in Section B.6 say nothing about whether the equilibrium prices and the equilibrium birthrates are positive. Our proof of the existence of a competitive equilibrium for the truncated economy will be not be complete unless we manage to prove these results, and this is the objective of Section B.7. The proofs of these results are long and will be established after a series of claims.

In what follows, we let $\xi_0 = X_0 / N_0^0$ denote the initial oil endowment per worker and $\kappa_0 = K_0 / N_0^0$ denote the initial capital/labor ratio. The energy endowment per worker in period 0 is thus given by $\xi_0 + \kappa_0$. Also, we let $\overline{q}_0 = \overline{Q}_0 / N_0^0$ denote the equilibrium oil input per worker in period 0. All the claims are stated and proved in terms of the production function per worker.

CLAIM 1: *All the prices in* $(\overline{\phi}_0, \overline{\varphi}_0, \overline{\omega}_0, \overline{\rho}_0, \overline{p}_0)$ *are positive.*

PROOF OF CLAIM 1: First, note that if $\overline{p}_0 = 0$, then in period 0 the demand for the consumption good by the young as well as the old generation will be unbounded, and the bounding constraint (B.3.4) as well as the bounding constraints (B.5.2) must be binding. However, at the end of Section B.6 we have shown that none of the bounding constraints is binding. Hence $\overline{p}_0 > 0$. From now on we shall normalize the equilibrium price system in period 0 by choosing the consumption good in this period to be the numeraire. Thus we will set $\overline{p}_0 = 0$ and adjust the other prices in period 0 accordingly. Next, note that if the price of an input in period 0 is equal to 0, then its demand by the firms in the consumption good sector will be infinite, and again the bounding constraint corresponding to this input will be binding. ■

Now note that the price of energy in period 0 is given by

(B.7.1) $\min\{\overline{\phi}_0, \overline{\rho}_0\} = f'(\overline{q}_0 + \kappa_0)$.

To see why, note that if the initial stock of backstop capital is equal to 0, i.e., if $\kappa_0 = 0$, then we must have $\overline{q}_0 > 0$, which means that $\overline{\phi}_0 = f'(\overline{q}_0) \leq \overline{\rho}_0$, and (B.7.1) must hold. In this case, we can set $\overline{\rho}_0 = \overline{\phi}_0$ without disturbing the equilibrium for the truncated economy being considered. On the other hand, if $\kappa_0 > 0$, then $\overline{\phi}_0 \geq \overline{\rho}_0 = f'(\overline{q}_0 + \kappa_0)$, with $\overline{\phi}_0 = \overline{\rho}_0 = f'(\overline{q}_0 + \kappa_0)$, when $\overline{q}_0 > 0$, and (B.7.1) also holds in this case.

CLAIM 2: *The price of energy in period* 0 *is strictly less than* $\rho^{\max}$. *More precisely, we have,*

(B.7.2) $$f'(\xi_0 + \kappa_0) \le \bar{\phi}_0 = \bar{\rho}_0 = f'(\bar{q}_0 + \kappa_0) < \rho^{\max}.$$

PROOF OF CLAIM 2: Because $\bar{q}_0 \le \xi_0$, we must have $f'(\xi_0 + \kappa_0) \le f'(\bar{q}_0 + \kappa_0) = \bar{\phi}_0 = \bar{\rho}_0$. To prove the strict inequality in (B.7.2), suppose the contrary, i.e., suppose $f'(\bar{q}_0 + \kappa_0) \ge \rho^{\max}$. Then the wage rate in period 0, namely $\bar{\omega}_0$, will be less than or equal to the critical level $\omega^{\min}$ that induces the young generation of period 0 not to raise any offspring, with the ensuing consequence that there is no young generation – and a fortiori no workers – in period 1. Without any workers, the firms in the consumption good sector will not be able to produce any new consumption good. The young generation of period 0 – which will become the old generation of period 1 and which will be the only generation that exists in period 1 – will obtain its old-age consumption by consuming what remains of the stock of backstop capital after allowing for depreciation. Any oil remaining at the beginning of period 1 will find no use in the production process and thus will have zero price. Under such a scenario, it will not be rational for a young individual of period 0 to invest in oil. However, if $f'(\bar{q}_0 + \kappa_0) \ge \rho^{\max}$, then we must have $\bar{q}_0 + \kappa_0 \le [f']^{-1}(\rho^{\max}) = e^{\min}$, and in equilibrium the oil investment made by a young individual of period 0 is given by

(B.7.3) $$\bar{x}_1 = \xi_0 - \bar{q}_0 \ge \xi_0 + \kappa_0 - e^{\min} > 0,$$

with the strict inequality in (B.7.3) being due to Assumption 4. ■

CLAIM 3: *The equilibrium birthrate in period* 0 *is strictly positive.*

PROOF OF CLAIM 3: To prove the lemma, suppose the contrary, say $\bar{b}_0 = 0$. Then a young individual of period 0 will not invest in oil, which means that all the oil resources will be depleted in period 0, and we must have

(B.7.4) $$\bar{\phi}_0 = \bar{\rho}_0 = f'(\bar{q}_0 + \kappa_0) = f'(\xi_0 + \kappa_0) < \rho^{\max}.$$

It follows from (B.7.4) that the equilibrium wage rate in period 0, namely $\bar{\omega}_0$, will exceed the critical level $\omega^{\min}$. Furthermore, because a young individual of period 0 only invests in backstop capital, the argument leading to Lemma 1 can be used to show that the birthrate in period 0 is strictly positive and the rental rate of backstop capital in period 1 will be given by $\bar{\rho}_1 = G'(\bar{\rho}_0)$. These results contradict the assumption $\bar{b}_0 = 0$. ■

According to Claim 3, the equilibrium birthrate in period 0 is positive, which means that there will be a young generation in period 1. The argument used to show that all the equilibrium prices in period 0 are positive can be repeated to show that all the equilibrium prices in period 1 are also positive. Again, we normalize the equilibrium price system in period 1 by choosing the consumption good in this period to be the numeraire, i.e., we set $\bar{p}_1 = 1$, then adjust the remaining prices accordingly. In what follows, we shall let $\bar{\xi}_1 = \bar{x}_1 / \bar{b}_0$ and $\bar{\kappa}_1 = \bar{k}_1 / \bar{b}_0$ denote, respectively, the equilibrium oil endowment/worker ratio and the equilibrium capital/labor ratio in period 1. Also, let $\overline{q}_1 = \overline{Q}_1 / N_0^0 \overline{b}_0$ denote the equilibrium oil input per worker in period 1.

CLAIM 4: *The equilibrium price of energy in period* 1 *is given by*

(B.7.5) $$\min\{\overline{\phi}_1, \overline{\rho}_1\} = f'(\overline{q}_1 + \overline{\kappa}_1) = \overline{\rho}_1 \le \overline{\phi}_1.$$

PROOF OF CLAIM 4: If $\overline{q}_1$ and $\overline{\kappa}_1$ are positive, then the price of oil and the price of renewable energy in period 1 are both equal, and their common value is the price of energy in that period, i.e.,

(B.7.6) $$\min\{\overline{\phi}_1, \overline{\rho}_1\} = f'(\overline{q}_1 + \overline{\kappa}_1) = \overline{\rho}_1 = \overline{\phi}_1.$$

The claim is established in this case. If $\overline{q}_1 > 0$, but $\overline{\kappa}_1 = 0$, then $f'(\overline{q}_1 + \overline{\kappa}_1) = f'(\overline{q}_1) = \overline{\phi}_1$ and $\overline{\rho}_1 \ge \overline{\phi}_1$. In this case, we can set $\overline{\rho}_1 = \overline{\phi}_1$ without disturbing the competitive equilibrium of the truncated economy being considered, and (B.7.6) also holds in this case. If $\overline{\kappa}_1 > 0$, but $\overline{q}_1 = 0$, then $f'(\overline{q}_1 + \overline{\kappa}_1) = f'(\overline{\kappa}_1) = \overline{\rho}_1 \le \overline{\phi}_1$, which is (B.7.5). ■

CLAIM 5: *The equilibrium energy endowment per worker ratio in period* 1 *is higher than the critical level* $e^{\min}$; *that is,* $\overline{\xi}_1 + \overline{\kappa}_1 > e^{\min}$.

PROOF OF CLAIM 5: If a young individual of period 0 does not invest in oil, then her backstop capital investment is equal to her saving, i.e., $\overline{x}_1 = 0$ and $\overline{k}_1 = s(\overline{\omega}_0, \overline{r}_1)$. The rental rate of backstop capital in period 1 is then given by $G(\overline{\rho}_0) = f'(\overline{k}_1 / \overline{b}_0) = f'(\overline{\kappa}_1) < \rho^{\max}$, and the claim is proved in this case.

If a young individual of period 0 invests in oil, then we must have $\overline{\phi}_1 / \overline{\phi}_0 \ge 1 - \delta + \overline{\rho}_1$ and $\overline{\rho}_1 \ge \overline{\phi}_1$, which together imply that $\overline{\phi}_1 / \overline{\phi}_0 \ge 1 - \delta + \overline{\phi}_1$. The last inequality can only hold if $\overline{\phi}_0 < 1$. To prove the claim in this case, suppose the contrary, i.e., suppose $\overline{\xi}_1 + \overline{\kappa}_1 \le e^{\min}$. If this is the case, then in period 1 the equilibrium price of oil and the equilibrium price of renewable energy satisfy the following relation: $\overline{\phi}_1 \ge \overline{\rho}_1 = f'(\overline{q}_1 + \overline{\kappa}_1) \ge f'(\overline{\xi}_1 + \overline{\kappa}_1) \ge \rho^{\max}$. In period 1, the rate of return to oil investment thus satisfies the following condition:

(B.7.7) $$\overline{\phi}_1 / \overline{\phi}_0 \ge 1 - \delta + \rho^{\max}.$$

Now imagine an economy without oil resources in which the equilibrium rental rate of backstop capital in period 0 is given by $\overline{\rho}_0$. Then according to Lemma 1 and Assumption 4 the equilibrium rental rate of backstop capital in period 1 of this economy is given by $G(\overline{\rho}_0) \le G^{\max} < \rho^{\max}$. The rate of return to backstop capital investment obtained in period 1 is then given by

(B.7.8) $$1 - \delta + G(\overline{\rho}_0) \le 1 - \delta + \rho^{\max}.$$

Together, (B.7.7) and (B.7.8) allow us to write

(B.7.9) $$1 - \delta + G(\overline{\rho}_0) \le 1 - \delta + \rho^{\max} \le \overline{\phi}_1 / \overline{\phi}_0,$$

which means that in period 0 the rate of return to capital investment in the economy without oil is less than or equal to that in the economy with oil resources. This last result, according to Assumption 4, implies that the saving of a young individual of period 0 is higher in the former economy than in the latter economy. Because the saving/offspring ratio of a young individual in the former economy is always higher than $e^{\min}$, we must then have $s(\overline{\omega}_0,\overline{r}_1)/\overline{b}_0 > e^{\min}$, and the energy endowment/young individuals ratio in period 1 satisfies the following relation:

$$\begin{aligned}\overline{\xi}_1+\overline{\kappa}_1 &= \frac{s(\overline{\omega}_0,\overline{r}_1)-\overline{k}_1}{\overline{\phi}_0\overline{b}_0}+\frac{\overline{k}_1}{\overline{b}_0}\\ &> \frac{s(\overline{\omega}_0,\overline{r}_1)-\overline{k}_1}{\overline{b}_0}+\frac{\overline{k}_1}{\overline{b}_0}\\ &= \frac{s(\overline{\omega}_0,\overline{r}_1)}{\overline{b}_0} > e^{\min}.\end{aligned}$$

■

B.8. *Completion of the Existence Proof*

An induction argument on t then shows that (i) the energy endowment per worker in each period $t=0,...,T,$ is strictly higher than the critical level $e^{\min}$; (ii) the equilibrium price of energy is strictly less than the critical level $\rho^{\max}$ for each period $t=0,...,T$; (iii) all the prices in $\overline{\mathsf{P}}^T$ are positive; and (iv) the equilibrium birth rate in each period $t=0,...,T-1,$ is positive. Finally, using (iii); (iv) the result that all the prices in $\overline{\mathsf{P}}^T$ are positive; and (v) Walras' law, namely (B.5.5), we can conclude that $\overline{Z}_t=0, t=0,...,T,$ i.e., all markets clear in each period. We have just established the existence of a competitive equilibrium for a truncated with time horizon T.

APPENDIX C: PROOF OF LEMMA 6

To prove Lemma 6, we need to establish some preliminary results.

LEMMA C.1: *If the initial energy endowment per worker is not too high, then its equilibrium value in period* 1 *will be higher. More precisely, there exists a small positive number* ε *such that if* $\rho^{\max}-\varepsilon \le f'(\xi_0+\kappa_0) < \rho^{\max}$, *then* $f'(\xi_1+\kappa_1) < f'(\xi_0+\kappa_0)$.

PROOF: Let n be a positive integer such that $e^{\min} < \xi_0+\kappa_0 < e^{\min}+1/n.$ We have

$$f'(e^{\min}+1/n) < f'(\xi_0+\kappa_0) \le f'(q_0+\kappa_0) < \rho^{\max}.$$

Now when $n\to+\infty$, the equilibrium price of energy in period 0 must tend to $\rho^{\max}$, which in turn implies that b_0, the number of offspring raised by a young individual of period 0, will tend to 0. On the other hand, $\phi_0 x_1 + k_1$, the saving of such an individual is bounded below and away from 0. Hence when $n\to+\infty$, we must have

$$\frac{\phi_1 x_1 + k_1}{b_0} = \phi_1 \xi_1 + \kappa_1 \to +\infty.$$

Now recall that in the argument leading to Proposition 5 we have shown that the price of oil in period 1 is bounded above, regardless of the value of the initial energy endowments per worker. Hence we must have $\xi_1 + \kappa_1 \to +\infty$ when $\phi_1 \xi_1 + \kappa_1 \to +\infty.$ Thus when $n \to +\infty,$ the energy endowment per worker in period 1, namely $\xi_1 + \kappa_1,$ will be large; that is, $f'(\xi_1 + \kappa_1) < f'(\xi_0 + \kappa_0)$ when $n \to +\infty.$ ■

LEMMA C.2: *For any positive number* $\varepsilon,$ *let* $\mathsf{I}(\varepsilon)$ *be the set of all positive integers n such that* $1/n < \varepsilon.$ *There exists an integer* $n \in \mathsf{I}(\varepsilon)$ *with the following property: if the energy endowment per worker in period* 0 *is such that* $f'(\xi_0 + \kappa_0) \le \rho^{\max} - 1/n,$ *then the energy endowments per worker in period* 1 *also satisfies the same condition, i.e.,* $f'(\xi_1 + \kappa_1) \le \rho^{\max} - 1/n.$

PROOF: Indeed, if the claim is not true, then for each large positive integer *n*, we can find a number $\xi_0(n),$, a number $\kappa_0(n),$ and a competitive equilibrium $(\mathsf{P}(n), \mathsf{A}(n))_{n=1}^{\infty}$ that has $\xi_0(n)$ and $\kappa_0(n),$ respectively, as its initial oil endowment per worker and its initial capital/labor ratio. Furthermore, under this competitive equilibrium, we have $f'(\xi_0(n) + \kappa_0(n)) \le \rho^{\max} - 1/n,$ but $f'(\xi_1(n) + \kappa_1(n)) > \rho^{\max} - 1/n.$

As in the proof of Proposition 5, we can also use Cantor's diagonal trick here to assert that the sequence of competitive equilibria $(\mathsf{P}(n), \mathsf{A}(n))_{n=1}^{\infty}$ has a convergent subsequence; that is, there is an increasing map $\psi : n \to \psi(n)$ such that the sequence $(\mathsf{P}(\psi(n)), \mathsf{A}(\psi(n)))_{n=1}^{\infty}$ converges in the product topology to a competitive equilibrium that we denote by (P, A).. There are three cases to consider.

First, if $f'(\xi_0(\psi(n)) + \kappa_0(\psi(n)))$ tends to 0 when *n* tends to infinity, then $\xi_0(\psi(n)) + \kappa_0(\psi(n))$ will be large when *n* is large, and according to Lemma 4, $\xi_1(\psi(n)) + \kappa_1(\psi(n))$ will also be large, which means that $f'(\xi_1(\psi(n)) + \kappa_1(\psi(n)))$ will be low, contradicting the hypothesis of the reductio ad absurdum argument.

Second, if $f'(\xi_0(\psi(n)) + \kappa_0(\psi(n)))$ tends to $\rho^{\max}$, then according to Lemma C.1, we will have $f'(\xi_1(\psi(n)) + \kappa_1(\psi(n))) < f'(\xi_0(\psi(n)) + \kappa_0(\psi(n))),$ a contradiction to the reductio ad absurdum hypothesis.

Third, if $f'(\xi_0(\psi(n)) + \kappa_0(\psi(n)))$ tends to a limit $\rho \in (0, \rho^{\max}),$ then $\xi_0(\psi(n)) + \kappa_0(\psi(n))$ will tend to $\kappa = [f']^{-1}(\rho) > e^{\min},$ and the equilibrium wage rate in period 0 will tend to $\omega = f(\kappa) - \kappa\rho > \omega^{\min}.$ Let $\xi_0 = \lim_{n \to +\infty} \xi_0(\psi(n))$ and $\kappa_0 = \lim_{n \to +\infty} \kappa_0(\psi(n))$ denote, respectively, the initial oil endowment per worker and the initial capital/labor ratio of the limiting economy. Also, let $\xi_1 = \lim_{n\to+\infty} \xi_1(\psi(n))$ and $\kappa_1 = \lim_{n\to+\infty} \kappa_1(\psi(n))$ denote, respectively, the oil endowment per worker and the capital/labor ratio – both in period 1 – of the limiting economy. According to Proposition 5, we have $\xi_1 + \kappa_1 > e^{\min}$, which means that $f'(\xi_1 + \kappa_1) < \rho^{\max}$. However, according to the reduction ad absurdum hypothesis, we have $f'(\xi_1(\psi(n)) + \kappa_1(\psi(n))) > \rho^{\max} - 1/n$. In the limit, this inequality becomes $f'(\xi_1 + \kappa_1) \geq \rho^{\max}$, which contradicts $f'(\xi_1 + \kappa_1) < \rho^{\max}$. ■

LEMMA C.3: *Let ε be a positive number and $n^+(\varepsilon)$ be the smallest integer in $\mathsf{I}(\varepsilon)$that satisfies the conclusion of Lemma C.2. There exists a positive integer n that satisfies the following conditions*: (i) $1/n < \rho^{\max} - 1/n^+(\varepsilon)$ *and* (ii) *if the initial energy endowment per worker satisfies the condition* $1/n \leq f'(\xi_0 + \kappa_0) \leq \rho^{\max} - 1/n^+(\varepsilon)$, *then the equilibrium energy endowment per worker in period* 1 *also satisfies the same condition i.e.,* $1/n \leq f'(\xi_1 + \kappa_1) \leq \rho^{\max} - 1/n^+(\varepsilon)$.

PROOF: If the claim is not true, then for each positive integer *n,* we can find a number $\xi_0(n)$, , a number $\kappa_0(n)$, and a competitive equilibrium $(\mathsf{P}(n), \mathsf{A}(n))_{n=1}^{\infty}$ that has $\xi_0(n)$ and $\kappa_0(n)$, respectively, as its initial oil endowment per worker and its initial capital/labor ratio. Furthermore, under this competitive equilibrium, we have $1/n \leq f'(\xi_0(n) + \kappa_0(n)) \leq \rho^{\max} - 1/n^+(\varepsilon)$, but $f'(\xi_1(n) + \kappa_1(n)) < 1/n$.

As in the proof of Proposition 5, we can also use Cantor's diagonal trick here to assert that the sequence of competitive equilibria $(\mathsf{P}(n), \mathsf{A}(n))_{n=1}^{\infty}$ has a convergent subsequence; that is, there is an increasing map $\psi: n \to \psi(n)$ such that the sequence $(\mathsf{P}(\psi(n)), \mathsf{A}(\psi(n)))_{n=1}^{\infty}$ converges in the product topology to a competitive equilibrium that we denote by (P, A).. There are two cases to consider.

First, if $f'(\xi_0(\psi(n)) + \kappa_0(\psi(n)))$ tends to 0 when n tends to infinity, then $\xi_0(\psi(n)) + \kappa_0(\psi(n))$ will be large when n is large, and according to Lemma 4, $\xi_1(\psi(n)) + \kappa_1(\psi(n))$ will be lower than $\xi_0(\psi(n)) + \kappa_0(\psi(n)$, which implies that $1/n \leq f'(\xi_0(\psi(n)) + \kappa_0(\psi(n))) < f'(\xi_1(\psi(n)) + \kappa_1(\psi(n)))$, contradicting the hypothesis of the reductio ad absurdum argument.

Second, if $f'(\xi_0(\psi(n)) + \kappa_0(\psi(n)))$ tends to a limit $\rho \in (0, \rho^{\max})$, then $\xi_0(\psi(n)) + \kappa_0(\psi(n))$ will tend to $\kappa = [f']^{-1}(\rho) > e^{\min}$, and the equilibrium wage rate in period 0 will tend to $\omega = f(\kappa) - \kappa\rho > \omega^{\min}$. Let $\xi_0 = \lim_{n\to+\infty} \xi_0(\psi(n))$ and $\kappa_0 = \lim_{n\to+\infty} \kappa_0(\psi(n))$ denote, respectively, the initial oil endowment per worker and the initial capital/labor ratio of the limiting economy. Also, let $\xi_1 = \lim_{n\to+\infty} \xi_1(\psi(n))$ and $\kappa_1 = \lim_{n\to+\infty} \kappa_1(\psi(n))$ denote, respectively, the oil endowment per worker and the capital/labor ratio – both in period 1 – of the limiting economy. According to Proposition 5, we have $\xi_1 + \kappa_1 < +\infty$, which means that $f'(\xi_1 + \kappa_1) > 0$. However, according to the reduction ad absurdum hypothesis, we have $f'\big(\xi_1(\psi(n)) + \kappa_1(\psi(n))\big) < 1/n$, which implies that $\lim_{n\to+\infty} f'\big(\xi_1(\psi(n)) + \kappa_1(\psi(n))\big) = f'(\xi_1 + \kappa_1) = 0$, which is not consistent with $f'(\xi_1 + \kappa_1) > 0$. ■

In what follows, we shall denote by $n^-(\varepsilon)$ the smallest integer that satisfies the conclusion of Lemma C.3.

Proof Of Lemma 6: Let $\varepsilon(\delta) = \rho^{\max} - f'(\xi_0 + \kappa_0)$. Applying Lemmas C.2 and C.3, we can assert the existence of two integers $n^+(\varepsilon(\delta))$ and $n^-(\varepsilon(\delta))$ such that

$$1/n^-(\varepsilon(\delta)) \le f'(\xi_t + \kappa_t) \le \rho^{\max} - n^+(\varepsilon(\delta)), \qquad (t = 0,1,...).$$

The set of confining intervals for $f'(\xi_t + \kappa_t), t = 0,1,...,$ is thus not empty. Let $[\varepsilon^-(\delta), \rho^{\max} - \varepsilon^+(\delta)]$ denote the intersection of all the confining intervals of $f'(\xi_t + \kappa_t)$. Then $[\varepsilon^-(\delta), \rho^{\max} - \varepsilon^+(\delta)]$ is the smallest confining interval for $f'(\xi_t + \kappa_t), t = 0,1,...$ Next, let

$$\rho^+(\delta) = \sup_{t\le 0} f'(q_t + \kappa_t).$$

Observe that in the definition of $\varepsilon^-(\delta)$, $\varepsilon^+(\delta)$, and $\rho^+(\delta)$ we have made their dependence on the rate of capital depreciation explicit. We claim that $\rho^+(\delta) < \rho^{\max}$. Indeed, if this is not true, then we can find a period t such that $f'(q_t + \kappa_t)$ is in a small left neighborhood of $\rho^{\max}$, and this will lead to a large value for the energy endowment per worker in period $t+1$, which implies $f'(\xi_t + \kappa_t) < \varepsilon^-(\delta)$, contradicting the fact that $[\varepsilon^-(\delta), \rho^{\max} - \varepsilon^+(\delta)]$ is a confining interval for $f'(\xi_t + \kappa_t), t = 0,1,...$ Letting $\rho^-(\delta) = \varepsilon^-(\delta)$, we have

$$0 < \rho^-(\delta) \le f'(\xi_t + \kappa_t) \le f'(q_t + \kappa_t) \le \rho^+(\delta) < \rho^{\max}, \qquad (t = 0,1,...).$$

Finally, let

$$I = \bigcup_{0\le\delta\le 1}[\rho^-(\delta), \rho^+(\delta)],$$

$$\rho^- = \inf I,$$

$$\rho^+ = \sup I.$$

Then $[\rho^-, \rho^+]$ is a confining interval for $f'(q_t + \kappa_t), t = 0,1,...,$ regardless of the value of δ. A limiting argument – similar to the ones used in the proofs of Lemmas C.2 and C.3 – can be used to show that $0 < \rho^- < \rho^+ < \rho^{\max}$. ■

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