An Atkinson-Gini family of social evaluation functions

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Abstract

We propose a class of social evaluation functions and of inequality indices which merge the useful features of the family of Atkinson (1970) and of S-Gini (Donaldson and Weymark (1980, 1983), Yitzhaki (1983) and Kakwani (1980)) indices. These social evaluation functions can be interpreted as average utility corrected for relative deprivation in individual welfare. They can also be understood as averages of altruistic welfare in the population, and have a simple and useful graphical interpretation.

Keywords Social evaluation functions, inequality indices, relative deprivation, altruism.

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1 Introduction

We propose a class of social evaluation functions and of inequality indices which merge the features of the two most popular indices of inequality. Our social evaluation functions, denoted by $W_{\rho,\epsilon}$, are indeed a combination of the family of Atkinson (1970) indices, characterised by a normative parameter ϵ of relative inequality aversion, and of the family of S-Gini (or Single-parameter Gini) indices of Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) (also see Kakwani (1980)), characterised by an analogous normative parameter ρ of aversion to rank inequality. The ethical criteria of our social evaluation functions correspondingly rely on the use of decreasing (individual or social marginal) utilities of incomes to capture the dispersion of incomes around their mean value, and on the use of rank-dependent ethical weights to capture the dispersion of ranks in the population.

We introduce these social evaluation function and their associated inequality indices in the next section, where they can be alternatively expressed in a discrete or in a continuous framework. We then justify the formulation and review the nature of the axioms which characterise S-Gini indices. Proposition 1 subsequently shows that for social evaluation functions to fulfill these axioms and to be homothetic in incomes, they must take the form of $W_{\rho,\epsilon}$. Section 3 shows how our social evaluation functions can be interpreted as average utility corrected for relative deprivation in individual welfare. The paper ends by linking the evaluation functions to averages of altruistic welfare in the population.

2 A class of social evaluation functions

For the discrete setting, we suppose that there are H individuals in the population, with (positive) incomes denoted by y_i and ordered such that $y_1 \leq y_2 \leq ... \leq y_H$. For the continuous setting, we denote by $y(p)$ the p-quantile of the distribution of income ($y(p)$ is the inverse distribution function $p = F(y)$, assumed for simplicity to be strictly increasing in y). A general social evaluation function can then be discretely defined as:

$$W_\rho = \sum_{h=1}^{H} \frac{[(H+1-h)^\rho - (H-h)^\rho]U(y_h)}{H^\rho} \tag{1}$$

where $U(y)$ is a function of y. For W_ρ to obey the Dalton principle of income transfers, it is necessary and sufficient that $\rho \in [1, \infty[$ and that $U(y)$ be concave in income. For a continuous distribution, W_ρ is given by:

$$W_\rho = \int_0^1 k(p,\rho)U(y(p))dp \tag{2}$$

where $k(p,\rho) = \rho(1-p)^{\rho-1}$. These formulations replace incomes in the S-Gini formulation of Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) by functions of incomes. As Sen (1973, p.39) argues, $U(y_i)$ can be an individual utility function, or it can be the "component of social welfare corresponding to person i, being itself a strictly concave function of individual utilities". He also adds that "it is fairly restrictive to think of social welfare as a sum of individual welfare components" (p.39), and that one might feel that "the social value of the welfare of individuals should depend crucially on the levels of welfare (or incomes) of others" (p.41). As we will see clearly later, the formulation of equations (1) and (2) does this by applying rank-dependent weights on each individual welfare component $U(y_i)$. Moreover, as Ben Porath and Gilboa (1994, p.445) note, "the most salient drawback of linear measures [e.g., S-Gini's] is that the effect on the social welfare of a transfer of income from one individual to another depends only on the ranking of the incomes but not on their absolute levels". Equations (1) and (2) also escape this drawback since $U(y)$ does not have to be affine in incomes.

The above equations are also linked to rank-dependent expected utility theory, as noted in Chew and Epstein (1989) and Ben Porath and Gilboa (1994). S-Ginis are members of Single-Series Ginis (Donaldson and Weymark (1980)), which are themselves members of Generalised Gini indices (Mehran (1976) and Weymark (1981)). These classes of indices share interesting properties. Mehran (1976) shows that members of the class of Generalised Ginis can be easily graphically interpreted as weighted areas between Lorenz curves and lines of perfect equality. Weymark (1981) shows that the (absolute version of the) family of generalised Gini indices is the only one which obeys an axiom of weak independence of income source ("when the distribution of income from all but one source of income is the same in two distributions, overall inequality is determined by the inequality of the last source"). Ben Porath and Gilboa (1994) and Weymark (1995) also show that the class of generalised Gini indices is the only one which obeys an axiom of order-preserving-transfer. This axiom requires for our purposes that a common transfer of individual welfare $U(y)$ made simultaneously in two distributions between pairs of individuals who occupy adjacent ranks in the income distributions should preserve the pre- and post-transfer social evaluation ranking of the distributions. Blackorby et al. (1994) show that the members of the family of Generalised Gini indices provide a family of solutions to cooperative bargaining – solutions which respond by the same constant to a constant addition to one agent's component in the feasible set of utility vectors. For the subclass of Single-Series Ginis, the weights on the individual $U(y_i)$ arranged in decreasing order are independent of population size (see Theorems 1 and 2 of Donaldson and Weymark (1980)). Bossert (1990) shows that this property is needed if an axiom of separability of the welfare of the rich from the rest of the population is to be respected. Finally, S-Ginis, besides obeying all previously described axioms, are the only family of Single-Series Ginis to satisfy the Dalton Population Principle.

Yitzhaki (1983, p.264) also shows how each of the two families incorporated in equations (1) and (2), the Atkinson and the S-Gini indices, have a common dual graphical interpretation as a weigthed distance of the cumulative distribution function curve from the ordinate (for the Atkinson indices) and from the abscissa (for the S-Ginis). We generalise this interpretation below.

For a social evaluation function to yield a relative inequality index, it must be homothetic (Blackorby and Donaldson (1978)). This in turn implies a restriction on the form of our social evaluation functions W_ρ, as shown by Proposition 1.

Proposition 1: For a social evaluation function to satisfy all of the above mentioned characterising S-Ginis, and for them to be homothetic, they must take the form of [1]:

$$W_{\rho,\epsilon} \;=\; \sum_{h=1}^{H} \frac{[(H+1-h)^{\rho} - (H-h)^{\rho}]U_{\epsilon}(y_h)}{H^{\rho}} \tag{3}$$

where U_ϵ is defined as:

$$U_{\epsilon}(y) \;=\; \begin{cases} \frac{y^{1-\epsilon}}{1-\epsilon} & if \quad \epsilon \neq 1 \\ \\ ln(y) & if \quad \epsilon = 1 \end{cases} \tag{4}$$

Proof: The proof is immediate from Donaldson and Weymark (1980) and from a well-known result of Pratt (1964). ∎

As is conventional in the literature, we then define $\xi_{\rho,\epsilon}$ as an equally distributed equivalent (EDE) income (a money-metric measure of social welfare):

$$\xi_{\rho,\epsilon} \;=\; U_{\epsilon}^{-1}(W_{\rho,\epsilon}) \tag{5}$$

where the inverse function $U_{\epsilon}^{-1}(x)$ is defined as

$$U_{\epsilon}^{-1}(x) \;=\; \begin{cases} ((1-\epsilon)x)^{\frac{1}{1-\epsilon}} & if \quad \epsilon \neq 1 \\ \\ exp(x) & if \quad \epsilon = 1 \end{cases} \tag{6}$$

We can draw on well-known results to state that

$$\frac{\partial \xi_{\rho,\epsilon}}{\partial \rho} \leq 0 \quad ; \quad \frac{\partial \xi_{\rho,\epsilon}}{\partial \epsilon} \leq 0 \tag{7}$$

The corresponding class of inequality indices is then:

$$G_{\rho,\epsilon} \;=\; 1 \;-\; \frac{\xi_{\rho,\epsilon}}{\mu_y} \tag{8}$$

[1] An analogous reformulation holds for the continuous case.

where μ_y stands for the mean income of y. As can be easily checked, these indices are homogeneous of degree 0 in incomes, since their underlying social evaluation functions were devised to be homothetic. The indices for $\rho > 1$ and $\epsilon > 0$ also satisfy strictly the Dalton principles of transfers and population, by which a transfer of $1 from a rich to a poor will decrease inequality, and by which the addition of a replica of each individual in the population will not affect the inequality indices. The indices also lie between 0 and 1.

Integration by parts of equation (2) yields an alternative way of computing the social evaluation functions, their EDE income, and the associated indices of inequality:

$$W_\rho \ = \ \int_0^1 \rho(\rho-1)(1-p)^{\rho-2} GL_U(p)dp \tag{9}$$

where $GL_U(p) \ = \ \int_0^p U(y(q))dq$ is the well-known generalised Lorenz curve (see Shorrocks (1983)) of utilities. It can also be shown that a convenient way to compute the indices is through a simple covariance formula:

$$W_\rho \ = \ \rho cov(U(y(p)), (1-p)^{\rho-1}) + \mu_U \tag{10}$$

3 Relative deprivation

Let $\delta(p_i, p_j)$ represent the relative welfare deprivation of an individual at rank p_i in the distribution of income, when comparing himself with an individual at rank p_j in the same distribution:

$$\delta(p_i, p_j) \ = \ max \ [0 \ , \ U(y(p_j)) \ - \ U(y(p_i))] \tag{11}$$

Aggregating this relative deprivation over all individuals j, we find the following expected relative deprivation $d(p_i)$ for the individual at rank p_i:

$$d(p_i) \ = \ \int_0^1 \delta(p_i, p)dp \tag{12}$$

If we were then to take an average of $d(p_i)$ across all individuals i, and were to weight each such expected deprivation by the ethical weight $k(p_i, \rho)$, we would find D_ρ:

$$D_\rho \ = \ \frac{1}{\rho}\int_0^1 d(p)k(p,\rho)dp \tag{13}$$

The following proposition can then be shown:

Proposition 2: A social evaluation function W_ρ is average utility corrected by the average relative deprivation in utility in the population:

$$W_\rho \ = \ \mu_u \ - \ D_\rho \tag{14}$$

Proof: The proof is straightforward from Proposition 1 in Duclos (1998) (which deals with relative income deprivation in the context of standard S-Ginis). ∎

4 Altruism

We can also interpret our social evaluation functions as average feelings of altruism in the population. Let an individual at rank p randomly compare himself with $\rho - 1$ other individuals in the population. Denote the incomes of these random individuals by $y(p_1), y(p_2), ..., y(p_{\rho-1})$. Let an enlarged altruistic welfare function of the individual at rank p equal his egoistic utility function $(U(y(p)))$ minus the difference between that same egoistic utility function and the minimum of the utilities of the $\rho - 1$ individuals he randomly observes. In other words, an individual's altruistic welfare is a function of the welfare of the least well-off person he randomly comes across in the society. Denoting the altruistic individual welfare function as $a_\rho(p)$, it then equals:

$$a_\rho(p) = U(y(p)) - \{U(y(p)) - min\,[U(y(p_1)), .., U(y(p_{\rho-1}))]\} \quad (15)$$

$$= min\,[U(y(p)), U(y(p_1)), .., U(y(p_{\rho-1}))] \quad (16)$$

We then have:

Proposition 3: The social welfare function W_ρ is the average of altruistic welfare in the population:

$$W_\rho = \int_0^1 a_\rho(p)dp \quad (17)$$

Proof: For the proof, note first that by definition of the altruistic individual welfare function, we have that:

$$a_\rho(p) = (1-p)^{\rho-1}U(y(p)) + \int_0^p (\rho-1)(1-q)^{\rho-2}U(y(q))dq \quad (18)$$

Equation (18) says that $a_\rho(p)$ is a weighted average of his egoistic utility function and of the utility of those that are poorer than him. The weight $(1-p)^{\rho-1}$ is the probability that the individual with rank p finds himself the least well-off in his comparison with the $\rho-1$ other individuals. The weight $(\rho-1)(1-q)^{\rho-2}$ is the probability (more precisely, the density of the event) that an other individual (with utility $U(y(q))$) in the population is the least well-off in the comparison.

Let then $\bar{U}_\rho(p) = \int_0^p (\rho-1)(1-q)^{\rho-2}U(y(q))dq$. From (18), this yields:

$$\int_0^1 a_\rho(p)dp = \int_0^1 (1-p)^{\rho-1}U(y(p))dp + \int_0^1 \bar{U}_\rho(p)dp \quad (19)$$

Integrating by parts the last term of (19):

$$\int_0^1 \bar{U}_\rho(p)dp \;=\; p\bar{U}_\rho(p)|_0^1 - \int_0^1 p(\rho-1)(1-p)^{\rho-2}U(y(p))dp \tag{20}$$

$$= \int_0^1 (\rho-1)(1-p)^{(\rho-1)}U(y(p))dp \tag{21}$$

Hence, we find the result of Proposition 3:

$$\int_0^1 a_\rho(p)dp \;=\; \int_0^1 \rho(1-p)^{\rho-1}U(y(p))dp \;=\; W_\rho \quad \blacksquare \tag{22}$$

Figure 1 finally shows that our social evaluation functions $W_{\rho,\epsilon}$ have a nice graphical interpretation. Population ranks p are shown on the horizontal axis, and the utility quantiles $y(p)^{1-\epsilon}/(1-\epsilon)$ are shown on the vertical axis. The curve $y(p)^{1-\epsilon}/(1-\epsilon)$ is thus the inverse distribution function of utilities $U_\epsilon(y)$. The contribution of each individual in the computation of $W_{\rho,\epsilon}$ is then the area of a rectangle, with size equal to the product of utility $U(y(p))$ and of the distance between his rank p and the top rank (1). Social welfare is then simply the sum of the area of all such rectangles. For the traditional Gini social evaluation function, $W_{2,0}$, this equals twice the size of all of the above-defined rectangles under the curve $y(p)$. For the traditional Atkinson social evaluation function, $W_{1,\epsilon}$ is the simple integral of the height of the rectangles, $y(p)^{1-\epsilon}/(1-\epsilon)$. Loosely speaking, for given ρ and ϵ, social welfare is largest when there exists no negative correlation between the vertical and horizontal lengths of the individual rectangles. For instance, when rectangles are all of equal sizes, $W_{\rho,\epsilon}$ reaches its maximum value of $\mu^{1-\epsilon}/1-\epsilon$.

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Figure 1