# RELATIVE PERFORMANCE, RELATIVE DEPRIVATION AND GENERALISED GINI INDICES OF INEQUALITY AND HORIZONTAL INEQUITY

par

Jean-Yves **DUCLOS**

**Cahier de recherche 95??**

**D É P A R T E M E N T D' É C O N O M I Q U E**
**Faculté des Sciences Sociales**

Relative Performance, Relative Deprivation
and Generalised Gini Indices of Inequality
and Horizontal Inequity

par

Jean-Yves DUCLOS*

Cahier 95??

Cahier de recherche 95-??
du Centre de Recherche en Économie
et Finance Appliquées
CRÉFA

June 1995

**Address:** Département d'économique, Pavillon De Sève, Université Laval, Québec, Canada, G1K 7P4; tel.: (418) 656-7096; fax: (418) 656-7798; email: jduc@ecn.ulaval.ca

* This research was financed by grants from the Social Sciences and Humanities Research Council of Canada and from the Fonds FCAR of the Province of Québec. I am grateful to Martin Tabi for his research assistance.

## Résumé

Il est possible d'interpréter une classe d'indices d'iniquité horizontale comme représentant un regroupement de moyennes normatives de sentiments individuels de performance relative dans la distribution des taxes et des transferts. Un membre très connu de cette classe est l'indice d'iniquité horizontale d'Atkinson et de Plotnick. On peut interpréter de façon similaire des indices généralisés de Gini comme étant des moyennes normatives de sentiments individuels de privation relative. La combinaison de ces deux classes d'indices peut nous permettre de mieux soupeser les objectifs concurrents de réduction d'inégalité et d'équité horizontale. Les résultats portant sur l'équité horizontale sont illustrés à l'aide de la distribution des taxes et des transferts au Canada en 1981 et en 1990. Nous trouvons que les sentiments de mauvaise performance relative ainsi que l'iniquité horizontale ont augmenté de façon statistiquement significative dans les années 1980.



**Mots clés:** Équité horizontale, inégalité, bien-être social



## Abstract

A class of horizontal inequity indices is interpreted as an ethically sensitive function of individual feelings of relative performance in the tax and benefit allocation. A well-known member of that class is the Atkinson-Plotnick index of horizontal inequity. Generalised Gini coefficients are analogously found to be ethically sensitive functions of individual feelings of relative deprivation. Combining these two classes can provide an ethically sensitive basis for weighing the twin objectives of inequality reduction and horizontal equity. We illustrate the horizontal inequity results using the 1981 and 1990 Canadian distributions of taxes and benefits, and find that relative ill-performance feelings and horizontal inequity have witnessed a statistically significant increase in the 1980's.



**JEL Numbers:** H23, I30
**Keywords:** Horizontal inequity, inequality, social welfare

## I. Introduction

The principle of horizontal equity generally requires that individuals in identical relevant circumstances should be treated identically by the state and should therefore face identical tax and benefit schedules[1]. This classical requirement also implies that the tax and benefit schedule should not alter the initial rank order of individuals in the distribution of welfare. Because of the empirical rarity of observing exact equals in the initial distributions of incomes or welfare, a number of applied studies have focussed on "reranking" approaches to determine the extent of horizontal inequity operated by various tax and benefit systems[2]. A popular tool for the measurement of reranking and the computation of indices of horizontal inequity has been the comparison of Lorenz and concentration curves for the distribution of economic well-being, the concentration curve being computed on the basis of the pre-tax and pre-benefit ranking of individuals[3]. Very similar indices also serve to decompose the Gini coefficient (and Generalised Gini coefficients) into between-groups and within-groups contributions, by showing the intensity of group overlapping[4].

This paper interprets these indices of reranking and horizontal inequity as aggregates of individual feelings of relative performance in the allocation of taxes and benefits. Suppose that the movement from a gross (*i.e.*, *pre-tax and benefit*) income distribution to a net (*post-tax and benefit*) income distribution involves a reranking of individuals in the dimension of their incomes. Let

[1] See, for instance, Musgrave (1959) and Simons (1950).

[2] For discussions and comparisons of the classical and reranking approaches, see, *inter alia*, Feldstein (1976), Atkinson (1979), Plotnick (1982), Jenkins (1988), Musgrave (1990), Aronson, Johnson and Lambert (1994), Lambert and Ramos (1995), and Kakwani and Lambert (1995).

[3] This is done, for instance, in Atkinson (1979), Plotnick (1981), Kakwani (1984), Duclos (1993), and Palme (1994).

[4] See Mooklerjee and Shorrocks (1982), Silber (1989), and Lambert and Aronson (1993).

individuals assess their relative ill-performance in the tax and benefit reranking as the sum of the incomes of those by whom they have been overpassed minus the sum of the incomes of those individuals they succeed in overpassing. The average feeling of ill-performance over the population can then be interpreted as an index of horizontal inequity since a horizontally equitable redistribution must not alter the rank order of individuals. We show in Section II that this index of horizontal inequity is the one proposed by Atkinson (1979) and Plotnick (1981), an index which equals twice the area between the familiar Lorenz and concentration curves for net incomes.

Following[5] Hey and Lambert (1980), it is also possible to define an individual feeling of deprivation as the difference between the individual's income and the income of richer individuals in the income distribution. Computing the mean feeling of deprivation over the population, we confirm that it is equal to the well-known Gini coefficient of inequality. We generalise this exercise in Section III by applying ethically sensitive weights to individual feelings of deprivation. This leads to the generalised Gini indices of inequality proposed in Donaldson and Weymark (1980) and in Yitzhaki (1983).

Applying the same general class of weights onto individual feelings of relative ill-performance, we find in Section IV a class of ethically sensitive indices of horizontal inequity. This is in the spirit of Plotnick's (1981) observation that any "inequity measure would embody a weighting scheme, the validity of which would also depend ultimately upon a normative judgement" (p.285). Our analysis expresses this normative dependence as an explicit function of individual feelings of ill-performance. Because such individual feelings give ethical status to the pre-tax distribution, they can contribute to the definition of a social welfare ordering in a manner analogous to that, for instance, of King (1983) and Chakravarty (1985), where both inequality reduction and the minimisation of horizontal inequities feature in the social welfare objectives of the state. Using the 1981 and 1990 Canadian distributions of taxes and benefits,

---

[5] See also Yitzhaki (1979).

we compute in Section V the average feelings of relative ill-performance at different points of the Canadian income distribution. Using recent developments in the application of statistical inference to Lorenz and concentration curves, we find that these feelings of relative ill-performance and the associated indices of horizontal inequity underwent a statistically significant increase in the 1980's. Section VI concludes.

## II. Relative Performance and Horizontal Inequity

Let X be gross income and let N(X) be the associated level of net income that is given by the state's allocation of taxes and benefits. N(X) is thus a mapping of the gross income distribution onto a net income distribution[6]. We assume that these incomes can range from 0 to infinity, and have a finite mean and variance. Denote by $p=F_X(t)$ and $q=F_N(t)$ their respective differentiable distributions[7], with $F_X^{-1}(p)$ and $F_N^{-1}(q)$ their inverse distribution functions. Define G(q) as

$$G(q) = F_X \left\{ N^{-1}\left[F_N^{-1}(q)\right] \right\} \qquad \textbf{(1)}$$

G(q) is the function that gives the initial rank p in the gross income distribution of someone whose rank in the net income distribution is q.

The Lorenz curve for $F_X$ is $L_X(p)$, such that:

$$L_X(p) = \frac{1}{\mu_X} \int_0^y t \, dF_X(t) \text{ , with } p = F_X(y) \qquad \textbf{(2)}$$

[6] For an explicit definition of the transition from a continuous income distribution to another, see Kanbur and Stromberg (1988).

[7] The assumption of continuous distributions in the main text is made for expositional ease. The corresponding and analogous results for the case of discrete distributions are derived in the appendix. Our examples in the text will also sometimes assume a discrete income distribution with a total of H individuals.

where $\mu_X$ is the mean of $F_X$. $L_X(p)$ shows the proportion of total income held by those individuals with rank p or less in the distribution $F_X$. Define in an analogous way the Lorenz curve for $F_N$ as $L_N(q)$. We refer to the concentration curve for $F_N$ as $C_N(q)$, with

$$C_N(q) = \frac{1}{\mu_N} \int_0^y N[(t)]\ dF_X(t) \ , \text{ with } q = F_X(y) \tag{3}$$

The Lorenz and the concentration curves for net income differ in the order in which observations of net incomes appear; for the Lorenz curve, N appears in increasing values of itself, but for the concentration curve, N appears in increasing values of its associated gross income.

Let the feeling of ill-performance of individual i comparing himself to individual j be positive when i is overpassed by j and negative when i overpasses j. Assume, moreover, that the intensity of the performance feeling is given by the net income level of j, $N_j$. The feeling of ill-performance of i, relative to j, is then given by $\rho^*(i,j)$:

$$\rho^*(i,j) = \begin{cases} -N_j \ , & \text{if} \quad p_i < p_j \text{ and } q_i > q_j \\ N_j \ , & \text{if} \quad p_i > p_j \text{ and } q_i < q_j \\ 0 \ , & \text{otherwise} \end{cases} \tag{4}$$

Now integrate $\rho^*(i,j)$ over the whole distribution of individuals j, for i with post-tax position $q_i$. We can check that this yields an average feeling $r^*(q_i)$ of ill-performance for i such that

$$r^*(q_i) = \int_{q_i}^{1} F_N^{-1}(q)\ dq - \int_{G(q_i)}^{1} N\left[X(p)\right]\ dp \tag{5}$$

This average feeling of ill-performance for i may be positive or negative. If, for instance, $q_i < G(q_i)$ because i has been outranked by j in the tax and benefit allocation, then, for a discrete distribution of H individuals,

$$r^{*}(q_i) = \frac{1}{H}\left[N_j\right] > 0 \qquad \textbf{(6)}$$

If $q_i > G(q_i)$ because the tax and benefit allocation has made i jump above k, then

$$r^{*}(q_i) = \frac{1}{H}\left[-N_k\right] < 0 \qquad \textbf{(7)}$$

This negative feeling of ill-performance $\rho^*$ indicates that i feels relatively favored by the tax and benefit allocation. We can also have that $q_i = G(q_i)$, with i simultaneously overpassing k and being overpassed by j. The average feeling of ill-performance of i would then be:

$$r^{*}(q_i) = \frac{1}{H}\left[N_j - N_k\right] > 0 \qquad \textbf{(8)}$$

We can provide an alternative specification of ill-performance feeling that will be useful for a generalisation procedure in Section IV. Define $\rho(i,j)$ as

$$\rho(i,j) = \begin{cases} -N_j, & \text{if } q_j < q_i < p_j \\ N_j, & \text{if } p_j < q_i < q_j \\ 0, & \text{otherwise} \end{cases} \qquad \textbf{(9)}$$

That is, with $\rho(i,j)$, i is assumed to monitor whether an individual j jumped below or above i's net income position. For a jump above i, $\rho$ is positive, and for a jump below, $\rho$ is negative. We note that $\rho(i,j)$ differs from $\rho^*(i,j)$ in that $\rho^*(i,j)$ registers all rank changes involving i, whereas $\rho(i,j)$ only reacts to rank changes that influence the distribution of those with a net income greater than that of i. For instance, if the only reranking disturbance were a jump of i above k, $\rho(i,j)$ would be zero for all j, but $\rho^*(i,k)$ would be negative.

Let $r(q_i)$ be the average value of $\rho(i, j)$ over the whole distribution of individuals j. We then find that:

$$r(q_i) = \int_{q_i}^{1} \left\{F_N^{-1}(q)\ dq - N\left[\ X(p)\ \right] dp\right\} = \mu_N \left[C_N(q_i) - L_N(q_i)\right] \qquad \textbf{(10)}$$

This can be interpreted as a resentment index for the presence of some "newly rich" in the group of individuals richer than i. This average feeling of resentment $r(q_i)$ indicates by what amount the income of the richer class exceeds what the income of the richer class would have been if no "new rich" had displaced "old rich". $r(q_i)$ is always non-negative.

Atkinson (1979) and Plotnick (1981) proposed an index of horizontal inequity, R, that equals twice the area between the Lorenz and the concentration curves for net income. They note that $C_N(p) \geq L_N(p)$ for all p, with strict inequality somewhere if and only if there is reranking in moving from X to N. Using equations (5) and (10), we can show that this index R is the average feeling of ill-performance or resentment over the whole population, normalised by mean income:

$$\begin{aligned} R &= \frac{2}{\mu_N} \int_0^1 r(p_N)\ dp_N \\ &= \frac{2}{\mu_N} \int_0^1 \breve{r}(p_N)\ dp_N \end{aligned} \qquad \textbf{(11)}$$

We illustrate this with a simple example.

<u>Example 1:</u>

Let a mean-preserving transfer occur between i and j, with $X_i<X_j$ and $N_i=N_j+a$, $a>0$. Assume that no other individual is reranked. For a discrete distribution, we then have that $r(q_j)=a/H$ and that $r(q_k)=0$, $\forall k \neq j$. With that rank-reversing transfer, the R index would equal

$$\frac{2 \cdot a}{H^2 \cdot \mu_N} \qquad \textbf{(12)}$$

Although clearly horizontally inequitable, this transfer would reduce inequality if and only if $N_i < X_j$ or, equivalently, if and only if $a< X_j$-$N_j$.

## III. Generalised Gini Indices of Inequality and Feelings of Deprivation

Let the relative deprivation felt by i with income $X(p_i)$, when comparing himself with j with income $X(p_j)$, be given[8] by the difference $X(p_j)$-$X(p_i)$ if j is richer than i. Otherwise, i feels no relative deprivation. For i comparing himself with j, relative deprivation then equals:

$$\gamma\,(p_i,\ p_j) = \min\left[0,\ X(p_j) - X(p_i)\right] \tag{13}$$

The average feeling of deprivation felt by i over the whole population of individuals j is:

$$c(p_i) = \int_0^1 \gamma(p_i,\ p)\ dp \tag{14}$$

Some manipulation shows that this can be rewritten as:

$$c(p_i) = \mu_X \left[1 - L_X\,(p_i)\right] - \left[X(p_i)\,(1-p_i)\right] \tag{15}$$

which is the gap between the total income of the richer than i and their total income if they all had $X(p_i)$ instead. Averaging $c(p_i)$ over the whole population and multiplying by $(2/\mu_X)$, we would obtain the Gini coefficient, as shown below.

We may, however, generalise this averaging process by applying an ethically sensitive distributional weight to $c(p_i)$. We focus here on one such class

[8] For a general discussion of this specification and of the interpretation of k(p) later in this Section, see Lambert (1993), p. 123-129.

of ethical weights[9], the class leading to a generalisation of the Gini coefficient by Donaldson and Weymark (1980) and by Yitzhaki (1983)[10].

The probability that all incomes in a random and independent draw of (v-1) incomes from F be larger than x is $[1-F(x)]^{v-1}$, with v>1. The distribution function for a minimum income x in a random draw of (v-1) incomes is then $1-[1-F(x)]^{v-1}$, and the density of this minimum x in draws of (v-1) incomes then becomes $(v-1)\bullet[1-F(x)]^{(v-2)}$. Increasing v increases the density of lower incomes relative to the density of higher incomes. Let $k(p)=(v-1)(1-p)^{v-2}$ and define an ethically sensitive average of c(p) as:

$$G_X(v) = \frac{1}{\mu_X} \int_0^1 c(p)\, k(p)\, dp \tag{16}$$

$G_X(v)$ is thus a weighted average feeling of deprivation in the population. This weighted average measures the expected deprivation feeling of the most deprived individual in a random draw (from F) of (v-1) individuals. Through integration by parts, we can show that (16) can be rewritten as

$$G_X(v) = v \int_0^1 \left[p - L_X(p)\right] k(p)\, dp \tag{17}$$

which is the Generalised Gini coefficient of Donaldson and Weymark (1980) and Yitzhaki (1983). Equation (17) also indicates that G(2) yields the standard Gini coefficient, with $k(p)\equiv 1$. Thus, if and only if we give equal ethical weight to the

---

[9] Other weights to the individual valuation of relative deprivation in Section III and of relative performance in Section IV can be used if they are allowed by the general linear class of inequality measures defined in Mehran (1976). Duclos (1993) shows how that class generates a general class of horizontal inequity indices. A different option for the computation of indices of inequality and horizontal inequity would be to define increasing and convex functions of c(p) and r(p), as suggested by Chakravarty and Chakraborty (1984). Berrebi and Silber (1985) suggest how we may express a number of other inequality indices as variously-defined functions of relative deprivation feelings.

[10] See Donaldson and Weymark (1983) for the correspondence between the continuous and the discrete versions of that generalisation of the Gini coefficient.

feelings of relative deprivation c(p) of all individuals, do we find the standard Gini coefficient.

The ethical weights k(p) in (16) and (17) vary with the parameter v. The greater the value of v, the more emphasis is placed on the deprivation feelings of the worst-off individuals since it is then more likely that they will emerge as the most deprived individual among (v-1) randomly drawn individuals. We can show that

$$\lim_{v \to 1} G_X(v) = 0 \qquad \textbf{(18)}$$

at which limit we become ethically insensitive towards inequality since we only care about the relative deprivation feeling of the richest individual, which is zero. When 1<v<2, in computing the mean feeling of deprivation over the population, more weight is placed upon the individuals higher up in the income distribution. As mentioned above, at v=2, equal weight is awarded to all feelings of deprivation c(p). As v increases above 2, more and more relative weight is put on the poor; for a discrete distribution, we can also show that

$$\lim_{v \to \infty} G_X(v) = \frac{1}{\mu_X} c\left(\frac{1}{H}\right) \qquad \textbf{(19)}$$

where the Generalised Gini coefficient of inequality equals the relative deprivation felt by the most deprived individual in the distribution.

## <u>IV. Generalised Horizontal Inequity Indices</u>

Now define:

$$I_N(v) = v \int_0^1 \left[q - C_N(q)\right] k(q)\, dq \qquad \textbf{(20)}$$

This is equivalent to the Generalised Gini coefficient $G_N(v)$ for the distribution of net income N except for the feature that the concentration curve $C_N$ is used for $I_N(v)$ instead of the Lorenz curve $L_N$ for $G_N(v)$. Subtracting $I_N(v)$ from $G_N(v)$, we obtain a v-sensitive class of horizontal inequity indices:

$$R(v) = G_N(v) - I_N(v) = v \int_0^1 \left[C_N(q) - L_N(q)\right] k(q)\, dq \qquad \textbf{(21)}$$

From (10) we note that this equals

$$R(v) = \frac{v}{\mu_N} \int_0^1 r(q)\, k(q)\, dq \qquad \textbf{(22)}$$

which is simply v times the average feeling of ill-performance r(q), pondered by the ethically sensitive distributional weights k(q), and normalised by mean income. R(v) satisfies the three properties which Plotnick (1982) proposes as desirable for an index of horizontal inequity, that is, the property of independence from mean income, the property of anonymity, and the property by which a reversal of initial ranks increases horizontal inequity.

R(v) generalises the Atkinson-Plotnick index of horizontal inequity, which is given by R(2). In particular, R(2) attaches equal weight to the ill-performance feeling of every individual in the population. This contradicts the claim of Plotnick (1981,p.285) that his index "attaches greater weight to inequities occurring among units with high preredistribution ranks than to those affecting lower ranking units". We can interpret the role of v as an ethical parameter in the same manner as in Section III. In particular, we have that

$$\lim_{v \to 1} R(v) = 0 \qquad \textbf{(23)}$$

where the ethical focus is entirely upon the feeling of ill-performance of the richest individual in the net income distribution, which is necessarily zero. The greater the value of v, the more weight is granted to the feeling of ill-performance of the poor in assessing horizontal inequity. For a discrete distribution, we find that, as v tends to infinity, the index R(v) equals the feeling of relative ill-performance of the poorest:

$$\lim_{v \to \infty} R(v) = \frac{1}{\mu_N}\, r\left(\frac{1}{H}\right) \qquad \textbf{(24)}$$

For the discrete reranking instance of Example 1 in Section II, we have that

$$R(v) = \frac{a}{H^2 \mu_N} v\ (v-1) \left(1-q_j\right)^{v-2} \qquad \textbf{(25)}$$

The assessed importance of horizontal inequity then depends on a, the importance of the inequity, on $q_j$ , the rank of the outranked individual, and on the ethical parameter v.

## V. Illustration

We illustrate some of the above results through the use of the 1981 and 1990 Canadian Surveys of Consumer Finances. These surveys provide, respectively, 37,779 and 45,461 observations on the distributions of earned and unearned incomes, income taxes, cash transfers, and a number of household characteristics. From this, we computed levels of family gross and net incomes; net incomes include, *inter alia*, personal income taxes, child tax credits, old age transfers and public pensions, and social assistance and unemployment insurance benefits. We use the OECD equivalence scale to take into account the heterogeneity of family size and composition. Families with negative gross or net incomes were removed from the samples. The asymptotic standard errors of the relative ill-performance feelings at different points of the income distribution and of the aggregate indices of total horizontal inequity were computed using the recent statistical inference results of Davidson and Duclos (1995).

Table 1 shows the relative ill-performance feelings r(q), normalised by mean incomes, for individuals at deciles 1 to 9 in the distributions of net incomes in 1981 and in 1990. Ill-performance is highest at the first deciles, and decreases continuously. Although this is not a *necessary* trend, it is not unexpected since the density of incomes is higher around the lower deciles, and it is also at these lower deciles that the benefit system is most operative. Ill-performance feelings at the highest deciles must largely be caused by personal income taxation, and the income density and reranking possibilities are also smaller at those higher income levels. Feelings of ill-performance range between 0.26% and 0.97% of average income in 1981, and exceed 1% of average income for the first four deciles of 1990. These feelings are also significantly and statistically higher in 1990 than in 1981, except for individuals at the ninth decile.

Table 2 aggregates the feelings of relative ill-performance at different points in the income distribution by applying the ethically-sensitive weights k(p) of Section IV. The asymptotic standard errors of the resulting indices are also shown in parentheses. In the light of the results of Table 1, it is not surprising to find that, as the parameter v increases and as the ethical focus is shifted from the feelings of relative ill-performance of the rich to those of the poor, the aggregate indices of horizontal inequity increase, both in 1981 and in 1990. It is also clear that horizontal inequity in 1990 is, statistically, significantly greater in 1990 than in 1981, regardless of the values of v shown in Table 3. Numerically, horizontal equity is about 50% larger in 1990 than in 1981. At v=2, we have the Atkinson-Plotnick index of horizontal inequity, which equals twice the average feeling of ill-performance across the population. This average feeling is, again, approximately 50% greater in 1990 than in 1981, and equals 0.51% and 0.74% of average incomes in 1981 and 1990, respectively.

## VI. Conclusion

We assume that individuals assess their relative performance in the state's tax and benefit allocation by observing those by whom they are overpassed and those they succeed in outranking in the income distribution. We measure the intensity of these relative performance feelings by the incomes of those individuals against whom performance is compared. A class of horizontal inequity indices is then defined as an ethically sensitive function of average ill-performance feelings. We show that a well-known member of that class is the Atkinson-Plotnick index of horizontal inequity. This also helps interpret and understand the popular reranking approach to measuring horizontal inequity.

Generalised Gini coefficients are analogously found to be ethically sensitive functions of individual feelings of deprivation, when deprivation is appraised relative to what others have in the income distribution. In assessing the population's overall feeling of ill-performance and relative deprivation, we can increase the value of the parameter v to shift the ethical focus from greater weight onto the better off, to equal weight onto all individuals, to greater weight onto the worst off. An illustration using Canadian data indicates that horizontal inequity has increased significantly between 1981 and 1990, and that the intensity of relative ill-performance feelings decreases as we move from the bottom to the top deciles of the Canadian income distribution.

## VII. Appendix: Relative Performance and Horizontal Equity with Discrete Distributions

Let a discrete income distribution contain H individuals, with finite and non-negative incomes. A class of Generalised Gini (or S-Gini, for Single-Parameter Gini) coefficients is then defined as [see Donaldson and Weymark (1980)]:

$$G_X(v) = 1 - \sum_{i=1}^{H} \left[ \frac{(H-i+1)^v - (H-i)^v}{H^{\,v}} \right] \frac{X_i}{\mu_X} \tag{26}$$

with v>1 and where the $X_i$ have been ordered such that $X_1 \le X_2 \le \ldots \le X_H$. If we define $Z=-(1-p)^v$, with $\Delta Z=[(H-i+1)/H]^v - [(H-i)/H]^v$ and $dZ=v(1-p)^{(v-1)}dp$, it can be seen that (26) is the discrete analogue of

$$G_X(v) = 1 - \int_0^1 v\ (1-p)^{v-1}\ \frac{X(p)}{\mu_X}\ dp \tag{27}$$

If the distribution function is a step function, the two specifications are identical. Integrating by parts equation (27), we obtain:

$$\begin{aligned} G_X(v) &= 1 - \int_0^1 L_X(p)\ d\left[-v(1-p)^{(v-1)}\right] \\ &= 1 - v(v-1) \int_0^1 L_X(p)\ (1-p)^{(v-2)}\ dp \end{aligned} \tag{28}$$

The second line of (28) is identical to the definition of the continuous distribution Generalised Gini coefficient in (17). The first line of (28) suggests that an alternative discrete distribution specification of the Generalised Gini would be:

$$1 - \sum_{i=1}^{H} L_{X_i}\ v \left[ \left( \frac{H-i+1}{H} \right)^{v-1} - \left( \frac{H-i}{H} \right)^{v-1} \right] \tag{29}$$

where $L_{Xi}$ is the Lorenz curve ordinate at quantile $X_i$ for a discrete distribution.

The tax and benefit system gives net incomes $N_i$ as a function of $X_i$, yielding the couples $(X_i,N_i)$ such that if $i<j$, it must be that $X_i \leq X_j$. The distribution of net incomes can also be ranked, yielding an ordered distribution $N_{[i]}$ such that $N_{[1]} \leq N_{[2]} \leq ... \leq N_{[H]}$. Consider then the v-sensitive class of horizontal inequity indices for discrete distributions, defined analogously to the continuous specification. From (29), R(v) would be:

$$R(v) = \sum_{i=1}^{H} \left( C_{N_i} - L_{N_i} \right) v \left[ \left( \frac{H-i+1}{H} \right)^{v-1} - \left( \frac{H-i}{H} \right)^{v-1} \right] \tag{30}$$

where $C_{Ni}$ and $L_{Ni}$ are the concentration and Lorenz curves ordinates at quantiles $N_i$ and $N_{[i]}$, respectively. We note that this is simply the discrete analogue of the weighted average feeling of ill-performance $\rho(i,j)$ [defined in (10)] over all i and all j, normalised by average net incomes.

From (26), R(v) can also be defined as:

$$R(v) = \sum_{i=1}^{H} \left[ \frac{(H-i+1)^v - (H-i)^v}{H^v} \right] \frac{\left( N_i - N_{[i]} \right)}{\mu_X} \tag{31}$$

For v=2, this reduces to:

$$\begin{aligned} R(2) &= \frac{2}{H^2 \mu_N} \cdot \sum_{i=1}^{H} i \left( N_{[i]} - N_i \right) \\ &= \frac{2}{H^2 \mu_N} \cdot \sum_{i=1}^{H} \sum_{j=i}^{H} \left( N_{[j]} - N_j \right) \end{aligned} \tag{32}$$

Let the function G(i) give the gross income rank of someone with net income rank i. Then

$$\sum_{j=G(i)}^{H} N_j \tag{33}$$

is the sum of the incomes of those who should have been ahead of the individual with post-tax and benefit rank i. Since there is a one-to-one correspondence between i and G(i), R(2) in (32) also equals:

$$R(2) = \frac{2}{H\,\mu_N} \cdot \sum_{i=1}^{H} \left[ \frac{\sum_{j=i}^{H} N_{[j]} - \sum_{j=G(i)}^{H} N_j}{H} \right] \tag{34}$$

which is simply twice the average feeling of relative ill-performance $\rho^*(i,j)$ [defined in (4)] over all j and over all i, normalised by mean income.

## References


Aronson, J.R., P. Johnson and P.J. Lambert (1994), "Redistributive Effect and Unequal Income Tax Treatment", Economic Journal, vol. 104, 423:262-270.

Atkinson, A. B.(1979), "Horizontal Equity and the Distribution of the Tax Burden", ch. 1 in The Economics of Taxation, ed. H. J. Aaron and M. J. Boskin, Brookings Institution, Washington DC, pp. 3-18.

Berrebi, Z.M., and Jacques Silber (1985), "Income Inequality Indices and Deprivation: A Generalization", Quarterly Journal of Economics, August, 807-810.

Chakravarty, S.R. (1985), "Normative Indices for the Measurement of Horizontal Inequity", Journal of Quantitative Economics, vol.1, #1, January, 81-89.

Chakravarty, S.R., and A.B. Chakraborty (1984), "On Indices of Relative Deprivation", Economics Letters, vol. 14, pp. 283-287.

Davidson, Russell, and Jean-Yves Duclos (1995), "Statistical Inference for the Measurement of Progressivity, Redistribution, and Horizontal Equity", mimeo.

Donaldson, D., and J. A. Weymark (1980), "A Single Parameter Generalization of the Gini Indices of Inequality", Journal of Economic Theory, #22, pp. 67-86.

Donaldson, D. and J.A. Weymark (1983), "Ethically Flexible Gini Indices for Income Distributions in the Continuum", Journal of Economic Theory, 29, 2:353-358.

Duclos, J.-Y. (1993), "Progressivity, Redistribution and Equity, with Application to the British Tax and Benefit System", Public Finance /Finances Publiques, vol. 48, 3:350-365.

Feldstein, M. (1976), "On the Theory of Tax Reform", Journal of Public Economics, vol. 6, pp.77-104.

Hey, J.D., and P.J. Lambert (1980), "Relative Deprivation and the Gini Coefficient: Comment", Quarterly Journal of Economics, vol. 95, pp. 567-573.

Jenkins, S.P. (1988), "Empirical Measurement of Horizontal Inequity", Journal of Public Economics, 37, pp. 305-329.

Kakwani, N.C. (1984), "On the Measurement of Tax Progressivity and Redistributive Effect of Taxes with Applications to Horizontal and Vertical Equity", Advances in Econometrics, vol. 3, pp. 149-168.

Kakwani, N.C. and P.J. Lambert (1995), "Vertical Equity and Horizontal Inequity: A New Approach to Measurement", IFS Working Paper Series No. W95/2, London.

Kanbur, S.M. Ravi, and Jan-Olov Stromberg (1988), "Income Transitions and Income Distribution Dominance", Journal of Economic Theory, vol. 45, 408-416.

King, Mervyn A. (1983), "An Index of Inequality: With Applications to Horizontal Equity and Social Mobility", Econometrica, vol. 51, #1, January, pp. 99-116.

Lambert, Peter (1993), The Distribution and Redistribution of Income: a Mathematical Analysis, 2nd edition, Manchester University Press, Manchester, 306 p.

Lambert, Peter J., and J. Richard Aronson (1993), "Inequality Decomposition Analysis and the Gini Coefficient Revisited", Economic Journal, vol. 103, #420, September, 1221-1227.

Lambert, P.J. and X. Ramos (1995), "Vertical Redistribution and Horizontal Inequity", IFS Working Paper Series no. W95/1, London.

Mehran, Farhad (1976), "Linear Measures of Income Inequality", Econometrica, vol. 44, #4, July, pp. 805-809.

Mookherjee, D., and A.F. Shorrocks (1982), "A Decomposition Analysis of the Trend in U.K. Income Inequality", Economic Journal, vol. 92, 886-902.

Musgrave, Richard A. (1959), The Theory of Public Finance, McGraw-Hill, New York.

Musgrave, Richard A. (1990), "Horizontal Equity, Once More", National Tax Journal, vol. 43, #2, June, 113-122.

Palme, Marten (1994), "Income Distribution Effects of the Swedish 1991 Tax Reform: An Analysis of a Microsimulation Using Generalized Kakwani Decomposition", Working Paper #5, January, Stockholm School of Economics.

Plotnick, Robert (1981), "A Measure of Horizontal Inequity", The Review of Economics and Statistics, vol. LXII, #2, May, pp. 283-288.

Plotnick, Robert (1982), "The Concept and Measurement of Horizontal Inequity", Journal of Public Economics, vol.17, April, pp. 373-391.

Silber, Jacques (1989), "Factor Components, Population Subgroups and the Computation of the Gini Index of Inequality", The Review of Economics and Statistics, vol. 71, #1, February, 107-115.

Simons, H. (1950), Federal Tax Reform, University of Chicago Press.

Yitzhaki, S. (1979), "Relative Deprivation and the Gini Coefficient", Quarterly Journal of Economics, vol. 93, pp. 321-324.

Yitzhaki, S., (1983), "On an Extension of the Gini Index", International Economic Review, #24, pp. 617-628.

## Table 1

**Feelings of Relative Ill-Performance at Different Deciles**

**Canadian Tax and Benefit Systems, 1981 and 1990**

**(asymptotic standard errors in parentheses)**

| **q** | **0.1** | **0.2** | **0.3** | **0.4** | **0.5** | **0.6** | **0.7** | **0.8** | **0.9** |
|---|---|---|---|---|---|---|---|---|---|
| **r(q) in 1981** | 0.0097 | 0.0074 | 0.0066 | 0.0063 | 0.0056 | 0.0053 | 0.0044 | 0.0038 | 0.0026 |
| | (0.0004) | (0.0006) | (0.0006) | (0.0006) | (0.0005) | (0.0005) | (0.0005) | (0.0004) | (0.0004) |
| **r(q) in 1990** | 0.0129 | 0.0117 | 0.0110 | 0.0109 | 0.0098 | 0.0088 | 0.0066 | 0.0050 | 0.0026 |
| | (0.0004) | (0.0006) | (0.0006) | (0.0006) | (0.0006) | (0.0005) | (0.0005) | (0.0004) | (0.0003) |

**Table 2**

**Aggregate Indices of Horizontal Inequity**

**Canadian Tax and Benefit Systems, 1981 and 1990**

**(asymptotic standard errors in parentheses)**

| **v** | **1.2** | **1.4** | **1.6** | **1.8** | **2.0** | **2.2** | **2.4** | **2.6** | **2.8** | **3.0** |
|---|---|---|---|---|---|---|---|---|---|---|
| **1981 Indices of Horizontal Inequity** | 0.0024<br>(0.0002) | 0.0046<br>(0.0004) | 0.0067<br>(0.0005) | 0.0086<br>(0.0007) | 0.0103<br>(0.0008) | 0.0120<br>(0.0009) | 0.0136<br>(0.0010) | 0.0152<br>(0.0011) | 0.016<br>(0.001) | 0.0182<br>(0.0012) |
| **1990 Indices of Horizontal Inequity** | 0.0035<br>(0.0002) | 0.0069<br>(0.0004) | 0.0101<br>(0.0005) | 0.0130<br>(0.0007) | 0.0159<br>(0.0008) | 0.0186<br>(0.0009) | 0.0211<br>(0.0010) | 0.0236<br>(0.0011) | 0.025<br>(0.001) | 0.0282<br>(0.0013) |