Bank value and financial fragility

(*Draft*)

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Abstract

We propose a valuation model for a bank which faces a bankruptcy risk. Banks are identified with a possibly infinite random sequence of net benefits. A bank is solvent as long as its benefits remain non-negative. To preserve distressed banks from destruction, banks will be pooled within a financial coalition. When possible, those with current positive balance sheet will refinance those in need of liquidity. Banks are refinanced to the extent that their current needs for liquidity do not exceed their expected endogenous continuation value. This value itself is affected by future refinancing possibilities. We provide a recursive formula to compute this value when there is an aggregate liquidity constraint.

1 Introduction

Correctly valuing a firm (or a project) is a central issue in finance. The value of a firm is typically equal to the expected discounted value of its future benefits. This assessment leads to an easy computation of a firm value as long as discount rates and the distribution of future benefits is known and stable. The difficulty lies in the assessment of this value when

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there is a risk of financial distress and refinancing is potentially available. A financier must be able to compare the future profitability of the firm with its current need for refinancing.

In a dynamic context, the flow of future benefits in the firm is conditioned by the possibility of financial distress, that is, net current benefits may be negative, which jeopardize the ability of the firm to continue operating. Suppose no refinancing is available. If there is a positive probability of distress, the firm will eventually stop operating, and the computation of its value should take this probability into account. As long as the occurrence of distress is well defined in the dynamic process of benefits, the computation of the firm's value remains trivial. The probability of bankruptcy enters into the "effective" discount rate which is used to discount future benefits. However, it is reasonable to assume that, in the event of financial distress, the firm may seek refinancing. Bankruptcy is then endogenous to current and future refinancing possibilities, and the computation of the firm's value becomes a non-trivial exercise.

The issue of endogenous bankruptcy has already been studied in the literature on optimal capital structure. Using a no-arbitrage argument, Merton (1974) computes the value of a firm's equity when its benefits follow a diffusion-type stochastic process. Merton (1974) assumes that the firm issues a zero-coupon bond with maturity at time T. If the assets' value is less than the debt's face value at T, the firm is bankrupt and the equity is worth 0. This makes the equity value resemble a European call option, which is valued using the Black and Scholes' (1973) formula. Merton's formula per se does not consider bankruptcy as an endogenous event. It can be used, however, to price any claim on a firm whose benefits are described by a diffusion process.

Leland (1994) considers a more complex type of debt with a continuous coupon, and computes the equity value when bankruptcy is either exogenous or endogenous. Bankruptcy is exogenous when it is triggered by the assets' value falling below a predetermined exogenous target level. Bankruptcy is endogenous when it is triggered by the impossibility to pay the coupon by issuing additional equity. In this case, there is a minimum value V_B of the firm's assets below which equity is worth 0 and the firm is bankrupt. The firm chooses this lower bound to maximize the total value of the firm. On the one hand, the lower bound V_B must be low enough to minimize the occurrence of bankruptcy; on the other hand, it cannot be too low since equity must remain positive for assets' value above the bound. Leland (1994) finds that the lower bound V_B on the assets' value that triggers bankruptcy is proportional to the debt coupon, independent of current assets' value, increasing in the risk-free rate of interest and decreasing in the volatility of the assets' value process.

Leland (1994) assumes that the firm can always refinance on the market as long as its equity value is positive. This translates into an environment of perfect financial markets. Liquidity constraints alone cannot be responsible for a firm declaring bankruptcy since there are no liquidity constraints facing the firm as long as its value remains positive. In this model, bankruptcy is said to be efficient.

We present here a model of firm's valuation when financial markets are imperfect. Financial imperfections may affect refinancing possibilities which in turn may affect the firm's value. Consider a firm in autarky which suffers a negative shock to its current benefit. Without additional financing, it has to declare bankruptcy since it does not have the necessary liquidities to continue operating. When refinancing is available, the firm can obtain additional financing, effectively selling claims on future benefits. If markets are perfect, the firm can always find refinancing as long as its value is greater than its liquidity needs. This case is studied in Leland (1994).

When financial markets are imperfect, the refinancing decision itself depends on future refinancing possibilities, hence on future financial imperfections. In a first step, we focus on a limited aggregate supply of liquidity as a source of market imperfections. A firm may not be able to obtain financing even though it would be profitable to do so because the aggregate supply of liquidity is bounded. It turns out that this assumption can limit the extent of refinancing a firm can obtain.

Specifically, we frame our analysis within the context of a bank. Each period, the bank receives a net benefit. For example, this benefit is its net cash flow which consists of the sum of interest income, paid back loans, and new deposits minus interest payments, loans and withdrawals. If this benefit is below a threshold level (normalized to zero), the bank needs refinancing to pursue its activities. Without financing, it cannot honour its financial liabilities and must declare bankruptcy. The bank seeks financing on the interbank loan market. It can raise new financing promising claims on its future benefits. The objective is to study the impact of the supply of aggregate liquidity on the value of a bank and its financial fragility.

Our main contribution in this paper is to propose a formula for banks' valuation when there is a potential aggregate shortage of liquidity. We suppose that there is no deep-pocket financier that could refinance all banks each time this is optimal to do so. Instead, we have a finite number of independent banks which can provide financing to each other when they have enough liquidity to do so. As long as a the value of the bank is greater than its liquidity needs, it is optimal to refinance it.[1] This may not be possible, however, if the other existing banks do not generate enough liquidity to refinance the distressed bank. A bank may become financially vulnerable because the aggregate supply of liquidity in the economy may be low, not because its potential value per se falls below zero. Of course, the "true" value of the bank is influenced by the current (and future) aggregate supply of liquidity since it may go bankrupt even though it would not if there was no liquidity shortage. We then say that the bank is "financially fragile".

Within this context, we study two specific questions.

1. What is the lower bound on the value of a bank so that it is optimal to let it file for bankruptcy?

2. If the aggregate supply of liquidity is limited, which banks should be rescued?

[1] Even this statement hides many difficulties since the value of the bank itself depends on future refinancing possibilities.

The analysis of the impact of aggregate liquidity constraints on firms is the more relevant the larger are the firms under study. In most industrial countries, financial actors such as banks are among the largest firms in the economy. Furthermore, financial distress at a bank can have significant consequences given the extent of their activities (in every firm and household). It is therefore important to study the determinants of banks' financial fragility.

In section 2, we introduce the model and notation. Section 3 computes the value of a bank when there is a potentially infinite supply of liquidity. In this case, a bank goes bankrupt only if it is not profitable for it to be refinanced. Bankruptcy is then said to be efficient. In section 4, we assume that the aggregate supply of liquidity is finite. The supply of liquidity is endogenized with the introduction of a finite number of banks in the economy. Hence, it may not always be possible for a bank to refinance itself even though it may be profitable to do so. The bank then effectively faces a liquidity constraint. We compute the value of a bank in this context, and study the occurrence of bankruptcy. The conclusion follows.

2 The model

Consider a stationary environment given by an infinite random sequence of i.i.d. states $(s_n)_{n\in\mathbb{N}}$ where n is a time subscript. Each state s_n is drawn in (S, σ, μ) where S is a compact set of states and μ is a probability measure. In what follows, the time subscript is dropped whenever this does not create any confusion. Hence s usually refers to the current state. A risk-neutral *bank* is described by a measurable continuous function $y : S \to \mathbb{R}$ which associates to each state s_n the random benefits $y(s_n)$ it generates in state s_n for all n. Benefits can be positive or negative.

A bank is *solvent* if the realization of a random variable $x : S \to \mathbb{R}$ is non-negative. If $x(s) < 0$, the bank is said to be bankrupt. The function x simply defines the criterion by which a bank is said to be solvent or not. It will be referred below as a "survival policy function". If a bank has a negative net benefit and it declares bankruptcy, we assume that it can forego its financial liabilities, effectively making its net benefit zero. This is standard in bankruptcy procedures, and we maintain this assumption throughout the paper. A bankrupt bank can never be reactivated so that if it goes bankrupt in period n, the bank brings a benefit of zero in period n and all subsequent periods.

The following mathematical notation will be used below. Let f be a measurable function on S. We may partition S into two measurable sets

$$f^+ \equiv \{s \in S | f(s) \geq 0\},$$
$$f^- \equiv \{s \in S | f(s) < 0\},$$

so that f^+ is the subset of states for which $f(s)$ is non-negative and $f^- = S \setminus f^+$ is its complement in S. The conditional expectation of z given any event P_i of positive measure is simply noted $\mathbb{E}\{z|P_i\}$.[2]

Suppose for now that the bank has no access to refinancing. In this case, the survival policy function x is equal to y since the bank is bankrupt if its benefit y is negative. The *value* of an autarcic bank is the expected discounted sum of its current and future benefits taking into account that the firm is bankrupt if its benefit $y(s)$ becomes negative. Stated differently, we may say that the value of a bank is the sum of its current benefit (which could be negative) and of its discounted *continuation value* where the latter stands for the expected value of future benefits. Notice that up to a bankruptcy episode, benefits are stationary. Hence, this continuation value is either zero if the bank is bankrupt or some constant non-negative expected discounted value if the bank is solvent.

Under the assumption of stationarity of the benefit function y, the value of the bank depends only on the current state, and may be defined as a random variable $v_{x=y}(y) : S \to \mathbb{R}$ with expected value $V_{x=y}(y)$. Notice that the value of the bank depends on the survival policy function x which is equal to y in autarky.

[2]See Lucas, Stokey, and Prescott (1989):205.

To compute the expected value, consider that

$$v_y(y)(s) = \begin{cases} y(s) + \delta V_y(y) & \text{if } s \in y^+, \\ 0 & \text{if } s \in y^-, \end{cases} \tag{1}$$

where $\delta \in (0,1)$ is the (subjective) discount rate. Because benefits are stationary, the expected continuation value of a bank is constant. Hence, taking the expectation on (1) yields

$$\begin{aligned} V_y(y) &= \mathbb{E}\{v_y(y)\}, \\ &= \mu(y^+)\mathbb{E}\{y + \delta V_y(y)|y^+\}, \\ &= \frac{1}{1-\delta\mu(y^+)}\mu(y^+)\mathbb{E}\{y|y^+\}. \end{aligned} \tag{2}$$

Equation (2) yields a formula for the valuation of a currently solvent bank that has a constant probability $\mu(y^-)$ of becoming bankrupt. This value depends on the survival policy function $x = y$ and on its benefit y.

In the next section, we introduce the possibility for refinancing.

3 Bank value without an aggregate liquidity constraint

Consider a risk-neutral bank with discount rate δ whose benefits are given by $y(s)$. As long as $s \in y^+$, the bank can get its benefit $y(s)$. Yet, if $\mu(y^-) > 0$, then sooner or later its benefit will be negative, and the bank will be in financial distress. If the bank cannot secure additional financing, it has to declare bankruptcy. As seen in the previous section, without external financing, the value of the bank is then $V_y(y)$.

If $y(s) < 0$, refinancing the bank makes economic sense if its continuation value is greater than its current liquidity requirement $-y(s)$. Thus, current and future refinancing can increase the value of the bank. But what is this value is not clear anymore: if it is possible to finance the bank today, then it is probably possible to refinance it in the future. Hence, the probability that the bank fails again in the future is not necessarily $\mu(y^-)$, and $V_y(y)$ is no longer the expected future value of the bank. What is then the value of a bank when the possibility of refinancing is taken into account?

It turns out that, despite the apparent difficulties, the answer is relatively easy to compute when the bank deals with a deep-pocket investor, that is, when the supply of liquidity appears infinite from the bank's point of view. Suppose that $y(s) < 0$ and that the bank raises an amount F from an investor to relax its liquidity constraint. The liquidity constraint in the current period is then

$$y(s) - t(s) + F \geq -t(s),$$

where $t(s)$ is a payment owed to the investor and determined in a past period. This constraint says that the net benefit of the bank plus financial transfers cannot be inferior to the amount owed to the investor. If this constraint is satisfied, the bank has enough liquidity to pay for its liabilities and can thus continue operating. If it is not satisfied, then the bank must declare bankruptcy and the investor gets nothing since $y(s) < 0$. This implies that the current debt $t(s)$ is irrelevant for the decision to refinance or not the bank. The liquidity constraint then becomes

$$y(s) + F \geq 0,$$

that is, the bank has to raise at least as much liquidity as to cover current liabilities $y(s)$ (excluding financial liabilities $t(s)$). The next step is to determine the maximal amount of financial capital the firm can raise. This amount is equal to the expected discounted value of all future benefits, again taking into account future possibilities of bankruptcy and refinancing. This maximal amount corresponds to the bank effectively selling the firm to an investor that is not liquidity constrained.

Define by S^* the set of states in which the firm is successfully refinanced and therefore solvent. Since the decision to refinance is independent of current financial liabilities and since benefits are stationary, the set S^* is time independent. Using similar computations as those in the previous section, we obtain that the expected discounted value of all future benefits is given by

$$\frac{\delta}{1-\delta\mu(S^*)}\mu(S^*)\mathbb{E}\{y|S^*\}.$$

This is the maximal amount of financial capital the bank can raise. The liquidity constraint then becomes

$$y(s)+\frac{\delta}{1-\delta\mu(S^*)}\mu(S^*)\mathbb{E}\{y|S^*\}\geq 0. \tag{3}$$

The set S^* corresponds to all these states s for which condition (3) is satisfied. It is easy to see that, if a state $s\in S^*$, then all states s' such that $y(s')\geq y(s)$ are also in S^*. This implies that there exists some lower bound y^* such that the firm is solvent for all $y(s)\geq y^*$. Note that this lower bound y^* must be negative, that is, it is never optimal to declare bankruptcy when the current benefit is positive. The set of solvent states is given by $(y-y^*)^+$. We can rewrite condition (3) as

$$y(s)+\frac{\delta}{1-\delta\mu((y-y^*)^+)}\mu((y-y^*)^+)\mathbb{E}\{y|(y-y^*)^+\}\geq 0.$$

The lower bound y^* solves

$$y^*+\frac{\delta}{1-\delta\mu((y-y^*)^+)}\mu((y-y^*)^+)\mathbb{E}\{y|(y-y^*)^+\}=0. \tag{4}$$

This solution implicitly defines the set $S^*=\{s\in S|y(s)\geq y^*\}$.

The survival policy function associated with S^* is $x_{y^*}:S\to\mathbb{R}$ such that

$$x_{y^*}(s)=\begin{cases} y(s) & \text{if } y(s)<y^*,\\ 0 & \text{else.}\end{cases}$$

We can now compute the expected value of the bank. In any period and state s, we have

$$v_{x_{y^*}}(y)(s)=\begin{cases} y(s)+\delta V_{x_{y^*}}(y) & \text{if } s\in(y-y^*)^+,\\ 0 & \text{if } s\in(y-y^*)^-.\end{cases} \tag{6}$$

Taking the expectation on (6) yields

$$\begin{aligned} V_{x_{y^*}}(y) &= \mathbb{E}\{v_{x_{y^*}}(y)\}, \\ &= \mu((y-y^*)^+)\mathbb{E}\{y+\delta V_{x_{y^*}}(y)|(y-y^*)^+\}, \\ &= \frac{1}{1-\delta\mu((y-y^*)^+)}\mu((y-y^*)^+)\mathbb{E}\{y|(y-y^*)^+\}. \end{aligned} \tag{7}$$

Equation (7) gives the expected value of the bank when it has access to a deep-pocket investor. This value depends on the time-independent survival policy function which characterizes the efficient bankruptcy rule. For $y(s) \geq y^*$, it is profitable to keep the bank operating. Bankrupting it would destroy value since its future value is larger than the amount of liquidity required to keep it solvent. For $y(s) < y^*$, it is optimal to bankrupt the bank since its future value is smaller than the amount of liquidity required to keep it solvent.

When the bank has access to a deep-pocket investor, it can raise funds up to its discounted expected value taking into account the probability of bankruptcy. In some cases, this amounts to selling the bank to the investor.

With a deep-pocket investor, the value of the bank is $V_{x_{y^*}}(y)$. It can be compared to its value in autarky, which is $V_y(y)$. This last expression corresponds to the case of $y^* = 0$. It is easily shown that $V_{x_{y^*}}(y) \geq V_y(y)$, and therefore the availability of a deep-pocket investor raises the value of the bank.

We now present an example to illustrate these results. Consider a benefit function $y(s) = s$ with s uniformly distributed over $S = [-a, a+2]$. We have that $\mathbb{E}\{y\} = 1$. The worst realization of benefits is decreasing in a, and the variance is increasing in a. If

$$-a + \frac{\delta}{(1-\delta)}\mathbb{E}\{y\} \geq 0,$$

then $S^* = S$, and it is always profitable to refinance the bank regardless of the level of current benefits $y(s)$. In what follows, we then restrict our attention to the case where $a > \delta/(1-\delta)$. Suppose first that $\delta = 0.9$ and $a = 10$. In autarky, the value of the bank is $V_y(y) = 3.51$. The financed bank has a value of $V_{x_{y^*}}(y) = 9.38$ with $y^* = -8.44$. The probability of bankruptcy is about 7%. Now suppose that $a = 20$. The autarcic value is now $V_y(y) = 5.71$, while the financed bank has value $V_{x_{y^*}}(y) = 11.06$ with $y^* = -9.95$, and the probability of bankruptcy is about 24%. As expected, as the variance of benefits increases, the value of the bank increases in both cases. With the possibility of bankruptcy, the benefit function of the bank is convex, which implies that an increase in variance increases the net benefit the bank gets. The probability of bankruptcy is also increasing in the variance. Finally, suppose that $\delta = 0.1$ and $a = 10$. The autarcic value is now $V_y(y) = 1.89$, while the financed bank has value $V_{x_{y^*}}(y) = 1.92$ with $y^* = -0.19$, and the probability of bankruptcy is about 45%. As the future becomes less important, the difference between the two values shrinks. Refinancing is hard to get and the probability of bankruptcy is high.

The results of this section are based on the assumption that the investor is not financially constrained. This implies that the bank can always be refinanced (when it is optimal to do so) regardless of what happens in the rest of the economy. This is a strong assumption. Suppose instead that there are many banks in the economy. It is then possible that the possibilities for refinancing depend on the demand for liquidity by other banks. If, for example, the economy is in a deep recession, it may be harder for a bank to refinance than it is when the economy is in a boom, that is, the demand for liquidity may be larger in the former case than in the latter case since, presumably, there may be more banks in financial distress in a recession. To address these issues, we introduce in the next section an aggregate liquidity constraint.

4 Bank value with an aggregate liquidity constraint

We relax the assumption that liquidity is always available to refinance a distressed profitable bank. This means that there might be some states where the bank should optimally be refinanced but there is simply not enough liquidity to do so. For example, there may be states s and s' such that $y(s) = y(s')$ but the bank is solvent in state s and bankrupt in state s' although its current liquidity requirement and future expected returns are *the same* in both states.[3] *Liquidity constraints* may bind at the aggregate level so that states s and s' differ in the sense that it is easier for the bank to get refinancing in state s than in state s'. To the extent that these contingencies are anticipated by investors, liquidity constraints increase the probability that a bank fails and reduce its value. Hence, two banks y and y' that have the same marginal distribution of returns could have different expected discounted values if one is more likely to be bankrupt than the other. This is what we mean by "financial fragility" although a formal definition would be premature at this point.

In standard portfolio theory, risk-aversion or a changing discount rate is needed to get a similar result. With risk-averse investors with sufficient aggregate purchasing power, two assets with the same marginal distribution of returns have different values if one allows the formation of a risk-balanced portfolio and the other one does not. It turns out that with liquidity constraints, risk aversion is not necessary for similar effects to take place. For tractability reasons, it is convenient to assume that banks are risk neutral.

If the discount rate is changing, a dollar in state s may be valued differently than a dollar in state s'. Hence, to preserve a given flow of returns, spending $-y(s)$ dollars to save a bank y distressed in state s might make sense[4] while spending the same amount in another state s' might be uneconomical. To a large extent, our model fits this story: exogenous shocks on the total supply of liquidity affect the "effective" discount rate different banks face since they affect their probability of bankruptcy. This can be contrasted with standard macroeconomic models where changes in the "effective" discount rate are driven by exogenous technological shocks.

We abstract from capital or financial accumulation considerations. This is a first step for characterizing financial contracts and financial fragility for a set of banks. To do so, we must simplify the model. In a more general model, it is conceivable that financially-constrained banks may accumulate physical or financial capital to better protect themselves

[3]Since a state s is a description of the whole economy, it is conceivable that a bank may have the same benefit in two different states, while benefits of other banks differ in these two states.

[4]When the interest rate is low so that the relative price of a current dollar is low.

against future adverse liquidity shocks. Notice, however, that financial arrangements allow banks to save by lending to other banks. Such saving is subject to potential financial distress of the borrowing bank.

Although the effects of accumulation are left aside, we focus on another neglected dynamic issue that is relevant in the short run. We analyze the risk that a series of bad shocks could have a domino effect and leave the economy in disarray. When a bank y goes bankrupt, the aggregate flow of liquidity in the future is reduced and this could jeopardize the solvency of some other bank y' in the future.

Consider an economy populated with many banks. These banks may trade on the inter-bank market to refinance themselves if need be. When there is an aggregate constraint on the supply of liquidity, the ability to refinance may be limited. We are interested in characterizing the outcome in such a world. In what follows, we derive a recursive formula to compute the value of a *coalition* of banks. Our approach is to maximize the current expected value of the coalition's benefits. This is done through a complex financial "contract" that pools the coalition's benefits which can be associated to a specific survival policy function. We therefore abstract, at this point, from the existence of a market to characterize the optima of such an economy. Whether the optimal allocation induced by this complex contract can be decentralized with a set of prices will be the object of future research.

The coalition model

Consider a stationary economy like before with a population y of $M \geq 1$ finitely many labeled (with y) banks. For a sub-coalition z of y, $z(s)$ is the set of labeled benefits generated by each bank in z in state s. The sum of the elements of $z(s)$ is simply noted $\Sigma\, z(s)$. If z is a coalition with benefits $z(s)$, then $z(s)^+$ is the subset of these benefits that are non-negative, and z_s^+ is the sub-coalition of y obtained using the labels associated to the values of $z(s)^+$. $z(s)^-$ and z_s^- are defined the same way, $z(s)^-$ being the subset of strictly negative benefits of $z(s)$, and z_s^- being the coalition of banks associated to these values.

As before, the net benefit of any bank may be modified with financial transfers. In state s, let c_s be the coalition of banks in financial distress ($y(s) < 0$) and that are profitable to refinance. Let $c(s)$ be the set of (negative) benefits associated to c_s in state s. Then the aggregate liquidity constraint eventually binds if the set of states

$$\left\{s \in S \middle| -\Sigma\, c(s) > \Sigma\, y_s^+(s)\right\}, \tag{8}$$

is of strictly positive measure, that is, if there is a positive probability that the liquidity requirements of the sub-coalition c_s of profitable but distressed banks exceed the liquidity generated by solvent banks.

This economy is similar to Lucas' (1978) tree economy where each bank is a "tree", that is, an asset that provides a stream of non-storable endowments. In any given period, a tree may bear fruits ($y(s) \geq 0$) like in Lucas' model or require care ($y(s) < 0$). In Lucas' model, trees are eternal. Here, a distressed tree survives only if a gardener takes care of the tree. It costs $-y(s)$ in fruits for the gardener to take care of a distressed tree for which $y(s) < 0$. An alternative to taking care of that tree is to let it die and to eat the $-y(s)$ fruits instead.[5] In a given period, the total amount of fruits may not be sufficient to take care of all distressed trees. If all trees are always profitable and if there is always a sufficient amount of fruits to take care of all distressed trees, the set (8) is of measure zero and the valuation of trees is the same as in Lucas' model. If the aggregate liquidity constraint sometimes binds, the computation of the value of a tree is more involved. We now take up that task.

We generalize the definition of a stationary survival policy function x to work for coalitions. Let $x^y : S \to \mathbb{R}^M$ determine whether bank y is solvent ($x^y(s) \geq 0$) or bankrupt ($x^y(s) < 0$) in state s. It is not necessarily possible that all banks survive one period to the other. In any given state s, we assume that all banks in y_s^+ are solvent. For a coalition z to remain solvent in state s, the distressed banks z_s^- must be financed. This is possible if and only if the total liquidity requirement of these distressed banks does not exceed the total liquidity generated by the solvent banks. Hence, a survival policy function is *feasible* if and

[5]The technology is discussed in the appendix.

only if it satisfies the following properties.

Admissibility (AD)**:** If a bank y has a non-negative benefit in state s, then it is solvent in state s, and $x^y(s) \geq 0$. Equivalently, if z is solvent in state s, then $y_s^+ \subseteq z$.

Budget Balance (BB)**:** If x allows the coalition z to survive in state s, then

$$-\Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s).$$

Admissibility precludes any mechanism where a solvent bank would be artificially indebted to finance other distressed banks. Budget balance is the standard definition adapted to our framework. Now, if x allows the coalition z to survive in state s, then $x(s)$ may be trivially defined as

$$x^y(s) = \begin{cases} 1 & \text{if bank } y \text{ belongs to } z, \\ -1 & \text{otherwise.} \end{cases}$$

Suppose that we know how to compute the expected value of an arbitrary coalition of banks z of size less than or equal to $M \geq 1$. Let $V(z)$ be this expected value. In what follows, we show how to compute the value of an arbitrary coalition y of $M+1$ banks. Let 2^y be the power set of sub-coalitions of y. Suppose that initially the set of active banks is y. In state s, an optimal survival policy function selects a coalition that solves

$$\begin{aligned} \text{Program 1}: \quad & \max_{z \in 2^y} \quad \Sigma\, z(s) + \delta V(z), \\ & \text{s.t.} \quad y_s^+ \subseteq z, && \text{(AD)} \\ & \qquad -\Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s). && \text{(BB)} \end{aligned}$$

This problem is well defined by assumption up to $V(y)$ which is unknown, that is, the expected value $V(z)$ of all coalitions z of no more than M banks is known by assumption, but the expected value of the (current) grand coalition y of $M+1$ banks is unknown.

By admissibility (AD), for all states such that y_s^- is empty, the instrument set contains only y and Program 1 reduces to

$$\Sigma\, y(s) + \delta V(y). \tag{9}$$

Consider now the states for which y_s^- is not empty. The following restricted program is now well defined.

$$\begin{aligned}
\text{Program 1a}: \quad \max_{z \in 2^y} \quad & \Sigma\, z(s) + \delta V(z), \\
\text{s.t.} \quad & y_s^+ \subseteq z, & \text{(AD)} \\
& -\Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s), & \text{(BB)} \\
& z \neq y.
\end{aligned}$$

By construction, we know how to solve Program 1a since $V(y)$ need not to be evaluated.

Program 1 can be represented as a dynamic program where, if y_s^- is non-empty, one decides first if y should survive and, in the case where it should not, which coalition z should survive. Define the random variable $\nu : S \to \mathbb{R}$ that takes the value of Program 1a. The value $v(s)$ of Program 1 then becomes

$$v(s) = \begin{cases} \Sigma\, y(s) + \delta V(y), & \text{if } y_s^- = \emptyset, \\ \max\left\{\Sigma\, y(s) + \delta V(y), \nu(s)\right\}, & \text{if } y_s^- \neq \emptyset \text{ and } \Sigma\, y(s) \geq 0, \\ \nu(s), & \text{otherwise.} \end{cases}$$

Since this is a stationary value, $V(y) = \mathbb{E}(v)$.

Now let

$$S^* = \left\{s \in S \middle| \, \Sigma\, y(s) + \delta V(y) \geq \nu(s) \text{ and } \Sigma\, y(s) \geq 0\right\}.$$

This is the set of states where the full coalition y survives, either because y_s^- is empty, or because it is feasible and profitable to refinance all distressed banks. In what follows, we assume that $\mu(S^*) \in (0, 1)$. The following lemmas will be useful in the next section.

Lemma 1. $\{s \in S | y_s^- = \emptyset\} \subseteq S^* \subseteq \{s \in S | \, \Sigma\, y(s) \geq 0\}$.

Proof. The first part comes directly from admissibility (AD). The second part, directly from the budget balance condition (BB). *Q.E.D.*

Lemma 2. *Monotonicity. Let $s \in S^*$ and consider s'. If, for all banks, $y(s') \geq y(s)$, then $s' \in S^*$.*

Proof. If $y_s^- = \emptyset$, the result is obvious. If $y_s^- \neq \emptyset$, then the question becomes: *given that we manage to keep all banks solvent, would we want to drop a bank now that aggregate liquidity has risen?* The answer is "No". Suppose that in state $s \in S^*$ the coalition z survives, and that banks $w \subset z$ are bankrupt in state $s' \in S^*$. This implies that

$$\Sigma\, z(s) + \delta V(z) \geq \Sigma\, z \setminus w(s) + \delta V(z \setminus w). \tag{10}$$

In state s', y increases for all banks. Given stationarity, this affects only the first term on each side of condition (10). Since there are more banks in z than in $z \setminus w$, this condition must also be satisfied in state s'. Hence, it is not optimal to bankrupt more banks in s' than in s. *Q.E.D.*

We now compute the value of the grand coalition y.

$$V(y) = \mu(S^*)(\mathbb{E}(\Sigma\, y|S^*) + \delta V(y)) + (1 - \mu(S^*))\mathbb{E}(\nu|\neg S^*), \tag{11}$$

or,

$$V(y) = \frac{\mu(S^*)\mathbb{E}(\Sigma\, y|S^*) + (1 - \mu(S^*))\mathbb{E}(\nu|\neg S^*)}{1 - \delta\mu(S^*)}. \tag{11'}$$

For any given y, the value $V(y)$ is a real number that solves either (11) or (11′).

We have shown in section 2 that a coalition $y = \{y\}$ composed of a single bank ($M = 1$) has an expected value of $V(y) = V_y(y)$. We have shown that if we know how to compute the expected value of M banks or less, we may compute the value of $M+1$ banks. By induction, we can therefore compute the expected value of an arbitrary but finite coalition of banks. In the next section, we do so explicitly for a coalition of 2 banks.

4.1 A two-bank coalition

Let $y = \{y_1, y_2\}$. We know that $V(y_1) = V_{y_1}(y_1)$ and that $V(y_2) = V_{y_2}(y_2)$. We want to compute $V(y)$. To do so, we need to identify S^*.

By Lemma 1, we need only to identify those states where only one bank is distressed and it makes economic sense to refinance it. If $y_j(s) > 0 > y_i(s)$ and $y_i(s) + y_j(s) \geq 0$, then bank y_i will be rescued if

$$y_i(s) + y_j(s) + \delta V(y) \geq y_j(s) + \delta V_{y_j}(y_j),$$

that is, if

$$y_i(s) \geq \delta(V_{y_j}(y_j) - V(y)) \equiv y_i^{**}.$$

Hence, both banks remain solvent as long as $\Sigma\, y(s) \geq 0$ and each $y_i(s)$ is at least equal to some endogenous stationary value y_i^{**} that depends on $V(y)$. $V(y)$ may be obtained as the solution to $(11')$ where

$$S^* = \{s \in S | y_1(s) \geq y_1^{**}, y_2(s) \geq y_2^{**} \text{ and } y_1(s) + y_2(s) \geq 0\}.$$

It is now possible to compute the value of a single bank within coalition y. Denote the value of bank $y_i \in y$ by

$$V^{y_i}(y) = \mu(S^*)V^{y_i}(y) + \mu(S_{y_i})V_{y_i}(y_i),$$

where

$$S_{y_i} = \{s \in S \setminus S^* | y_i(s) \geq 0\}$$

is the set of states for which only bank y_i is solvent. The value of bank y_i depends implicitly on the value of the whole coalition through its dependence on the set S^*. This underlines the type of externality that an aggregate liquidity constraint can create. This will have to be taken into account when decentralizing this allocation.

Consider figure 1. The thick curve represents the image $y(S)$. It passes through four different regions. The first region is the negative orthant were both banks are necessarily bankrupt. The subset $y(\{s \in S | y_s^+ = \emptyset\})$ is the portion of the thick curve that goes through that region. The second region is tinted in dark gray. There both banks survive. Again, $y(S^*)$ is the portion of the thick curve that goes through that region. In the two L-shaped region tinted in light gray, only one of the bank survives.

Notice the fact that y_i^{**} is independent of s is an artefact of the two-bank coalition. In general, this threshold value depends on the state s. For example, suppose there are three banks. Further assume that only one bank is solvent (say bank 1) and that it can only refinance one of the two distressed banks. Whether say bank 2 is refinanced or not depends not only on the net future payoff of doing so (as it is the case with two banks), but also on the opportunity cost of bankrupting bank 3. This cost depends on the current amount of liquidity needed to refinance bank 3. Hence, the threshold value depends on the state s.

5 Conclusion

A bank remains solvent as a long as its benefit is positive or it is profitable to refinance it. It turns out that the decision to refinance is complicated by the fact that it depends on future refinancing opportunities. Furthermore, if there is an aggregate liquidity constraint, the opportunity cost of refinancing one bank has to take into account the possibility of bankrupting another bank. So far, our analysis shows how to compute the value of a given number of banks in an economy plagued with an aggregate liquidity constraint. We then illustrate our results for a two-bank coalition. Important issues remain to be explored at this point.

First, it would be important to fully characterize the value of a two-bank coalition, and then to extend it to more banks. As argued in the text, with more than two banks, the decision to refinance a bank or not is not independent of what happens to the other banks. This complicates the analysis, but it also makes it more interesting.

Secondly, our approach so far has been one of maximizing the aggregate surplus in the

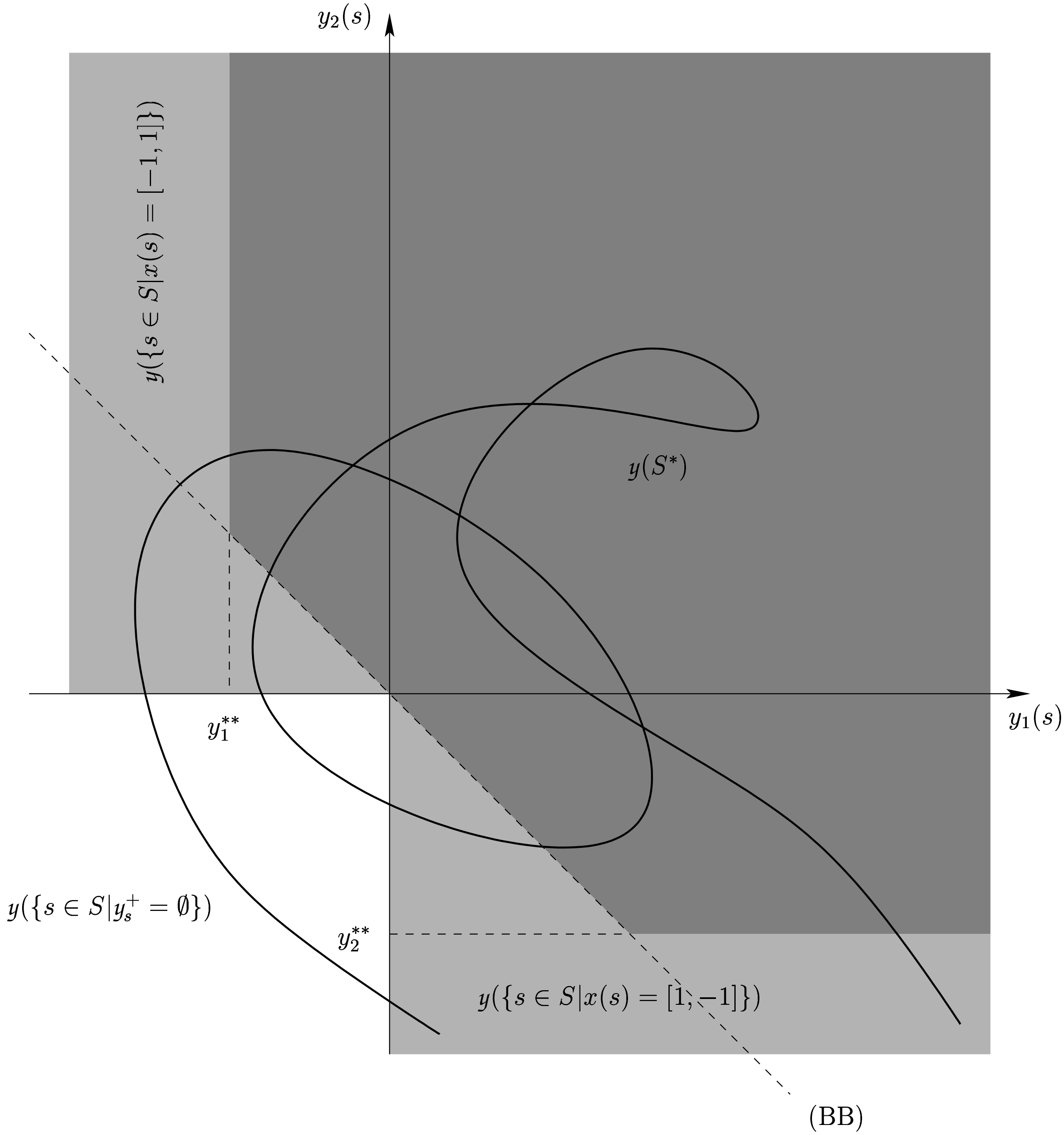

Figure 1: Optimal financing with liquidity constraints.

economy. It is important to see whether the maximizing allocation can be decentralized through a set of competitive financial markets.

Finally, the only source of financial imperfection has been a potential shortage of liquidity at the aggregate level. If markets cannot decentralize the maximizing allocation, banks may have to use long-term contracts. It would then be interesting to characterize the nature of these contracts when they suffer from other imperfections such as non-commitment.

The analysis can lead to a theory of financial fragility. A more complete characterization should provide insights on the incidence of financial distress, its causes and consequences.

Our modelling approach raises another important issue. In the macroeconomic interpretation of our model, "liquidity" is a composite good that is available in a limited amount in any given period. This liquidity constraint is itself a reduced form of some underlying financial phenomenon: someone out there requires immediate payment for (otherwise idle) resources provided to keep the firm solvent. For some unspecified reason, this agent would not accept a standard I.O.U. for payment. Hence, our model implicitly includes two kinds of investors: some very sophisticated who can accept complex financial arrangements (that could end up in a transfer of ownership) and some who insist on immediate payment. The first type could be financiers, while the second type could be workers of the firm or suppliers that cannot afford to finance the distressed firm since they have their own immediate need for liquidity. To make sense of these behavioral differences, we need to understand why some investors are ready to cooperate while others are not.

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A The tree technology

In this appendix we discuss some features of the technology we use in this model. In particular, we show that the technology is not necessarily convex so that the existence of a price equilibrium or the Welfare Theorems may fail to hold in this economy.

To relate the present model to existing literature, we recast it in the neo-classical framework. A *technology* is a representation of the abilities of producers to transform inputs into outputs. It is usually described in terms of *production sets*, that is, sets of combinations of inputs and outputs (called *netputs*) that are assumed feasible notwithstanding the availability of these inputs.[6]

At any given time, the technology for this economy involves two types of goods, the trees and their fruits. Fruits are an homogenous good, but trees are not homogenous since they are affected by specific exogenous factors like local weather, soil irrigation, etc. For a tree labeled y, the value $y(s)$ results from the effect of these factors in a given state s. Hence, if there are M trees, we consider $M + 1$ goods.

There are $2(M+1)$ goods to define a *production plan* which distinguishes the availability of goods *before* and *after* production takes place. Let $(-w, 0)$ be the netput before production where w is a M-dimensional vector in some space K^M (to be defined below) that represents

[6] We follow below the treatment given in Mas-Colell, Whinston, and Green (1995) §20.C.

the various kinds of trees. Let $x \in \mathbb{R}$ be the amount of fruits produced fo consumption. So, (w', x) is the netput after production.

The shock $y(s)$ is related to trees of type y that are used in quantity[7] w_y in the process. y_s^+ and y_s^- are the subsets of y's that are respectively positive or negative. In state s, the production set is

$$Y(s) = \{(-w, 0, w', x) : 0 \leq x \leq \sum w'_y y(s)\}$$
$$\cap \{(-w, 0, w', x) : w'_y = w_y, \forall y \in y_s^+; w'_y \leq w_y, \forall y \in y_s^-\}$$
$$\cap \{(-w, 0, w', x) : w \geq 0, w' \geq 0\}.$$

The first subset in the r.h.s. expresses budget balance and ensures the non-negativity of consumption. The second subset combines the admissibility assumption and the idea that the population of any kind of tree cannot increase. The third subset ensures that the population of trees remains non-negative.

Using these sets, one may then construct a *production path* where the production plan of one period overlaps with that of the next. The properties of the set of these production paths depend heavily on those of the $Y(s)$. In particular, notice that if K was $\mathbb{R}$, then $Y(s)$ would be convex. In our setting, K is the set of natural numbers and this renders the technology non-convex. Starting with a total supply (endowment) of 1 unit of each kind of tree (inherited from the past), this number can only stay at 1 or go to zero.

Hence, it is only to the extent that we have assumed that the number of banks is an integer (either 0 or 1) that non-convexity comes to play in the model. Intuitively, if it was possible to downsize the activities of any distressed bank to an arbitrary fine level, then the cost of renouncing to the future benefits of this bank could always be adjusted at the margin to the benefit of preserving the size of another bank. Marginal cost would reflect marginal benefit irrespectively of the distribution of the ownership of banks, and the Welfare Theorems would apply. In short, bankruptcy matters when size matters.

[7] Remember that neo-classical theory defines technology notwithstanding the availability of inputs.