# Choosing from a Weighted Tournament*

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**Abstract**

A voting situation, in which voters are asked to rank all candidates pair by pair, induces a tournament and a weighted tournament, in which the strength of the majority matters. Each of these two tournaments induces in turn a two-player zero-sum game for which different solution concepts can be found in the literature. Four social choice correspondences for voting situations based exclusively on the simple majority relation, and called $C1$, correspond to four different solution concepts for the game induced by the corresponding tournament. They are the top cycle, the uncovered set, the minimal covering set, and the bipartisan set. Taking the same solution concepts for the game induced by the corresponding weighted tournament instead of the tournament and working backward from these solution concepts to the solutions for the corresponding weighted tournament and then to the voting situation, we obtain the $C2$ counterparts of these correspondences, i.e. correspondences that require the size of the majorities to operate. We also perform a set-theoretical comparison between the four $C1$ correspondences, their four $C2$ counterparts and three other $C2$ correspondences, namely the Kemeny, the Kramer-Simpson, and the Borda rules. Given two subsets selected by two correspondences, we say whether it always belongs to, always intersects or may not intersect the other one.

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**Résumé**

Un vote à la majorité où les candidats sont comparés deux à deux induit un tournoi basé sur la relation majoritaire et un tournoi pondéré, où la taille de la majorité compte. Chacun de ces tournois induit à son tour un jeu à somme nulle pour lesquels on dispose de différents concepts de solution. Quatre correspondances de choix social applicables à la relation majoritaire, dites de type $C1$, correspondent à quatre concepts de solution différents pour le jeu induit par le tournoi correspondant. Ce sont le top cycle, le uncovered set, le minimal covering set et le bipartisan set. En utilisant les mêmes concepts de solution pour les jeux induits par les tournois pondérés équivalents, plutôt que par les tournois, et en allant des solutions pour les jeux aux tournois pondérés et ensuite aux relations majoritaires (votes), nous obtenons l'équivalent de type $C2$ des quatre correspondances de type $C1$, i.e. des correspondances qui exigent la dimension de la majorité pour opérer. Nous effectuons également une comparaison entre les quatre correspondances de type $C1$, leurs quatre équivalents de type $C2$ et trois autres correspondences de type $C2$, soit les règles de Kemeny, de Simpson-Kramer et de Borda. De façon plus précise, étant donné les ensembles de décision produits par deux correspondances de choix social, nous répondons aux questions : Est-ce qu'un de ces ensembles est toujours inclus dans l'autre? Si non, y a-t-il toujours intersection entre les deux ou, au contraire, peut-il arriver que leur intersection soit vide?

# 1 Introduction

A tournament is a competition in which every contestant meets every other contestant in turn.[1] In some tournaments, contestants may meet each other more than once, say $q$ times. In each of the $q$ encounters, one of the opponents beats the other one or they tie. We call them $q$-weighted tournaments or simply weighted tournaments, a term that we may trace back to Moulin (1988). More generally, a (weak) tournament is defined by a complete binary relation on a finite set. A (weak) weighted tournament is defined by a non-negative number $q$ and a list of numbers or weights $n(x, y)$, such that $n(x, y) + n(y, x) = q$ for all pairs of different contestants. The number $n(x, y)$ is the share of $q$ that contestant $x$ gets when opposed to $y$. Weighted tournaments may be seen as tournaments with additional information as to how strong each contestant is with respect to every other contestant.

Tournaments may arise in many contexts other than sports. Our main interest is in voting situations where voters are asked to rank all candidates pair by pair as imagined by Condorcet (1785) or equivalently to furnish their complete preference over the set of all candidates. In this paper, we define a (voting) situation as a set of candidates together with a list of preferences over this set. A situation induces a $q$-weighted tournament on the set of candidates, where $q$ is the number of voters and $n(x, y)$ is the number of voters ranking $x$ ahead of $y$. By retaining only the majority relation, i.e. whether $n(x, y) \geq n(y, x)$, the opposite or both, we get an unweighted tournament.

In Guénoche, Vanderputte-Riboud and Denis (1994), one finds 0-weighted tournaments where $n(x, y)$ represent the difference between the yield of rapeseed varieties $x$ and $y$ in a given territory. Guénoche (1995) applies similar techniques to marketing. He obtains a weighted tournament by comparing 9 brands of computers on a pairwise basis according to a poll of consumers. This is not very different from voting.

[1] In sports, this is called a round robin tournament.

A question that arises naturally in voting situations and thus in tournaments is how to choose a winner. This question has been the subject of many articles and books since the celebrated controversy between Borda (1784) and Condorcet (1785). Laslier (1997) provides an illuminating synthesis of these various contributions. In the language of the theory of social choice, a rule that selects a set of candidates in any voting situation is called a social choice correspondence. Condorcet advocated selecting a contestant that defeats all others more than half of the time in the $q$ encounters. Such a candidate is called a Condorcet winner. However, Condorcet was well aware that there might be a cycle in the majority relation preventing the existence of a Condorcet winner. When this happens, some other rule must be called to the rescue to break these cycles or to select some subset of contestants. A social choice correspondence that selects exclusively the Condorcet winner whenever it exists is called a Condorcet consistent social choice correspondence. Many of the social choice correspondences that have been proposed in the literature, including most of those that will be considered here, are of this sort. The above concepts can be transposed to tournaments where the social choice correspondences become solutions.

Following Fishburn (1977), we classify social choice correspondences on the basis of their data requirements. A social choice correspondence belongs to the class $C1$ if it is based exclusively on the simple majority relation, i.e. on the outcomes of all pairwise majority comparisons. The $C2$ social choice correspondences require more data, namely the size of the majorities or the numbers $n(x, y)$. Social choice correspondences requiring more data form the class $C3$. We could define another class contained in $C2$ and containing $C1$, say the class $C1.5$, consisting of social choice correspondences that use the numbers $n(x, y) - n(y, x)$ instead of the separate numbers $n(x, y)$ and $n(y, x)$. However, all $C2$ social choice correspondences that have been considered in the literature and those that we shall propose in this paper use only the numbers $n(x, y) - n(y, x)$. Thus Fishburn might have defined the class $C2$ as consisting of all correspondences that are based on the numbers $n(x, y) - n(y, x)$. This is the definition that we shall adopt here. This is particularly useful since the numbers $n(x, y) - n(y, x)$ define a 0-weighted tournament, which are easier to work with than other $q$-weighted tournaments.

Four of the most studied $C1$ social choice correspondences are: Schwartz's (1972) top cycle ($TC$), Fishburn's (1977) and Miller's (1980) uncovered set ($UC$), Dutta's (1988) minimal covering set ($MC$) and Laffond, Laslier, and Le Breton's (1993) bipartisan set ($BP$). Laffond, Laslier, and Le Breton (1995) compare these $C1$ choice correspondences from a set-theoretical point of view. Given a pair of social choice correspondences, and more precisely the choice sets of these correspondences, they determine which of the following three propositions holds:

- One always contains the other.
- They always intersect but, in some situations, none of them contains the other one.
- In some situations, they have an empty intersection.

Moulin (1986) also discusses some of these relations.

Laffond, Laslier, and Le Breton (1994) introduce the plurality (weighted) bipartisan set ($BP_w$). However, one does not find elsewhere in the literature $C2$ counterparts of the three other $C1$ correspondences mentioned above. Our purpose in this paper is twofold. First we fill the gap between $C1$ and $C2$ correspondences by introducing the weighted top cycle ($TC_w$), the weighted uncovered set ($UC_w$), and the weighted minimal covering set ($MC_w$).

There is probably a good reason why these concepts have not been proposed before. There is no obvious way of defining them directly in the context of voting situations. Our strategy is the following. As explained above, a voting situation induces a tournament and a 0-weighted tournament. Each of these two tournaments induces in turn a two-player zero-sum game for which different solution concepts can be found in the literature. It turns out that the four $C1$ social choice correspondences mentioned above correspond to four different solution concepts for the game induced by the corresponding tournament. Taking the same solution concepts for the game induced by the corresponding 0-weighted tournament instead of the tournament and working backward from these solution concepts to the solutions for the corresponding 0-weighted tournament and then to the voting situation, we obtain the desired $C2$ social choice correspondences.

Our second objective is to perform the set-theoretical comparisons, as in Laffond, Laslier, and Le Breton (1995), between the four $C1$ correspondences, their four $C2$ counterparts and three other $C2$ correspondences frequently encountered in the literature, namely the Kemeny ($Ke$), the Kramer-Simpson ($SK$) and the Borda ($Bor$) rules. Some of these comparisons are borrowed from the existing literature. The other ones, and specially those involving the new $C2$ concepts, have to be done from scratch. Our findings are summarized in Table 2 at the beginning of Section 3. One can view these comparisons as a preliminary step towards the definition of a metric on the space of social choice correspondences. Such a metric would give a more complete picture of the possible disagreements between different correspondences.

We restrict ourselves to (asymmetric) tournaments, in which no tie is allowed either in the individual encounters or in the split of the victories between contestants. In other words we assume that the binary relation defining an unweighted tournament is asymmetric and that the numbers $n(x, y)$ defining a $q$-weighted tournament satisfy $n(x, y) \neq n(y, x)$ for all pairs of different contestants. As is the tradition in the literature, we reserve the term *tournament* for the latter. The more general ones are *weak tournaments*. Dutta and Laslier (1997) consider these weak weighted tournaments, where $n(x, y) = n(y, x)$ is allowed. They call them *comparison functions*. Peris and Subiza (1998) also study weak tournaments. Most of our results carry over to weak tournaments. We shall indicate which ones do not in the Conclusion.

Here are some highlights of our results. Concerning the relations between $TC, UC, MC, BP$ and their weighted counterparts $TC_w, UC_w, MC_w, BP_w$, we show that $UC \subseteq UC_w$ and $MC \subseteq MC_w$.[2] We know from Laffond, Laslier and Le Breton (1994) that possibly $BP \cap BP_w = \emptyset$. As for the top cycles, we obtain instead that $TC_w \subseteq TC$, i.e. the weighted top cycle is a refinement of the top cycle. It is well known that $BP \subseteq MC \subseteq UC \subseteq TC$. In the weighted case the relations are more intricate. It is easy to see that $BP_w \subseteq UC_w$ and $MC_w \subseteq UC_w$. We show that $BP_w \subseteq TC_w$ and $BP_w \subseteq MC_w$ and also that none of $TC_w$ and $MC_w$ is a superset of the other.

[2] $A \subseteq B$ signifies that $A$ is a subset of $B$ with the possibility that $A = B$ and $\subset$ indicates a strict inclusion.

Our main results concerning the Kemeny rule are: $Ke$ belongs to $TC$ and $UC_w$; otherwise this set may have an empty intersection with any of the six remaining sets and notably with $UC$ and $TC_w$. As for the Kramer-Simpson rule, we prove that $SK$ intersects $MC_w$ and therefore $UC_w$. Otherwise, $SK$ may have an empty intersection with any of the six remaining sets and notably with the top cycle. Finally the set of Borda winners $Bor$ may have an empty intersection with all other sets except with $UC_w$. Actually $Bor$ belongs to $UC_w$.

The paper is organized as follows. In section 2, we present the definitions concerning tournaments and voting situations, some non standard game-theoretical notions, and the eleven solutions for tournaments or weighted tournaments discussed and compared in this paper. In section 3, we proceed to the set-theoretical comparison of these eleven solutions. Finally, we conclude with some lessons that can be drawn from these comparisons and we offer some directions for further investigation.

# 2 Definitions

## 2.1 Tournaments, weighted tournaments and situations.

A *tournament* is a pair $(X, T)$ where $X$ is a finite set and $T$ is an asymmetric and complete binary relation over $X$. Let $\hat{X}^2 \equiv \left\{(x, y) \in X^2 : x \neq y\right\}$. Thus, in a tournament, $\forall (x, y) \in \hat{X}^2$, we have either $xTy$ or $yTx$, where $xTy$ may be interpreted as: $x$ beats $y$.

Let $q$ be a non negative real number. A *$q$-weighted tournament* is a pair $(X, N)$ where $X$ is a finite set and $N$ is a matrix $N \equiv [n(x, y)]_{x,y \in X}$ such that $n(x, y) + n(y, x) = q \; \forall x, y \in X$ and $n(x, y) \neq n(y, x) \; \forall (x, y) \in \hat{X}^2$. This implies that $n(x, x) = q/2 \; \forall x \in X$. A $q$-weighted tournament $(X, N)$ induces the tournament $\left(X, T^N\right)$ where $T^N$ is defined by: $\forall (x, y) \in \hat{X}^2 : xT^N y$ if and only if $n(x, y) > n(y, x)$. Thus a $q$-weighted tournament may be seen as a tournament with additional information as to how strong $x$ is with respect to $y$.

In this paper, we focus on 0-weighted tournaments. A $q$-weighted tournament $(X, N)$ induces the 0-weighted tournament $\left(X, M^N\right)$ where the elements of $M^N$ are defined by:

$$m^N(x,y) \equiv n(x,y) - n(y,x) = 2n(x,y) - q$$

Conversely, given any $q \geq 0$, a 0-weighted tournament $(X, M)$ induces the $q$-weighted tournament $\left(X, N^M\right)$ where the elements of $N^M$ are defined by:

$$n^M(x,y) \equiv \frac{m(x,y) + q}{2}$$

Clearly, $N^{M^N} = N$ and $M^{N^M} = M$. For any $q \geq 0$, there is thus a one-to-one correspondence between 0-weighted tournaments and $q$-weighted tournaments. A tournament $(X, T)$ may be seen as a 0-weighted tournament $(X, M)$ where, $\forall (x,y) \in \hat{X}^2,\ m(x,y) \in \{-1, 1\}$.

Let $\mathcal{T}$ be the set of all tournaments over all finite sets of alternatives. A *solution* for $\mathcal{T}$ is a multivalued mapping $S_{\mathcal{T}} : \mathcal{T} \to X$ that assigns a nonempty *choice set* $S_{\mathcal{T}}(X, T) \subseteq X$ to each tournament $(X, T)$. Similarly if we let $\mathcal{N}$ be the set of weighted tournaments, a solution for $\mathcal{N}$ is a multivalued mapping $S_{\mathcal{N}} : \mathcal{N} \to X$ that assigns a nonempty *choice set* $S_{\mathcal{N}}(X, N) \subseteq X$ to every weighted tournament $(X, N)$. A solution $S_{\mathcal{N}}$ is *Condorcet consistent* if, for each $q$-weighted tournament $(X, N)$, we have $S_{\mathcal{N}}(X, N) = \{x\}$ whenever $n(x,y) > \frac{q}{2}$ for all $y \in X,\ \ y \neq x$. Alternative $x$ is then called a *Condorcet winner*.

Consider a solution $S_{\mathcal{M}}$ for the subset $\mathcal{M}$ of 0-weighted tournaments. Given the one-to-one correspondence between 0-weighted tournaments and $q$-weighted tournaments for a particular $q$, we can extend the solution $S$ for $\mathcal{M}$ to all other weighted tournaments in $\mathcal{N}$ by $S(X, N) \equiv S(X, M^N)$. The contrary is not possible. A solution for $\mathcal{N}$ may associate two different subsets of $X$ to two different $N$ even if the latter induces the same $M$.

To show that this abstract framework covers voting, we define a (*voting*) *situation* as a pair $(X, P)$ where $X$ is a finite set of alternatives and $P$ is a finite list $(P_1, P_2, \ldots, P_i, \ldots)$ of linear orders on $X$. $\#P$ is the number of linear orders in $P$. These linear orders may be interpreted as the preferences of $\#P$ individuals or voters. Since only preferences matter, we do not introduce voters explicitly in the model. Given a subset $Y \subseteq X$, $P|Y$ represents the restriction of $P$ to $Y$. Given a profile $P$, we define:

$$
\begin{aligned}
\forall (x,y) &\in \hat{X}^2 : n_P(x,y) \equiv \# \{P_i \in P : xP_iy\} \\
\forall x &\in X : n_P(x,x) \equiv \#P/2 \\
N_P &\equiv [n_P(x,y)]_{x,y \in X}
\end{aligned}
$$

Let $\Omega$ be the set of all possible situations. A *social choice correspondence* is a multi-valued mapping $\Gamma : \Omega \to X$ that associates a non-empty *choice set* $\Gamma(X,P) \subseteq X$ to every situation $(X,P)$. As for tournaments, a social choice correspondence is *Condorcet consistent* if, for each situation $(X,P)$, we have $\Gamma(X,P) = \{x\}$ whenever $n(x,y) > \frac{\#P}{2}$ for all $y \in X$, $y \neq x$. We concentrate on the subclass $\mathcal{D} \subset \Omega$ of situations such that the matrix $N_P$ defines a $\#P$-weighted tournament. We write $M_P$ for $M^{N_P}$ and $T_P$ for $T^{N_P}$. $(X,T_P)$, $(X,N_P)$ and $(X,M_P)$ are respectively the tournament, the $\#P$-weighted tournament, and the 0-weighted tournament *induced* by profile $P$ on $X$.

The relation from $\mathcal{D}$ to $\mathcal{T}$ defined by $T \equiv T_P$ is onto. Indeed, McGarvey (1953) shows that, for any tournament $(X,T)$, there exists a profile $P$ such that $T = T_P$. A similar result by Debord (1987) asserts that, for any anti-symmetric matrix $M$, there exists a profile $P$ such that $M = M_P$ if and only if all the off-diagonal entries of $M$ have the same parity. Thus the relation from $\mathcal{D}$ defined by $M \equiv M_P$ is not onto the whole set $\mathcal{M}$. Following Barthelemy, Guénoche and Hudry (1989), we call *voting tournaments* the 0-weighted tournaments $(X,M)$ such that all the off-diagonal entries of $M$ have the same parity. The characterization of the relation from $\mathcal{D}$ to $\mathcal{N}$ is more intricate and will not be addressed here.[3]

[3] This is the well known binary stochastic choice problem. See Fishburn (1992) for a description of the state of the art on the latter.

A social choice correspondence $\Gamma$ is $C1$ if, for all pairs of situations $(X, P)$ and $(X, P')$, $T_P = T_{P'}$ implies $\Gamma(X, P) = \Gamma(X, P')$. Similarly a social choice correspondence $\Gamma$ is $C2$ if, for all pairs of situations $(X, P)$ and $(X, P')$ , $M_P = M_{P'}$ implies $\Gamma(X, P) = \Gamma(X, P')$. Put differently, a social choice correspondence $\Gamma$ is $C1$ if there exists a solution $S_{\mathcal{T}}$ for $\mathcal{T}$ such that, for all situations $(X, P)$ , $\Gamma(X, P) = S_{\mathcal{T}}(X, T_P)$ . Similarly a social choice correspondence $\Gamma$ is $C2$ if it is not $C1$ and if there exists a solution $S$ for $\mathcal{M}$ such that, for all situations $(X, P)$ , $\Gamma(X, P) = S_{\mathcal{M}}(X, M_P)$ . As explained in the Introduction, Fishburn (1977), to whom one owes this classification, defines the $C2$ class with respect to $N_P$ instead of $M_P$. We justified our use of the above definition in the introduction.

It has just been seen that the choice sets of $C1$ and $C2$ social choice correspondences are actually the choice sets of solutions for respectively the tournaments and 0-weighted tournaments induced by the voting situations. We shall go one step further and work with zero-sum two persons games that can be defined from these tournaments.[4] To prepare for this task, we review in the next subsection some non standard solution concepts for zero-sum two-person games.

### 2.2 Solutions for two-player zero-sum games

This subsection presents solution concepts for two-player zero-sum games. It is based on Shapley (1964) and Duggan and Le Breton (1997a)[5].

Let $G \equiv (X_1, X_2, u)$ be a finite two-player zero-sum game where $X_1$ and $X_2$ are the sets of pure strategies of players 1 and 2 respectively and $u : X_1 \times X_2 \to \Re$ is the payoff function of player 1. The payoff function of player 2 is $-u$. A game $G$ is symmetric if $X_1 = X_2$ and $u(x, y) + u(y, x) = 0$ for all $(x, y) \in X_1 \times X_2$. For any subset $A_i \subseteq X_i$, $\Delta(A_i)$ denotes the set of probability distributions over $A_i$. Given some vector of probability $p \in \Delta(A_i)$, the *support* of $p$ is the set $\text{Supp}(p) \equiv \{x \in A_i : p(x) > 0\}$ .

[4] Laffond, Laslier and Le Breton (1994) give an interpretation of these games as Downsian games in Political Science.

[5] We restrict ourselves to two-player zero-sum games but these notions can be extended to any finite constant sum n-player game.

Let $A_2$ be a subset of strategies for player 2. The following notions of dominance are classical.

- $x_1$ is strictly dominated by $y_1$ relative to $A_2$ if $u(y_1, z_2) > u(x_1, z_2)$ for all $z_2 \in A_2$.

- $x_1$ is weakly dominated by $y_1$ relative to $A_2$ if $u(y_1, z_2) \geq u(x_1, z_2)$ for all $z_2 \in A_2$ with a strict inequality for at least one $z_2 \in A_2$ .

- $x_1$ is strictly dominated in the mixed sense by $p_1 \in \Delta(X_1)$ relative to $A_2$ if:

  $\sum_{y_1 \in X_1} p_1(y_1)u(y_1, z_2) > u(x_1, z_2)$ for all $z_2 \in A_2$.

The same three dominance relations are defined similarly for player 2. The following terminology is borrowed from Shapley (1964) and Duggan and Le Breton (1997a).

**Definition 1** *A Generalized Saddle Point (GSP) for a game G is a product* $A_1 \times A_2 \subseteq X_1 \times X_2$ *such that:*

$$\begin{aligned} \forall x_1 &\notin A_1 : \exists y_1 \in A_1 \textit{such that } x_1 \textit{ is strictly dominated by } y_1 \textit{ relative to } A_2 \\ \forall x_2 &\notin A_2 : \exists y_2 \in A_2 \textit{such that } x_2 \textit{ is strictly dominated by } y_2 \textit{ relative to } A_1 \end{aligned}$$

*A GSP that does not contain other GSP is a Saddle.*

**Definition 2** *A Weak Generalized Saddle Point (WGSP) for a game G is a product* $A_1 \times A_2 \subseteq X_1 \times X_2$ *such that:*

$$\begin{aligned} \forall x_1 &\notin A_1 : \exists y_1 \in A_1 \textit{such that } x_1 \textit{ is weakly dominated by } y_1 \textit{ relative to } A_2 \\ \forall x_2 &\notin A_2 : \exists y_2 \in A_2 \textit{such that } x_2 \textit{ is weakly dominated by } y_2 \textit{ relative to } A_1 \end{aligned}$$

*A WGSP that does not contain other WGSP is a Weak Saddle.*

**Definition 3** *A Mixed Generalized Saddle Point (MGSP) for a game G is a product* $A_1 \times A_2 \subseteq X_1 \times X_2$ *such that:*

$$
\begin{aligned}
\forall x_1 &\notin A_1 : \exists p_1 \in \Delta(A_1) \textit{ such that } x_1 \textit{ is strictly dominated in} \\
&\textit{the mixed sense by } p_1 \textit{ relative to } A_2 \\
\forall x_2 &\notin A_2 : \exists p_2 \in \Delta(A_2) \textit{ such that } x_2 \textit{ is strictly dominated in} \\
&\textit{the mixed sense by } p_2 \textit{ relative to } A_1
\end{aligned}
$$

*A MGSP that does not contain other MGSP is a Mixed Saddle.*

Since $X_1$and $X_2$ are assumed to be finite, saddles, weak saddles, and mixed saddles exist. Shapley (1964) proves that there exists a unique saddle and Duggan and Le Breton (1997a) prove that there is a unique mixed saddle. A two-player zero-sum game may have several weak saddles. However, Duggan and Le Breton (1996) prove that, if $G$ is symmetric and $u(x, y) \neq 0$ for all $x \neq y$, then there is a unique weak saddle. Moreover, if $G$ is symmetric the saddle, the weak saddle (if unique) and the mixed saddle are symmetric, i.e. of the form $A \times A$. In this case, we shall refer to $A$ as the saddle, weak saddle or mixed saddle.

**Definition 4** *An equilibrium in mixed strategies for a game $G$ is a pair of probability distributions $p_1 \in \Delta(X_1)$ and $p_2 \in \Delta(X_2)$ that satisfy the following inequalities:*

$$
\begin{aligned}
\sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) p_2(x_2) u(x_1, x_2) &\geq \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} q_1(x_1) p_2(x_2) u(x_1, x_2), \ \forall q_1 \in \Delta(X_1) \\
\sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) p_2(x_2) u(x_1, x_2) &\leq \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) q_2(x_2) u(x_1, x_2), \ \forall q_2 \in \Delta(X_2)
\end{aligned}
$$

The following result, known as the *Minmax Theorem,* provides a characterization of equilibria in mixed strategies. See Owen (1982) for a proof.

**Lemma 1** *Let* $\vartheta \equiv \min_{p_2 \in \Delta(X_2)} \max_{p_1 \in \Delta(X_1)} \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) p_2(x_2) u(x_1, x_2)$. *A pair* $(p_1, p_2) \in \Delta(X_1) \times \Delta(X_2)$ *is an equilibrium in mixed strategies if and only if :*

$$
\begin{aligned}
\sum_{x \in X_1} p_1(x) u(x, y) &\geq \vartheta, \ \forall y \in X_2 \\
\sum_{y \in X_2} p_2(y) u(x, y) &\leq \vartheta, \ \forall x \in X_1
\end{aligned}
$$

The scalar $\vartheta$ in the above lemma is the *value of the game*. If $G$ is symmetric then $\vartheta = 0$. Laffond, Laslier and Le Breton (1997) prove that if $G$ is symmetric and $u(x, y)$ is an odd integer for all $x \neq y$, then $G$ has a unique Nash equilibrium in mixed strategies.

### 2.3 Tournament Games and C1 Choice Correspondences

A tournament $(X, T)$ induces a symmetric two-player zero-sum game $(X, X, u_T)$ where:

$$u_T(x, y) \equiv \begin{cases} 1 & \text{if} \quad xTy \\ -1 & \text{if} \quad yTx \\ 0 & \text{if} \quad x = y \end{cases}$$

We call this game the *tournament game* induced by $(X, T)$. The relation between tournaments and tournament games is clearly one-to-one. Since a tournament game is symmetric and $u_T(x, y)$ is an odd integer for all $x \neq y$, we know that it has a unique saddle, a unique weak saddle, a unique mixed saddle, and a unique equilibrium in mixed strategies. Duggan and Le Breton (1997b) point out that if $T$ has no Condorcet winner, then the saddle of the tournament game consists of the whole set $X$, which is not very discriminating. We shall therefore focus on the other two saddles.

Next, we introduce three $C1$ solution concepts for tournaments and we mention their respective connections with a solution of the corresponding tournament game. The first of these solutions is Schwartz's (1972) top cycle.

**Definition 5** *The top cycle $TC(X, T)$ of a tournament $(X, T)$ is the set of all outcomes that beat directly or indirectly any other outcome in $X$:*

$$TC(X, T) \equiv \left\{ \begin{array}{c} a \in X : \forall b \in X / \{a\}, \; \exists a_1, a_2, \ldots, a_k \in X \\ \textit{such that } a = a_1 T a_2 T \ldots T a_k = b \end{array} \right\}$$

Duggan and Le Breton (1997b) prove that the mixed saddle of the tournament game induced by $(X, T)$ is the top cycle of $(X, T)$.

Fishburn (1977) and Miller (1980) propose another tournament solution called the uncovered set. They first define the *covering relation* $C$ relative to a subset $Y \subseteq X$ as: $\forall x, y \in X,\ xC(T)y$ relative to $Y \subseteq X$ if $xTy$ and if $yTw \Rightarrow xTw,\ \forall w \in Y$.

**Definition 6** *The uncovered set* $UC(X,T)$ *of a tournament* $(X,T)$ *is the set of maximal elements of the covering relation* $C(T)$ *relative to* $X$*:*

$$UC(X,T) \equiv \{a \in X :\ \nexists b \in X : bC(T)a \text{ relative to } X\}$$

The covering relation is also the transposition, to tournaments, of the weak dominance relation of the corresponding tournament game. Thus, the uncovered set of a tournament $(X,T)$ is also the set of weakly undominated strategies (relative to $X$) in the corresponding tournament game $(X, X, u_T)$.

We can refine this solution concept by iteration. Let $UC^0(X,T) \equiv X$ and, for all integers $t \geqq 1$, let:

$$UC^t(X,T) \equiv UC\left(UC^{t-1}(X,T), P|UC^{t-1}(X,T)\right)$$

Let $k$ be the smallest integer such that $UC^{k+1}(X,T) = UC^k(X,T)$ and let $\overline{UC}(X,T) \equiv UC^k\ (X,T)$. Clearly, $\overline{UC}(X,T) \subseteq UC(X,T)$ for any tournament.

Dutta (1988) proposes another solution concept, the minimal covering set, also based on the covering relation $C(T)$. First, he defines a *covering set* of a tournament $(X,T)$ as a subset $A \subseteq X$ satisfying:

$$UC(X,T|A) = A \text{ and } \forall b \notin A :\ b \notin UC\left(X,T|A \cup \{b\}\right)$$

We can reformulate these two conditions as stability conditions. The first is an *internal stability* condition:

$$\forall x, y \in A,\ \ \text{not}\, xC(T)y \text{ relative to } A \text{ and } \text{not}\, yC(T)x \text{ relative to } A$$

i.e. no option in $A$ is covered relative to $A$. The second is an *external stability* condition:

$$\forall b \notin A,\ \exists y \in A : yC(T)b \text{ relative to } A \cup \{b\}$$

It is easy to see that $\overline{UC}(X,T)$ is a covering set. We also define a *weak covering set* of a tournament $(X,T)$ as a subset $A \subseteq X$ such that $\forall x \notin A,\ \exists y \in A : yC(T)x$ on $A \cup \{x\}$. A weak covering set satisfies external stability but not necessarily internal stability. Clearly, a weak covering set is a WGSP of the corresponding tournament game.

Next, Dutta defines a *minimal covering set* of a tournament $(X,T)$ as a covering set of $(X,T)$ that does not contain other covering sets of $(X,T)$. Since $\overline{UC}(X,T)$ is a covering set and since $X$ is finite, minimal covering sets do exist. Dutta proves that there is in fact a unique minimal covering set that will be denoted $MC(X,T)$. Thus, we have the following definition.

**Definition 7** *The minimal covering set $MC(X,T)$ of a tournament $(X,T)$ is the unique covering set of $(X,T)$ that does not contain other covering sets of $(X,T)$.*

Duggan and Le Breton (1996) show that $MC(X,T)$ is the weak saddle of the tournament game induced by $(X,T)$. The last $C1$ solution to be introduced in this subsection is due to Laffond, Laslier, and Le Breton (1993). It is transposed from the solution of the corresponding tournament game.

**Definition 8** *The Bipartisan set of a tournament $(X,T)$ is the support $BP(X,T)$ of the unique Nash equilibrium in mixed strategies of the tournament game $(X,X,u_T)$ induced by this tournament.*

## 2.4 Plurality Games and C2 Choice Correspondences

As for tournaments, a 0-weighted tournament $(X,M)$ induces a symmetric two-player zero-sum game $(X,X,u_M)$ where:

$$u_M(x,y) \equiv m(x,y) \text{ for all } x,y \in X.$$

We call this game the *plurality game* induced by $(X, M)$. The relation between 0-weighted tournaments and plurality games is clearly one-to-one. The plurality game has a unique saddle and, in contrast to the tournament game, the saddle may be a proper subset of $X$. We shall not discuss this set in this paper, to keep a symmetric treatment of the two kinds of games.

The result concerning the mixed saddle of the tournament game suggests the following definition, for which there is no obvious intuition.

**Definition 9** *The weighted top cycle $TC_w(X, M)$ of a 0-weighted tournament $(X, M)$ is the mixed saddle of the plurality game induced by $(X, M)$.*

We now transpose the covering relation of the previous subsection to 0-weighted tournaments. We define the *weighted covering relation* $C_w(M)$ relative to a subset $Y \subseteq X$ as follows:

$$\forall x, y \in X,\ xC_w(M)y \text{ relative to } Y \subseteq X \text{ if } m(x, y) > 0 \text{ and if } m(x, z) \geq m(y, z), \forall z \in Y$$

**Definition 10** *The weighted uncovered set $UC_w(X, M)$ of a 0-weighted tournament $(X, M)$ is the set of maximal elements of the weighted covering relation $C_w(M)$ relative to $X$:*

$$UC_w(X, M) \equiv \{a \in X :\ \nexists b \in X : bC_w(M)a \textit{ relative to } X\}$$

The weighted covering relation is also the transposition, to 0-weighted tournaments, of the weak dominance relation of the corresponding plurality game. Thus, the weighted uncovered set of a 0-weighted tournament $(X, M)$ can also be defined as the set of weakly undominated strategies (relative to $X$) in the corresponding plurality game $(X, X, u_M)$.

We can also define the iterates of this set, namely $UC_w^1(X, M), UC_w^2(X, M)$, etc. as we did for $UC(X, T)$. The limit, $\overline{UC}_w(X, M)$, of this series is the set of strategies that remain after iterative elimination of weakly dominated strategies.

We use the weighted covering relation $C_w$ to introduce the weighted equivalent of the minimal covering set. A *weighted covering set* of a 0-weighted tournament $(X, M)$ is a subset $A \subseteq X$ satisfying:

$$UC_w(X,\ M \mid A) = A \text{ and } \forall x \notin A:\ x \notin UC_w(X,\ M \mid A \cup \{x\})$$

Note that $\overline{UC}_w(X, M)$ is a weighted covering set. We also define a *weak weighted covering set* of a 0-weighted tournament $(X, M)$ as a subset $A \subseteq X$ such that $\forall x \notin A, \exists y \in A : yC_w(M)x$ relative to $A \cup \{x\}$. Clearly, a weak weighted covering set of $(X, M)$ is a WGSP of the plurality game induced by $(X, M)$. The following lemma establishes the relation between weak weighted covering sets and weak covering sets.

**Lemma 2** *If $A$ is a weak weighted covering set of a 0-weighted tournament $(X, M)$, then $A$ is also a weak covering set of the tournament $\left(X, T^M\right)$.*

**Proof.** Let $A$ be a weak weighted covering set of a 0-weighted tournament and take any $x \notin A$. By definition, $\exists y \in A : m(y, z) \geq m(x, z), \forall z \in A \cup \{x\}$. Thus $xT^M z \Rightarrow m(x, z) > 0 \Rightarrow m(y, z) > 0 \Rightarrow yT^M z$. This means that $yC(T^M)x$ relative to $A \cup \{x\}$. ■

A *minimal weighted covering set* of a 0-weighted tournament $(X, M)$ is a weighted covering set that does not contain other covering sets of $(X, M)$. Since $\overline{UC}_w(X, M)$ is a weighted covering set and since $X$ is finite, minimal weighted covering sets do exist. In order to justify the next definition, we need to show the following result.

**Lemma 3** *A minimal (with respect to inclusion) weak covering set of a tournament is also a minimal covering set of the same tournament. Similarly, a minimal weak weighted covering set of a 0-weighted tournament is also a minimal weighted covering set of the same 0-weighted tournament.*

**Proof.** Let $A$ be a minimal weak covering set of a tournament. By definition, $A$ satisfies the external stability condition. We claim that it also satisfies the internal stability condition. Suppose not, i.e. $\exists x, y \in A : xCy$ relative to $A$. Since $C$ is transitive, $y$ can be removed from $A$ without sacrificing external stability, contradicting the assumption that $A$ is a minimal weak covering set.

Thus $A$ is a minimal covering set. The same arguments apply to minimal weighted weak covering sets. ■

Since the plurality game induced by a 0-weighted tournament $(X, M)$ has a unique weak saddle, it has a unique minimal weak weighted covering set, which must also be the unique minimal weighted covering set of $(X, M)$ by Lemma 3. Thus, we may state the following definition:

**Definition 11** *The minimal weighted covering set $MC_w(X, M)$ of a 0-weighted tournament $(X, M)$ is the unique weighted covering set of $(X, M)$ that does not contain other covering sets of $(X, M)$.*

Finally, under the additional restriction that the elements $m(x, y)$ of $M$ are odd integer for all $x \neq y$, the plurality game has a unique equilibrium in mixed strategies. The support of this equilibrium is another solution for 0-weighted tournaments.

**Definition 12** *The Weighted Bipartisan Set $BP_w(X, M)$ of a 0-weighted tournament $(X, M)$ is the support of the unique equilibrium in mixed strategies of the corresponding plurality game.*[6]

We list the different solution sets for tournament and plurality games in Table 1. Recall that $X$ is a saddle for a tournament only if there is no Condorcet winner.

| Game | Tournament | Plurality |
|---|---|---|
| Saddle | $X$ | |
| Weak Saddle | $MC$ | $MC_w$ |
| Mixed Saddle | $TC$ | $TC_w$ |
| Weakly Undominated Strategies | $UC$ | $UC_w$ |
| Support of the Unique Equilibrium in Mixed Strategies | $BP$ | $BP_w$ |

Table 1: Solution Concepts for Tournament and Plurality Games

[6] This definition has been proposed first by Laffond, Laslier, and Le Breton (1994).

### 2.5 Three Other C2 Choice Correspondences

We now present three $C2$ social choice correspondences that have been widely discussed in the literature and that are very often used. Their transposition as solutions for weighted tournament will be immediate.

The first concept is due to Kemeny (1959). Given a situation $(X,P)$, this correspondence first chooses the linear orders that are as close as possible to $P$ in a sense to be made precise below and then takes the top elements of these orders as the choice set. Given a set $X$ and two linear orders $O$ and $O'$ on $X$, let

$$\delta(O,O') \equiv \# \left\{(x,y) \in X^2, x \neq y, xOy \text{ and } yO'x\right\}.$$

In words, $\delta(O,O')$ is the numbers of inversions in the two orders $O$ and $O'$. Clearly $\delta$ is a distance over the set of orders on $X$. Kemeny then defines a "distance" between a linear order $O$ and a profile $P$ as :

$$d\left(O,P\right) \equiv \sum_{P_i \in P} \delta\left(O,P_i\right)$$

**Definition 13** *A Kemeny order for a situation* $(X,P)$ *is a linear order* $O^* \in \arg\min_{O \in L} d(O,P)$*. The Kemeny set* $Ke(X,P)$ *of a situation* $(X,P)$ *is the set of the top elements of the Kemeny orders for the situation.*

The following simple and well known lemma gives a useful characterization of the Kemeny choice correspondence.

**Lemma 4** $O^*$ *is a Kemeny order for a situation* $(X,P)$ *if and only if*

$$O^* \in \arg\max_{O \in L} \sum_{x \in X} \sum_{\substack{y \in X \\ xOy}} n_P(x,y)$$

It follows from this lemma that the Kemeny social choice correspondence is $C2$. This lemma inspires the following definition for weighted tournaments.

**Definition 14** *A Kemeny order for a q-weighted tournament* $(X,N)$ *is a linear order* $O^* \in \arg\max_{O\in L} \sum_{x\in X} \sum_{\substack{y\in X \\ xOy}} n(x,y)$. *The Kemeny set* $\widehat{Ke}(X,N)$ *of a q-weighted tournament* $(X,N)$ *is the set of the top elements of the Kemeny orders for the weighted tournament.*

Since $\arg\max_{O\in L} \sum_{x\in X} \sum_{\substack{y\in X \\ xOy}} n_P(x,y) = \arg\max_{O\in L} \sum_{x\in X} \sum_{\substack{y\in X \\ xOy}} m_P(x,y)$, the following relation holds for all situations:

$$Ke(X,P) = \widehat{Ke}(X,N_P) = \widehat{Ke}(X,M_P) \tag{1}$$

Since there is no risk on confusion, we shall write $Ke(X,N)$ instead of $\widehat{Ke}(X,N)$ in the remaining of the paper.

The Kemeny rule has been axiomatized by Young and Levenglick (1978). Young (1988) also provides a very nice foundation for this procedure. Suppose that there is a true ordering $O$ of the alternatives in $X$ and that, in any pairwise comparison, each voter chooses the better candidate with some fixed probability $p$ (the competence parameter), where $\frac{1}{2} < p < 1$ and $p$ is the same for all voters. Assume also that every voter's judgment on every pair of candidates is independent of his judgment on every other pair and that judgments are independent from one voter to another. Under these assumptions, it can be shown that $O^*$ is a Kemeny order if and only if it maximizes the likelihood of being the true order $O$ given the pattern of pairwise votes in the profile $P$. This is precisely the statistical framework used by Condorcet to justify his criterion.

Slater (1961) proposes a choice correspondence that may be seen as the $C1$ counterpart of the Kemeny rule. Indeed the Slater rule uses only the information in $T_P$. For this reason, we may define the concept directly for tournaments. A *Slater order* for a tournament $(X,T)$ is an order $O^* \in \arg\min_{O\in L} \delta(O,T)$. The *Slater set* $Sl(X,T)$ for a tournament $(X,T)$ is the set of top elements of Slater orders for the tournament.

Simpson (1969) and Kramer (1977) propose to take the set of outcomes whose maximal opposition is the weakest as the solution of a situation. Kramer (1977) shows that this set is an attraction point of a sequential electoral competition between two parties when platforms belong to some Euclidean space. Also, as pointed out by Young (1988), if instead of estimating the "true" ranking of candidates, we focus on the determination of which candidates are likely to be the best, then the Kramer-Simpson winners emerge when the competence parameter $p$ is close to 1.

**Definition 15** *The Simpson-Kramer set or the minmax set $SK(X,P)$ of a situation $(X,P)$ is:*

$$SK\,(X,P) \equiv \arg\min_{x\in X}\ \max_{y\in X\setminus\{x\}} n_P(y,x)$$

This social choice correspondence is also $C2$. We can thus transpose its definition to weighted tournaments as we did with the Kemeny rule and a relation similar to (1) holds for the minmax sets.

The last choice correspondence to be introduced is the Borda rule, a well-known scoring method. Given a situation $(X,P)$, we first define: $\forall P_i \in P,\ \forall x \in X,\ R\,(x,P_i) \equiv \#\,\{y \in X : xP_iy\}$. The Borda score of an element $x \in X$ is then defined as $B\,(x,P) \equiv \sum_{P_i\in P} R\,(x,P_i)$.

**Definition 16** *A Borda winner for a situation $(X,P)$ is any $x^* \in \arg\max_{x\in X} B\,(x,P)$. The Borda set $Bor(X,P)$ of a situation $(X,P)$ is the set of Borda winners for this situation.*

The following lemma is well known.

**Lemma 5** *For any situation $(X,P)$, $Bor(X,P) = \arg\max_{x\in X} \sum_{y\in X/\{x\}} n_P(x,y)$*

Thus, the Borda correspondence is $C2$. As for the Kemeny rule and the Kramer-Simpson rule, the definition of the Borda rule can be transposed to weighted tournaments and a relation similar to (1) holds for the Borda sets.

Many justifications have been given for the Borda rule. One of them emerges from the statistical framework of Condorcet. If, instead of searching the "true" ranking of the candidates, we focus on the determination of which candidates are likely to be the best, then as pointed out by Condorcet (1785) himself and by Young (1988) the Borda winners turn out to be these best candidates when the competence parameter $p$ is close to $\frac{1}{2}$.

Condorcet (1785) showed that the Borda choice correspondence does not always select the Condorcet winner when it exists. The Borda rule is not the only choice correspondence studied here not to be Condorcet consistent. The following example shows that $UC_w$ is not Condorcet consistent. Consider the 0-weighted tournament defined by the following table:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 1 | 1 | 1 |
| $b$ | -1 | 0 | 5 | -5 |
| $c$ | -1 | -5 | 0 | 5 |
| $d$ | -1 | 5 | -5 | 0 |

It is easy to see that $\{a\}$ is the Condorcet winner and that $UC_w(X, M) = \{a, b, c, d\}$. All other solutions defined in this section are Condorcet consistent.

## 3 Set-Theoretical Comparison of Solution Concepts

In this section, we examine the relationships between the eleven solutions for 0-weighted tournaments introduced in the previous section. Table 2 summarizes the results of these set-theoretical comparisons.

- A $\subseteq$ in a cell indicates that, for any 0-weighted tournament $(X, M)$, the solution set of the corresponding row is contained in the solution set of the corresponding column.

- A $\cap$ means that, for any 0-weighted tournament $(X, M)$, the solution set of the corresponding row intersects the solution set of the corresponding column but that there exists a voting tournament $(X, M)$ such that none of the solution sets of the row or the column is a subset of the other.

- A $\emptyset$ means that there exists voting tournament $(X, M)$ such that the intersection of the solution sets of the row and the column is empty.

| | $TC$ | $TC_w$ | $UC_w$ | $MC_w$ | $BP_w$ | $UC$ | $MC$ | $Ke$ | $SK$ | $BP$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $TC_w$ | $\subseteq$ | | | | | | | | | |
| $UC_w$ | $\cap$ | $\cap$ | | | | | | | | |
| $MC_w$ | $\cap$ | $\cap$ | $\subseteq$ | | | | | | | |
| $BP_w$ | $\subseteq$ | $\subseteq$ | $\subseteq$ | $\subseteq$ | | | | | | |
| $UC$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | $\cap$ | $\emptyset$ | | | | | |
| $MC$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | | | | |
| $Ke$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | | | |
| $SK$ | $\emptyset$ | $\emptyset$ | $\cap$ | $\cap$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | | |
| $BP$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | $\subseteq$ | $\emptyset$ | $\subseteq$ | $\subseteq$ | $\emptyset$ | $\emptyset$ | |
| $Bor$ | $\emptyset$ | $\emptyset$ | $\subseteq$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

Table 2: Comparison of Solution Concepts

**Remark 1** *The chain of inclusions $BP \subseteq MC \subseteq UC \subseteq TC$ is well known from the literature on tournament solutions. Fishburn (1977) shows that $Ke \subseteq TC$ and Laffond, Laslier and Le Breton (1994) prove that we may have $BP \cap BP_w = \emptyset$. The possible empty intersection of $Bor$ with $TC$, $TC_w$, $MC_w$, $BP_w$, $UC$, $MC$, $Ke$, $SK$, and $BP$ follows from the fact that these nine solutions are Condorcet consistent and that a Condorcet winner may fail to be a Borda winner.*[7]

The remaining of this section deals with the other entries in the table. Many of the inclusion results may be deduced from a glance at Table 2. These deductions are left to the reader. Except when dealing with $BP_w$, results such as $S \subseteq S'$ and $S \cap S' \neq \emptyset$, where $S$ and $S'$ are two solution sets, hold for any 0-weighted tournament. On the other hand, when showing that there exists a tournament such that $S \subset S'$ or $S \cap S' = \emptyset$, we do it by exhibiting a voting tournament. Since, by Debord's theorem, there always exists a voting situation underlying a voting tournament, these

[7] See the proof of Theorem 4 in Le Breton and Truchon (1997) for a general example.

results can be transposed to social choice correspondences. The two examples that follow will be used to establish some of these results.

**Example 1** *Consider the following voting tournament* $(X, M)$*:*

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | −5 | −3 | 1 |
| $b$ | −1 | 0 | 3 | −3 | 1 |
| $c$ | 5 | −3 | 0 | 3 | 1 |
| $d$ | 3 | 3 | −3 | 0 | 1 |
| $e$ | −1 | −1 | −1 | −1 | 0 |

*One can check that* $TC(X, T^M) = \{a, b, c, d\}$, $TC_w(X, M) = \{b, c, d\}$, $UC_w(X, M) = \{b, c, d, e\}$, $Ke(X, M) = Bor\,(X, M) = \{c\}$, *and* $SK\,(X, M) = \{e\}$.

**Example 2** *Consider the following voting tournament* $(X, M)$*:*

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 0 | 3 | −5 | −3 | 1 | −1 |
| $b$ | −3 | 0 | 3 | −3 | 1 | 1 |
| $c$ | 5 | −3 | 0 | 3 | 1 | −3 |
| $d$ | 3 | 3 | −3 | 0 | 1 | 3 |
| $e$ | −1 | −1 | −1 | −1 | 0 | −1 |
| $f$ | 1 | −1 | 3 | −3 | 1 | 0 |

*One can check that* $TC(X, T^M) = TC_w(X, M) = \{a, b, c, d, f\}$,
$UC(X, T^M) = \{b, c, d, f\}$, $UC_w(X, M) = \{b, c, d, e, f\}$,
$MC(X, T^M) = BP(X, T^M) = BP_w(X, M) = \{b, c, d\}$, $MC_w(X, M) = \{b, c, d, e\}$,
$Ke(X, M) = Bor\,(X, M) = \{d\}$, *and* $SK(X, M) = \{e\}$.

### 3.1 The Weighted Top Cycle ($TC_w$)

**Proposition 1** *$TC_w(X,M) \subseteq TC(X,T^M)$ for any 0-weighted tournament $(X,M)$.*

**Proof.** Consider the plurality game induced by $(X,M)$ restricted to $TC(X,T^M)$. Since the value of this game is zero, we deduce from the minmax theorem that there exists a $p \in \Delta(TC(X,T^M))$ such that $\sum_{x \in TC} p(x)m(x,y) \geq 0$ for all $y \in TC(X,T^M)$. Next take a $z \notin TC(X,T^M)$. From the definition of $TC(X,T^M)$, we have $m(z,y) < 0$ for all $y \in TC(X,T^M)$. Combining the two sets of inequalities, we deduce that $z$ is strictly dominated in the mixed sense relative to $TC(X,T^M)$. If there are no other strategies that are strictly dominated in the mixed sense relative to $TC(X,T^M)$ then $TC(X,T^M)$ is a MGSP. Otherwise, $TC(X,T^M)$ contains the MGSP. In any case, since $TC_w(X,T^M)$ is the unique mixed saddle, we therefore have $TC_w(X,M) \subseteq TC(X,T^M)$. ■

**Remark 2** *Example 1 shows that one can have $TC_w(X,M) \subset TC(X,T^M)$.*

**Proposition 2** *There exists a voting tournament $(X,M)$ such that $TC_w(X,M) \cap BP(X,T^M) = TC_w(X,M) \cap MC(X,T^M) = TC_w(X,M) \cap UC(X,T^M) = \emptyset$*

**Proof.** Consider the following voting tournament $(X,M)$, which Laffond, Laslier and Le Breton (1994) use to prove that we may have $BP \cap BP_w = \emptyset$:

| | $a$ | $b$ | $c$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|
| $a$ | 0 | 1 | −1 | 1 | −9 | 1 |
| $b$ | −1 | 0 | 1 | 1 | 1 | −9 |
| $c$ | 1 | −1 | 0 | −9 | 1 | 1 |
| $x$ | −1 | −1 | 9 | 0 | 5 | −5 |
| $y$ | 9 | −1 | −1 | −5 | 0 | 5 |
| $z$ | −1 | 9 | −1 | 5 | −5 | 0 |

The reader can check that $BP(X,T^M) = MC(X,T^M) = UC(X,T^M) = \{a,b,c\}$ and $BP_w(X,M) = TC_w(X,M) = \{x,y,z\}$[8]. ■

### 3.2 The Weighted Uncovered Set ($UC_w$)

It has been shown that the top cycle is a superset of the weighted top cycle. This order of inclusion is reversed for the uncovered set.

**Proposition 3** *$UC(X,T^M) \subseteq UC_w(X,M)$ for any 0-weighted tournament $(X,M)$.*

**Proof.** The proof is straightforward and left to the reader. ■

**Remark 3** *Example 2 shows that there exists a voting tournament $(X,M)$ such that $UC(X,T^M) \subset UC_w(X,M)$.*

**Proposition 4** *a) $UC_w(X,M) \cap TC\left(X,T^M\right) \neq \emptyset$ for any 0-weighted tournament $(X,M)$.*

*b) There exists a voting tournament $(X,M)$ such that $UC_w(X,M) \nsubseteq TC(X,T^M)$ and $TC(X,T^M) \nsubseteq UC_w(X,M)$.*

**Proof.** a) $Ke(X,M)$ is a subset of both $UC_w(X,M)$ and $TC(X,T^M)$ by respectively Proposition 10 below and Remark 1.

b) This follows from Example 2. ■

**Proposition 5** *$Bor(X,M) \subseteq UC_w(X,M)$ for any 0-weighted tournament $(X,M)$.*

**Proof.** If $UC_w(X,M) = X$, there is nothing to prove. Suppose $UC_w(X,M) \neq X$ and take any $y \notin UC_w(X,M)$. By transitivity of $C_w$, there must exist an $x \in UC_w(X,M)$ such that $xC_wy$ relative to $X$, i.e. such that $m(x,y) > 0$ and $m(x,z) - m(y,z) \geq 0, \forall z \neq x,y$. This implies $\sum_{z\in X} m(x,z) > \sum_{z\in X} m(y,z)$ and thus $y$ is not a Borda winner. ■

[8] $a$ is strictly covered in the mixed sense relative to $\{x,y,z\}$ by $p(x) = 0,\ p(y) = 3/10,\ p(z) = 7/10$.
$b$ is strictly covered in the mixed sense relative to $\{x,y,z\}$ by $p(x) = 7/10,\ p(y) = 0,\ p(z) = 3/10$.
$c$ is strictly covered in the mixed sense relative to $\{x,y,z\}$ by $p(x) = 3/10,\ p(y) = 7/10,\ p(z) = 0$.

**Remark 4** *Examples 1 and 2 both show that there exists a voting tournament* $(X, M)$ *such that* $Bor(X, M) \subset UC_w(X, M)$.

### 3.3 The Weighted Minimal Covering Set ($MC_w$)

We now show that the order of inclusion between the weighted and unweighted minimal covering sets is the same as for the top cycle.

**Proposition 6** $MC(X, T^M) \subseteq MC_w(X, M)$ *for any* 0*-weighted tournament* $(X, M)$.

**Proof.** From Lemma 2, we know that $MC_w(X, M)$ is a weak covering set and, from Lemma 3, that $MC(X, T^M)$ is the unique minimal weak covering set of $(X, M)$. Thus $MC(X, T^M) \subseteq MC_w(X, M)$. ■

**Remark 5** *Example 2 exhibits a voting tournament* $(X, M)$ *such that* $MC(X, T^M) \subset MC_w(X, M)$.

**Remark 6** *By definition,* $MC_w(X, M) \subseteq \overline{UC}_w(X, M) \subseteq UC_w(X, M)$. *Combining Proposition 6 with* $MC \subseteq UC \subseteq TC$ *of Remark 1 yields:* $MC_w(X, M) \cap UC(X, T^M) \neq \emptyset$, $MC_w(X, M) \cap TC\left(X, T^M\right) \neq \emptyset$ *for any* 0*-weighted tournament* $(X, M)$.

**Remark 7** *Example 2 shows that* $MC_w(X, M) \nsubseteq TC\left(X, T^M\right)$ *and* $TC\left(X, T^M\right) \nsubseteq MC_w(X, M)$. *Example ? shows that* $MC_w(X, M) \nsubseteq UC\left(X, T^M\right)$ *and* $UC\left(X, T^M\right) \nsubseteq MC_w(X, M)$.

### 3.4 The Weighted Bipartisan Set ($BP_w$)

Most of the results of this subsection are established for voting tournaments with off-diagonal odd entries.

**Proposition 7** *If* $(X, M)$ *is a voting tournament with odd* $m(x, y)$ $\forall x \neq y$, *then* $BP_w(X, M) \subseteq TC_w(X, M)$ *and* $BP_w(X, M) \subseteq MC_w(X, M)$.

**Proof.** From a result of Laffond, Laslier and Le Breton (1997), the game induced by this tournament has a unique Nash equilibrium in mixed strategies and this equilibrium is symmetric.

$BP_w(X,M)$ is the support of this equilibrium. Using results in Duggan and Le Breton (1997a), we may assert that, if $A_1 \times A_2$ is a WGSP or a MGSP, then $BP_w(X,M) \subseteq A_1$ and $BP_w(X,M) \subseteq A_2$. The set $TC_w(X,M)$ has been defined as a MGSP that contains no other MGSP and, as explained in subsection 2.4, $MC_w(X,M)$ is the unique minimal WGSP, hence the results. ■

**Remark 8** *Combining Proposition 7 with* $MC_w(X,M) \subseteq UC_w(X,M)$ *of Remark 6, we obviously have* $BP_w(X,M) \subseteq UC_w(X,M)$ *for any* 0*-weighted tournament* $(X,M)$.

**Remark 9** *Example 2 shows that one can have* $BP_w(X,M) \subset TC_w(X,M)$ *and* $BP_w(X,M) \subset MC_w(X,M)$.

**Proposition 8** *a)* $TC_w(X,M) \cap MC_w(X,M) \neq \emptyset$ *for any voting tournament* $(X,M)$ *with odd* $m(x,y)$ $\forall x \neq y$.

*b) There exists a voting tournament* $(X,M)$ *such that* $TC_w(X,M) \nsubseteq MC_w(X,M)$ *and* $MC_w(X,M) \nsubseteq TC_w(X,M)$.

**Proof.** a) By Proposition 7, both $TC_w(X,M)$ and $MC_w(X,M)$ contain $BP_w(X,M)$.

b) This follows from Example 2. ■

**Remark 10** *Combining Proposition 8 and* $MC_w(X,M) \subseteq UC_w(X,M)$ *of Remark 6, we obtain* $UC_w(X,M) \cap TC_w(X,M) \neq \emptyset$ *for any voting tournament* $(X,M)$ *with odd* $m(x,y)$ $\forall x \neq y$. *Example 2 shows that there exists a voting tournament* $(X,M)$ *such that* $TC_w(X,M) \nsubseteq UC_w(X,M)$ *and* $UC_w(X,M) \nsubseteq TC_w(X,M)$.

**Remark 11** *The example used in the proof of Proposition 2 shows that there exists a voting tournament* $(X,M)$ *for which* $BP_w(X,M) \cap UC(X,T^M) = BP_w(X,M) \cap MC(X,T^M) = BP_w(X,M) \cap BP(X,T^M) = \emptyset$.

### 3.5 The Kemeny Set ($Ke$)

**Proposition 9** *There exists a voting tournament* $(X,M)$ *such that* $Ke(X,M) \cap BP(X,T^M) = Ke(X,M) \cap MC(X,T^M) = Ke(X,M) \cap UC(X,T^M) = \emptyset$.

**Proof.** Consider the following voting tournament:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 3 | -1 | -1 |
| $b$ | -3 | 0 | 5 | -1 |
| $c$ | 1 | -5 | 0 | 3 |
| $d$ | 1 | 1 | -3 | 0 |

It is easy to see that $UC(X,T^M) = \{b,c,d\}$. Using Lemma 4, one can check that $(a,b,c,d)$ is the unique Kemeny order so that $Ke\,(X,M) = \{a\}$. Since $BP \subseteq MC \subseteq UC$, we have the result. ■

**Proposition 10** *$Ke(X,M) \subseteq UC_w(X,M)$ for any 0-weighted tournament $(X,M)$.*

**Proof.** We prove a stronger claim: $xC_w(M)y \Rightarrow xOy$ for every Kemeny order $O$. Suppose on the contrary that there exists a Kemeny order $O \equiv (\ \ldots, y, x_1, x_2, x_3, \ldots, x_k, x, \ldots)$ and consider the order $O'$ obtained by permuting $x$ and $y$ in $O$. Then,

$$\begin{aligned}\sum_{u\in O'}\sum_{\substack{v\in O' \\ uO'v}} m(u,v) - \sum_{u\in O}\sum_{\substack{v\in O \\ uOv}} m(u,v) &= \sum_{i=1}^{k}(m(x,x_i)-m(y,x_i)) + \sum_{i=1}^{k}(m(x_i,y)-m(x_i,x)) \\ +m(x,y)-m(y,x) &= 2\sum_{i=1}^{k}(m(x,x_i)-m(y,x_i)) + m(x,y)-m(y,x)\end{aligned}$$

Since $xC_w(M)y$, the first term on the right-hand-side is non negative and the last one is strictly positive. Therefore, by Lemma 4, $O$ is not a Kemeny order, a contradiction. ■

**Remark 12** *Examples 1 and 2 both show that there exists a voting tournament $(X,M)$ such that $Ke(X,M) \subset UC_w(X,M)$.*

**Proposition 11** *There exists a voting tournament $(X,M)$ such that $Ke(X,M) \cap MC_w(X,M) = Ke(X,M) \cap BP_w(X,M) = \emptyset$.*

**Proof.** Consider the following voting tournament $(X, M)$, which induces the tournament of Proposition 3.1 in Laffond, Laslier and Le Breton (1995):

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | 0 | 1 | 1 | -1 | -1 | 1 | 1 | 1 |
| $b$ | -1 | 0 | 1 | -1 | -1 | 1 | 1 | 1 |
| $c$ | -1 | -1 | 0 | 1 | 1 | 1 | 1 | 1 |
| $d$ | 1 | 1 | -1 | 0 | 1 | -1 | -1 | -1 |
| $e$ | 1 | 1 | -1 | -1 | 0 | 1 | 1 | 1 |
| $f$ | -1 | -1 | -1 | 1 | -1 | 0 | 1 | 1 |
| $g$ | -1 | -1 | -1 | 1 | -1 | -1 | 0 | 1 |
| $h$ | -1 | -1 | -1 | 1 | -1 | -1 | -1 | 0 |

It can be checked that $Ke(X, M) = \{e\}$ and $MC_w(X, M) = BP_w(X, M) = \{a, c, d\}$. ■

**Proposition 12** *There exists a voting tournament* $(X, M)$ *such that* $Ke(X, M) \cap TC_w(X, M) = \emptyset$.

**Proof.** Consider the following voting tournament:

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | 0 | 9 | 9 | -9 | -7 | 11 | 11 | 11 |
| $b$ | -9 | 0 | 9 | -9 | -7 | 11 | 11 | 11 |
| $c$ | -9 | -9 | 0 | 9 | 11 | 11 | 11 | 11 |
| $d$ | 9 | 9 | -9 | 0 | 9 | -7 | -7 | -7 |
| $e$ | 7 | 7 | -11 | -9 | 0 | 9 | 9 | 9 |
| $f$ | -11 | -11 | -11 | 7 | -9 | 0 | 9 | 9 |
| $g$ | -11 | -11 | -11 | 7 | -9 | -9 | 0 | 9 |
| $h$ | -11 | -11 | -11 | 7 | -9 | -9 | -9 | 0 |

It can be checked that $TC_w(X, M) = \{a, b, c, d\}$ and $Ke(X, M) = \{e\}$. ■

### 3.6 The Simpson-Kramer Minmax Set ($SK$)

**Proposition 13** *There exists a voting tournament* $(X, M)$ *such that* $SK(X, M) \cap BP_w(X, M) = SK(X, M) \cap TC_w(X, M) = SK(X, M) \cap BP(X, T^M) = SK(X, M) \cap MC(X, T^M) = SK(X, M) \cap UC(X, T^M) = SK(X, M) \cap TC(X, T^M) = SK(X, M) \cap Ke(X, M) = \emptyset$.

**Proof.** This follows from Example 2 and the fact that $BP \subseteq MC \subseteq UC \subseteq TC$ by Remark 1. ■

**Remark 13** *Examples 1 and 2 also show that a Simpson-Kramer winner may be a Condorcet loser, i.e. it can lose in pairwise comparisons against every other alternative.*

**Proposition 14** *a)* $SK(X, M) \cap MC_w(X, M) \neq \emptyset$ *for any* 0*-weighted tournament* $(X, M)$.

*b) There exists a voting tournament* $(X, M)$ *such that* $MC_w(M) \nsubseteq SK(M)$ *and* $SK(X, M) \nsubseteq MC_w(M)$.

**Proof.** a) Suppose $SK(X, M) \cap MC_w(X, M) = \emptyset$ and take any $k \in SK(X, M)$. Since $k \notin MC_w(X, M), \exists a \in MC_w(X, M) : aC_w(X, M)k$ relative to $MC_w(X, M) \cup \{k\}$, i.e. $m(X, a, k) > 0$ and

$$m(a, z) \geq m(k, z), \forall z \in MC_w(M)/\{a\} \tag{2}$$

We distinguish two cases:

Case 1: $\exists x \in MC_w(X, M) \cap \arg\min_{z \in X/\{a\}} m(a, z)$. By (2), $m(a, z) \geq m(k, z)$. Thus $a \in SK(X, M)$, a contradiction.

Case 2: $MC_w(X, M) \cap \arg\min_{z \in X/\{a\}} m(a, z) = \emptyset$. Take any $x \in \arg\min_{z \in X/\{a\}} m(a, z)$. Since $x \notin MC_w(X, M), \exists b \in MC_w(X, M) : bC_w(M)x$ relative to $MC_w(X, M) \cup \{x\}$. We must have $m(a, x) < 0$. Otherwise, $a$ would be a Condorcet winner and one would have $SK(X, M) = MC_w(X, M) = \{a\}$. Since $m(b, x) > 0$ and $m(a, k) > 0$, we thus have $b \neq a$ and $x \neq k$. From (2),

$$m(a, b) \geq m(k, b) \tag{3}$$

Moreover $bC_w(M)x$ relative to $MC_w(X,M) \cup \{x\}$ implies:

$$m(b,a) \geq m(x,a) \tag{4}$$

Combining (3) and (4) yields $m(a,x) = -m(x,a) \geq -m(b,a) = m(a,b) \geq m(k,b)$. Thus $a \in SK(X,M)$, a contradiction.

b) Consider the following voting tournament $M$:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 1 | -1 | 3 |
| $b$ | -1 | 0 | -1 | -1 |
| $c$ | 1 | 1 | 0 | -3 |
| $d$ | -3 | 1 | 3 | 0 |

One can easily check that $SK(X,M) = \{a,b\}$ and that $UC_w(X,M) = MC_w(X,M) = \{a,c,d\}$. ■

**Remark 14** *Combining Proposition 14 and* $MC_w \subseteq UC_w$ *of Remark 6, we get* $SK(X,M) \cap UC_w(X,M) \neq \emptyset$ *for any* 0*-weighted tournament* $(X,M)$*. The voting tournament used in the proof of Proposition 14 shows that* $UC_w(X,M) \nsubseteq SK(X,M)$ *and* $SK(X,M) \nsubseteq UC_w(X,M)$*.*

## 4 Conclusion

In this paper we have introduced three new solutions for weighted tournaments, which are the weighted equivalents of the Top Cycle, the Uncovered set, and the Minimal Covering set. We have also performed a comparison of eleven solution sets for weighted and unweighted tournaments in terms of inclusion, intersection or absence of the latter.

A first conclusion that can be drawn from these comparisons is that the weighted uncovered set $UC_w$ is a superset of most other solution sets with three exceptions. $UC_w$ always intersects $TC$, $TC_w$, and $SK$ without being a subset or a superset of the latter. $UC_w$ includes the Borda winners and the Condorcet winner whenever it exists, despite the fact that it is not Condorcet consistent. Thus $UC_w$ should rally both Condorcet and Borda advocates as being a set within which the choice of an alternative should be made.

A second conclusion that can be drawn from these comparisons is that the chain of inclusions $BP \subseteq MC \subseteq UC \subseteq TC$ established for tournaments is lost when we move to weighted tournaments. We still obtain $BP_w \subseteq MC_w \subseteq UC_w$ and $BP_w \subseteq TC_w$. However, while $TC_w$ and $UC_w$ always intersect, it can happen that none contains the other. The same is true of $MC_w$ and $TC_w$.

A third conclusion concerns the comparison between tournaments and weighted tournaments. Since weighted tournaments contain more information than tournaments, one could expect solution sets for weighted tournaments to refine their tournament counterparts. Surprisingly, this is the case only for $TC$. Indeed $TC_w \subseteq TC$ and, in many examples that we worked out, $TC_w$ is much smaller than $TC$. But this order of inclusion is reversed for $UC$ and $MC$ while $BP$ and $BP_w$ might not even intersect.

With a few exceptions, the results of this paper carry over to weighted weak tournaments. $TC_w$, $UC_w$, and $MC_w$ are still well defined. On the other hand, $BP_w$ has to be redefined as the set of outcomes played with a positive probability in some equilibrium. Dutta and Laslier (1997) prove that $BP_w \subseteq MC_w$. The inclusion $BP_w \subseteq TC_w$ can be proved in the same way. Duggan and Le Breton (1997b) prove that $BP \subseteq MC \subseteq UC$ and $BP \subseteq TC$. However, $TC$ and $MC$ are not nested anymore. Finally, Dutta and Laslier show that $UC \subseteq UC_w$ and $MC \subseteq MC_w$.

A final conclusion deals with the position of $Ke$ and $SK$ within the family of solutions examined in this paper. Since $Ke$ and $SK$ are defined without any reference to the solution concepts for the game induced by a weighted tournament, we might expect the absence of clear relations with the solution sets that have a relation with solution concepts for games. This is confirmed by our results. Except for $Ke \subseteq TC$, $Ke \subseteq UC_w$, $SK \cap UC_w \neq \emptyset$, and $SK \cap MC_w \neq \emptyset$, the intersection of $Ke$ and $SK$ with any of the other solution sets may be empty. In as much as the $TC_w$, $MC_w$, $BP_w$, $UC$, and $MC$ concepts have some significance, this casts some doubt on the pertinence of $SK$ and $Ke$ but, in this respect, the Borda rule does not fare better.

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