

Maximal Decompositions of Cost Games into Specific and Joint Costs*

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Abstract

The problem in which some agents joint together to realize a set of projects and must decide how to share its cost may be seen as a cooperative cost game. In many instances, total cost may naturally be decomposed into joint costs and costs that are specific to individual agents. We show that the maximal amount that can be attributed directly to each agent while yielding a problem for the joint cost that remains a cost game, is given by the minimal incremental cost of adding this agent to any of the possible coalitions of other agents. Thus, for concave games, it is given by the incremental cost of adding the agent to all others. We also show that a concave game yields a reduced game that is itself concave.

Keywords: Cost game; decomposition

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Résumé

Le problème où plusieurs agents entreprennent en commun un ensemble de projets et doivent décider du partage du coût total peut être vu comme un jeu coopératif. Dans certains cas, le coût total peut naturellement être décomposé en coûts joints et coûts spécifiques aux agents. On montre que le montant maximal qui peut être attribué directement à chaque agent, tout en donnant un problème de partage de coût joint qui constitue encore un jeu de coût, est donné par le minimum des coûts incrémentaux de l'adjonction de l'agent aux coalitions possibles des autres agents.

1 Introduction

The problem in which some agents joint together to realize a set of projects and must decide how to share its cost may be seen as a cooperative cost game, provided that the cost function satisfies certain properties.¹ Then, solution concepts from the theory of cooperative games may be applied to these problems. In many instances, total cost may naturally be decomposed into joint costs and costs that are specific to individual agents. When this is the case, a desirable property of a solution concept or cost sharing rule is that it be invariant to such a decomposition. More precisely, it must give the same cost shares whether the rule is applied to total costs or to joint costs alone, the share of which are then added to the specific costs of agents. The core, the Shapley value, and the nucleolus are examples of solution concepts that are invariant to such a decomposition, provided that the remaining joint costs problem still defines a cost game. The serial cost sharing rule, which is of a different variety, is also invariant to such a decomposition in the context of a single private homogenous good.

Given a solution concept invariant to decomposition, it may be desirable to reduce the problem to its essential part, the one that cannot be attributed directly to the agents as being specific. As noted by Young (1994, p.1203), “Sometimes the distinction between direct and joint costs is not so clear, however.” Our purpose in this note is simply to characterize the maximal amounts of total cost that can be formally declared as being specific to the different agents, while giving a problem for the joint costs that remains a cost game. We show that the maximal amount that can be attributed directly to each agent is given by the minimal incremental cost of adding this agent to any of the possible coalitions of other agents. Thus, for concave games, it is given by the incremental cost of adding the agent to all others. We also show that a concave game yields a reduced game that is itself concave.

¹See Young (1994) for an excellent survey on this approach.

2 Definitions and preliminary results

Let $N = \{1, \dots, n\}$ be the set of agents and \mathcal{N} be the family of all subsets or coalitions of N including N and \emptyset . Given $i \in N$, let $\mathcal{N}_{-i} = \{S \in \mathcal{N} : i \notin S\}$.

Definition 1 A *cost game* is a pair (c, N) where $c : \mathcal{N} \rightarrow \mathbb{R}_+$ is a mapping satisfying $c(\emptyset) = 0$ and the following conditions:

$$\forall S, T \in \mathcal{N} : S \subset T \Rightarrow c(S) \leq c(T) \quad (\text{Monotonicity})$$

$$\forall S, T \in \mathcal{N} : c(S \cup T) \leq c(S) + c(T) \quad (\text{Sub-additivity})$$

Remark 1 Monotonicity can be defined equivalently by:

$$\forall i \in N : \forall S \in \mathcal{N}_{-i} : c(S) \leq c(S \cup \{i\})$$

Remark 2 Sub-additivity implies:

$$c(N) \leq \sum_{i \in N} c(\{i\}) \quad (\text{Non-negative gain from cooperation})$$

Definition 2 Given a cost game (c, N) and a vector $d \in \mathbb{R}_+^n$, let $\hat{c}_d : \mathcal{N} \rightarrow \mathbb{R}_+$ be a mapping defined by:

$$\forall S \in \mathcal{N} : \hat{c}_d(S) = c(S) - \sum_{i \in S} d_i$$

with the convention that $\sum_{i \in S} d_i = 0$ if $S = \emptyset$. Then, (d, \hat{c}_d) is a *decomposition* of (c, N) if (\hat{c}_d, N) is itself a game, i.e. if \hat{c}_d is sub-additive and monotonic.

The next lemma shows that it is sufficient to check for the monotonicity of \hat{c}_d to make sure that (\hat{c}_d, N) is a game.

Lemma 1 *Given a decomposition (d, \hat{c}_d) of a game (c, N) , then (\hat{c}_d, N) is a game if \hat{c}_d is monotonic.*

Proof. Consider a game (c, N) and a decomposition (d, \hat{c}_d) such that \hat{c}_d is monotonic. Then,

$$\begin{aligned}
\hat{c}_d(S \cup T) &= \hat{c}_d(S \cup T \setminus S) = c(S \cup T \setminus S) - \sum_{i \in S} d_i - \sum_{i \in T \setminus S} d_i \\
&\leq c(S) + c(T \setminus S) - \sum_{i \in S} d_i - \sum_{i \in T \setminus S} d_i \\
&= \hat{c}_d(S) + \hat{c}_d(T \setminus S) \leq \hat{c}_d(S) + \hat{c}_d(T)
\end{aligned}$$

where the first inequality results from the sub-additivity of c and the last one from the monotonicity of \hat{c}_d . Thus \hat{c}_d is sub-additive. ■

3 Maximal decompositions

Given a decomposition (d, \hat{c}_d) of a game, d is usually seen as a vector of specific costs while $\hat{c}_d(N)$ is the joint or common cost. Except in trivial cases, it can be difficult to tell which costs are specific and which ones are common. The question addressed here is what are the maximal values that the components of d can take while respecting the constraint that (\hat{c}_d, N) be a game, i.e. that \hat{c}_d be monotonic.

Definition 3 A *maximal decomposition* of (c, N) is a decomposition (d^+, \hat{c}_{d^+}) such that for any other decomposition (d, \hat{c}_d) :

$$\exists i \in N : d_i > d_i^+ \Rightarrow \exists j \in N : d_j < d_j^+$$

Definition 4 A maximal decomposition of (c, N) for agent i is a decomposition (d^i, \hat{c}_{d^i}) such that for any other decomposition $(d, \hat{c}_d) : d_i \leq d_i^i$.

Lemma 2 (d^i, \hat{c}_{d^i}) is a maximal decomposition of (c, N) for agent i if and only if:

$$\begin{aligned}
d_i^i &= \min_{S \in \mathcal{N}_{-i}} c(S \cup \{i\}) - c(S) \\
0 \leq d_j^i &\leq c(S \cup \{j\}) - c(S) \quad \forall S \in \mathcal{N}_{-j}, \quad \forall j \neq i
\end{aligned} \tag{1}$$

Proof. By Lemma 1 and Remark 1, for (d^i, \hat{c}_{d^i}) to be a decomposition of (c, N) , it is necessary and sufficient that, for any $j \in N$ and any $S \in \mathcal{N}_{-j}$:

$$\hat{c}_{d^i}(S \cup \{j\}) - \hat{c}_{d^i}(S) = c(S \cup \{j\}) - c(S) - d_j^i \geq 0$$

This is precisely what the constraints require for all agents other than i . As for d_i^i , it is defined as the maximal value compatible with these constraints for agent i . ■

From the above lemma, we can guess immediately the content of the main proposition.

Proposition 1 *Any cost game (c, N) has a unique maximal decomposition (d^+, \hat{c}_{d^+}) where:*

$$\forall i \in N : d_i^+ = \min_{S \in \mathcal{N}_{-i}} c(S \cup \{i\}) - c(S).$$

Proof. It suffices to note that d^+ is a solution of (1), i.e. (d^+, \hat{c}_{d^+}) is a maximal decomposition for any i . ■

We can say more of concave games.

Definition 5 A cost function $c : \mathcal{N} \rightarrow \mathbb{R}_+$ is concave if:

$$\forall i \in N, \forall S, T \in \mathcal{N}_{-i} : S \subset T \Rightarrow c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$$

An equivalent definition is:

$$\forall S, T \in \mathcal{N} : c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$$

A concave cost function also satisfies sub-additivity and non-negative gain from cooperation.

It thus defines a cost game, which is said concave.

Proposition 2 *A decomposition (d, \hat{c}_d) of a concave game (c, N) yields a reduced game (\hat{c}_d, N) that is concave. In particular, the maximal decomposition of a concave game yields a concave game.*

Proof. Consider a concave game (c, N) and a decomposition (d, \hat{c}_d) . Since \hat{c}_d is monotonic, then:

$$\begin{aligned}
\hat{c}_d(S) + \hat{c}_d(T) &= c(S) + c(T) - \sum_{i \in S} d_i - \sum_{i \in T} d_i \\
&\geq c(S) + c(T) - \sum_{i \in S} d_i - \sum_{i \in T \setminus S} d_i - \sum_{i \in S \cap T} d_i \\
&= c(S \cup T) + c(S \cap T) - \sum_{i \in S} d_i - \sum_{i \in T \setminus S} d_i - \sum_{i \in S \cap T} d_i \\
&= \hat{c}_d(S \cup T) + \hat{c}_d(S \cap T)
\end{aligned}$$

Thus, \hat{c}_d is concave. ■

Remark 3 Clearly, the maximal decomposition (d^+, \hat{c}_{d^+}) of a concave game is given by $d_i^+ = c(N) - c(N \setminus \{i\}) \forall i$.

Remark 4 The results of this note can be extended to games where sub-additivity and concavity are replaced by super-additivity and convexity respectively. Then, the maximal decomposition of a convex game is given by $d_i^+ = c(\{i\}) \forall i$.

References

Young, H.P. (1994), “Cost Allocation”, in R.J.Aumann and S. Hart, eds, *Handbook of Game Theory*, Vol. II, Amsterdam, North Holland, Chap. 34, 1191-1235.