A BORDA MEASURE FOR SOCIAL CHOICE FUNCTIONS

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Abstract

The question addressed in this paper is the order of magnitude of the difference between the Borda rule and any given social choice function. A social choice function is a mapping that associates a subset of alternatives to any profile of individual preferences. The Borda rule consists in asking voters to order all alternatives, knowing that the last one in their ranking will receive a score of zero, the second lowest a score of 1, the third a score of 2 and so on. These scores are then weighted by the number of voters that support them to give the Borda score of each alternative. The rule then selects the alternatives with the highest Borda score. In this paper, a simple measure of the difference between the Borda rule and any given social choice function is proposed. It is given by the ratio of the best Borda score achieved by the social choice function under scrutiny over the Borda score of a Borda winner. More precisely, it is the minimum of this ratio over all possible profiles of preferences that is used. This "Borda measure" or at least bounds for this measure is also computed for well known social choice functions.

Résumé

Cet article se penche sur la distance entre la règle de Borda et n'importe quelle autre fonction de choix social. Ces dernières associent un sous-ensemble d'options possibles à tout profil ou configuration de préférences individuelles. La règle de Borda consiste à demander aux votants d'ordonner les options possibles, en leur disant que la dernière dans leur ordre recevra un score nul, l'avant-dernière un score égal à 1, celle qui vient au troisième pire rang un score égal à 2 et ainsi de suite. Ces scores sont ensuite pondérés par le nombre de votants qui les supportent pour donner le score de Borda de chaque option. La règle choisit les options qui ont reçu le score le plus élevé. Dans cet article, une mesure simple de la différence entre la règle de Borda et n'importe quelle autre fonction de choix social est proposée. Elle est donnée par le rapport du meilleur score de Borda obtenu par les options que sélectionne la fonction de choix social considérée sur le score de Borda d'un gagnant de Borda. De façon plus précise, c'est le minimum de ces rapports, sur l'ensemble des profils de préférences, qui est utilisé. Cette mesure de Borda ou, à tout le moins, un intervalle pour cette mesure est calculé pour un certain nombre de fonctions de choix social bien connues.

1. Introduction

The controversy between Borda (1784) and Condorcet (1785) is certainly the oldest in social choice theory. Young (1988, 1995) gives one of the most interesting account of this controversy. Condorcet advocated the selection of a candidate or alternative that defeats all others in a pair-wise majority vote, although he was aware of the possibility that a cycle in this majority relation prevent the use of such a procedure. When this happens, a more elaborate rule must be called to the rescue to break these cycles. Any social choice function or voting rule that selects a Condorcet winner, when it exists, is called a Condorcet social choice function.

Borda has been the protagonist of a rule that consists in asking voters to order all alternatives, knowing that the last one in their ranking will receive a score of zero, the second lowest a score of 1, the third a score of 2 and so on. These scores are then weighted by the number of voters that support them to give the Borda score of each alternative. The winners are the alternatives with the highest Borda score. One obtains other scoring methods by using a different sequence of non-decreasing scores. These methods are also called positional voting methods.

The modern axiomatic approach permits the understanding of the respective features of the two voting rules advocated by these two eighteenth century scientists. Moulin (1988) gives an account of the respective strengths and weaknesses of the two kinds of rules. Using an innovative geometric approach, Saari (1995) also puts into perspective the radically different properties of the two methods.

In two fundamental papers, Young (1974, 1988) proves that the scoring voting rules are the only social choice functions that satisfy an axiom called reinforcement and some extra mild axioms. Reinforcement requires that, if two different electorates select the same alternative, then the union of the two should also selects this alternative. This result provides an important argument in defense of the Borda rule.

On the other hand, it is well known that the scoring rules behave badly with respect to modifications in the choice set. This makes them prone to manipulation by large coalitions. They can also give rise to all kind of paradoxes. However Saari (1989) shows that, of all the scoring methods, the Borda rule or an extension of the latter is the one least susceptible to these paradoxes. Saari (1990) shows that it is the least susceptible to manipulation by small coalitions. In the same vein, Gehrlein et al. (1982) show that, among scoring rules, the Borda one is least likely to change the winner when a non-winner is removed from the list of alternatives.

Given the respective strengths of the Borda and Condorcet social choice functions, an institution that would opt for a Condorcet function would be well advised to choose one that is as close as possible to the Borda rule. In this paper, we propose a simple measure of the difference between the Borda rule and any given social choice function. This measure is given by the ratio of the best Borda score achieved by the social choice function under scrutiny over the Borda score of a Borda winner. More precisely, we take the minimum of this ratio over all possible profiles of preferences.

This "Borda measure" of a given social choice function is in the spirit of the Copeland measure of Laffond, Laslier, and Le Breton (1994). It is a measure that focuses on the worst profiles under which the alternate social choice function could operate when compared to the Borda rule. It gives a first indication as to how far from the Borda rule the alternate social choice function can be.

Instead, one could have opted for some average ratio, given some probability measure on the set of profiles. Such an approach would have been in the spirit of Condorcet’s work. But it would have posed the problem of selecting the probability measure on the set of profiles and it would have required more complex computations, because of the combinatorial nature of the problem. This is why we opt for the simpler measure proposed here.

We also compute this "Borda measure" or at least bounds for this measure for well known Condorcet social choice functions. Most of them are reviewed by Levin and Nalebuff (1995). The remainder of the paper is organized as follows: The main definitions are given in section 2. In section 3, we compute the "Borda measure" of the Condorcet rule by obviously considering only profiles for which a Condorcet winner exists. In section 4, we consider a familiar social choice function named after Kemeny (1959). We provide a close interval for the "Borda measure" of this rule. It turns out that the Kemeny rule almost achieves the "Borda measure" of the Condorcet rule. Young (1995) promotes the Kemeny rule because it is a natural extension of the maximum likelihood approach developed by Condorcet. Our result confirms this fact.

In sections 5 and 6, we focus on quite different social choice functions proposed by Copeland (1951), and Simpson (1969). We provide upper bounds for the "Borda measure" of these rules. They indicate that these rules do not do as well as the Kemeny rule. Saari and Merlin (1996) show that a Borda winner and a Copeland winner can be as far apart as one can imagine in terms of their relative rankings. Our results supplement theirs in showing how far from the Borda winner a Copeland winner can be in terms of the Borda scores.

In Sections 7 and 8, we study the top cycle introduced by Good (1971) and the uncovered set proposed by Fishburn (1977) and Miller (1980). The top cycle does as well as the Condorcet rule. We conjecture that this is also the case for the uncovered set. Section 9 concludes with a more detailed comparison of these functions in terms of their Borda measures.

2. Notation and preliminary results

Throughout this paper, N is the set of individuals or voters and X the set of alternatives. Their cardinality is respectively $|N| = n \geq 3$ and $|X| = m \geq 3$. Each individual $i \in N$ is assumed to have a transitive strict preference, i.e. a linear order P_i on X. Let L represent the set of linear orders on X. A **profile** is a $P = (P_1, ..., P_n) \in L^n$.

Given a profile $P \in L^n$ and a pair $(x, y) \in X$, let $n_{xy}(P) = |\{ i \in N : xP_iy \}|$. By convention, $n_{xx}(P) = 0$, $\forall\ x \in X$. The following lemma follows from transitivity of the preferences. It is stated without proof.

Lemma 1: For any profile P and any $x, y, z \in X$, $n_{xy}(P) \geq n_{xz}(P) + n_{zy}(P) - n$.

For a given profile P, a binary relation $M(P)$ on X is defined by $xM(P)y$ if and only if $n_{xy}(P) > n_{yx}(P)$. A profile $P \in L^n$ induces a **tournament** on X whenever $M(P)$ is complete. Obviously, when n is odd, any P induces a tournament on X.

A **social choice function (SCF)** or **voting rule** or simply a rule for the pair (X, N) is a mapping $\Gamma : L^n \to 2^X \backslash \{\varnothing\}$, where 2^X represents the family of all subsets of X. $\Gamma(P)$ is the set of alternatives selected by the SCF.

An alternative $x \in X$ is a **Condorcet winner** for a given profile P if $n_{xy}(P) > n_{yx}(P) \quad \forall\, y \neq x$. A SCF Γ is a **Condorcet type function** if $\Gamma(P) = \{x\}$ whenever x is the Condorcet winner. If the majority relation M yields an order over all alternatives, then this order is the **Condorcet order**.

Given an $x \in X$ and a profile $P \in L^n$, let $R(x, P_i) = |\{y \in X : xP_iy\}|$. The **Borda score** of x is defined as: $B(x, P) = \sum_{i=1}^{n} R(x, P_i)$. Let $B^*(P) = \max_{x \in X} B(x, P)$. Any $x \in \operatorname{argmax}_{x \in X} B(x, P)$ is a **Borda winner**. $Bor(P)$ is the set of Borda winners for profile P. The following useful lemma is well known and will not be proven.

Lemma 2: $B(x, P) = \sum_{y \in X} n_{xy}(P)$.

Given a SCF Γ, we propose the following measure for the discrepancy between Γ and the Borda SCF, which we call the **Borda measure of** Γ:

$$B_\Gamma(n, m) = \min_{P \in L^n} \frac{\max_{x \in \Gamma(P)} B(x, P)}{B^*(P)}$$

In plain words, this measure is given by the ratio of the best Borda score achieved by an alternative chosen by Γ over the Borda score of a Borda winner. More precisely, we take the minimum of this ratio over all possible profiles of preferences. This measure is invariant with respect to a linear transformation of the function $R(\cdot, \cdot)$. However, it is not invariant with respect to affine transformations.

Adding any positive constant to the $R(x, P_i)$, $x \in X$, would increase the value of our measure toward 1. Yet, there is much to be said in favour of the scores originally proposed by Borda as we did. Indeed, according to Lemma 2, the Borda score of alternative x is the total number of other alternatives defeated by x in all pairwise comparisons. Our measure is a ratio of such totals numbers, which have a natural interpretation.

The precise value of this measure or the bounds for this value that will be obtained for different SCFs will often depend on whether n or m is odd or even. Thus they will often involve the function $\nu : \mathbb{N} \to \{1, 2\}$ defined by:

$$\nu(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

3. Condorcet

In this section, we restrict ourselves to the subset of profiles for which there exists a Condorcet winner and we find the Borda measure B_{Con} of the SCFs selecting this winner.

Lemma 3: If $n \le m\nu(n)/(m - 2)$, i.e. for $(n, m) = (3, 3)$, $(4, 3)$, $(4, 4)$, and $(6, 3)$, the Condorcet winner is a Borda winner.

Proof. Without any loss of generality, let us concentrate on the profiles such that 1 is the Condorcet winner. The latter requires profiles such that $n_{1j}(P) \ge (n + \nu(n))/2$, $j = 2, ..., m$, which implies $n_{j1}(P) \le (n - \nu(n))/2$, $j = 2, ..., m$. For any P such that 1 is the Condorcet winner, we thus have $B(1, P) \ge (m - 1)(n + \nu(n))/2$ and $B(j, P) \le (m - 2)n + (n - \nu(n))/2 = ((2m - 3)n - \nu(n))/2$,
$j = 2, ..., m$. If $n \le m\nu(n)/(m - 2)$, we have $B(j, P) \le ((2m - 3)n - \nu(n))/2 \le$
$(m - 1)(n + \nu(n))/2 \le B(1, P)$, $j = 2, ..., m$. □

Theorem 4:

$$B_{Con}(n, m) = \begin{cases} 1 & \text{if } n \le \dfrac{m\nu(n)}{m-2} \\ \dfrac{(m-1)(n+\nu(n))}{(2m-3)n-\nu(n)} & \text{if } n > \dfrac{m\nu(n)}{m-2} \end{cases}$$

Proof. From Lemma 3, $B_{Con} = 1$ whenever $n \le m\nu(n)/(m-2)$. For the other case, we have, without any loss of generality:

$$B_{Con}(n, m) = \min_{P \in L^n} \frac{B(1, P)}{B(2, P)}$$

subject to alternative 1 being the Condorcet winner and 2 a Borda winner. From the proof of Lemma 3, for 1 to be the Condorcet winner, we need $B(1, P) \ge (m-1)(n+\nu(n))/2$ and $B(2, P) \le ((2m-3)n - \nu(n))/2$. These bounds are reached for profile P^1 defined by:

(1, 2, ..., m) for $(n + \nu(n))/2$ individuals,

(2, 3, ..., m, 1) for the remaining $(n - \nu(n))/2$ individuals.

Thus $B_{Con}(n, m) = \dfrac{B(1, P^1)}{B(2, P^1)} = \dfrac{(m-1)(n+\nu(n))}{(2m-3)n-\nu(n)}$. □

Corollary 5: $\lim\limits_{n\to\infty} B_{Con}(n, m) = \dfrac{m-1}{2m-3}$, $\lim\limits_{m\to\infty} B_{Con}(n, m) = \dfrac{1}{2} + \dfrac{\nu(n)}{2n}$, and $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} B_{Con}(n, m) = \dfrac{1}{2}$.

$B_{Con}(n, m)$ is compiled in Table 1 for $m \le 15$ and $n \le 40$. The limits of B_{Con} with respect to n and m appear respectively in the bottom row and the last column.

From now on, we shall allow for all profiles of preferences. Hence the existence of a Condorcet winner will not be guaranteed. The SCFs that we shall examine all pick the Condorcet winner when it exists but differ in their choice of a set of alternatives when there is no Condorcet winner.

4. Kemeny

Kemeny (1959) proposes a principle to order all alternatives, which works for any profile. This order agrees with the Condorcet order when the latter exists. A Kemeny order is closest to the given profile in the following sense. Given two linear orders P_1 and $P_2 \in L$, and two alternatives $x, y \in X$, define:

$$\delta_{xy}(P_1, P_2) = \begin{cases} 1 & \text{if } xP_1y \text{ and } yP_2x \\ 0 & \text{otherwise} \end{cases}$$

and $\Delta(P_1, P_2) = \sum_{x \in X} \sum_{y \in X} \delta_{xy}(P_1, P_2)$.

The function Δ is a distance on the set L. One can then define a "distance" d between an order O and a profile P by:

$$d(O, P) = \sum_{i=1}^{n} \Delta(O, P_i).$$

A **Kemeny order** for a profile P is an order belonging to $\operatorname{argmin}_{O \in L} d(O, P)$. A **Kemeny winner** is a top element of a Kemeny order. The Kemeny SCF, *Ke*, selects the Kemeny winners.

Young (1988) shows that the principle advocated by Kemeny is very much in the spirit of Condorcet’s endeavour. Take a situation where there is some sense in saying that an alternative is objectively better than another one. Suppose that the ranking of these two alternatives by each voter is his or her evaluation of the true ranking and that the probability that each voter be right is greater than 1/2. Then the Kemeny order is the one that has the maximum likelihood of being the true ranking over all alternatives. This is precisely what Condorcet was after. The following lemmas and corollaries list some of the properties of a Kemeny order.

Lemma 6: A Kemeny order for a profile P is an order belonging to $\underset{O \in L}{\operatorname{argmax}} \sum_{x \in X} \sum_{\substack{y \in X \\ xOy}} n_{xy}(P)$.

Proof. $d(O, P) = \sum_{x\in X}\sum_{y\in X}\sum_{i=1}^{n} \delta(O, P_i) = \sum_{x\in X}\sum_{\substack{y\in X\\ xOy}} n_{yx}(P) =$

$\sum_{x\in X}\sum_{\substack{y\in X\\ xOy}} (n - n_{xy}(P)) = \dfrac{m(m-1)n}{2} - \sum_{x\in X}\sum_{\substack{y\in X\\ xOy}} n_{xy}(P)$, hence the result. □

Lemma 7: Suppose $O = (1, 2, ..., m)$ is a Kemeny order for a given profile P. Then $n_{j,j+1}(P) \geq n_{j+1,j}(P)$, $j = 1, ..., m - 1$.

Proof. For any $j = 1, ..., m - 1$, consider the order $O^* = (1, 2, ..., j - 1, j + 1, j, j + 2, ..., m)$. By the proof of Lemma 6, $d(O^*, P) - d(O, P) = n_{j,j+1} - n_{j+1,j}$, which cannot be negative if O is a Kemeny order. □

Corollary 8: Suppose $O = (1, 2, ..., m)$ is a Kemeny order for a given profile P. Then, if there exists a Condorcet winner under this profile, this winner must be alternative 1.

Proof. From Lemma 7, $n_{j,j-1}(P) \leq n/2$, $j = 2, ..., \mathrm{m}$. Thus none of the alternatives $j = 2, ..., m$ may be a Condorcet winner. □

Lemma 9: If x is a (unique) Kemeny winner for a profile P, then $\sum_{y\in X} n_{xy}(P) \geq (m - 1)n/2$ (>).

Proof. Without any loss of generality, suppose $O = (1, 2, ..., m)$ is a (unique) Kemeny order and consider the order $O^* = (2, \ldots, m, 1)$. By the proof of Lemma 6, $d(O^*, P) - d(O, P) = \sum_{j=2}^{m} n_{1j}(P) - \sum_{j=2}^{m} n_{j1}(P) = \sum_{j=2}^{m} (2n_{1j}(P) - n) = 2\sum_{j=2}^{m} n_{1j}(P) - (m - 1)n \geq 0$ (>). □

Lemma 10: $(m - 1)n/((2m - 3)n - \nu(n)) \leq B_{Ke}(n, m) \leq B_{Con}(n, m)$.

Proof. From Corollary 8, the Kemeny SCF always selects the Condorcet winner when it exists. Hence $B_{Ke}(n, m)$ is bounded above by $B_{Con}(n, m)$. Turning to the lower bound, let us restrict ourselves, without any loss of generality, to profiles that yield alternative 1 as a Kemeny winner and 2 as a Borda winner. According to Lemma 9, for alternative 1 to be a Kemeny winner, we must have $B(1, P) \equiv \sum_{j=2}^{m} n_{1j}(P) \geq (m - 1)n/2$. We must also have $n_{2j}(P) \leq (n - \nu(n))/2$ for at least one $j \neq 2$. Otherwise 2 would be a Condorcet and a Kemeny winner. Thus, we must have $B(2, P) \leq (m - 2)n + (n - \nu(n))/2 =$

$((2m-3)n-\nu(n))/2$. Hence any profile P under which 1 is a Kemeny winner and 2 a Borda winner without being a Kemeny winner gives:

$$\frac{B(1,\ P)}{B(2,\ P)} \geq \frac{(m-1)n}{(2m-3)n-\nu(n)}$$

The right-hand side of the latter inequality is also a lower bound for $B_{Ke}(n, m)$. □

Corollary 11: $\lim_{n\to\infty} B_{Ke}(n, m) = (m-1)/(2m-3)$.

Proof. The limits of the left-hand and right-hand sides of the inequalities in Lemma 10 are both $(m-1)/(2m-3)$. □

According to the last corollary, the possible discrepancy between the Condorcet and the Kemeny SCFs vanishes as the number of voters increases. The next theorem shows that, for finite values of (n, m), B_{Ke} is actually lower than B_{Con}, except when $(n, m) = (3, 3)$, $(4, 3)$, $(4, 4)$, and $(6, 3)$. The discrepancy is highest when there are few voters. The exception obtains because the best profiles that we are able to find in the next theorem toward achieving the Borda measure of the Kemeny rule yield a Kemeny winner that is also a Borda winner. Recall that $B_{Con}(n, m) = 1$ for these pairs of (n, m). We are quite confident that the upper bounds given in this theorem cannot be improved.

Theorem 12: Let

$$F_{11}(n,\ m) = \frac{(m-1)n-\nu(m)+3}{(2m-3)n-(m-3)-\nu(m)}, \qquad F_{12}(n,\ m) = \frac{(m-1)n-\nu(m)+3}{2(m-2)n-2},$$

$$F_{21}(n,\ m) = \frac{(m-1)n+2}{2m(n-1)-3n+2}, \qquad F_{22}(n,\ m) = \frac{(m-1)n+2}{2(m-2)n-2}.$$

Then, $B_{Ke}(n, m) \leq F_{11}(n, m)$, for $n = 3$ and 5. For $n \geq 7$ and odd, $B_{Ke}(n, m) \leq \min(F_{11}(n, m), F_{12}(n, m))$. For $(n, m) \neq (4, 3)$ and n even, $B_{Ke}(n, m) \leq \min(F_{21}(n, m), F_{22}(n, m))$.

Proof. For n odd, consider profile P^{11} defined by:

$(1, 2, ..., m)$ for $(n-1)/2$ voters,

$(2, 3, ..., m, 1)$ for $(n-1)/2$ voters,

($m - 2 + \nu(m)$, ..., 6, 4, 1, 2, 3, 5, ..., $m + 1 - \nu(m)$) for the remaining voter.

Under this profile, (1, 2, 3, ..., m) is the unique Kemeny order, alternative 2 a Borda winner (ex-aequo with 1 for $m = n = 3$, unique otherwise) and $B(1, P^{11})/B(2, P^{11}) = F_{11}(n, m)$. Thus $F_{11}(n, m)$ is an upper bound for $B_{Ke}(n, m)$ when n is odd, the best that we could find for $n = 3$ and 5.

For $n \geq 7$ and odd, consider profile P^{12} defined by:

(1, 2, 3, ..., m) for $(n - 7)/2$ voters,

(1, 3, 2, 4, ..., m) for 3 voters,

(3, 2, 4, 5, ..., m, 1) for $(n - 3)/2$ voters,

(2, 4, 5, ..., m, 1, 3) for 1 voter,

(2, 4, 6, ..., $m - 2 + \nu(m)$, 1, 3, 5, 7, ..., $m + 1 - \nu(m)$) for the remaining voter.

Under this profile, (1, 3, 2, 4, ..., m) is now the unique Kemeny order, alternative 2 remains the unique Borda winner if $m > 3$, and $B(1, P^{12})/B(2, P^{12}) = F_{12}(n, m)$. Thus $F_{12}(n, m)$ is another upper bound for $B_{Ke}(n, m)$ when n is odd and $m > 3$. It is smaller than $F_{11}(n, m)$ if and only if $n < m - 5 + \nu(m)$. Since one must have $n \geq 7$ for P^{12} to be defined, this last condition implies $m > 12$.

For n even, consider profile P^{21} defined by:

(1, 2, ..., m) for $n/2$ voters,

(2, m, ..., 5, 4, 3, 1) for $(n - 2)/2$ voters,

(3, 4, 5, ..., m, 1, 2) for the remaining voter.

and profile P^{22} defined by:

(1, 2, 3, ..., m) for $(n - 4)/2$ voters,

(1, 3, 2, ..., m) for 2 voters,

(3, 2, 4, 5, ..., m, 1) for $(n - 2)/2$ voters,

(2, 4, 5, ..., m, 1, 3) for the remaining voter.

Note that $B(1, P^{21})/B(2, P^{21}) = F_{21}(n, m)$ and $B(1, P^{22})/B(2, P^{22}) = F_{22}(n, m)$. The unique Kemeny order is (1, 2, 3, ..., m) under P^{21} and (1, 3, 2, 4, ..., m) under P^{22}. For $(n, m) = (4, 3)$, both profiles P^{21} and P^{22} yield alternative 1 as the unique Borda winner as well as the Kemeny winner. Thus the bound given

$B_{Con}(4, 3) = 1$ cannot be improved with any of these two profiles. For $m = 3$, and any even n, alternative 1 is the unique Borda winner under P^{22} but, for $n \neq 4$, alternative 2 is a Borda winner under P^{21} (ex-aequo with 1 for $n = 6$, unique otherwise). Thus, for $m = 3$ and $n \neq 4$, $F_{21}(n, m)$ is an upper bound for $B_{\mathrm{Ke}}(n, m)$. For $m > 3$, both P^{21} and P^{22} make alternative 2 a Borda winner (ex-aequo with 1 for $m = n = 4$, unique otherwise). Thus, for n even and $(n, m) \neq (4, 3)$, $F_{21}(n, m)$ and $F_{22}(n, m)$ are two upper bounds for $B_{\mathrm{Ke}}(n, m)$. We have $F_{21}(n, m) > F_{22}(n, m) \Leftrightarrow m > (n + 4)/2$. □

Remark 13: From the proof of the last theorem, $F_{11}(n, m) > F_{12}(n, m)$ and $F_{21}(n, m) > F_{22}(n, m)$ when m is sufficiently large with respect to n. Thus, when m becomes large and n takes fixed values other than 3 and 5, the best upper bound for B_{Ke} is given by F_{12} or F_{22}. Since $\lim_{m\to\infty} F_{12}(n, m) = \lim_{m\to\infty} F_{22}(n, m) = 1/2$, B_{Ke} is bounded above by 1/2 when m increases. Also note that $\lim_{m\to\infty} F_{11}(n, m) = \lim_{m\to\infty} F_{21}(n, m) = n/(2n - 1)$. In particular, $\lim_{m\to\infty} F_{11}(3, m) = 3/5$ and $\lim_{m\to\infty} F_{11}(5, m) = 5/9$. The limit with respect to n for B_{Ke} has been established in Corollary 11. It is an exact value.

The best upper bounds for $B_{Ke}(n, m)$ are compiled in Table 2 for $m \leq 15$ and $n \leq 40$. The limits of B_{Ke} with respect to n and m appear respectively in the bottom row and the last column. Bold numbers are given by F_{12} or F_{22}. All others by F_{11} or F_{21}.

We are quite confident that the upper bounds given in Theorem 12 cannot be improved. Actually, we can be more affirmative in the case $m = 3$. For n odd, profile P^{11} reduces to:

(1, 2, 3) for $(n + 1)/2$ voters,

(2, 3, 1) for $(n - 1)/2$ voters.

This is clearly the best that can be done in order to give alternatives 1 and 2 respectively the lowest and highest possible Borda scores under the constraint that alternative 1 is a unique Kemeny winner.

For n even, profile P^{21} reduces to:

(1, 2, 3) for $n/2$ voters,

(2, 3, 1) for $(n - 2)/2$ voters,

(3, 1, 2) for the remaining voter.

By having one voter of type (3, 1, 2) instead of one more of type (1, 2, 3), one obtains a lower value for the $B(1, P)/B(2, P)$ ratio. But one cannot have more voters of the last type without making alternative 3 a Condorcet winner. This is again the best that can be done.

5. Copeland

Copeland (1951) proposes a different and simple way of selecting winning alternatives. The **Copeland SCF** is the function *Cop* that selects the alternatives that defeat a maximum number of other alternatives. Let $s(x, P) = |\{y \in X : n_{xy}(P) > n_{yx}(P)\}|$. This is the **Copeland score** of x. *Cop* is defined by: $Cop(P) = \text{argmax}_{x \in X}\, s(x, P)$.

For any tournament, the following holds:

$$\sum_{\substack{y \in X \\ y \neq x}} s(y, P) = \frac{m(m-1)}{2}.$$

Thus, if an alternative x is a Copeland winner under a profile P that induces a tournament, we must have $s(x, P) \geq (m-1)/2$. However, if this condition is satisfied with equality, then all other alternatives are also Copeland winners. The next lemma gives a necessary condition for an alternative to be a unique Copeland winner.

Lemma 15: Given a profile P that induces a tournament on X, if an alternative x is a unique Copeland winner, then $s(x, P) > m/2$, i.e. $s(x, P) \geq (m + \nu(m))/2$.

Proof. Suppose $s(x, P) \leq m/2$, i.e. $s(x, P) \leq (m + \nu(m) - 2)/2$. Then

$$\sum_{\substack{y \in X \\ y \neq x}} s(y, P) = \frac{m(m-1)}{2} - s(x, P) \geq \frac{m(m-1)}{2} - \frac{m + \nu(m) - 2}{2}$$

which implies $\exists\ y \in X : s(y, P) \geq \frac{m}{2} - \frac{m + \nu(m) - 2}{2(m-1)}$ which implies $s(y, P) \geq \frac{m-1}{2}$.

This in turn implies $s(y, P) \geq \frac{m + \nu(m) - 2}{2} \geq s(x, P)$. □

Corollary 16: For $m = 3$ or 4, a unique Copeland winner under a profile that induces a tournament is a Condorcet winner.

Using Lemma 15, it is easy to establish a lower bound for $B_{Cop}(n, m)$. This is the object of the next theorem.

Theorem 17: Let $G(n, m) = \dfrac{(m + \nu(m))(n + \nu(n))}{m(3n - \nu(n)) + n(\nu(m) - 4) + \nu(m)\nu(n)}$.

For any profile P that induces a tournament, $G(n, m) \leq B_{Cop}(n, m)$.

Proof. Without any loss of generality, consider a profile P such that alternative 1 is a unique Copeland winner while 2 is a Borda winner. From Lemma 15,

$B(1, P) \geq \dfrac{(m + \nu(m))}{2}\dfrac{(n + \nu(n))}{2}$. We must also have $s(2, P) \leq (m + \nu(m) - 2)/2$, from which

$B(2, P) \leq \dfrac{(m + \nu(m) - 2)n}{2} + \dfrac{(m - \nu(m))}{2}\dfrac{(n - \nu(n))}{2}$.

Combining these two inequalities yields the result. □

From Corollary 16, we know that $B_{Cop}(n, m) = B_{\mathrm{Con}}(n, m)$ for $m = 3$ or 4. The value of $B_{\mathrm{Con}}(n, m)$ has been established exactly in Theorem 4. We thus have an exact value of $B_{Cop}(n, m)$ for $m = 3$ or 4. In the next theorem, we establish upper bounds for $B_{Cop}(n, m)$ that will be lower than $B_{\mathrm{Con}}(n, m)$ for larger values of m and large enough values of n. We are again confident that it is not possible to lower the value of the lowest of these bounds when it is already smaller than $B_{\mathrm{Con}}(n, m)$. We actually show that the upper bound that is given for the subset of profiles that induce a tournament on X, and in particular for n odd, becomes exact when we take the limit with respect to m or n.

Theorem 18: Let $E(n, m) = 2(m + n - 1)/(mn - 2)$ and

$$D(n, m) = \frac{m(n + 3\nu(n)) + n\nu(m) - \nu(n)(\nu(m) + 4)}{(3m + \nu(m) - 4)(n - \nu(n)) + 4\alpha(m)\nu(n)} \quad \text{where } \alpha(m) = \begin{cases} 0 & \text{if } m < 8 \\ 1 & \text{if } m \geq 8. \end{cases}$$

Then $B_{Cop}(n, m) \leq D(n, m)$. Moreover, for n even, $B_{Cop}(n, m) \leq E(n, m) < D(n, m)$.

Proof. Consider the following profile over 15 alternatives:

(5, 7, 9, 11, 13, 15, 1, 2, 3, 4, 8, 6, 10, 12, 14) for $(n - 3\nu(n))/2$ voters,

(7, 5, 11, 9, 15, 13, 1, 2, 8, 10, 12, 14, 6, 3, 4) for $\nu(n)$ voters,

(2, 14, 12, 10, 6, 8, 4, 3, 15, 13, 11, 9, 7, 5, 1) for $(n - 3\nu(n))/2$ voters,

(2, 8, 10, 12, 14, 4, 6, 3, 5, 9, 7, 13, 11, 15, 1) for $\nu(n)$ voters,

(1, 3, 4, 6, 15, 13, 14, 11, 12, 9, 10, 7, 5, 2, 8) for the remaining $\nu(n)$ voters.

For $n = 9$, it yields the majority matrix of Table 5. To get the general matrix, replace 4 by $(n - \nu(n))/2$, 5 by $(n + \nu(n))/2$, 8 by $n - \nu(n)$ and 9 by n. This profile can be extended or reduced in an obvious way to any number of alternatives. Call the general profile P^{31}. For any m and for n odd, it yields a tournament in which 1 is the unique Copeland winner and 2 the unique Borda winner. Moreover $B(1, P^{31})/B(2, P^{31}) = D(n, m)$, hence the first statement of the theorem.

For n even, consider profile P^{32} defined by:

$(1, 2, ..., m)$ for one voter,

$(2, 3, ..., m, 1)$ for $(n - 2)/2$ voters,

$(4, ..., m, 1, 2, 3)$ for one voter,

$(m, ..., 4, 1, 2, 3)$ for the remaining $(n - 2)/2$ voters.

It gives $n_{12} = n_{13} = (n + 2)/2$, $n_{23} = n$, $n_{1j} = 1$, and $n_{2j} = n/2$, $j = 4, ..., m$, and $n_{ij} = n/2$, $j > i$, $i = 3, ..., m$. Thus 1 is a unique Copeland winner and 2 a Borda winner. For $m = 3$, alternative 1 is also a Condorcet winner. Moreover $B(1, P^{32})/B(2, P^{32}) = E(n, m)$, hence the second statement of the theorem. □

Corollary 19: For n even, $\lim_{n\to\infty} B_{Cop}(n, m) \le \lim_{n\to\infty} E(n, m) = \frac{2}{m}$, $\lim_{m\to\infty} B_{Cop}(n, m) \le \lim_{m\to\infty} E(n, m) = \frac{2}{n}$.

Within the subset of profiles that induce a tournament on X, and in particular for n odd,

$$\lim_{n\to\infty} B_{Cop}(n, m) = \frac{m + \nu(m)}{3m + \nu(m) - 4}, \quad \lim_{m\to\infty} B_{Cop}(n, m) = \frac{n + 3\nu(n)}{3(n - \nu(n))}, \quad \lim_{\substack{m\to\infty \\ n\to\infty}} B_{Cop}(n, m) = \frac{1}{3}.$$

Proof. The first assertion follows from Theorem 18, the second from the combination of Theorems 17 and 18 and the fact that

$$\lim_{n\to\infty} G(n, m) = \lim_{n\to\infty} D(n, m) = \frac{m + \nu(m)}{3m + \nu(m) - 4},$$

$$\lim_{m\to\infty} G(n, m) = \lim_{m\to\infty} D(n, m) = \frac{n + 3\nu(n)}{3(n - \nu(n))}, \text{ and}$$

$$\lim_{\substack{m\to\infty \\ n\to\infty}} G(n, m) = \lim_{\substack{m\to\infty \\ n\to\infty}} D(n, m) = \frac{1}{3}. \ \square$$

Remark 20: For some small values of m and n, profile P^1, which makes 1 a Condorcet hence a Copeland winner, gives a ratio $B(1, P^1)/B(2, P^1)$ that is smaller than both $D(n, m)$ and $E(n, m)$. Thus a least upper bound for $B_{Cop}(n, m)$ is either $B_{Con}(n, m)$ or $D(n, m)$ for n odd and $E(n, m)$ for n even. These bound appear in Table 3. Bold numbers are the same as for the Condorcet rule. For n even profile P^{32}, for which $E(n, m)$ is attained, does not yield a tournament. If we were to restrict ourselves to profiles that induce a tournament on X, as with P^{31}, then $D(n, m)$ would be the bound to use instead of $E(n, m)$. The resulting numbers would be slightly higher than the ones that can be infered by interpolating from those of Table 3 for n odd. The reason is that the 1 and $n - 1$ in the majority matrix would become 2 and $n - 2$ respectively. Two limits with respect to n are given. The first row gives the limit of $D(n, m)$, which is valid if we restrict ourselves to tournaments. The second row gives the limit of $E(n, m)$, which is valid if we restrict ourselves to even n and allow for profile P^{32} that does not induce a tournament.

In a recent paper, Saari and Merlin (1996) show that, for $m \geq 5$ and any two rankings of the candidates, there exist profiles where these rankings are, respectively, the Copeland and the Borda rankings. In particular, for any alternative, there exist profiles such that this candidate is the Copeland winner but occupies the last position in the Borda ranking. Profile P^{32} used in the proof of Theorem 18 is an illustration of this result. Indeed, one can check that $B(1, P^{32}) = m + n - 1$ while $B(3, P^{32}) = (n(m - 2) - 2)/2$. Thus $2B(3, P^{32}) - 2B(1, P^{32}) = n(m - 4) - 2m > 0$ if and only if $m \geq 5$ and $n > 2m/(m - 4)$. Alternative 3 being the more serious other contender for the last place in the Borda ranking, this shows that alternative 1 is the Borda looser for values of m and n satisfying the above inequalities.

Moreover our results supplement the ordinal finding of Saari and Merlin by showing how far from the Borda winner a Copeland winner can be in terms of the Borda scores. Given the intrinsic interest in the Borda scores, this is useful information. Our results show that the Copeland winner can do pretty bad when n is even and when n and m become large, the limit being 0. The problem is less dramatic when n is odd. For small values of n and m, the Copeland rule does as most as well as the Condorcet rule when it does not perform exactly as the latter.

6. Kramer-Simpson

Simpson (1969) proposes a quite different method of avoiding the Condorcet paradox in Euclidean spaces. Adapting his principle to our finite context gives: The alternative s of voter maximum agreement should be one where the maximum number of voters wishing to move to any other alternative is as small as possible over the range of alternatives. It is the democratic hope so to speak, that the number wishing an alternative other than s is small. If it less than half the voters, democracy is in good luck and s tops all other alternatives. To be more precise, s should belong to $\text{argmin}_{x \in X} \max_{y \in X} n_{yx}(P)$. Kramer (1977) justifies this principle by showing that the minmax set is the equilibrium of sequential electoral competition between two parties whose platforms belong to an Euclidean space.

We call this function the **Kramer-Simpson SCF** and we define it equivalently by $KS(P) = \text{argmax}_{x \in X} \min_{y \in X \backslash x} n_{xy}(P)$. An element of $KS(P)$ is a Kramer-Simpson winner.

Lemma 23: $B_{KS}(n, m) \leq B_{Con}(n, m)$, $\forall\ n, m \geq 3$.

Proof. Suppose $x \in X$ is a Condorcet winner. Then $n_{xy}(P) > n_{yx}(\mathrm{P})$ $\forall\ y \in X\backslash\{x\}$. Thus clearly x is a Kramer-Simpson winner, giving the upper bound for B_{KS}. □

The next three theorems establish smaller upper bounds for large enough values of m or n.

Theorem 24: $B_{KS}(n, m) \leq 4(m - 1)/(m^2 - m - 4)$, $\forall\ m \geq 6$, $\forall\ n \geq 2(m - 1)$ or $n = m - 1$.

Proof. Consider the $m - 1$ following orders:

$O^2 = (2, 3, ..., m - 2, 1, m - 1, m)$

$O^3 = (3, 4, ..., m - 1, 1, m, 2)$

....

....

$O^m = (m, 2, ..., m - 3, 1, m - 2, m - 1)$.

Then let $p = \lfloor n/(m-1) \rfloor$ where $\lfloor a \rfloor$ represents the largest integer smaller or equal to a, $k = n - p(m-1)$, $n_j = p$, $j = 2, ..., m - k$, and $n_j = p + 1$, $j = m - k + 1, ..., m$. Since $\sum_{j=2}^{m} n_j = n$, one can partition the set of voters into $m - 1$ subsets containing n_j individuals, for $j = 2, ..., m$. Next, consider the profile P^2 defined by giving preference O^j to the n_j individuals of subset j, for $j = 2, ..., m$. In plain words, each order O^j is shared by p different individuals and the k remaining individuals, if any, are distributed among the last k orders. A parameter α appears in the remaining of the proof so that the latter be reusable for the next two theorems. Set $\alpha = 2$ for the present theorem. Under profile P^2 just constructed,

$$\min_{j \in X} n_{1j}(P^2) \geq \alpha p \quad \text{and}$$

$$\max_{i \neq 1} \min_{j \in X} n_{ij}(P^2) = \begin{cases} p & \text{if } k = 0 \\ p + 1 & \text{if } k > 0 \end{cases}$$

Since $n \geq (4 - \alpha)(m - 1)$ implies $p \geq 4 - \alpha$ and $n = m - 1$ implies $p = 1$ and $k = 0$, alternative 1 turns out to be the unique Kramer-Simpson winner. Turning to Borda scores, $B(1, P^2) = \alpha n$ and $B(m, P^2) = (m - 1)mp/2 - \alpha p + km - k(k + 1)/2 - \max(0, \alpha + k + 1 - m)$. One can check that m is the Borda winner in the set of alternatives $\{2, ..., m\}$ (unique if and only if $k > 0$, i.e. $p < n/(m - 1)$). In the case where $p = n/(m - 1)$ and $k = 0$, we have $B(m, P^2) = nm/2 - \alpha n/(m - 1)$. Hence,

$$B(m, P^2) - B(1, P^2) = n\left(\frac{m(m-1) - 2\alpha}{2(m-1)} - \alpha\right) = n\left(\frac{m(m - 1 - 2\alpha)}{2(m-1)}\right) > 0 \iff m > 1 + 2\alpha.$$

Thus, in this case, m is a Borda winner and $\dfrac{B(1, P^2)}{B(m, P^2)} = \dfrac{2\alpha(m-1)}{m^2 - m - 2\alpha}$.

In the more general case, $B(m, P^2) \geq nm/2 - \alpha/(m-1)$. Since there is no loss of generality in letting 1 be a Kramer-Simpson winner and m a Borda winner, rather than any other pair of alternatives, we have $B_{KS}(n, m) \leq 2\alpha(m-1)/(m^2 - m - 2\alpha)$, $\forall\ m \geq 2(\alpha + 1)$, $\forall\ n \geq (4-\alpha)(m-1)$ or $n = m - 1$. □

The next theorem establishes another upper bound on $B_{KS}(n, m)$ for the case $m \geq 8$ and $n \geq m - 1$.

Theorem 25: $B_{KS}(n, m) \leq 6(m-1)/(m^2 - m - 6)$, $\forall\ m \geq 8$ and $\forall\ n \geq m - 1$.

Proof. Raise alternative 1 by one position in all orders of profile P^2 and set $\alpha = 3$ in the proof of Theorem 23. □

The next theorem establishes another upper bound for $B_{KS}(n, m)$, which, although larger than the ones obtained in the last two theorems, is smaller than B_{Con} for some (n, m) such that $m \geq 7$ and $n < m - 1$.

Theorem 26: $B_{KS}(n, m) \leq 8/(2m - n - 1 - 2\max(0, n - m + 5)/n)$, for $m = 7$ and $n = 3$, $m = 8$ and $n \leq 5$, or $m \geq 9$ and $n \leq m - 2$.

Proof. Set $\alpha = 4$ in the proof of Theorem 23 and raise alternative 1 by two positions in all orders of profile P^2. We are left with orders $O^k, \ldots, O^m$ only in this profile, where $k = n$ in this case. Call P^{42} the modified profile. It yields $B(m, P^{42}) = (2m - n - 1)n/2 - \max(0, n - m + 5)$ and one can check that this score is larger than $B(1, P^{42})$ for the values of m and n given in the statement of the theorem and only for these values. Hence the result. □

Remark 27: The proofs of Theorems 24 and 25 give upper bounds on B_{KS} that are smaller than the ones given in the statement of these theorems. The general formula for these upper bound is: $\alpha n/((m-1)mp/2 - \alpha p + km - k(k+1)/2 - \max(0, \alpha + k + 1 - m))$, with $\alpha = 2$, 3 or 4 according to whether $n \geq 2(m-1)$ (or $n = m - 1$), $m - 1 < n < 2(m-1)$ or $n < m - 1$. For $\alpha = 2$ or 3, the restriction on m may also vary slightly from the one given in Theorems 24 and 25. Table 4 gives the best of the four bounds established in Lemma 23 and Theorems 24, 25, and 26.

Remark 28: From Theorem 26, $\lim_{m\to\infty} B_{KS}(n, m) = 0$. Since $p > n/(m - 1) - 1$, the upper bound given in Remark 27 for $\alpha = 2$ is bounded above by

$$\frac{4n}{(m^2 - m - 4)\left(\frac{n}{m - 1} - 1\right) + (2m - 1)k - k^2 - 2\max(0,\ 3 + k - m)}$$

which is equal to

$$\frac{4(m - 1)}{(m^2 - m - 4)\left(1 - \frac{m - 1}{n}\right) + \frac{(m - 1)}{n}((2m - 1)k - k^2 - 2\max(0,\ 3 + k - m))}.$$

Hence $\lim_{n\to\infty} B_{KS}(n, m) \le \frac{4(m - 1)}{(m^2 - m - 4)}$.

7. The Top Cycle

The SCFs examined in the previous sections propose different ways of breaking cycles in $M(P)$. Instead of breaking these cycles, we might settle for a cruder SCF that leaves us with a set of alternatives that form a cycle and defeat all other alternatives under the relation $M(P)$. Such a set is called the top cycle. The name of Schwartz (1972) is often associated with this concept because of the axiomatization he makes of this rule. However, Good (1971) is probably the first to introduce this concept, which he calls the Condorcet set, in the literature. Miller (1980) calls it the minimal undominated set.

For a given profile P, let $M^*(P)$ be the transitive closure of $M(P)$, i.e. $xM^*(P)y$ if there exists a sequence $x = x_0, x_1, ..., x_k = y$ such that $x_i M^*(P) x_{i+1}$, $i = 0, ..., k - 1$. The **top cycle** of X with respect to P is defined by $TC(P) = \{x \in X : xM^*(P)y, \forall\ y \in X\backslash\{x\}\}$. It always exists by transitivity of M^*.

Lemma 29: $Ke(P) \subseteq TC(P)$.

Proof. Let $x \in Ke(P)$. From Lemma 6, $xM^*(P)y$, $\forall\ y \in X\backslash\{x\}$. Thus $x \in TC(P)$. □

Corollary 30: $B_{Ke}(n, m) \le B_{TC}(n, m) \le B_{Con}(n, m)$.

Actually we can do better and show that $B_{TC}(n, m) = B_{Con}(n, m)$. We begin with the following lemma, the contain of which is well known. See Moulin (1988, Exercise 9.10) or Miller (1980). Parts b and c give the substence of this concept.

Lemma 31: a) $TC(P) = \{x\} \Leftrightarrow x$ is the Condorcet winner;

b) $TC(P)$ is the smallest subset $Y \subseteq X$ such that $y \in Y, x \in X\backslash Y \Rightarrow yM(P)x$;

c) $|TC(P)| \neq 2$;

d) If $|TC(P)| \ge 3$, then one can order $TC(P)$ as $\{x_1, ..., x_k\}$ such that $x_i M(P) x_{i+1}$, $i = 1, ..., k - 1$, $x_k M(P) x_1$ and such that this cycle is of maximal length.

Lemma 32: For $m = 3$ and 4, the Borda winners belong to the top cycle whenever the latter is not a singleton.

Proof. Obvious for $m = 3$ from Lemma 31. For $m = 4$, let $\{1, 2, 3\}$ be the top cycle with $1M2$, $2M3$, and $3M1$. In order that the Borda score of all members of the top cycle be as small as possible, suppose $n_{12} = n_{23} = n_{31} > n/2$. This implies $n_{21} = n_{32} = n_{13}$, $n_{12} + n_{13} = n_{21} + n_{23} = n_{31} + n_{32} = n$, $n_{14} = n_{24} = n_{34} > n/2$, and $n_{41} = n_{42} = n_{43} < n/2$. Thus $\{1, 2, 3\}$ is the set of the Borda winners. □

Theorem 33: $B_{TC}(n, m) = B_{Con}(n, m)$.

Proof. For $m = 3$ and 4, the result follows from Lemma 32. For $m \ge 5$, consider a profile P such that $TC(P) = \{1, ..., k\}$ with $k \ge 3$. By Lemma 31, $n_{ij}(P) \ge (n + \nu(n))/2$ for $i \le k < j$, which implies $n_{ji}(P) \le (n - \nu(n))/2$ for $i \le k < j$. Since $\sum_{i=1}^{k}\sum_{j=1}^{k} n_{ij}(P) = (k(k-1)n)/2$, we have $\max_{i \in TC(P)} \sum_{j=1}^{k} n_{ij}(P) \ge (k-1)n/2$. Let $B_1(P) = \max_{x \in TC(P)} B(x, P)$ and $B_2(P) = \max_{x \notin TC(P)} B(x, P)$. From what precedes, $B_1(P) \ge (k-1)n/2 + (m-k)(n + \nu(n))/2$ and $B_2(P) \le k(n - \nu(n))/2 + (m - k - 1)n$. If $B_1(P) \ge B_2(P)$, then $B^*(P) = B_1(P)$ and $B_1(P)/B^*(P) = 1 \ge B_{Con}(n, m)$. Otherwise let

$$A = \frac{(k-1)n + (m-k)(n+\nu(n))}{k(n-\nu(n)) + 2(m-k-1)n}$$

$$C = \frac{(k-1)\nu(n) + (k-1)n + (m-k)(n+\nu(n))}{n - \nu(n) + (k-1)(n-\nu(n)) + (k-1)(n+\nu(n)) + 2(m-k-1)n}$$

$$D = \frac{(k-1)(n+\nu(n)) + (m-k)(n+\nu(n))}{n - \nu(n) + 2(k-1)n + 2(m-k-1)n}$$

$$E = \frac{(m-1)(n+\nu(n))}{n - \nu(n) + 2(m-2)n}$$

and recall that B_{Con} is attained for profile P^1 and that $n_{1j}(P^1) = (n + \nu(n))/2$, $j = 2, ..., m$, $n_{21}(P^1) = (n - \nu(n))/2$, and $n_{2j}(P^1) = n$, $j = 3, ..., m$. Thus we have $B_1(P)/B^*(P) = B_1(P)/B_2(P) \geq A > C = D = E = B(1, P^1)/B(2, P^1) = B_{Con}(n, m)$. The first inequality follows from the definition of $B_1(P)$ and $B_2(P)$. As for $A > C$, let a be the numerator of C, b its denominator, $\alpha = (k-1)\nu(n)$ and $\beta = (k-1)(n+\nu(n))$. Then $A = (a-\alpha)/(b-\beta)$ and $A > C$ follows from the fact that $\alpha/\beta < 1/2 < a/b$. □

8. The Uncovered Set

Unfortunately, the top cycle is in general very large and may contain Pareto-inefficient alternatives. It is not difficult to find example where it is the whole set X. For these reasons, Fishburn (1977) and Miller (1980) have independently come up with a more decisive SCF called the uncovered set.

For a given profile P, let us consider the binary relation defined on X by $xC(P)y$ if $xM(P)y$ and if $yM(P)z$ implies $xM(P)z$, $\forall z \in X$. It is called the covering relation of X with respect to P. The **uncovered set** $UC(P)$ of X with respect to P is the set of maximal elements of the covering relation, i.e. $UC(P) = \{x \in X : \nexists\, y \in X : yC(P)x\}$. This set always exits since the covering relation is transitive. Throughout this section we shall confine ourselves to profiles that induce tournaments on X. We have the following lemma the content of which can also be found in Moulin (1988, Exercise 9.11) or Miller (1980).

Lemma 34: a) $x \in UC(P) \Leftrightarrow \forall y \in X, xMy$ or $\exists z \in X : xMz$ and zMy.

b) $UC(P) = \{x\} \Leftrightarrow x$ is the Condorcet winner.

c) $|UC(P)| \neq 2$.

d) $Cop(P) \subseteq UC(P) \subseteq TC(P)$.

Corollary 35: For $m = 3$, the Borda winners belong to $UC(P)$ whenever the latter is not a singleton.

Lemma 36: For $n = 4$, $UC(P)$ is a singleton for any profile P that induces a tournament on X.

Proof. Consider any $x \in UC(P)$. If $UC(P)$ is not a singleton, then, by Lemma 34 b), there must exist a $y \in X$ such that not xMy, i.e. $n_{xy} \leq 1$. Then, by Lemma 34 a), there must exist a $z \in X$ such that xMz and zMy, i.e. $n_{xz} \geq 3$ and $n_{zy} \geq 3$. By Lemma 1, this implies $n_{xy} \geq 2$, a contradiction. □

Corollary 37: $B_{UC}(4, m) = B_{Con}(4, m)$.

Conjecture 38: $B_{UC}(n, m) = B_{Con}(n, m)$.

For $m = 3$ or $n = 4$, there is nothing to prove. For $n \neq 4$ and $m \geq 4$, let

$$H(n, m) = \frac{(m-3)(n+\nu(n)) + 2n}{(2m-8)n + 4(n-\nu(n))}$$

and consider profile P^{51} defined by:

$(4, 5, ..., m, 3, 2, 1)$ for $(n - 3\nu(n))/2$ voters,

$(2, 4, 5, ..., m, 3, 1)$ for $\nu(n)$ voters,

$(3, 1, 4, 5, ..., m, 2)$ for $\nu(n)$ voters,

$(1, 2, 4, 5, ..., m, 3)$ for $(n - \nu(n))/2$ voters.

Under this profile, $\{1, 2, 3\}$ is the uncovered set, alternative 4 is the unique Borda winner when $(m-3)n > (m-1)\nu(n))$, and $B(x, P^{51})/B(4, P^{51}) = H(n, m)$, for $x = 1, 2$. However, one can write $H(n, m) = (a - \alpha)/(b - \beta)$ where a is the numerator of C in the previous section, b its denominator, $\alpha = 2\nu(n)$ and $\beta = n + 3\nu(n)$. Since $\alpha/\beta < 1/2 < a/b$, it follows that $a/b < H(n, m)$. Note that $a/b = B_{Con}(n, m)$ when $(m-3)n > (m-1)\nu(n)$. When this condition is not satisfied, alternatives 1 and 2 are the Borda winners giving us a potential value of 1 for $B_{UC}(n, m)$, which again is as least as large as $B_{Con}(n, m)$. Hence, profile P^{51} gives us an upper bound for B_{UC} that is as least as large as $B_{Con}(n, m)$.

We conjecture that it is not possible to improve upon profile P^{51} if the uncovered set must not be a singleton. First of all notice that, under P^{51}, the Copeland score of alternative 4 is just one below the one of alternatives 1 and 2. This gap must be maintained for alternative 4 to remain outside of the uncovered set. The numbers of votes for alternatives 1 and 2 are as small as possible given the constraint that must be maintained on the Copeland scores. The numbers of votes for alternative 4 are also as large as possible in view of the same constraint and Lemma 1.

One will note that $n_{1j} = n_{2j} = (n + \nu(n))/2$ and $n_{3j} = \nu(n)$, $j \geq 4$. One might think that interchanging some of these values could reduce the numerator in $H(n, m)$. Again this is not possible because of the same constraint or Lemma 1.

A profile yielding an uncovered set larger than 3 would not do any good either. It would reduce the Borda score of the Borda winner while keeping the largest Borda score in the uncovered set almost unchanged.

Remark 39: From Lemma 34 d), $B_{\text{Cop}}(n, m)$ constitutes a lower bound for $B_{UC}(n, m)$. A better bound could be obtained if we had $Ke(P) \subset UC(P)$, as with the Top cycle, and this would reinforce the preceding conjecture and make it easier to prove it. Unfortunately, the following example shows that one may have $Ke(P) \cap UC(P) = \varnothing$ and $Ke(P) \cap Cop(P) = \varnothing$. Let $m = 4$, $n = 9$ and consider the profile defined by:

(1, 2, 3, 4) for 4 voters,

(4, 2, 3, 1) for 3 voter,

(3, 4, 1, 2) for 2 voters.

It can be checked that (1, 2, 3, 4) is the unique Kemeny order so that $Ke(P) = \{1\}$, that $UC(P) = \{2, 3, 4\}$, and that $Cop(P) = \{3, 4\}$.

For the reason explained in the preceding remark, it would be interesting to have a stronger covering relation C^* leading to an uncovered set that would satisfy $Ke(P) \subseteq UC^*(P)$. Since the Kemeny and the Condorcet rules are close in terms of our measure, the chance that $B_{UC^*}(n, m) = B_{Con}(n, m)$ should be better than in the case of UC or, at least, it should be easier to prove it. Such a relation is introduced and analyzed in De Donder, Le Breton and Truchon (1996). Actually this relation is so strong that we have $Bor(P) \subseteq UC^*(P)$, $\forall\ P$, and, as a corollary, $B_{UC^*}(n, m) = 1$. This means that UC^* is not a Condorcet type function. $UC^*(P)$ contains the Condorcet winner when it exists but it may contain other alternatives as well, namely the Borda winners, any of which need not be the Condorcet winner.

The relation C^* is defined by $x\,C^*(P)\,y$ if and only if $x\,M(P)\,y$ and $n_{xz}(P) \geq n_{yz}(P)\ \ \forall\ z \in X\backslash\{x, y\}$. Let $UC^*(P)$ be the set of maximal elements of the covering relation C^*. It always exists since C^* is transitive. Let $Par(P)$ be the set of Pareto efficient alternatives. The following lemma is proven in De Donder et al. (1996).

Lemma 40: a) $UC(P) \subseteq UC^*(P) \subseteq Par(P)$;

b) Supose x is the Condorcet winner. Then $x \in UC^*(P)$;

c) $Ke(P) \subseteq UC^*(P)$;

d) $Bor(P) \subseteq UC^*(P)$.

Corollary 41: $B_{UC^*}(n, m) = 1$.

9. Conclusion

The uncovered set UC^* analyzed at the end of the last section has a perfect Borda score of 1. This is because the Borda winners always belong to this set. Unfortunately, this set, which is very interesting in other respects, may contain alternatives other than the Condorcet winner when the latter exists.

For all other SCFs analyzed in this paper, the Borda measure decreases monotonically with respect to the number m of alternatives and the number n of voters when this last number is odd or even. However, monotonicity does not hold on the entire set of integers. Thus all rules do best for m and n small, even achieving the maximum value of 1 for very small numbers of alternatives and voters. This is because there is a Borda winner among the alternatives that win according to the rule under scrutiny.

The rules that fare better are the Condorcet SCF, the top cycle, and possibly the uncovered set. They all have the same Borda measures. In the case of the uncovered set, this remains a conjecture. The Borda measure of these rules is 0.5 in the limit with respect to both m and n. The Kemeny SCF fares almost as well as the Condorcet SCF and the top cycle. This is not surprising since the Kemeny rule is much in the spirit of the Condorcet one. What is more surprising is the fact that the top cycle has the same measure as the Condorcet one instead of the Kemeny one. Allowing for profiles that yield cycles does not worsen the Borda measure. One may see the explanation in the fact that the top cycle contains not only the Kemeny winners but possibly many other alternatives. While Kemeny seeks to break the cycles that may obtain under the majority relation by picking an order over all alternatives, with the top cycle, one is left with the largest set of alternatives that defeat all other alternatives without being defeated by the latter. This is not a very decisive rule. Actually, the top cycle may leave us with the whole set of alternatives to choose from. The same remarks apply to the uncovered set.

The Copeland and the Kramer-Simpson SCF are two different ways of resolving the presence of cycles in the majority relation. For small values of m and n, for example for $m = n = 5$, they seem to do better than the Kemeny SCF and almost as well as the Condorcet SCF but recall that, except in the limit, we have only been able to show that the numbers given for these two SCFs are upper bound for their Borda measure. The performance of these two SCFs deteriorates rapidly as the values of these parameters increase. The Borda measure of the Kramer SCF goes to 0 as m tends to infinity, whatever the value of n. The limit of the Borda measure with respect to both m and n is 1/3 for the Copeland SCF if we restrict ourselves to profiles that yield a tournament and 0 otherwise. It would thus appear that the Kemeny SCF is the one that does the best in terms of resolving the presence of cycles in the majority relation and in terms of the Borda measure that we propose in this paper.

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Table 1: Condorcet

n\m	3	4	5	6	7	8	9	10	11	12	13	14	15	∞
3	1.00	0.86	0.80	0.77	0.75	0.74	0.73	0.72	0.71	0.71	0.71	0.70	0.70	0.67
4	1.00	1.00	0.92	0.88	0.86	0.84	0.83	0.82	0.81	0.80	0.80	0.80	0.79	0.75
5	0.86	0.75	0.71	0.68	0.67	0.66	0.65	0.64	0.64	0.63	0.63	0.63	0.63	0.60
6	1.00	0.86	0.80	0.77	0.75	0.74	0.73	0.72	0.71	0.71	0.71	0.70	0.70	0.67
7	0.80	0.71	0.67	0.65	0.63	0.62	0.62	0.61	0.61	0.60	0.60	0.60	0.60	0.57
8	0.91	0.79	0.74	0.71	0.70	0.69	0.68	0.67	0.67	0.66	0.66	0.66	0.65	0.63
9	0.77	0.68	0.65	0.63	0.61	0.60	0.60	0.59	0.59	0.59	0.58	0.58	0.58	0.56
10	0.86	0.75	0.71	0.68	0.67	0.66	0.65	0.64	0.64	0.63	0.63	0.63	0.63	0.60
11	0.75	0.67	0.63	0.61	0.60	0.59	0.59	0.58	0.58	0.57	0.57	0.57	0.57	0.55
12	0.82	0.72	0.68	0.66	0.65	0.64	0.63	0.62	0.62	0.62	0.61	0.61	0.61	0.58
13	0.74	0.66	0.62	0.60	0.59	0.58	0.58	0.57	0.57	0.57	0.56	0.56	0.56	0.54
14	0.80	0.71	0.67	0.65	0.63	0.62	0.62	0.61	0.61	0.60	0.60	0.60	0.60	0.57
15	0.73	0.65	0.62	0.60	0.59	0.58	0.57	0.57	0.56	0.56	0.56	0.56	0.55	0.53
16	0.78	0.69	0.65	0.63	0.62	0.61	0.61	0.60	0.60	0.59	0.59	0.59	0.59	0.56
17	0.72	0.64	0.61	0.59	0.58	0.57	0.57	0.56	0.56	0.56	0.55	0.55	0.55	0.53
18	0.77	0.68	0.65	0.63	0.61	0.60	0.60	0.59	0.59	0.59	0.58	0.58	0.58	0.56
19	0.71	0.64	0.61	0.59	0.58	0.57	0.56	0.56	0.56	0.55	0.55	0.55	0.55	0.53
20	0.76	0.67	0.64	0.62	0.61	0.60	0.59	0.59	0.58	0.58	0.58	0.57	0.57	0.55
21	0.71	0.63	0.60	0.59	0.57	0.57	0.56	0.56	0.55	0.55	0.55	0.55	0.54	0.52
22	0.75	0.67	0.63	0.61	0.60	0.59	0.59	0.58	0.58	0.57	0.57	0.57	0.57	0.55
23	0.71	0.63	0.60	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.55	0.54	0.54	0.52
24	0.74	0.66	0.63	0.61	0.60	0.59	0.58	0.58	0.57	0.57	0.57	0.57	0.56	0.54
25	0.70	0.63	0.60	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.52
26	0.74	0.66	0.62	0.60	0.59	0.58	0.58	0.57	0.57	0.57	0.56	0.56	0.56	0.54
27	0.70	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.54	0.52
28	0.73	0.65	0.62	0.60	0.59	0.58	0.57	0.57	0.57	0.56	0.56	0.56	0.56	0.54
29	0.70	0.63	0.59	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.54	0.52
30	0.73	0.65	0.62	0.60	0.59	0.58	0.57	0.57	0.56	0.56	0.56	0.56	0.55	0.53
31	0.70	0.62	0.59	0.58	0.56	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.54	0.52
32	0.72	0.65	0.61	0.59	0.58	0.57	0.57	0.56	0.56	0.56	0.56	0.55	0.55	0.53
33	0.69	0.62	0.59	0.57	0.56	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.52
34	0.72	0.64	0.61	0.59	0.58	0.57	0.57	0.56	0.56	0.56	0.55	0.55	0.55	0.53
35	0.69	0.62	0.59	0.57	0.56	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.51
36	0.72	0.64	0.61	0.59	0.58	0.57	0.57	0.56	0.56	0.55	0.55	0.55	0.55	0.53
37	0.69	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.53	0.51
38	0.71	0.64	0.61	0.59	0.58	0.57	0.56	0.56	0.56	0.55	0.55	0.55	0.55	0.53
39	0.69	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.53	0.51
40	0.71	0.64	0.60	0.59	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.55	0.55	0.53
∞	0.67	0.60	0.57	0.56	0.55	0.54	0.53	0.53	0.53	0.52	0.52	0.52	0.52	0.50

Table 2: Kemeny

n\m	3	4	5	6	7	8	9	10	11	12	13	14	15	∞
3	1.00	0.83	0.78	0.73	0.71	0.69	0.68	0.67	0.67	0.65	0.66	0.65	0.65	0.60
4	1.00	1.00	**0.82**	**0.73**	**0.68**	**0.65**	**0.63**	**0.61**	**0.60**	**0.59**	**0.58**	**0.57**	**0.57**	**0.50**
5	0.86	0.73	0.69	0.65	0.64	0.62	0.62	0.61	0.60	0.60	0.60	0.59	0.59	0.56
6	1.00	0.83	0.76	**0.70**	**0.66**	**0.63**	**0.61**	**0.60**	**0.58**	**0.58**	**0.57**	**0.56**	**0.56**	**0.50**
7	0.80	0.69	0.65	0.62	0.61	0.60	0.59	0.58	0.58	**0.57**	**0.57**	**0.55**	**0.56**	**0.50**
8	0.90	0.76	0.71	0.68	**0.64**	**0.62**	**0.60**	**0.59**	**0.58**	**0.57**	**0.56**	**0.56**	**0.55**	**0.50**
9	0.77	0.67	0.63	0.61	0.60	0.58	0.58	0.57	0.57	0.56	0.56	**0.55**	**0.55**	**0.50**
10	0.85	0.73	0.68	0.65	0.63	**0.61**	**0.59**	**0.58**	**0.57**	**0.57**	**0.56**	**0.55**	**0.55**	**0.50**
11	0.75	0.65	0.62	0.60	0.59	0.57	0.57	0.56	0.56	0.55	0.55	0.55	0.55	**0.50**
12	0.81	0.70	0.66	0.63	0.62	0.61	**0.59**	**0.58**	**0.57**	**0.56**	**0.56**	**0.55**	**0.55**	**0.50**
13	0.74	0.65	0.61	0.59	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.54	0.54	**0.50**
14	0.79	0.69	0.64	0.62	0.61	0.60	0.59	**0.58**	**0.57**	**0.56**	**0.56**	**0.55**	**0.55**	**0.50**
15	0.73	0.64	0.61	0.58	0.58	0.56	0.56	0.55	0.55	0.55	0.54	0.54	0.54	**0.50**
16	0.77	0.68	0.63	0.61	0.60	0.59	0.58	0.57	**0.57**	**0.56**	**0.55**	**0.55**	**0.55**	**0.50**
17	0.72	0.63	0.60	0.58	0.57	0.56	0.56	0.55	0.55	0.54	0.54	0.54	0.54	**0.50**
18	0.76	0.67	0.63	0.61	0.59	0.58	0.57	0.57	0.57	**0.56**	**0.55**	**0.55**	**0.55**	**0.50**
19	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.54	**0.50**
20	0.75	0.66	0.62	0.60	0.59	0.58	0.57	0.57	0.56	0.56	**0.55**	**0.55**	**0.54**	**0.50**
21	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	**0.50**
22	0.74	0.65	0.62	0.60	0.58	0.57	0.57	0.56	0.56	0.55	0.55	**0.55**	**0.54**	**0.50**
23	0.71	0.63	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.53	**0.50**
24	0.74	0.65	0.61	0.59	0.58	0.57	0.56	0.56	0.56	0.55	0.55	0.55	**0.54**	**0.50**
25	0.70	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	0.53	**0.50**
26	0.73	0.65	0.61	0.59	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.54	0.54	**0.50**
27	0.70	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.53	0.53	0.53	**0.50**
28	0.73	0.64	0.61	0.59	0.57	0.57	0.56	0.55	0.55	0.55	0.55	0.54	0.54	**0.50**
29	0.70	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.53	0.53	0.53	**0.50**
30	0.72	0.64	0.60	0.58	0.57	0.56	0.56	0.55	0.55	0.55	0.54	0.54	0.54	**0.50**
31	0.70	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.53	0.53	0.53	0.53	**0.50**
32	0.72	0.64	0.60	0.58	0.57	0.56	0.56	0.55	0.55	0.54	0.54	0.54	0.54	**0.50**
33	0.69	0.62	0.59	0.57	0.56	0.55	0.55	0.54	0.54	0.53	0.53	0.53	0.53	**0.50**
34	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.54	**0.50**
35	0.69	0.62	0.59	0.57	0.56	0.55	0.54	0.54	0.54	0.53	0.53	0.53	0.53	**0.50**
36	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54	0.54	0.54	**0.50**
37	0.69	0.62	0.59	0.57	0.56	0.55	0.54	0.54	0.54	0.53	0.53	0.53	0.53	**0.50**
38	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.54	**0.50**
39	0.69	0.61	0.59	0.57	0.56	0.55	0.54	0.54	0.54	0.53	0.53	0.53	0.53	**0.50**
40	0.71	0.63	0.60	0.58	0.57	0.56	0.55	0.55	0.54	0.54	0.54	0.54	0.53	**0.50**
∞	0.67	0.60	0.57	0.56	0.55	0.54	0.53	0.53	0.53	0.52	0.52	0.52	0.52	**0.50**

Bold numbers have been obtained from F_{12} or F_{22}.

Table 3: Copeland

n\m	3	4	5	6	7	8	9	10	11	12	13	14	15	∞
3	**1.00**	**0.86**	**0.80**	**0.77**	**0.75**	**0.74**	**0.73**	**0.72**	**0.71**	**0.71**	**0.71**	**0.70**	**0.70**	**0.67**
4	**1.00**	**1.00**	0.89	0.82	0.77	0.73	0.71	0.68	0.67	0.65	0.64	0.63	0.62	0.50
5	**0.86**	**0.75**	**0.71**	**0.68**	**0.67**	**0.66**	**0.65**	**0.64**	**0.64**	**0.63**	**0.63**	**0.63**	**0.63**	**0.60**
6	**1.00**	0.82	0.71	0.65	0.60	0.57	0.54	0.52	0.50	0.49	0.47	0.46	0.45	0.33
7	**0.80**	**0.71**	**0.67**	**0.65**	**0.63**	**0.62**	**0.62**	**0.61**	**0.61**	**0.60**	**0.60**	**0.60**	0.59	0.56
8	**0.91**	0.73	0.63	0.57	0.52	0.48	0.46	0.44	0.42	0.40	0.39	0.38	0.37	0.25
9	**0.77**	**0.68**	**0.65**	**0.63**	0.61	0.60	0.57	0.58	0.56	0.57	0.55	0.56	0.54	0.50
10	**0.86**	0.68	0.58	0.52	0.47	0.44	0.41	0.39	0.37	0.36	0.34	0.33	0.32	0.20
11	**0.75**	**0.67**	**0.63**	**0.61**	0.58	0.57	0.54	0.55	0.53	0.53	0.52	0.52	0.51	0.47
12	**0.82**	0.65	0.55	0.49	0.44	0.40	0.38	0.36	0.34	0.32	0.31	0.30	0.29	0.17
13	**0.74**	**0.66**	0.61	**0.60**	0.56	0.55	0.52	0.53	0.51	0.51	0.50	0.50	0.49	0.44
14	**0.80**	0.63	0.53	0.46	0.42	0.38	0.35	0.33	0.32	0.30	0.29	0.28	0.27	0.14
15	**0.73**	**0.65**	0.60	0.59	0.54	0.54	0.51	0.52	0.49	0.50	0.48	0.49	0.47	0.43
16	**0.78**	0.61	0.51	0.45	0.40	0.37	0.34	0.32	0.30	0.28	0.27	0.26	0.25	0.13
17	**0.72**	**0.64**	0.58	0.58	0.53	0.53	0.49	0.50	0.48	0.49	0.47	0.48	0.46	0.42
18	**0.77**	0.60	0.50	0.43	0.39	0.35	0.33	0.30	0.29	0.27	0.26	0.25	0.24	0.11
19	**0.71**	**0.64**	0.57	0.57	0.52	0.52	0.49	0.50	0.47	0.48	0.46	0.47	0.45	0.41
20	**0.76**	0.59	0.49	0.42	0.38	0.34	0.31	0.29	0.28	0.26	0.25	0.24	0.23	0.10
21	**0.71**	**0.63**	0.57	0.56	0.51	0.51	0.48	0.49	0.46	0.47	0.45	0.46	0.45	0.40
22	**0.75**	0.58	0.48	0.42	0.37	0.33	0.31	0.28	0.27	0.25	0.24	0.23	0.22	0.09
23	**0.71**	**0.63**	0.56	0.56	0.51	0.51	0.47	0.48	0.46	0.47	0.45	0.46	0.44	0.39
24	**0.74**	0.57	0.47	0.41	0.36	0.33	0.30	0.28	0.26	0.24	0.23	0.22	0.21	0.08
25	**0.70**	**0.63**	0.56	0.55	0.50	0.50	0.47	0.48	0.45	0.46	0.44	0.45	0.43	0.39
26	**0.74**	0.57	0.47	0.40	0.36	0.32	0.29	0.27	0.25	0.24	0.23	0.22	0.21	0.08
27	**0.70**	**0.63**	0.55	0.55	0.50	0.50	0.46	0.48	0.45	0.46	0.44	0.45	0.43	0.38
28	**0.73**	0.56	0.46	0.40	0.35	0.32	0.29	0.27	0.25	0.23	0.22	0.21	0.20	0.07
29	**0.70**	**0.63**	0.55	0.54	0.49	0.50	0.46	0.47	0.45	0.46	0.43	0.44	0.43	0.38
30	**0.73**	0.56	0.46	0.39	0.35	0.31	0.28	0.26	0.24	0.23	0.22	0.21	0.20	0.07
31	**0.70**	**0.62**	0.54	0.54	0.49	0.49	0.46	0.47	0.44	0.45	0.43	0.44	0.42	0.38
32	**0.72**	0.56	0.46	0.39	0.34	0.31	0.28	0.26	0.24	0.23	0.21	0.20	0.19	0.06
33	**0.69**	**0.62**	0.54	0.54	0.49	0.49	0.46	0.47	0.44	0.45	0.43	0.44	0.42	0.38
34	**0.72**	0.55	0.45	0.39	0.34	0.30	0.28	0.25	0.24	0.22	0.21	0.20	0.19	0.06
35	**0.69**	**0.62**	0.54	0.54	0.48	0.49	0.45	0.46	0.44	0.45	0.43	0.44	0.42	0.37
36	**0.72**	0.55	0.45	0.38	0.34	0.30	0.27	0.25	0.23	0.22	0.21	0.20	0.19	0.06
37	**0.69**	**0.62**	0.54	0.53	0.48	0.49	0.45	0.46	0.44	0.45	0.42	0.43	0.42	0.37
38	**0.71**	0.55	0.45	0.38	0.33	0.30	0.27	0.25	0.23	0.22	0.20	0.19	0.18	0.05
39	**0.69**	**0.62**	0.54	0.53	0.48	0.49	0.45	0.46	0.43	0.44	0.42	0.43	0.42	0.37
40	**0.71**	0.54	0.44	0.38	0.33	0.30	0.27	0.25	0.23	0.21	0.20	0.19	0.18	0.05
∞	**0.67**	**0.60**	0.50	0.50	0.44	0.45	0.42	0.43	0.40	0.41	0.39	0.40	0.38	0.33
∞	**0.67**	0.50	0.40	0.33	0.29	0.25	0.22	0.20	0.18	0.17	0.15	0.14	0.13	0.00

Bold numbers are the same as the Condorcet numbers.

Table 4: Kramer-Simson

n\m	3	4	5	6	7	8	9	10	11	12	13	14	15	∞
3	1.00	0.86	0.80	0.77	0.75	0.67	0.57	0.50	0.44	0.40	0.36	0.33	0.31	0.00
4	1.00	1.00	0.92	0.88	0.86	0.76	0.62	0.53	0.47	0.42	0.38	0.35	0.32	0.00
5	0.86	0.75	0.71	0.68	0.67	0.66	0.65	0.57	0.50	0.44	0.40	0.36	0.33	0.00
6	1.00	0.86	0.80	0.77	$\underline{0.63}$	0.74	0.73	0.63	0.53	0.47	0.42	0.38	0.35	0.00
7	0.80	0.71	0.67	0.65	0.63	$\underline{0.54}$	0.62	0.61	0.58	0.50	0.44	0.40	0.36	0.00
8	0.91	0.79	0.74	0.71	0.70	0.69	$\underline{0.47}$	0.67	0.64	0.54	0.47	0.42	0.38	0.00
9	0.77	0.68	0.65	0.63	0.61	0.60	0.60	$\underline{0.42}$	0.59	0.59	0.51	0.44	0.40	0.00
10	0.86	0.75	0.71	0.68	0.67	0.66	**0.63**	**0.59**	$\underline{0.38}$	0.63	0.55	0.48	0.42	0.00
11	0.75	0.67	0.63	0.61	0.60	0.59	0.59	**0.56**	**0.53**	$\underline{0.34}$	0.57	0.51	0.45	0.00
12	0.82	0.72	0.68	0.66	0.63	0.64	**0.61**	**0.55**	**0.51**	**0.49**	$\underline{0.32}$	0.55	0.48	0.00
13	0.74	0.66	0.62	0.60	0.59	0.58	0.58	**0.54**	**0.49**	**0.46**	**0.45**	$\underline{0.29}$	0.51	0.00
14	0.80	0.71	0.67	0.65	0.57	0.54	0.62	**0.55**	**0.49**	**0.45**	**0.43**	**0.42**	$\underline{0.27}$	0.00
15	0.73	0.65	0.62	0.60	0.57	0.51	0.57	**0.56**	**0.49**	**0.45**	**0.42**	**0.40**	**0.39**	0.00
16	0.78	0.69	0.65	0.63	0.57	0.49	0.47	**0.58**	**0.49**	**0.44**	**0.41**	**0.39**	**0.37**	0.00
17	0.72	0.64	0.61	0.59	0.58	0.49	0.45	0.56	**0.50**	**0.45**	**0.41**	**0.38**	**0.36**	0.00
18	0.77	0.68	0.65	0.63	0.61	0.49	0.43	0.42	**0.52**	**0.45**	**0.41**	**0.38**	**0.36**	0.00
19	0.71	0.64	0.61	0.59	0.58	0.49	0.43	0.40	**0.55**	**0.46**	**0.41**	**0.38**	**0.35**	0.00
20	0.76	0.67	0.64	0.62	0.59	0.51	0.43	0.39	0.38	**0.48**	**0.42**	**0.38**	**0.35**	0.00
21	0.71	0.63	0.60	0.59	0.57	0.54	0.43	0.38	0.36	**0.50**	**0.43**	**0.38**	**0.35**	0.00
22	0.75	0.67	0.63	0.61	0.59	0.52	0.44	0.38	0.35	0.34	**0.44**	**0.39**	**0.35**	0.00
23	0.71	0.63	0.60	0.58	0.57	0.51	0.45	0.38	0.35	0.33	**0.46**	**0.40**	**0.36**	0.00
24	0.74	0.66	0.63	0.61	0.60	0.50	0.47	0.38	0.34	0.32	0.32	**0.41**	**0.37**	0.00
25	0.70	0.63	0.60	0.58	0.57	0.50	0.45	0.39	0.34	0.32	0.30	**0.43**	**0.37**	0.00
26	0.74	0.66	0.62	0.60	0.59	0.50	0.44	0.40	0.34	0.31	0.30	0.29	**0.38**	0.00
27	0.70	0.63	0.60	0.58	0.57	0.52	0.44	0.42	0.35	0.31	0.29	0.28	**0.40**	0.00
28	0.73	0.65	0.62	0.60	0.59	0.54	0.44	0.41	0.35	0.31	0.29	0.28	0.27	0.00
29	0.70	0.63	0.59	0.58	0.57	0.52	0.44	0.40	0.36	0.32	0.29	0.27	0.26	0.00
30	0.73	0.65	0.62	0.60	0.59	0.51	0.44	0.39	0.38	0.32	0.29	0.27	0.26	0.00
31	0.70	0.62	0.59	0.58	0.56	0.51	0.46	0.39	0.37	0.32	0.29	0.27	0.25	0.00
32	0.72	0.65	0.61	0.59	0.58	0.51	0.47	0.39	0.36	0.33	0.29	0.27	0.25	0.00
33	0.69	0.62	0.59	0.57	0.56	0.51	0.46	0.39	0.35	0.34	0.29	0.27	0.25	0.00
34	0.72	0.64	0.61	0.59	0.58	0.52	0.45	0.40	0.35	0.33	0.30	0.27	0.25	0.00
35	0.69	0.62	0.59	0.57	0.56	0.54	0.45	0.41	0.35	0.33	0.31	0.27	0.25	0.00
36	0.72	0.64	0.61	0.59	0.58	0.53	0.44	0.42	0.35	0.32	0.32	0.27	0.25	0.00
37	0.69	0.62	0.59	0.57	0.56	0.52	0.45	0.41	0.36	0.32	0.31	0.28	0.25	0.00
38	0.71	0.64	0.61	0.59	0.58	0.51	0.45	0.40	0.36	0.32	0.30	0.28	0.25	0.00
39	0.69	0.62	0.59	0.57	0.56	0.51	0.46	0.40	0.37	0.32	0.30	0.29	0.26	0.00
40	0.71	0.64	0.60	0.59	0.58	0.52	0.47	0.40	0.38	0.32	0.30	0.29	0.26	0.00
∞	0.67	0.60	0.57	0.56	0.55	0.54	0.47	0.42	0.38	0.34	0.32	0.29	0.27	0.00

The numbers in the upper-right corner and those in the limit column come from Theorem 26, underlined numbers and those in the lower-right corner from Theorem 24, and bold numbers from Theorem 25. All others are the same as the Condorcet numbers.

Table 5: The majority matrix of profile P^{31} for $n = 9$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	5	5	5	1	5	1	5	1	5	1	5	1	5	1
2	4	0	8	8	4	8	4	9	4	8	4	8	4	8	4
3	4	1	0	5	5	4	5	4	5	4	5	4	5	4	5
4	4	1	4	0	5	5	5	4	5	4	5	4	5	4	5
5	8	5	4	4	0	4	4	5	5	4	5	4	5	4	5
6	4	1	5	4	5	0	5	4	5	4	5	4	5	4	5
7	8	5	4	4	5	4	0	5	4	4	5	4	5	4	5
8	4	0	5	5	4	5	4	0	4	5	4	5	4	5	4
9	8	5	4	4	4	4	5	5	0	5	4	4	5	4	5
10	4	1	5	5	5	5	5	4	4	0	4	5	4	5	4
11	8	5	4	4	4	4	4	5	5	5	0	5	4	4	5
12	4	1	5	5	5	5	5	4	5	4	4	0	4	5	4
13	8	5	4	4	4	4	4	5	4	5	5	5	0	5	4
14	4	1	5	5	5	5	5	4	5	4	5	4	4	0	4
15	8	5	4	4	4	4	4	5	4	5	4	5	5	5	0