FIGURE SKATING
AND
THE THEORY OF SOCIAL CHOICE

by

Michel Truchon

CRÉFA and Département d'économique
Pavillon J.A. de Sève
Université Laval
Québec, Canada, G1K 7P4
tél.: 418-656-5102
fax: 418-656-7798
email: mtru@ecn.ulaval.ca

cahier 9814

cahier 98-16
du Centre de Recherche en Économie et Finance Appliquées
CRÉFA

October 1998

I am most grateful to John Galbraith and Bryan Campbell for allowing me to work with a set of data that they had collected on 24 olympic competitions. I would also like to thank Monique Cashman, Philippe De Donder, Stephen Gordon, Jacques Robert, and Mohamed Sidi Drissi-Bakhkhat for their helpful comments on an earlier version of this paper. However, I remain responsible for any errors or weaknesses. Finally, I acknowledge the financial support of the Social Sciences and Humanities Research Council of Canada.

Abstract

The rule used by the United States Figure Skating Association and the International Skating Union, hereafter the ISU Rule, to aggregate individual rankings of the skaters by the judges into a final ranking, is an interesting example of a social welfare function. This rule is examined thoroughly in this paper from the perspective of the modern theory of social choice.

The ISU Rule is based on four different criteria, the first being median ranks of the skaters. Although the median rank criterion is a majority principle, it is completely at odd with another majority principle introduced in this paper and called the Extended Condorcet Criterion. It may be translated as follows: If a competitor is ranked consistently ahead of another competitor by an absolute majority of judges, he should be ahead in the final ranking. Consistency here refers to the absence of a cycle in the majority relation involving these two skaters. There are actually many cycles in the data of four Olympic Games that were examined. The Kemeny rule may be used to break these cycles. This is not only consistent with the Extended Condorcet Criterion but the latter also proves useful in finding Kemeny orders over large sets of alternatives, by allowing decomposition of these orders.

The ISU, the Kemeny, the Borda rankings and the ranking according to the raw marks are then compared on 24 olympic competitions. The four rankings disagree in many instances. Finally it is shown that the ISU Rule may be very sensitive to small errors on the part of the judges and that it does not escape the numerous theorems on manipulation. Some considerations are also offered as to whether the ISU Rule is more or less prone to manipulation than others.

Résumé

La règle utilisée par la United States Figure Skating Association et l’International Skating Union, ci-après la règle de l’ISU, pour agréger les classements des patineurs par chacun des juges en un classement final, est un exemple intéressant de fonction de bien-être social. Cette règle est examinée en détail dans cet article du point de vue de la théorie moderne des choix sociaux.

Cette règle repose sur quatre critères, le premier étant le rang médian des patineurs. Bien que ce critère soit en fait un principe majoritaire, il va à l’encontre d’un autre principe majoritaire introduit ici et appelé le Critère de Condorcet généralisé. Il peut être traduit ainsi: Si un compétiteur est classé avant un autre de manière cohérente par une majorité de juges, il devrait l’être dans le classement final. La cohérence réfère à l’absence de cycle dans la relation majoritaire impliquant ces deux compétiteurs. De fait, plusieurs cycles ont été rencontrés dans les données de quatre Jeux olympiques qui ont été examinées. La règle de Kemeny peut être utilisée pour briser ces cycles. Non seulement cette règle est-elle cohérente avec le Critère de Condorcet généralisé mais ce dernier s’avère utile dans la recherche d’ordres de Kemeny sur un grand nombre d’alternatives, en permettant la décomposition de ces ordres.

Les classements des patineurs selon les règles de l’ISU, de Kemeny, de Borda et selon les notes brutes sont ensuite comparés pour 24 compétitions olympiques. Les quatre classements sont souvent différents. Finalement, il est démontré que la règle de l’ISU peut être très sensible à de petites erreurs de la part des juges et qu’elle n’échappe pas aux nombreux théorèmes d’impossibilité sur la manipulation. Quelques remarques sont aussi offertes sur la plus ou moins grande susceptibilité de cette règle à la manipulation par rapport à d’autres règles.

1. Introduction

An extensive literature has been devoted to the design of social welfare functions, i.e. rules for aggregating individual preferences or rankings on a set of alternatives into a collective preference or a final ranking. Yet, there are not very many instances in real life where the preoccupation is to arrive at a collective preference, as opposed to merely choosing an alternative or a subset of alternatives. A notable exception is professional sport, where a wide variety of methods are used to aggregate individual rankings into a final ranking. These methods are especially prominent in judged sports such as diving, synchronized swimming, gymnastics, and figure skating, but we also find examples of their use in some professional sports where ranking could be made from objective data. See for instances Jech (1983), Benoit (1992), and Levin and Nalebuff (1995).

Among judged sports, figure skating has probably become the most popular. Almost everybody knows that skaters are ranked from the scores that they receive from a panel of judges. However, very few people know exactly how the aggregation procedure works. Given the complexity of this procedure, this is quite understandable. Bassett and Persky (1994), hereafter BP, analyze the rule used by the United States Figure Skating Association (1992) and the International Skating Union (1994), hereafter the ISU Rule, to come up with a final ranking of the skaters. They stress the fact that figure skating uses median ranks for determining placement. This is indeed what paragraphs 1, 2, and 6 of Rule 371 of the ISU, reproduced in the Appendix, amount to.

BP show that this system responds positively to increased marks by each judge and respects the view of a majority of judges when this majority agrees on a skater's rank. They also demonstrate that the Median Rank Principle is the only one to possess these two properties. Moreover, they claim that this principle provides strong safeguards against manipulation by a minority of judges. Finally, in a Monte Carlo study, they find that the ISU Rule outperforms the simple aggregation of raw marks in picking the true winner, when judges' marks are subject to errors and significantly skewed toward an upper limit.

BP's analysis is mostly confined to the Median Rank Principle. Yet, there are three other principles that are also used by the ISU Rule to break ties left by the Median Rank Principle. One of them is the mean rank, which is equivalent to the Borda (1784) principle or Borda Count. The latter has received much attention in the literature. The ISU also uses a weighted sum of the ranks to aggregate the rankings in the different programs of a competition into a final ranking. This is similar to using the Borda rule. This paper examines more thoroughly all aspects of the ISU Rule from the perspective of the modern theory of social choice. It also compares four different ranking rules on the data of 24 olympic competitions: men, women, and couple, short and free programs, for 1976, 1988, 1992 and 1994.

Since the ISU Rule combines many ranking principles, which are characterized by different properties, it is not surprising that, as a whole, it violates many properties that are often judged desirable for social welfare functions. This paper shows which properties the ISU Rule satisfies and which it violates. In particular, like most social welfare functions encountered in the literature, the ISU Rule does not satisfy the monotonicity condition usually found in the theory of social choice. This is why BP use a weaker monotonicity condition, namely the positive response of the rule to increased marks by each judge, to characterize the Median Rank Principle.

The median rank of a skater being the best rank that he or she obtains from a majority of judges, clearly the Median Rank Principle translates a majority principle. As compelling as this majority principle may seem, it is completely at odds with another prominent majority principle advocated by Condorcet (1785), who was a colleague of Borda at the French Académie des sciences. This principle prescribes that if a competitor is ranked ahead of another competitor by an absolute majority of judges, he should be ahead in the final ranking. Unfortunately, this principle may fail to give a consistent ranking because of a cycle in the majority relation, a possibility that Condorcet was well aware of. For example, A may be judged better than B by a strict majority of judges, who may be judged better than C, who may be judged better than D, and D may be judged better than A. This probably explains why the Condorcet Criterion that we find nowadays in the literature simply says that if a competitor is ranked ahead of all other competitors by an absolute majority of judges, he should be first in the final ranking. Again such a competitor may fail to exist because of a cycle. When it exists, it is called the Condorcet winner.

Despite the possibility of cycles, there is still something to be drawn for other ranks from Condorcet's prescription. A more general criterion, called the Extended Condorcet Criterion, is proposed in this paper. Loosely speaking, it says that if a competitor is ranked consistently ahead of another competitor by an absolute majority of judges, he or she should be ahead in the final ranking. Consistency here refers to the absence of cycle involving these two skaters.

What is the probability of encountering cycles? We find many computations and estimations of the fractions of profiles of votes leading to cycles. Computer simulations by Campbell and Tullock (1965) give an estimate of .305 for 7 voters and 7 alternatives. This fraction increases with the number of voters. It is .342 for 9 voters. It also increases with the number of alternatives, going to .464 with 9 voters and 11 alternatives. However, one must be cautious not to interpret these fractions as probabilities applicable to real life situations since the different profiles might not be equiprobable. When the number of the voters becomes large, there can be strong correlations in their rankings, thus reducing the probability of cycles. For more on this topic, see Fishburn (1973) and Kelly (1986).

In figure skating, if the judges abide by the ISU Rules and try to be as objective as possible, we should expect a strong correlation between their rankings of the skaters and thus no or few cycles. Thus it was somewhat surprising to find 15 cycles in the data of the 24 olympic competitions. Note that many of these cycles occurred for the weak majority relation instead of the strict one, i.e. they involved some ties between skaters. Ties could not be neglected despite the complication that they usually bring. Indeed, two skaters may tie even if the number of judges is odd because a judge may give the same rank to more than one skater. Ties may create cycles in the weak majority relation and increase the length of existing ones.

The cycles that have been found in the olympic games involved as many as nine skaters. These were often middle ranked skaters, as if the disagreements between the judges occur mainly for competitors who are not medal contenders. However, in one case these were famous skaters. In another case, one of the skaters in the cycle actually obtained the third place in the ISU ranking. These cycles never prevented the occurrence of a Condorcet winner, except in one instance where two skaters tied for this title.

In the presence of cycles, the Extended Condorcet Criterion gives only a partial ranking of the competitors. A partial ranking may be completed in different ways but there is one method to accomplish this task that is perfectly in line with Condorcet's quest. The best way to introduce this method is to turn to an interesting question raised by BP.

Are judging systems such as the one that prevails in figure skating intended to reconcile the conflicting views of the judges or are they intended to furnish a final ranking that is most likely to be the true ranking of the competitors, based on their relative merits? Obviously, the answer depends on whether the rankings of the judges represent their preferences or an evaluation of the relative merits of the competitors according to given criteria. The regulations of the ISU are very clear in this respect. The judges are supposed to give an objective evaluation of the relative merits of the competitors in terms of scores. The different scores attributed to the different elements of the competition are then aggregated and the ranking of each judge is determined from these aggregated scores.

Of course, these instructions do not preclude cheating by judges who may hope that their marks will result in their most preferred ranking. For instance, Campbell and Galbraith (1996) find strong evidence of the presence of a small national bias in the results of the 24 olympic competitions analyzed in this paper. Yet, if we take the rankings of the different judges as independent evaluations of the true ranking of the competitors according to the established rules, we are led to the question: which final ranking is most likely to be the true ranking of the competitors? This is precisely the question addressed by Condorcet (1785), whose objective was to justify the usual majority principle.

If the pairwise ranking of the competitors under the majority rule does not involve any cycle, then Condorcet showed that it yields a complete ranking that has maximum likelihood of being the true ranking, under the assumption that every judge chooses the best of two competitors with a probability larger than one half and that this judgment is independent between pairs and judges. This makes the Condorcet majority principle more compelling than the ISU principle.

Condorcet gave indications on how to break cycles that might occur. However, his prescription is not completely clear. Young (1988) shows that a correct application of Condorcet's maximum likelihood approach leads to a ranking that has the maximum pairwise support from the voters (judges). Such a ranking is often called a Kemeny ranking because it involves the minimum number of pairwise inversions with the individual rankings. Kemeny (1959) proposes this number as a distance between an order and a profile of individual rankings. Such a ranking is also a median ranking for those composing a profile. In this sense, it represents a best compromise between the possibly conflicting views of the judges. This paper examines the Kemeny-Young approach and advocates this method in the context of figure skating.

Not only does a Kemeny order satisfy the Extended Condorcet Criterion but the latter turns out to be of great help in constructing Kemeny orders. The precise method is described in Truchon (1998).

Would the use of a different rule by the ISU, such as the Kemeny or the Borda rule, give significantly different rankings of the skaters in real competitions? Levin and Nalebuff (1995), using data from 30 British Union elections, find that many different electoral systems would not have given different top choices. The systems differed in the ranking of the lower candidates. They suggest that, when voters' preferences are sufficiently similar, a variety of voting systems lead to similar choices, and these choices have desirable properties. The difficulties in aggregating preferences would arise when there is a lack of consensus. In this case, the choice of an electoral system can make the greatest difference.

In the case of skating, many disagreements have been found between the ISU, the Borda, the Kemeny rankings and the ranking according to raw marks for the 24 olympic competitions. The differences between the ISU and the Kemeny rankings include the conflicts between the Extended Condorcet Criterion and the Median Rank Principle. A summary of these comparisons appears in Table 4. The measure of the disagreement between two rankings is the number of pairs of competitors for which the relative ranks are inverted. For example, there are two inversions between the orders *cab* and *abc* namely one for the pair $\{a, c\}$ and another one for the pair $\{b, c\}$. If two competitors obtain the same rank in one ranking and different ranks in another ranking, this is counted as half a difference.

Many of the differences occurred for middle places but disagreements for these places may be important since participation in future competitions may depend on being ranked in the first ten places. In one case, the ISU Rule gave a fourth place while the Kemeny rule gave a third place. In two instances, the Kemeny rule gave a tie for the first rank instead of ranks 1 and 2. In one instance, it gave a tie for the second place instead of ranks 2 and 3. Finally, in four instances, it gave a tie for the third place instead of ranks 3 and 4. Thus the choice of a rule is not merely a theoretical question. It can have a real impact on the results.

The last issue treated in this paper is manipulation, or the misrepresentation of one's true ranking by a judge in order to change the final ranking for one that he or she prefers. For example, a judge may prefer rankings that favour a particular competitor. There is a famous impossibility theorem in this context, due to Gibbard (1973) and Satthertwaite (1975), which says that all social choice functions, i.e. functions that select a winner, are manipulable. More generally, a judge may prefer a ranking to another, not just because of the winner, but because of the whole ranking. Bossert and Storcken (1992) extend the Gibbard-Satthertwaite theorem to this context, i.e. to social welfare functions or ranking rules.

The ISU Rule does not escape these theorems. This paper shows, by means of examples, how the different principles on which rests the ISU Rule, including the Median Rank Principle, are prone to manipulation. It also shows that the ISU Rule may be very sensitive to small errors on the part of the judges. This is troublesome if we are after the best evaluation of a true ranking.

Some ranking rules may be more prone to manipulation than others. The question is then whether or not the ISU Rule does well in this respect. Some considerations on this matter are offered, drawing on recent work by Saari (1990). However, the present state of research does not permit a clear-cut answer on this subject.

The paper is organized as follows. Section 2 introduces the notation including the formal description of the ISU Rule. Section 3 presents BP's characterization of the Median Rank Principle, shows that the last steps of the ISU Rule consist in applying the Borda criterion and examines the whole ISU Rule from the perspective of properties that are often encountered in the theory of social choice. It shows which ones are satisfied by the ISU Rule and which ones are violated. Section 4 defines the Extended Condorcet Criterion and shows that this criterion as well as the ordinary Condorcet Criterion are at odds with the ISU Rule. Section 5 illustrates the possible conflicts between the Condorcet principle, the Borda criterion and the ISU Rule. The quest for a true ranking, which culminates in the Kemeny rule, is the object of Sections 6 and 7. The comparison of four different rules on olympic data is done in Section 8. Manipulability is taken up is Section 9. A summary of the paper and concluding remarks on the choice of a ranking rule are offered in Section 10.

2. Notation

Let X be the set of competitors, skaters or alternatives, with cardinality $|X| = m$, and N be the set of judges or voters, with $|N| = n$, an odd number. The terms competitor, skater and alternative will be used interchangeably, depending on the context. The first two are more appropriate to our context while the term alternative is more usual in the theory of social choice. This term will often be used when referring to this theory. The same kind of remark applies to the terms judge and voter.

From the scores given to the competitors by judge j, we obtain a *weak order* or *ranking* r^j of the competitors in X. The element r_s^j of this vector is the rank of skater s. A ranking with no tie for a rank is an *order* on X. An order can be represented alternatively as a sequence $s_1 s_2 ...$, where s_1 and s_2 are respectively the competitors with ranks 1 and 2, etc.

Let $\mathfrak{R}$ be the set of all possible rank vectors r. A *profile of rankings* is an $m \times n$ matrix $R = (r^1, ..., r^n) \in \mathfrak{R}^n$. A *ranking rule* is a mapping $FR : \mathfrak{R}^n \to \mathfrak{R}$. $FR(R)$ is the final ranking resulting from profile R. In the language of the theory of social choice, FR is a *social welfare function*.

Next, let us define:

$$N_{is}(R) = \{j \in N : r_s^j \leq i\}, \quad n_{is}(R) = |N_{is}(R)|, \quad \bar{n}_{is}(R) = |\{j \in N : r_s^j = i\}|,$$

$$\rho_s(R) = \min i \in \{1, ..., m\} \text{ such that } n_{is}(R) > n/2.$$

For the sake of simplicity, let:

$$N_{\rho s}(R) = N_{\rho_s s}(R) \text{ and } n_{\rho s}(R) = n_{\rho_s s}(R).$$

Next, let:

$$B_{\rho s}(R) = \textstyle\sum_{j \in N_{\rho s}} r_s^j, \quad B_s(R) = \sum_{j \in N} r_s^j,$$

$$\nu_{st}(R) = |\{j \in N : r_s^j < r_t^j\}|.$$

$N_{is}(R)$ is the set of judges who rank skater s at rank i or better, $n_{is}(R)$ their number and $\bar{n}_{is}(R)$ the number of those placing skater s exactly at rank i. $\rho_s(R)$ is the median rank of skater s, $B_{\rho s}(R)$ is the sum of the ranks that s obtains from the judges who gave him or her the median rank or better, $B_s(R)$ is the sum of the ranks that s obtains from all judges, and $\nu_{st}(R)$ is the number of judges who rank skater s ahead of skater t. Note that $B_s(R)/n$ is the mean rank of s.

Finally, we define the complete binary relation M on X by $sMt \Leftrightarrow \nu_{st} \geq \nu_{ts}$. We write $\bar{M}$ for the asymmetric component of M, i.e. the relation defined by $s\bar{M}t \Leftrightarrow \nu_{st} > \nu_{ts}$ and T for the symmetric component of M, i.e. the relation defined by $sTt \Leftrightarrow \nu_{st} = \nu_{ts}$. They are respectively the *majority relation*, the *strict majority relation* and the *ex aequo relation* on X. We can read $s\bar{M}t$ as s defeats t and sTt as s ties with t. A cycle of M is a subset $\{x_1, ..., x_k\} \subset X$ such that $x_i M x_{i+1}$, $i = 1, ..., k-1$, and $x_k M x_1$. One defines similarly cycles of $\bar{M}$ and T. Equivalently, a cycle of T is a subset $S \subset X$ such that $sTt \;\forall\, s,t \in S$. Cycles of $\bar{M}$ and T are obviously cycles of M.

Rule 371 of the ISU Regulations, which prescribes how the final ranks of the competitors are determined, is reproduced in the Appendix. The *ISU ranking rule*, as we shall call it, can be formally defined as follows. Its definition involves five steps incorporating four principles or criteria. When a principle has been applied, the next one is used only if there remain ties between some competitors.

The ISU Rule: $\forall R \in \Re^n, \forall s, t \in X,$

a) $\rho_s(R) < \rho_t(R) \Rightarrow FR_s(R) < FR_t(R)$

b) $\rho_s(R) = \rho_t(R)$ and $n_{\rho s}(R) > n_{\rho t}(R) \Rightarrow FR_s(R) < FR_t(R)$

c) $\rho_s(R) = \rho_t(R)$, $n_{\rho s}(R) = n_{\rho t}(R)$ and $B_{\rho s}(R) < B_{\rho t}(R) \Rightarrow FR_s(R) < FR_t(R)$

d) $\rho_s(R) = \rho_t(R)$, $n_{\rho s}(R) = n_{\rho t}(R)$, $B_{\rho s}(R) = B_{\rho t}(R)$ and $B_s(R) < B_t(R) \Rightarrow FR_s(R) < FR_t(R)$

e) $\rho_s(R) = \rho_t(R)$, $n_{\rho s}(R) = n_{\rho t}(R)$, $B_{\rho s}(R) = B_{\rho t}(R)$ and $B_s(R) = B_t(R) \Rightarrow FR_s(R) = FR_t(R)$

In plain words, a) skaters are first ranked according to their median ranks $\rho_s(R)$. b) If two skaters, say *s* and *t*, tie for a rank, one tries to break the tie according to the respective numbers of judges $n_{\rho s}(R)$ and $n_{\rho t}(R)$ who gave them their respective median rank or a better rank. c) If criterion b) is not sufficient to break all ties, one then takes into consideration the sum of the ranks obtained from the judges who gave the median rank or a better rank to these competitors. d) If there still remains some ties after criterion c) has been applied, one uses the sum of the ranks obtained from all judges or equivalently the mean rank as a breaking criterion. e) Finally, if all ties are not resolved after principles a) − d) have been applied, competitors who tie for a rank obtain the same rank.

As pointed out by BP, principle a), which they call the *Median Rank Principle* (MRP), translates paragraphs 1, 2, and 6 of rule 371. Criteria b), c), and d) translate respectively paragraphs 3, 4, and 5 of rule 371. Moreover, they take into account paragraphs 7, 8, and 9. Finally, criterion e) translates the first part of paragraph 10. The second part, which says that if two competitors tie for the first place, the next place to be awarded is third place (not second) and so on, shall not be formalized.

The example of Table 1, reproduced from the ISU Regulations book, illustrates the application of the ISU Rule. The final rank, as given by this rule appears under the heading ISU. Note that A is ranked ahead of B by using principle a) alone. There is a tie between I and J after application of principle a). It is resolved in favour of I by reverting to principle b). One must go as far as to principle c) to rank

D ahead of E and to principle d) to rank B ahead of C. However, there remains a tie between K and L even after applying the first four principles.

3. Properties of the ISU Rule

We start with an interesting characterization of principle a) of the ISU Rule due to Bassett and Persky (1994). We then show that principles d) and e) consist in applying the Borda criterion to the skaters that remain tied after steps a) − c). Principle c) is a sort of restricted Borda criterion. Finally, we examine the whole ISU procedure from the perspective of properties often found in the theory of social choice.

BP show that (MRP) or step a) of the ISU Rule is the only criterion to satisfy simultaneously a certain majority principle and a weak monotonicity condition, which they also misname incentive compatibility. The logical propositions defining these two conditions, as the ones to follow later in this section, must hold for all profiles R, $\tilde{R} \in \mathfrak{R}^n$ and all pairs s, $t \in X$ such that $s \neq t$.

Rank Majority Principle (RMP)

$$[\bar{n}_{is}(R) > n/2,\ \bar{n}_{kt}(R) > n/2, \text{ and } i < k] \Rightarrow FR_s(R) < FR_t(R)$$

Bassett and Persky Monotonicity (BPM)

$$[\tilde{r}^j_s \leq r^j_s \text{ and } \tilde{r}^j_t \geq r^j_t,\ \ \forall\, j \in N] \Rightarrow$$
$$[FR_s(R) < FR_t(R) \Rightarrow FR_s(\tilde{R}) < FR_t(\tilde{R}) \text{ and } FR_s(R) = FR_t(R) \Rightarrow FR_s(\tilde{R}) \leq FR_t(\tilde{R})]$$

(RMP) says that if two skaters obtain a majority for two different ranks, this ought to be reflected in their relative final ranking. (BPM) says that if all judges were to improve or maintain the ranking of skater *s* while diminishing or maintaining the ranking of skater *t*, then the final ranking of skater *s* should remain at least as good as the one of skater *t* if it was so before the change.

Theorem 1 (Bassett and Persky): (MRP) ⇔ [(BPM) and (RMP)].

Proof. Suppose (MRP) holds. For any $s, t \in X$, if there exist ranks i and k such that $i < k$, $\bar{n}_{is}(R) > n/2$ and $\bar{n}_{kt}(R) > n/2$, then $\rho_s(R) = i < k = \rho_t(R)$, which implies $FR_s(R) < FR_t(R)$ by (MRP). Thus (RMP) holds. That (BPM) holds is immediate from the definitions.

Conversely, suppose (BPM) and (RMP) hold but that (MRP) fails, i.e. $\exists\ R \in \Re^n$, $s, t \in X$ such that $s \neq t$, and positive integers i, k such that $i = \rho_s(R) < \rho_t(R) = k$ but $FR_s(R) \geq FR_t(R)$. Then consider another profile $\tilde{R} \in \Re^n$ such that, $\forall\ j \in N$, $r_s^j \leq i \Rightarrow \tilde{r}_s^j = i$, $r_s^j > i \Rightarrow \tilde{r}_s^j = r_s^j$, $r_t^j \geq k \Rightarrow \tilde{r}_t^j = k$, and $r_t^j < k \Rightarrow \tilde{r}_t^j = r_t^j$. This gives $\bar{n}_{is}(\tilde{R}) = n_{\rho s}(\tilde{R}) > n/2$ and $\bar{n}_{kt}(\tilde{R}) = n_{\rho t}(\tilde{R}) > n/2$. Thus, by (RMP), we must have $FR_s(\tilde{R}) < FR_t(\tilde{R})$. By (BPM), we should also have $FR_s(R) < FR_t(R)$, a contradiction. □

Remark 2: (RMP) alone does not imply (MRP). Indeed, consider a ranking rule defined in the following way. First, rank all competitors according to the $B_s(R)$. Next, if (RMP) is violated for two competitors s and t, move the one with the smallest (better) rank according to a majority of judges in front of the other and adjust the rank of those who were between s and t accordingly. Applying this rule to the data of Example 2 in Table 2 would give the ranking headed BC to start with. Applying (RMP) would then change this ranking for the ISU final ranking. Now, if judge 2 were to interchange her ranking of B and E, then B would loose the majority for rank 2 and the final ranking would thus be changed back from the ISU to the BC ranking, in violation of (MRP). This simple interchange also shows that this rule violates (BPM). Given Theorem 1, this was to be expected.

The ranking of the skaters according to the $B_s(R)$, headed BC in the examples, is the Borda ranking. Indeed, the Borda ranking relies on the Borda scores or counts defined as follows. The last competitor in a judge's ranking receives a score of zero, the second lowest a score of 1, the third a score of 2 and so on. Summing these scores over all judges yields the Borda score of each competitor. These score are then used to rank the competitors.

In our notation, the score that skater s obtains from judge j in a profile R is given by $m - r_s^j$. The Borda score of skater s is thus given by $mn - B_s(R)$. The *Borda criterion* (BC) is thus equivalently defined by:

$$\forall\ s, t \in X,\ B_s(R) \leq B_t(R) \Leftrightarrow FR_s(R) \leq FR_t(R)$$

In short, the Borda criterion consists in ranking competitors according to their mean ranks. (BC) is also equivalent to:

$$\forall\ s, t \in X, [B_s(R) < B_t(R) \Leftrightarrow FR_s(R) < FR_t(R)] \text{ and } [B_s(R) = B_t(R) \Leftrightarrow FR_s(R) = FR_t(R)]$$

The two terms within brackets are the two principles used respectively in steps d) and e) of the ISU Rule.

Principle c) is a variant of the Borda criterion. It compares the $B_{\rho s}(R)$, which can be seen as the Borda scores computed from the restricted set of judges who give competitor s his or her value $\rho_s(R)$. However, these sets are not necessarily the same from one skater to the other.

In the theory of social choice, other kinds of properties are often imposed on ranking rules or deemed desirable. The most famous are probably the four to follow. We analyze the ISU Rule from their perspective. The first one is a monotonicity condition that is stronger than (BPM).

Monotonicity (M)

$$[r_s^j < r_t^j \Rightarrow \tilde{r}_s^j < \tilde{r}_t^j \text{ and } r_s^j = r_t^j \Rightarrow \tilde{r}_s^j \leq \tilde{r}_t^j,\ \ \forall\ j \in N] \Rightarrow$$
$$[FR_s(R) < FR_t(R) \Rightarrow FR_s(\tilde{R}) < FR_t(\tilde{R}) \text{ and } FR_s(R) = FR_t(R) \Rightarrow FR_s(\tilde{R}) \leq FR_t(\tilde{R})]$$

Binary Independence (BI)

$$[r_s^j \leq r_t^j \Leftrightarrow \tilde{r}_s^j \leq \tilde{r}_t^j \text{ and } r_s^j \geq r_t^j \Leftrightarrow \tilde{r}_s^j \geq \tilde{r}_t^j,\ \ \forall\ j \in N] \Leftrightarrow [FR_s(R) \leq FR_t(R) \Leftrightarrow FR_s(\tilde{R}) \leq FR_t(\tilde{R})]$$

Weak Pareto (WP)

$$r_s^j < r_t^j,\ \forall\ j \in N \Rightarrow FR_s(R) < FR_t(R)$$

Non-Dictatorship (ND)

$$\text{There exists no } j \in N \text{ such that } FR(R) = r^j,\ \forall\ R \in \Re^n$$

(M) has the same conclusion as (BPM) but a weaker premise. It is thus stronger. (M) says that if all judges were to maintain or improve the relative ranking of skater s with respect to skater t, then the final ranking of skater s should remain at least as good as the one of skater t if it was so before the change. A judge may maintain or improve such a relative ranking by increasing the mark of skater s or diminishing the one of skater t, as with (BPM). However, he might also do so while increasing or diminishing both skaters' marks, which is permitted by (M).

(BI) says that only the relative rankings of two skaters should matter in establishing the final relative ranking of these two skaters. (WP) says that if all judges are unanimous on the relative rankings of two skaters, the final relative ranking of these two skaters should agree with the unanimous view of the judges. Finally, (ND) prescribes that no judge be able to impose his or her ranking as the final ranking in all circumstances.

Lemma 3: (M) $\Rightarrow$ (BPM).

Proof. $[\tilde{r}^j_s \leq r^j_s$ and $\tilde{r}^j_t \geq r^j_t, \ \forall\, j \in N] \Rightarrow [r^j_s < r^j_t \Rightarrow \tilde{r}^j_s < \tilde{r}^j_t$ and $r^j_s = r^j_t \Rightarrow \tilde{r}^j_s \leq \tilde{r}^j_t, \ \forall\, j \in N] \Rightarrow$ $[FR_s(R) < FR_t(R) \Rightarrow FR_s(\tilde{R}) < FR_t(\tilde{R})$ and $FR_s(R) = FR_t(R) \Rightarrow FR_s(\tilde{R}) \leq FR_t(\tilde{R})]$. □

That (M) is stronger than (BPM), i.e. that the inverse implication does not hold follows from the proof of Theorem 4.

Theorem 4: The ISU ranking rule satisfies (ND), (WP), (RMP) and (BPM) but neither (M) nor (BI).

Proof. That the ISU Rule satisfies (ND), (WP), (RMP) and (BPM) can be checked readily from the definitions. It is not difficult to find instances of violations of (M) and (BI). In Example 2, if judge 2 were to change the ranks given to A, B, and C for respectively 4, 3, and 2, then their final rank would become respectively 2, 3, and 1. Thus B, who was originally ranked before A, would move after this competitor despite the fact that her relative position with respect to A has not changed in any of the judges' opinion. This is a violation of (M) and (BI). In the ISU Example, if judge 9 were to interchange his ranking of B and E, this would not change his ranking of C relative to B. Yet, C would move ahead of B in the final ranking. This is another violation of (M) and (BI). □

There is a famous impossibility theorem in the theory of social choice due to Arrow (1951), which says that there is no ranking rule that satisfies (BI), (ND), (WP) and (M). This theorem has been reinforced in many ways by weakening some of the conditions imposed on the ranking rule. As put by Kelly (1978, p.3), for each of Arrow's conditions, there is now an impossibility theorem not employing that condition. One of them, due to Muller and Satterthwaite (1977), asserts that there is no ranking rule that satisfies (ND), (WP) and (M). Since the ISU Rule satisfies (ND) and (WP), we could not expect (M) to hold. This explains the use of the weaker condition (BPM) by BP. Since (M) and (BI) have much in common, the violation of (BI) also comes as no surprise.

4. Which majority principle ?

Paragraphs 1, 2, and 6 of rule 371, which translate into (MRP), clearly reflect a majority principle. The notion of median rank itself is based on a majority condition. Moreover, Theorem 1 states a clear relationship between (MRP) and (RMP), a majority principle. However, (RMP) and thus (MRP) conflict with the following well known criterion:

Condorcet Criterion (CC)

$$\forall\ s \in X,\ \forall\ t \in X,\ t \neq s,\ s\overline{M}t \Rightarrow FR_s(R) = 1 \text{ and } FR_t(R) > 1$$

In plain words, if a competitor is ranked ahead of all other competitors by an absolute majority of judges, he or she should be ranked first. An s satisfying (CC) may not exist because of a cycle in the majority relation M. This is the case in Example 3 of Table 3. There is a cycle of $\overline{M}$ over the whole set X. When there exists an $s \in X$ satisfying (CC), this s is called the *Condorcet winner*. Note that there may exist a Condorcet winner even if there are cycles over some subsets of competitors. There is an instance of this in the ISU Example. That (MRP) and (RMP) are completely at odds with (CC) may be seen from Example 2. Competitor A obtains the first rank according to (CC) but is ranked after B according to (MRP).

Condorcet was preoccupied not only with the winner but also with the whole ranking of the alternatives, i.e. the competitors in our context. This may pose a problem since the majority relation may contain cycles. Yet, a partial extension of (CC) to other ranks can be done as follows: If a competitor is consistently ranked ahead of another competitor by an absolute majority of judges, he should be ahead in the final ranking. The term "consistently" refers to the absence of cycles involving these two competitors. Formally, let $\wp_0(X)$ be the class of partitions $\boldsymbol{X} = \{X_1, ..., X_p\}$ of X, satisfying:

$$\forall\ X_\alpha, X_\beta \in \boldsymbol{X} \text{ with } \alpha < \beta,\ \forall\ s \in X_\alpha,\ \forall\ t \in X_\beta : s\overline{M}t$$

Notice that, if there is a cycle of M over some subset of alternatives, then these alternatives must belong to a same subset X_α of any partition in $\wp_0(X)$. In particular, this must be the case for two alternatives s and t such that $s\,T\,t$. In the finest partition of this class, the sets X_α are cycles of maximal length of M or singletons. X_1 is also called the *top cycle* of M or the *Condorcet set*, a solution concept introduced by Good (1971) and Schwartz (1972) for the strict majority relation. X_2 is the top cycle on $X \setminus X_1$, etc.

Extended Condorcet Criterion (XCC)

For any partition $\boldsymbol{X} \in \wp_0(X)$, the following must hold:

$$\forall\ X_\alpha,\ X_\beta \in \boldsymbol{X} \text{ with } \alpha < \beta,\ \forall\ s \in X_\alpha,\ \forall\ t \in X_\beta : FR_s(R) < FR_t(R)$$

It will be shown below that a maximum likelihood or a Kemeny order satisfies (XCC). Recall that it was precisely Condorcet's objective to show that the majority principle leads to a maximum likelihood order. Hence, this justifies calling the above principle an *Extended Condorcet Criterion*.

If $M = \overline{M}$ and if $\overline{M}$ contains no cycles, then all X_α of the finest partition of $\wp_0(X)$ are singletons and a final ranking $FR(R)$ satisfying (XCC) is a complete order. When M contains a cycle, (XCC) does not say how to rank alternatives within an X_α of the finest partition in $\wp_0(X)$. In particular, it does not imply that $FR_s = FR_t$ if $s\,T\,t$. In other words, (XCC) yields only a partial order in these circumstances. We shall see in section 6 how a complete final ranking can be obtained with the maximum likelihood approach.

In the different examples of this paper the (possibly partial) final ranking obtained from the extended Condorcet criterion is headed by XCC. Competitors who belong to a same X_α and who cannot be ranked by (XCC), because they belong to a same set of the finest possible partition of X, are simply marked by a "?". This is the case with the set {I, M, L, K} in the ISU Example and with the whole set X in Example 3. There is a cycle I $\overline{M}$ M $\overline{M}$ L $\overline{M}$ K $\overline{M}$ I over the set {I, M, L, K} in the ISU Example and a cycle of $\overline{M}$ over the whole set X in Example 3.

5. Condorcet versus Borda versus ISU

Since (XCC) implies (CC), the conflict between (CC) and (MRP) of Example 2 is also a conflict between (XCC) and (MRP). There is another violation of (XCC) by the ISU Rule in the ISU Example: Competitor J should be ranked ahead of I according to (XCC) but the ISU Rule ranks them in the reverse order.

(XCC) and (BC) may also conflict as can be seen from Example 2. This possibility of conflict has been known since the lifetime of Borda and Condorcet, who debated passionately over the respective merits of their rules. Nonetheless, it is well known that a Borda winner (loser) can never be a Condorcet loser (winner).

Finally, the ISU Rule may violate (BC) despite the fact that principle d) is based on this criterion. There are instances of such conflicts in all three examples. Worse, a Borda winner can be a loser according to the ISU Rule. In Example 3, if the rankings of judges 4, 5, and 6 were changed for respectively (6, 3, 4, 1, 5, 2), (6, 3, 4, 2, 5, 1), and (6, 3, 4, 2, 5, 1), then none of the criteria ρ_s and $n_{\rho s}$ would be modified and thus the ISU final ranking would remain the same. Yet E, who comes last in the ISU final ranking, would become the Borda winner. The reason for these conflicts is precisely that (BC) is called to the rescue only when the other three principles fail in ranking two competitors.

It would be interesting to know why the ISU seems reluctant to use exclusively the Borda criterion to aggregate the rankings of the judges. Surprisingly, it does not hesitate to use a similar criterion to aggregate the rankings FR^S obtained for the short program, FR^I for interpretive program and FR^F for free skating into a single ranking. Indeed, the final ranking is established according to a weighted sum of the ranks in the different FR. For example, if the three programs just mentioned are present in a competition, the final ranks are determined according to the values of the components of the vector $.3\,FR^S + .2\,FR^I + .5\,FR^F$. This is equivalent to supposing that there are 3 judges with the ranking FR^S, 2 with FR^I, and 5 with FR^F, and using the Borda Criterion to aggregate them. More will be said on this criterion in the final section.

6. The quest for a true ranking

An interesting question that arises in relation to the judging systems such as the one that prevails in figure skating is whether a profile *R* represents the **preferences** of the judges on the set *X* or an **evaluation** of the relative merits of the competitors according to given criteria. The regulations book of the ISU is very clear in this respect. The judges do not furnish a ranking per se but scores for the different elements of the competition. They are instructed on how to subtract points for different types of mistakes. The intent of the rules is clearly to have judges furnish an evaluation of the relative merits of the competitors in terms of scores. The different scores attributed to the different elements of the competition are then aggregated and the ranking r^j of each judge is determined from these aggregated scores.

Assuming that the rankings r^j in some profile *R* are independent evaluations of the true ranking of the competitors according to the established rules, an interesting question is: which final ranking $FR(R)$ is most likely to be the true ranking of the competitors? This is precisely the question addressed by Condorcet (1785). His objective was to justify the majority principle. On this, he was certainly inspired by Rousseau (1762) in his *Social Contract*, for whom the opinion of the majority is legitimate because it expresses the "general will."

> When in the popular assembly a law is proposed, what the people is asked is not exactly whether it approves or rejects the proposal, but whether it is in conformity with the general will, which is their will. Each man, in giving his vote, states his opinion on that point; and the general will is found by counting votes. When therefore the opinion that is contrary to my own prevails, this proves neither more nor less that I was mistaken, and that what I thought to be the general will was not so. [Rousseau (1913), p. 93]

Condorcet's objective was to formulate this proposition rigorously, using the calculus of probability, which was new at that time. There is a best alternative, a second best, etc. Voters may have different opinions because they are imperfect judges. However, if they are right more often than they are wrong, then the opinion of the majority should yield the true order of the alternatives.

Condorcet's approach is one of the first applications of statistical hypothesis testing and maximum likelihood estimation. He assumes that every voter chooses the best of two alternatives with a probability p satisfying $1/2 < p \leq 1$, and that this judgment is independent between pairs and voters. If the binary relation M is an order on X, then it is the solution to his problem, i.e. the most probable order on X. If M contains a cycle, Condorcet's prescription is to eliminate some of the propositions ($s M t$ is a proposition), starting with the one with the weakest majority and so on until the cycle disappears. This works fine if $m = 3$ but may give ambiguous results or a partial order for $m \geq 4$.

Young (1988) develops a correct application of Condorcet's maximum likelihood approach. In the case $m = 3$, it goes as follows. If the true order on the set $\{a, b, c\}$ is *abc*, then, neglecting a multiplicative constant and the argument R in $\nu_{st}(R)$, the conditional probability of observing a profile of votes R is given by:

$$p^{\nu_{ab}}(1-p)^{\nu_{ba}}p^{\nu_{ac}}(1-p)^{\nu_{ca}}p^{\nu_{bc}}(1-p)^{\nu_{cb}} = p^{\nu_{ab}+\nu_{ac}+\nu_{bc}}(1-p)^{\nu_{ba}+\nu_{ca}+\nu_{cb}}$$

The probability of observing the same profile, conditional on the true order being *acb*, is given by:

$$p^{\nu_{ac}}(1-p)^{\nu_{ca}}p^{\nu_{ab}}(1-p)^{\nu_{ba}}p^{\nu_{cb}}(1-p)^{\nu_{bc}} = p^{\nu_{ac}+\nu_{ab}+\nu_{cb}}(1-p)^{\nu_{ca}+\nu_{ba}+\nu_{bc}}$$

Hence if $p > ½$, *abc* is more probable than *acb* as on order if and only if $\nu_{ab}+\nu_{ac}+\nu_{bc} > \nu_{ac}+\nu_{ab}+\nu_{cb}$. More generally, let:

$$K(r, R) = \sum_{s \in X} \sum_{\substack{t \in X \\ r_s < r_t}} \nu_{st}(R).$$

An order r^* on X is a solution of Condorcet's problem or a *maximum likelihood order* if it is a solution of $\max_{r \in \Re} K(r, R)$. The value of $K(r, R)$ may be seen as the total number of pairwise supports for r in profile R, i.e. the total number of voters who rank pairs of alternatives as in r. A maximum likelihood order is thus one that has the maximum total support from the judges as expressed in R.

This problem may be given a different expression using a notion of distance for orders proposed by Kemeny (1959). It is presented here in a slightly modified form to accommodate the fact that weak orders may be found in profiles of rankings. Given an order r, a weak order $r^j \in \Re$, and two competitors $s, t \in X$, define:

$$\delta_{st}(r,\ r^j) = \begin{cases} 1 & \text{if } r_s < r_t \text{ and } r_t^j \leq r_s^j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \Delta(r,\ r^j) = \Sigma_{s \in X} \Sigma_{t \in X} \delta_{st}(r,\ r^j)$$

The value of $\delta_{st}(r, r^j)$ indicates whether there is a disagreement in the relative ranking of s and t between r and r^j. $\Delta(r, r^j)$ is the total number of such disagreements between r and r^j. The function Δ is a distance on the set $\mathfrak{R}$, with the restriction that its first argument must be an order. One can then define a "distance" d between an order r and a profile R by: $d(r, R) = \sum_{j=1}^{n} \Delta(r, r^j)$. In plain words, $d(r, R)$ is the total number of disagreements between an order r and all the rankings in profile R.

A *Kemeny order* for a profile R is an order r^K solving $\min_{r \in \mathfrak{R}} d(r, R)$, i.e. an order that is closest to the given profile according to the "distance" d or an order that has the minimum number of disagreements with the profile. A Kemeny order is also a median order for the rankings in the profile. As such, it represents the best compromise between the different opinions of the judges or voters.

The following lemmas and corollaries give some of the properties of this order. The first one asserts that a Kemeny order is a maximum likelihood order. From this lemma, we can reassert that a maximum likelihood order is one that has the maximum number of agreements with the profile.

Lemma 5: A Kemeny order for a profile R is an order solving $\max_{r \in \mathfrak{R}} K(r, R)$.

Proof. $$d(r,\ R) = \sum_{s \in X} \sum_{t \in X} \sum_{j=1}^{n} \delta_{st}(r,\ r^j) = \sum_{s \in X} \sum_{\substack{t \in X \\ r_s < r_t}} (n - \nu_{st}(R)) = \frac{m(m-1)n}{2} - \sum_{s \in X} \sum_{\substack{t \in X \\ r_s < r_t}} \nu_{st}(R),$$

hence the result. □

Lemma 6: Suppose $r^K = (1, 2, ..., m)$ is a Kemeny order for a given profile R. Then $\nu_{s,\, s+1}(R) \geq \nu_{s+1,\, s}(R)$, $s = 1, ..., m - 1$, or, equivalently, $1 \, M \, 2 \, M \ldots M \, m$.

Proof. For any $s = 1, ..., m - 1$, consider the order $r = (1, 2, ..., s - 1, s + 1, s, s + 2, ..., m)$. By the proof of Lemma 5, $d(r^K, R) - d(r, R) = \nu_{s,\, s+1} - \nu_{s+1,\, s}$, which cannot be negative if r is a Kemeny order. □

Corollary 7: Given a Kemeny order for a given profile R, if there exists a Condorcet winner under this profile, it must be the competitor ranked first in the Kemeny order.

Proof. Let $r^K = (1, 2, ..., m)$ be a Kemeny order. From Lemma 6, $\nu_{s-1,s}(R) \geq \nu_{s,s-1}(R)$, $s = 2, ..., m$. Thus none of the alternatives $s = 2, ..., m$ may be a Condorcet winner, leaving 1 as the Condorcet winner. □

Corollary 8: A Kemeny order r^K satisfies (XCC).

Proof. Suppose r^K violates (XCC), i.e. there exists a partition $\boldsymbol{X} \in \wp_0(X)$, $X_\alpha, X_\beta \in \boldsymbol{X}$ with $\alpha < \beta$, $s \in X_\alpha$ and $t \in X_\beta$ such that $r^K_t \leq r^K_s$. By definition of $\wp_0(X)$, we must have $s\bar{M}t$. Thus, by Lemma 6, there must exist other skaters, say $a, ..., k$, between t and s in the Kemeny order. Using Lemma 6 again, we must have $tMaM...MkMs$. Since we also have $s\bar{M}t$, there is a cycle over the set $\{t,a, ..., b,s\}$. Using the definition of $\wp_0(X)$ again, $\{t,a, ..., b,s\}$ should belong to the same set of the partition $\boldsymbol{X}$. We thus have a contradiction since, at the outset, s and t belonged to different X_α and X_β. □

The next result provides an easy way to find complete Kemeny orders. In essence, it says that the latter can be constructed by the concatenation of Kemeny orders on each of the sets of a partition $\boldsymbol{X} \in \wp_0(X)$. Recall that an order can take the form x or r, where x_i is the competitor whose rank is i while r_s is the rank of competitor s.

Theorem 9: Take any partition $\boldsymbol{X} = \{X_1, ..., X_p\} \in \wp_0(X)$ and an order $x^* = \tilde{x}_1...\tilde{x}_p$ where $\tilde{x}_\alpha$ is a Kemeny order on X_α under profile R restricted to X_α, $\alpha = 1, ..., p$. Then x^* or equivalently the corresponding r^* is a Kemeny order on X.

Proof. Suppose that there exists an order r on X such that $K(r, R) > K(r^*, R)$. Then r cannot be different from r^* in respect only to competitors who belong to the same X_α since this would violate the assumption that x_α is a Kemeny order on X_α. Thus there exist $X_\alpha, X_\beta \in \boldsymbol{X}$ with $\alpha < \beta$, $s \in X_\alpha$ and $t \in X_\beta$ such that $r_t \leq r_s$ instead of $r_s < r_t$ as in r^*. By Corollary 8, r cannot be a Kemeny order since it violates (XCC). If there were orders r such that $K(r, R) > K(r^*, R)$, there would be a Kemeny order among them. Thus there is no such order and r^* is a Kemeny order. □

If $M \neq \overline{M}$, we can go one step further in partitioning X. The details and an algorithm based on such a partition can be found in Truchon (1998). For the time being, let us stay with the class $\wp_0(X)$. The finest partition of $\wp_0(X)$ for the ISU Example is:

$$\{\{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}, \{H\}, \{J\}, \{K, I, M, L\}, \{N\}, \{O\}\}.$$

The subset {I, K, L, M} cannot be broken because of a cycle on this subset. There are 4!, i.e. 24, possible orders over this subset. However, using Lemma 6, one can eliminate all orders involving the terms LI, IK, KL, LM, MI, and MK. This leaves the five orders given in the table below together with their values $K(r, R_{IKLM})$ where R_{IKLM} is the restriction of R to the set {I, K, L, M}.

Order r	$K(r, R_{IKLM})$
KIML	30
IMLK	29
ILKM	29
LKIM	27
MLKI	26

Hence the unique Kemeny order over the set {I, K, L, M} is KIML. Thus, according to Theorem 9, the complete and unique Kemeny order over all competitors is ABCDEFGHJKIMLNO. The corresponding r appears in the last column of Table 1. Within this order, we have JKIML. This is quite different from IJ<u>KL</u>M (KTL) given by the ISU Rule. The Borda rule gives: JIM<u>KL</u>.

Example 3 is a more striking illustration of the conflict that could exist between the ISU and the Kemeny rules. There is a cycle over the whole set of competitors: E $\overline{M}$ A $\overline{M}$ B $\overline{M}$ C $\overline{M}$ D $\overline{M}$ F $\overline{M}$ E. The Kemeny rule gives EABCDF as the unique final order while the ISU Rule gives ABDCFE. E is the Kemeny winner but the worst competitor according to the ISU Rule.

A Kemeny order is not necessarily unique. The following rule can be applied to handle the occurrence of multiple Kemeny orders. Given a set $\{r^1, ..., r^k\}$ of Kemeny orders, consider the weak order r^m defined by:

$$\forall\ s,t \in X : r_s^m \leq r_t^m \Leftrightarrow \sum_{q=1}^{k} r_s^q \leq \sum_{q=1}^{k} r_t^q$$

This weak order is a ranking according to the mean ranks of alternatives over all Kemeny orders. It will be called the *mean Kemeny ranking* if it weakly agrees with at least one order in $\{r^1, ..., r^k\}$, i.e. if there exists an order $r^q \in \{r^1, ..., r^k\}$ such that:

$$\forall\ s,t \in X : r_s^q < r_t^q \Rightarrow r_s^m \leq r_t^m$$

If $r_s^m < r_t^m$, this means that there are more Kemeny orders in which *s* is ranked ahead of *t* than Kemeny orders in which *s* is placed after *t*. Thus, if a Kemeny order is chosen at random, the probability that *s* be ranked ahead of *t* is higher than the probability that it be ranked after *t*. In r^m, alternatives are thus ranked according to these probabilities. In particular, two alternatives obtain the same rank if $r_s^m = r_t^m$. Thus, choosing r^m over other Kemeny orders makes sense if r^m weakly agrees with one Kemeny order. However, it would be inconsistent with the Kemeny-Young approach to choose r^m if it is not a Kemeny order, since it is then less probable than any Kemeny order. In this case, a Kemeny order could be chosen at random or according to some other criterion.

With this approach, we look for Kemeny orders but we may end up with a weak order as a final choice. An alternative approach would consist in working with the set of weak orders instead of orders at the outset but this would be costly. For example, there are 75 weak orders on a set of four alternatives compared to 24 orders. The above approach is thus more practical.

One case in which r^m gives the same rank to two alternatives *s* and *t* is when $\nu_{st} = \nu_{ts}$ and when in addition *s* and *t* are adjacent in any Kemeny order. Indeed, in this case, for any Kemeny order in which *s* is ahead of *t*, there is another one in which the only difference is that the positions of *s* and *t* are interchanged. In particular, all competitors of a cycle in T will be declared ex aequo under r^m.

There are other instances in which some competitors could be declared ex aequo. In Example 3, if judge 3 were to interchange her ranking of C and F, there would then be three Kemeny orders: EABFCD, EABDFC, and EABCDF. This means that CDF, DFC and FCD have the same likelihood. Not surprisingly, with this change, there is a cycle C $\bar{M}$ D $\bar{M}$ F $\bar{M}$ C over the set {C, D, F}. There is good ground here to declare these three competitors ex aequo since they have the same likelihood of being in any of the last three positions and this is what happens with r^m.

The modification of Example 3 introduced in Section 5, by changing the rankings of judges 4, 5, and 6 for respectively (6, 3, 4, 1, 5, 2), (6, 3, 4, 2, 5, 1), and (6, 3, 4, 2, 5, 1), provides another example of multiple Kemeny orders. There are six Kemeny orders: EADFBC, EADBCF, EABCDF, DFEABC, DEAFBC, DEABCF. Here, there is no mean Kemeny order since r^m does not agree weakly with any of the Kemeny orders.

Interestingly, in this case, the Borda criterion would rank the competitors in the following way: (4, 2, 6, 2, 1, 4). Thus B and D would tie for the second rank and A and F would tie for the fourth rank. Moreover, only one point in the Borda scores separates adjacent ranks, thus confirming the close competition between all skaters. Recall that the ISU ranking in this case is (1, 2, 4, 3, 6, 5), completely at odds with the Borda and the six Kemeny orders.

7. Other properties of the Kemeny Rule

As expected, the Kemeny rule does not satisfy (BI) and (M). In Example 3, if A and B had not shown up for the competition or equivalently if all the judges had ranked them in the last two places without changing the marks of all other competitors, then the unique Kemeny and Condorcet order would become CDFE. Hence E would become the Condorcet loser, in violation of both (BI) and (M).

However, Young and Levenglick (1978) show that the Kemeny rule satisfies a weaker independence condition that they call *local independence of irrelevant alternatives* (LIIA). Its definition involves the concept of interval for orders. An *interval for an order* is any subset of alternatives that occurs in succession in that order. For example, *abcd, bcd, bc, cd, cde* are intervals of the order *abcde*. (LIIA) requires that the ranking of alternatives within any interval be unaffected by the presence of alternatives outside this interval. This condition implies that the ranking of alternatives toward the top of the list is unaffected by the removal of those at the bottom, etc.

The Kemeny rule is also *symmetric* (it puts all judges on an equal footing), *neutral* (it treats all skaters in the same way). Moreover, it satisfies (WP) and *reinforcement* (whenever two distinct groups of judges both reach the same ranking of the skaters, this ranking is also the consensus for the two groups of judges merged together). The Borda rule and more general positional methods to be defined in Section 9 also satisfy these properties but the Kemeny rule is the only one to satisfy (LIIA) as well. In Example 3, if B was dropped from the list, then C would take the first place from A in the Borda ranking. In Example 2, we have an illustration of the violation of (LIIA) by the ISU Rule. Dropping D from the list causes A to move ahead of B in the ISU ranking.

The ISU Rule may come far from giving the most probable ranking because of its extreme sensitivity to the data. In Example 3, the ISU Rule gives E the last rank because 4 judges out of 7 gave E rank 5. Yet it gives A the first rank while 3 judges gave this competitor rank 6. It appears that the ISU Rule may be very sensitive to small perturbations and thus to errors in the judges' rankings or attempts at manipulation. For instance, in Example 3, suppose that judge 7 made a mistake and gave a ranking of 1 to A and 5 to E instead of the other way around. The effect of this error on the final ranking is dramatic. It gives A the first rank and E the last rank, just the reverse of what would have happened without this error. If one adheres to the point of view that judges try to evaluate all competitors as objectively as possible but may err in doing so, this sort of sensitivity should be avoided.

This extreme sensitivity of the ISU Rule to the data is due to an incomplete use of available information by (MRP). For example, a skater has the same median rank 3 whether he obtains ranks (3, 3, 3, 3, 6, 6, 6) or (1, 1, 1, 3, 3, 3, 3) from a panel of seven judges. If he is alone with this median rank, he will be ranked according to this median rank and will possibly obtain the same final rank with any of the two profiles. In these circumstances, a change in the rank given by the middle judge in any of the two rankings could have a big impact on the median rank of this skater and hence on his final ranking.

8. Comparison of Four Ranking Rules in 24 Olympic Competitions

The ISU, the Borda, the Kemeny rules have been applied to the data of the 24 olympic competitions: men, women, and couples, short and free programs, for 1976, 1988, 1992 and 1994. The rankings of the judges were constructed from the raw marks and the three rules were then used to aggregate these individual rankings into a final ranking. Summing the raw marks of all the judges provided a fourth ranking. The Kemeny orders were found with the algorithm described in Truchon (1998). In the case of multiple Kemeny orders, the mean Kemeny order r^m was chosen when it existed. Otherwise, the Kemeny order closest to the ISU ranking was retained so as to minimize the disagreement between the two.

The result of applying the four ranking rules to these competitions is summarized in Table 4 with a measure of the disagreement between the rankings. This measure is essentially the Kemeny measure with the difference that a tie between two competitors in a ranking instead of a strict relation in the other is counted as half a complete reversal. More precisely, given two rankings, i.e. two weak orders r and $r^* \in \Re$ and two competitors $s, t \in X$, define:

$$\gamma_{st}(r, r^*) = \begin{cases} 1 & \text{if } r_s < r_t \text{ and } r_t^* < r_s^* \\ \frac{1}{2} & \text{if } r_s = r_t \text{ and } r_t^* < r_s^* \\ \frac{1}{2} & \text{if } r_s < r_t \text{ and } r_t^* = r_s^* \\ 0 & \text{otherwise} \end{cases}$$

The measure of the disagreement between r and r^* is defined by: $G(r, r^*) = \sum_{s \in X} \sum_{t \in X} \gamma_{st}(r, r^*)$

The last column of Table 4 complements this measure by giving some details on the presence of cycles and on the disagreements between the Kemeny and the ISU orders. A k-cycle is a cycle over k competitors. All ties that are mentioned in these remarks are those found in the mean Kemeny rankings. These results have been summarized in the Introduction. More can also be found in the Conclusion. Note that all the computations needed for each pair of competitions (short, free) or each row of Table 4 have taken less than 8 seconds on a Pentium 200 running Mathematica and a procedure written by the author.

Finally Table 5 gives the details of the women short competition of the 1988 Olympic Games. The names have been changed for A, B, C, ... The majority relation contains a cycle over the subset {O, P, Q, R, S, T}. The unique Kememy order over this subset is quite different from the other three orders. But there are differences involving other skaters as well, for instance skater L.

9. On the Manipulability

BP contend that "Median ranks provide strong safeguard against manipulation by a minority of judges." Manipulation means misrepresentation of one’s true ranking by a judge in order to change the final ranking for one that he or she prefers. A judge might be interested only in the winner of a competition and could thus try to manipulate the ranking procedure with this objective in mind. We could then see the ranking procedure as a *social choice function*, i.e. a function that selects a winner.

There is another famous impossibility theorem in this context, due to Gibbard (1973) and Satthertwaite (1975), which says that all non-dictatorial (ND) social choice functions are manipulable. Muller and Satthertwaite (1977) also establish that a social choice function is not manipulable if and only if it satisfies (M), a condition violated by the ISU, the Borda and the Kemeny rules. There is a strong link between these impossibility results and the one of Arrow.

A judge may prefer a ranking to another one, not just because of the winner, but because of the whole ranking. One way to formalize this preference would be to use the Kemeny distance between two rankings defined in Section 6 to represent the preference of a judge over the set of all possible rankings. Bossert and Storcken (1992) extend the Gibbard-Satthertwaite theorem to this context. The ISU Rule does not escape these theorems.

For instance, in the ISU Example, judge 9 could improve the final ranking of his most favourite competitor C by simply interchanging his ranking of B and E, without changing anything else in the final ranking. In this case, the possibility of manipulation rests upon the use of the (BC) criterion or principle d). However, (MRP) or principle a) lends itself to manipulation. In Example 2, if judge 5 were to interchange his ranking of A and C, he would make A move ahead of B in the final ranking, which would agree with his own relative ranking of the two competitors.

Some ranking rules may be more prone to manipulation than others. For instance, the Borda rule has been considered as highly manipulable for a long time, mainly because it violates (BI). Many authors have constructed examples showing that a coordinated action by many voters may have a dramatic impact on the final Borda ranking. Already in the 19th Century, the French mathematician Laplace (1812) pointed out that even honest voters could be tempted to give the last ranks to the strongest candidates in order to favour their own candidate. This, in his opinion, would give a great advantage to mediocre candidates. He added that, for this reason, this rule had been abandoned by institutions that had previously adopted it. Borda's answer is well known: My method should be used only with honest people.

The last word on this question, for the present time at least, is probably due to Saari (1990). He shows that, at least for $m = 3$, it is the Borda rule that maximizes the expected strategic impact of manipulation. However, the impact of manipulation is just one aspect of the question. To have any impact, manipulation must be successful. The second question is thus opportunity. How often can a judge be successful at manipulating the final ranking? An analysis of manipulation requires combining both the impact of manipulation and an accounting of how often manipulation can succeed.

Saari develops a measure of susceptibility to manipulation for positional voting procedures. A *positional voting procedure* is one in which m specified weights $w_1, ..., w_m$, with $w_s \geq w_{s+1}$, $w_1 > w_m = 0$, are used to tally the ballots. The competitors are then ranked according to the total number of points that they receive. For example, (1, 0, ..., 0) corresponds to the plurality vote and $(m - 1, m - 2, ..., 1, 0)$ to the Borda method.

His study is done under the following assumptions. It is equally likely for any pair of alternatives to be the target of a manipulation attempt; all profiles of rankings are equally likely; and it is equally likely that a strategic voter or a small coalition of strategic voters has any particular ranking.

His most surprising finding is that, among all positional voting procedures, the Borda rule is the method that either minimizes, or comes close to minimizing, the likelihood of a successful manipulation by a small group of individuals. This result is essentially due to the fixed value for the successive differences between the weights w_i used in the Borda method. However, this same property makes this rule vulnerable to carefully coordinated manipulation by large groups. The worst procedures, i.e. the ones most susceptible of being manipulated by a small coalition, are the plurality (1, 0, ..., 0) and the anti-plurality (1, 1, ..., 1, 0) rules.

What can be said of the ISU Rule in this respect? It can be shown that the ISU Rule is the combination of many positional procedures. Indeed, we can write the ISU Rule as a ($m - 1$) iteration procedure: At iteration $i < m - 1$, rank all competitors who have not been ranked in previous iterations according to $n_{is}(R)$ and retain those for whom $n_{is}(R) > n/2$. If there are ties between some of the competitors who have just been ranked, try to break the ties using the $B_{\rho s}(R)$ first and the $B_s(R)$ if there still remain ties. If some of the competitors have not been ranked after this iteration, then go to the next one, i.e. increase i by one. If iteration $m - 1$ is reached, retain the ranking given by $n_{is}(R)$. Note that there can be only one competitor with rank m and that no tie breaking rule is necessary to find him or her.

This iterative process can be seen as an election with runoffs. At the first iteration, the competitors who get a majority for the first rank are given this rank. The remaining competitors are then ranked in a new election and so on. Formally, at iteration i, the positional rule defined by (1, ..., 1, 0, ..., 0), i.e. a vector with the first i components equal to 1 and all others equal to 0, is actually used in the first step. Each of these positional rules is an approval voting method: List your i most preferred competitors and each of them will get one point in the counting process. According to Saari's findings, each of these procedures is more susceptible to manipulation than the Borda rule. The procedures used in the first and last iterations are respectively the plurality and the anti-plurality ones, the worst possible. The restricted Borda and the Borda procedure itself, which are also two positional procedures, are used only to break ties. The restricted Borda procedure is a special one in that not all votes are counted.

The ISU Rule is a combination of many procedures that are susceptible to manipulation. Is the ISU Rule more or less susceptible to manipulation than any of its components? There is probably no clear cut answer to this question. On the one hand, the fact that manipulation within a positional procedure may imply the recourse to a different one may reduce the opportunities for manipulation within the first procedure. On the other hand, this same fact may open additional possibilities to the manipulators. Whether these additional possibilities will compensate for the lost ones is an open question to me. The answer probably depends on the circumstances. We would like to know what it gives on average.

However, in as much as the Borda rule is rarely used in this context, it is safe to assert that the performance of the whole ISU Rule cannot be as good as that of the Borda rule in terms of susceptibility to manipulation.

The Kemeny rule does not escape the Gibbard-Satthertwaite and the Bossert-Storcken theorems either. Whether it is more or less susceptible to manipulation than the ISU Rule or any positional rule seems to remain an open question. However, the fact that it satisfies the local independence condition certainly limits the possibilities of its manipulation. Moreover, the fact that it involves complex computations certainly does not make it easy to manipulate. Susceptibility to manipulation and ease of manipulation are two different things.

Under (XCC) and the Kemeny rule, if a subset of judges is unable to arrive at a cycle in the majority relation whatever their way of ranking the competitors, then they are unable to manipulate the final ranking since the latter will be given by the majority relation alone. Under these circumstances, the best way for these judges to make sure that the final ranking resembles their own ranking is to report the latter.

If a subset of judges can produce a cycle by strategically ranking the competitors, thus forcing the use of the Kemeny rule per se, then this strategic behaviour, to be successful, must at the same time produce a Kemeny order preferable to the final ranking that would be obtained otherwise. This may be impossible or would require a lot of sophistication from these judges. Their task would be easier if there was a cycle in the majority relation before any attempt at manipulation. However, they would have to be aware of the presence of this cycle before hand.

Of course, this begs the question of the possible lack of objectivity by the judges. In their empirical study of the results of the 24 olympic competitions analyzed in this paper, Campbell and Galbraith (1996) find strong evidence of the presence of a small national bias, the latter being more marked for medal contenders than for less strong competitors.

10. Conclusion

This paper shows that the occurrence of cycles in the majority relation is not only a theoretical possibility. It also shows that the choice of a rule is not merely an academic question. Aside from the sensitivity to manipulation, it can have a real impact on the results. Many disagreements have been found between the ISU, the Borda, the Kemeny rankings and the one according to raw marks in 24 olympic competitions. Many of the differences occurred for middle places but in many instances, the Kemeny rule gave a tie for the first, second and third ranks instead of a strict order.

The choice of a particular ranking rule should be based on its properties. While the ISU puts forward a majority principle as the basis of its ranking rule, this principle is at odds with the usual Condorcet criterion, which says that if a competitor is ranked ahead of all other competitors by an absolute majority of judges, he should be first in the final ranking. The ISU rule also conflicts with an extension of this principle to other ranks, which goes as follows: If a competitor is ranked consistently ahead of another competitor by an absolute majority of judges, he or she should be ahead in the final ranking. Consistency refers to the absence of cycle involving these two skaters.

The Extended Condorcet Criterion defined in this paper may give a complete ranking of all competitors in ideal circumstances. In the case of cycles, an additional criterion is however needed to break cycles while retaining Condorcet's objective in mind, which was to find a ranking with the highest probability of being the true ranking. The Kemeny rule fulfils this objective if the judges are able to choose the better of two competitors with a probability larger than one half. It thus seems especially appropriate when the decision is being taken by a group of experts as is the case in skating.

Young (1995) advocates this rule for social decision purposes. Le Breton and Truchon (1997) propose a measure of how far from the Borda rule a social choice function may be. They find that the Kemeny rule fares best, as compared to other rules satisfying the Condorcet criterion. It does almost as well as the Condorcet rule. This paper shows that, for most practical applications, Kemeny orders can be found easily with a laptop computer.

Another alternative to the ISU Rule would be to use the Borda rule from the beginning of the process through the obtention of the global ranking. This rule seems to be used in many local competitions. The Borda rule has much to command for itself. It is a positional rule, which satisfies interesting requirements such as another *reinforcement* condition (different from the one defined in Section 7) and *participation*. The first one says that, if two distinct panels of judges select the same winner, then the joint panel should also select this winner. Participation means that if an additional judge succeeds in changing the winner, it can only be in the sense that he or she favours. There is no rule consistent with the Condorcet criterion, such as the Kemeny rule, that satisfies these two requirements. The reader is referred to Young (1974) or Moulin (1988) for more details on these rules. Based on the findings of Saari (1990), the Borda rule also appears to be less susceptible to manipulation by a minority of judges than the ISU Rule. Manipulation by a large coalition of judges is not really an issue in rating skating.

In their Monte Carlo experiment, BP contrast the ISU Rule with a method that consists in summing the raw marks of the judges. They start with true marks for all competitors that result in a complete ranking, their true ranking. The marks of each judge are then obtained by adding a random term to the true marks, truncating the result at 6.0. With true marks ranging from 4.8 to 5.8, they find that the ISU Rule outperforms the simple addition of the raw marks in picking the true winner (54% of the time versus 46%). It would be interesting to see how the Kemeny and the Borda rules would have performed in picking the true winner as compared to the ISU Rule. In the olympic data, the Kemeny rankings had, on a whole, slightly less disagreements and the Borda rankings had significantly less disagreements with the rankings according to the raw marks than the ISU rankings.

BP's results raise an interesting theoretical question: Is there a good theoretical basis for throwing away the raw marks of the judges and retaining only the relative rankings drawn from these marks? BP's simulations suggest that neglecting part of the information may play an important role in smoothing out errors in the judges' marks. We might also think of a role for this procedure in reducing the scope for manipulation. However, both contentions would need a firmer theoretical justification.

Table 1: The ISU Example

s	Judges									Criteria				Final rank			
	1	2	3	4	5	6	7	8	9	ρ_s	$n_{\rho s}$	$B_{\rho s}$	B_s	ISU	BC	XCC	K
A	1	1	1	1	1	3	1	4	4	1	6		17	1	1	1	1
B	3	3	2	2	1	1	2	3	3	2	5	8	20	2	2	2	2
C	2	2	4	3	3	2	3	1	1	2	5	8	21	3	3	3	3
D	6	6	5	6	4	7	5	2	2	5	5	18	43	4	4	4	4
E	4	4	6	4	7	6	8	5	5	5	5	22	49	5	5	5	5
F	4	5	8	7	5	4	4	8	8	5	5	22	53	6	6	6	6
G	8	8	3	9	9	5	6	6	6	6	5		60	7	7	7	7
H	7	7	7	5	6	9	7	7	7	7	8		62	8	8	8	8
I	11	10	12	12	10	11	12	11	11	11	6		100	9	10	?	11
J	12	9	13	11	8	8	13	9	12	11	5	45	95	10	9	9	9
K	10	12	11	10	13	10	14	10	15	11	5	51	105	11	12	?	10
L	13	11	14	8	11	13	11	14	10	11	5	51	105	11	12	?	13
M	9	13	9	14	15	14	9	12	9	12	5		104	13	11	?	12
N	15	15	10	13	12	12	10	15	13	13	6		115	14	14	14	14
O	14	14	15	15	14	15	15	13	14	14	5		129	15	15	15	15

Table 2: Example 2

s	Judges					Criteria				Final rank			
	1	2	3	4	5	ρ_s	$n_{\rho s}$	$B_{\rho s}$	B_s	ISU	BC	XCC	K
A	1	3	1	3	3	3	5		11	2	1	1	1
B	2	2	2	4	4	2	3		14	1	3	2	2
C	3	4	3	1	1	3	4		12	3	2	3	3
D	4	5	4	2	2	4	4		17	4	4	4	4
E	5	1	5	5	5	5	5		21	5	5	5	5

Table 3: Example 3

s	Judges							Criteria				Final rank			
	1	2	3	4	5	6	7	ρ_s	$n_{\rho s}$	$B_{\rho s}$	B_s	ISU	BC	XCC	K
A	2	2	2	6	6	6	1	2	4		25	1	3	?	2
B	3	3	3	1	1	1	6	3	6		18	2	1	?	3
C	4	4	4	4	4	4	2	4	7		26	4	4	?	4
D	5	5	6	3	3	2	3	3	4		27	3	5	?	5
E	1	1	1	5	5	5	5	5	7		23	6	2	?	1
F	6	6	5	2	2	3	4	4	4		28	5	6	?	6

Table 4: Comparison of Four Ranking Rules in 24 Olympic Competitions

Competition	Short Program	Kemeny	Borda	Marks	Free Program	Kemeny	Borda	Marks	Remarks
Men 1976	ISU	4.5	2.5	3.5	ISU	2.	0	0	A 6-cycle, tie for ranks 3-4, inversions in 5-9 in short pr. A 3-cycle, tie for 2-3 in free pr.
	Marks	3.	2.		Marks	2.	0		
	Borda	3.			Borda	2.			
Men 1988	ISU	2.5	3.	7.5	ISU	2.	3.	5.	Three 3-cycles in short program.
	Marks	7.	4.5		Marks	4.	2.		
	Borda	2.5			Borda	3.			
Men 1992	ISU	0.5	1.	2.	ISU	0	1.5	0.5	A 3-cycle in short program. A 5-cycle and a 3-cycle in free program.
	Marks	2.5	1.		Marks	0.5	2.		
	Borda	1.5			Borda	1.5			
Men 1994	ISU	2.	1.5	3.5	ISU	1.5	3.5	5.5	No cycle. Tie for ranks 3-4 in short program.
	Marks	1.5	2.		Marks	4.	5.		
	Borda	1.5			Borda	3.			
Women 1976	ISU	2.5	1.	4.	ISU	1.	3.5	3.5	No cycle. Inversion in ranks 3-4, tie for ranks 5-6, 7-8, 9-10 in short program.
	Marks	4.5	3.		Marks	2.5	2.		
	Borda	1.5			Borda	3.5			
Women 1988	ISU	8.	5.	8.	ISU	2.	2.5	3.	A 9-cycle (ranks 12-20) in short program. Inversions starting at rank 12 in both programs.
	Marks	7.	3.		Marks	3.	3.5		
	Borda	6.			Borda	3.5			
Women 1992	ISU	7.5	7.	6.5	ISU	1.5	4.	3.5	Two 3-cycles, inversion in ranks 7-8, tie for 3-4 in short program. A 3-cycle, tie for ranks 1-2, 4-5, 10-11 in free program.
	Marks	4.	3.5		Marks	5.	1.5		
	Borda	5.5			Borda	4.5			
Women 1994	ISU	4.5	3.	4.	ISU	2.	2.	5.5	Two 3-cycles, inversion for ranks 9-10 in short program. Tie for ranks 1-2 in free program
	Marks	6.5	3.		Marks	5.5	3.5		
	Borda	4.5			Borda	2.			
Couples 1976	ISU	2.	1.5	3.	ISU	1.	0	1.	A 4-cycle (ranks 10-13) in short program.
	Marks	3.	3.5		Marks	2.	1.		
	Borda	3.5			Borda	1.			
Couples 1988	ISU	0	1.	0.5	ISU	0.5	0	2.	No cycle. Tie for ranks 5-6 in free program.
	Marks	0.5	0.5		Marks	1.5	2.		
	Borda	1.			Borda	0.5			
Couples 1992	ISU	0	1.	1.	ISU	1.5	1.	2.	No cycle. Tie for ranks 3-4 in free program.
	Marks	1.	0		Marks	0.5	1.		
	Borda	1.			Borda	0.5			
Couples 1994	ISU	1.	2.	2.	ISU	0.5	1.	2.	No cycle.
	Marks	1.	0		Marks	1.5	1.		
	Borda	1.			Borda	0.5			

The numbers in the cells are measures of the disagreement between the rankings.

Table 5: Comparison of Four Ranking Rules in the Women Short Program of the 1988 Olympic Competition

Skater	ISU	Marks	Borda	Kemeny
A	1	1	1	1
B	2	2	2	2
C	3	3	3	3
D	4	4	4	4
E	5	5	5	5
F	6	6	6	6
G	7	8	7	7
H	8	7	7	8
I	9	9	9	9
J	10	10	10	10
K	11	11	11	11
L	12	14	14	14
M	13	12	12	12
N	14	13	13	13
O	15	16	15	18
P	16	18	17	15
Q	17	14	15	17
R	18	19	19	20
S	19	17	18	16
T	20	19	20	19
U	21	21	21	21
V	22	22	22	22
W	23	23	23	23

Appendix

Rule 371 of the ISU

Determination of results of each part of a competition

1. The competitor[1] placed first by the absolute majority of Judges in a part of the competition is first; he who is placed second or better by an absolute majority is second and so on.

2. For this purpose, the place numbers 1 and 2 count as second place; place numbers 1, 2 and 3 count as third place, and so on.

3. If two or more competitors have obtained a majority for the same place, the first among them is he who has been so placed by the greater number of Judges.

4. If such majorities are equal, then the lowest total of place numbers of those Judges forming the majority determines between them.

5. If the total of the place number is equal according to paragraph 4, the sum of the place numbers of all Judges determines the result; if this is also equal the competitors are tied.

6. If there is no absolute majority for a place, the result for such place must be ascertained by seeking the best majority for the following place; and if there is no such majority then by seeking the best majority for the next following place and so on.

7. If such majorities are equal under paragraph 6, the systems referred to in paragraphs 4 and 5 must be applied.

8. The ascertainment of each place must first be made in accordance with paragraphs 1 through 5, and thereafter according to paragraphs 6 and 7 in the above mentioned order.

9. a) If two or more competitors are temporarily tied with majorities for the same place, the place must be awarded to one of those competitors on the basis of paragraphs 3, 4 and 5. After awarding the place, the remaining temporarily tied competitor(s) must be awarded the next following place(s) on the basis of paragraphs 3, 4 and 5 without considering any additional competitors.

b) In awarding the subsequent places thereafter, the unplaced competitors with a majority for the lowest numbered place shall be given first consideration.

10. If the foregoing rules fail to determine the award of any place, then the competitors tied for that place must be announced as tied. If two competitors so tie for first place, the next place to be awarded is third place (not second). If two skaters so tie for second place, the next place to be awarded is fourth place (not third) and so on.

---

[1] Rule 371 adds "or the team" after every instance of "competitor". It has been omitted.

References

Arrow, K.J., *Social Choice and Individual Values*, New York: John Wiley and Sons, second edition, 1963.

Bassett, W. and J. Persky, "Rating Skating," *Journal of the American Statistical Association*, 89 (1994), 1075-1079.

Barthelemy, J.P, A. Guénoche, and O. Hudry, "Median Linear Orders: Heuristics and a Branch and Bound Algorithm," *European Journal of Operational Research*, 42 (1989), 313-325.

Benoit, J.P., "Scoring Reversals: A Major League Dilemma," *Social Choice and Welfare*, 9 (1992), 89-97.

Borda, J.C. de, "Mémoire sur les élections au scrutin," *Histoire de l'Académie royale des sciences*, 1784.

Bossert, W. and T. Storcken, "Strategy-proofness of Social Welfare Functions: The Use of the Kemeny Distance Between Preference Orderings," *Social Choice and Welfare*, 9 (1992), 345-360.

Campbell, D.C. and G. Tullock, "A Measure of the Importance of Cyclical Majorities," *Economic Journal*, 75 (1965), 853-857.

Campbell, B. and J.W. Galbraith, "Non-Parametric Tests of the Unbiasedness of Olympic Figure Skating Judgements," *Journal of the Royal Statistical Society Ser. D.: The Statistician*, 4 (1996), 521-526.

Condorcet, Marquis de, "Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix," Paris: Imprimerie royale, 1785. Also reproduced in Condorcet, *Sur les élections et autres textes*, edited by O. de Bernon, Fayard, 1986.

Fishburn, P.C., *The Theory of Social Choice*, Princeton: Princeton University Press, 1973.

Gibbard, A., "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41 (1973), 587-601.

Good, I.J., "A Note on Condorcet Sets," *Public Choice*, 10 (1971), 97-101.

International Skating Union, *Regulations*, 1994.

Jech, T., "The Ranking of Incomplete Tournaments: A Mathematician's Guide to Popular Sports," *American Mathematical Monthly*, 90 (1983), 246-266.

Kelly, J.S., *Impossibility Theorems*, New York: Academic Press, 1978.

Kelly, J.S., "Condorcet Winner Proportions," *Social Choice and Welfare*, 1 (1986), 311-314.

Kemeny, J. "Mathematics without Numbers," *Daedalus*, 88 (1959), 571-591.

Laplace, P.S., *Oeuvres*, Tome 7, Paris : Imprimerie Royale, 1912.

Le Breton, M. and M. Truchon, "A Borda Measure for Social Choice Functions", *Mathematical Social Sciences*, 34 (1997), 249-272.

Levin, J. and B. Nalebuff, "An Introduction to Vote-Counting Schemes," *Journal of Economic Perspectives*, 9 (1995), 3-26.

Moulin, H., *Axioms of Cooperative Decision Making*, Cambridge: Cambridge University Press, 1988.

Muller, E. and M.A. Satterthwaite, "The Equivalence of Strong Positive Association and "Strategy-proofness," *Journal of Economic Theory*, 14 (1977), 412-418.

Rousseau, J.J., *Du Contrat Social*, Paris: 1762.

Rousseau, J.J., *The Social Contract*, London: J.M. Dent & Sons, 1913.

Saari, D.G., "Susceptibility to Manipulation," *Public Choice*, 64 (1990), 21-41.

Satterthwaite, M.A., "Strategy-proofness and Arrow's Condition: Existence and Correspondence Theorem for Voting Procedures and Social Welfare Function," *Journal of Economic Theory*, 10 (1975), 187-217.

Schwartz, T., "Rationality and the Myth of the Maximum," *Noûs*, 6 (1972), 97-117.

Truchon M., "An Extension of the Condorcet Criterion and Kemeny Orders," Cahier de recherche 9813, Département d'économique, Université Laval, October 1998.

United States Figure Skating Association, *Rulebook*, 1992.

Young, H.P., "An Axiomatization of Borda's Rule," *Journal of Economic Theory*, 9 (1974), 43-52.

Young, H.P., "Condorcet's Theory of Voting," *American Political Science Review*, 82 (1988), 1231-1244.

Young, H.P., "Optimal Voting Rules," *Journal of Economic Perspectives*, 9 (1995), 51-64.

Young, H.P. and A. Levenglick, "A Consistent Extension of Condorcet's Election Principle," *SIAM Journal of Applied Mathematics*, 35 (1978), 285-300.