Endogenous value and Financial Fragility

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Abstract

We construct a model of valuation to assess the financial fragility of a set of firms in a closed economy. A firm is identified with a possibly infinite random sequence of benefits. Firms with negative benefits in a given period are said to be in distress and need liquidity to refinance their projects. Those liquidities must be obtained from firms with positive benefits (which represent excess liquidities). Distressed projects are refinanced to the extent that their need for liquidity does not exceed their endogenous continuation value. This value is, in turn, affected by current and future refinancing possibilities. We provide a recursive procedure to compute this value when there is an aggregate liquidity constraint. We compare the allocation under a centralized coalition of firms with that of a decentralized competitive liquidity market. We show that the competitive market is more fragile because it does not value the possibility that a currently distressed firm could become a provider of liquidity some period in the future. That is, the market value of a firm can diverge from its social value due to externalities involving the ability of that firm to refinance other distressed firms in the future.

1 Introduction

A system is financially fragile relative to another when its expected value in the steady state is lower due to an inability to manage liquidity in the system in a manner that is dynamically

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efficient. We show that a decentralized mechanism for allocating liquidity is more fragile than a centralized system due to the divergence in social and market values of firms when there is a potential for aggregate liquidity constraints to bind in any period. That is, a market mechanism is unable to correctly value firms in terms of their ability to provide liquidity to the system in the future, and hence can allow a firm to go bankrupt even though it is socially more valuable than another firm that is refinanced. This is because this potential to be a liquidity supplier in the future increases the values of other firms but this externality is not accounted for in the market value of firms.

Correctly valuing a firm (or a project) is a central issue in finance. The value of a firm is typically equal to the expected discounted value of its future benefits, conditioned on its survival. In the autarkic case where no refinancing is available, the firm will eventually go bankrupt when there is a positive probability of distress, and the computation of its value takes this probability into account. The probability of bankruptcy enters into the "effective" discount rate. The difficulty in the computation of the value arises when refinancing is potentially available but subject to an endogenous liquidity constraint. In this dynamic context, the flow of future benefits in the firm is conditioned by the possibility of financial distress and its ability to obtain refinancing in future periods, should it become distressed. Bankruptcy is then endogenous to current and future refinancing possibilities, and the computation of the firm's value becomes a non-trivial exercise.

In an environment of perfect financial markets, there are no liquidity constraints facing the firm as long as its value, net of its liquidity needs, remains positive. Firms are easy to value in this world, and bankruptcy, when it occurs, is efficient. We present a model of firm valuation when financial markets are imperfect. We focus on a limited aggregate supply of liquidity as a source of market imperfection. A firm may not be able to obtain financing even though it would be profitable to do so because the aggregate supply of liquidity is bounded. This assumption can limit the extent of refinancing a firm can obtain, and affect its current value. In addition, current and potential liquidity constraints create a divergence in a firm's social and market value, which causes a decentralized market for liquidity to be dynamically inefficient, or, financially fragile.

In this model, a firm is identified with an infinite random sequence of benefits, conditional on its survival. Each period, a firm realizes a net benefit. For example, this benefit is its cash flow consisting of revenues minus costs net of any new investment requirement. If this benefit is below a threshold level (normalized to zero), the bank is in distress and needs refinancing to pursue its activities. Without refinancing, it must declare bankruptcy. If this benefit is positive, the firm can choose to either consume its benefits, or use it to refinance a distressed firm. We obtain a procedure for valuing firms when there is a potential aggregate shortage of liquidity. We suppose that there is no deep-pocket financier that could refinance all firms whenever it is optimal to do so. Instead, we have a finite number of firms which can provide financing to each other when they have the liquidity to do so. As long as a the value of a firm is greater than its liquidity needs, it is optimal to refinance it. This may not be possible, however, if the other existing firms do not generate enough liquidity to refinance the distressed firm. A firm may become financially vulnerable because the aggregate supply of liquidity in the economy is low, and not because its net value falls below zero.

Within this context, we study two specific environments. In the first, we assume that all firms are part of a coalition in which financing decisions are centralized to maximize the value of this coalition. In each period, the set of surviving firms is chosen to maximize the future value of the coalition of surviving firms. If there is an aggregate liquidity constraint, some firms cannot be refinanced and must be shut down. The decision about which firms should survive in this case depends on the marginal contribution of firms to the future value of the coalition. This contribution depends on the ability of a given firm to "rescue" some other firms in the future. We compute a specific two-firm example to illustrate our results. In the second environment, we assume that, instead of a centralized decision-making mechanism, there exists a market for liquidity and distressed firms must borrow on this market at the equilibrium rate of interest. For each period, we characterize the equilibrium interest rate that determines which firms are refinanced. These are the firms that have the highest market value net of refinancing costs.

We then compare the efficiency of these two organizations. For each case, we show that the economy converges to a stable coalition of firms, a set in which no bankruptcies can occur. This limit set may be history dependent. More interestingly, we show that the two organizations can produce different sets of stable coalitions. Any stable coalition in a decentralized market is also stable in the centralized organization, but the converse is not true. In a decentralized market, the firms with the highest market value net of refinancing costs are refinanced. This value, however, does not include the impact that the firm may have on the future refinancing possibilities of other firms. When there is an aggregate liquidity constraint that is expected to be binding in some future period, each firm has a shadow value that depends on its potential for rescuing other firms in that period. That is, each firm has an externality on the value of other firms.[1] The market for liquidity cannot take this externality into account while a centralized organization can. For example, suppose that firm A has a higher net market value than firm B today, but that firm B is more likely to "rescue" from bankruptcy firm C in the future (maybe because its returns are negatively correlated with those of firm C). Suppose there is an aggregate liquidity constraint that prevents the refinancing of both firms A and B. A central planner may prefer to rescue firm B than firm A if this increases the value of firm C sufficiently. However, a decentralized market does take this externality into account when computing firms' value. In this sense, the market is not dynamically efficient. This is why the market is more fragile than a centralized organization. We use a simple numerical example to show how the market may fail to correctly compute firm's true value while a centralized coalitional organization would perform efficiently.

The issue of endogenous bankruptcy has already been studied in the literature on optimal capital structure. Using a no-arbitrage argument, Merton (1974) computes the value of a firm's equity when its benefits follow a diffusion-type stochastic process. Merton (1974) assumes that the firm issues a zero-coupon bond with maturity at time T. If the assets' value is less than the debt's face value at T, the firm is bankrupt and the equity is worth 0. This makes the equity value resemble a European call option, which is valued using the

[1] This externality vanishes when there is no aggregate liquidity constraint.

Black and Scholes' (1973) formula. Merton's formula per se does not consider bankruptcy as an endogenous event. It can be used, however, to price any claim on a firm whose benefits are described by a diffusion process.

Leland (1994) considers a more complex type of debt with a continuous coupon, and computes the equity value when bankruptcy is either exogenous or endogenous. Bankruptcy is exogenous when it is triggered by the assets' value falling below a predetermined exogenous target level. Bankruptcy is endogenous when it is triggered by the impossibility to pay the coupon by issuing additional equity. In this case, there is a minimum value V_B of the firm's assets below which equity is worth 0 and the firm is bankrupt. The firm chooses this lower bound to maximize the total value of the firm. On the one hand, the lower bound V_B must be low enough to minimize the occurrence of bankruptcy; on the other hand, it cannot be too low since equity must remain positive for assets' value above the bound. Leland (1994) finds that the lower bound V_B on the assets' value that triggers bankruptcy is proportional to the debt coupon, independent of current assets' value, increasing in the risk-free rate of interest and decreasing in the volatility of the assets' value process. Leland (1994) assumes that the firm can always refinance on the market as long as its equity value is positive. This translates into an environment of perfect financial markets. In this model, bankruptcy is said to be efficient.

Den Haan, Ramey and Watson (1999) also study the fragility of an economic system in which there is an aggregate liquidity constraint. Borrowers and lenders are matched and, in each period, lenders get a random liquidity endowment. The realized endowment affects the viability of a match. The main difference of this paper from our approach is that they assume that there is no short-run market for liquidity. Assuming that liquidities can flow across agents is a main feature of our analysis. We show that an economy may still be fragile despite having a short-run competitive market for liquidities.

In section 2, we introduce the model and notation. We then compute the value of a firm in two benchmark cases: in autarky and when there is a deep-pocket financier who supplies liquidity in each period. In the following sections, we assume that the aggregate supply of liquidity is finite and given by the cash flow realizations of all firms in the economy (that is, it is endogenous). In section 3, we develop our centralized coalitional model and illustrate our results with a two-firm example. In section 4, we assume a decentralized market for liquidity in each period, and characterize the market equilibrium. In section 5, we compare the efficiency of the two organizations and illustrate our results with a numerical two-firm example. The conclusion follows.

2 The model

Consider a multi-period, single good economy where all consumers have (risk-neutral) linear preferences with respect to random consumption paths. They discount future consumption by a common factor δ. Consumers are assumed to have rational expectations, that is, they perfectly anticipate future prices and coordinate on the same equilibrium if many equilibria can exist.

There is an infinite random sequence of i.i.d. states $(s_n)_{n\in\mathbb{N}}$ where n is a time subscript. Each state s_n is drawn from $(S, \mathcal{S}, \mu)$ where S is a compact set of states, $\mathcal{S}$ is a σ-algebra on S and μ is a probability measure. In what follows, the time subscript is dropped whenever this does not create any confusion. Hence, s usually refers to the current state.

There are N productive projects, owned by the consumers.[2] The number of projects can decrease in time with the occurrence of bankruptcy. However, we forbid the entry of new projects. Each period, projects generate random benefits measured in units of the consumption good. A project is described by a measurable continuous function $y : S \to \mathbb{R}$ which relates each state, s, to the random benefit, $y(s)$, the project generates in that state.

A negative benefit generated by a project represents a temporary shortage of liquidity that prevents it from investing in its technology in order to continue to create value in the

[2]In this paper, we use the terms "project" and "firm" interchangeably.

future. A negative benefit that is not refinanced results in the bankruptcy of the project. We assume limited liability so that if a project has a negative benefit and declares bankruptcy, it forgoes its financial liabilities. A bankrupt project can never be reactivated so that if it goes bankrupt in period n, it brings a benefit of zero in period n and all subsequent periods. A positive benefit, on the other hand, creates excess liquidities that can be used to refinance other projects or consumed by the owners of the project. There is no storage technology for transferring liquidities in the current period to a future period: all positive benefits created in the economy must be used in the same period.

A project is said to be in *financial distress* in state s if $y(s) < 0$. We say that the project is *solvent* in one period if its benefit is non-negative or if it can obtain refinancing to survive until next period. Since there is no storage technology, refinancing can only be obtained from positive benefits realized by other projects.

Let us denote the current population of N projects by y. For a subset z of the population y, $z(s)$ is the set of benefits generated by each project in z in state s. The sum of the elements of $z(s)$ is denoted by $\Sigma\, z(s)$. Furthermore, $z(s)^+$ is the subset of those benefits that are non-negative, and z_s^+ is the subset of z obtained using the labels associated with the values of $z(s)^+$. $z(s)^-$ and z_s^- are defined the same way.

Autarky

A project that lives in complete autarky has no access to any refinancing. It is solvent if and only if its benefit is non negative. The value of an autarkic project is then the expected discounted sum of its current and future benefits taking into account that it goes bankrupt whenever its benefit $y(s)$ is negative. Up to a bankruptcy episode, benefits are stationary. Hence, the continuation value is either zero if the project is bankrupt or some constant non-negative expected discounted value if the project is solvent.

Let us denote by y^+ (y^-), the set of states in which $y(s) \geq 0$ ($y(s) < 0$), that is,

$$y^+ \equiv \{s \in S | y(s) \geq 0\}, \quad \text{and} \quad y^- \equiv \{s \in S | y(s) < 0\}.$$

We will keep this notation for any other measurable function on S throughout the paper. Under the assumption of stationarity of the benefit function y, the value of the project only depends on the current state, and may be defined as a random variable $v_0(y) : S \to \mathbb{R}$,

$$v_0(y)(s) = \begin{cases} y(s) + \delta V_0(y) & \text{if } s \in y^+, \\ 0 & \text{if } s \in y^-, \end{cases} \tag{1}$$

where $\delta \in (0, 1)$ is the discount rate. Let us denote $V_0(y)$ the expected value of $v_0(y)$. Because benefits are stationary, this expected continuation value is constant. Hence, taking the expectation on (1) yields

$$\begin{aligned} V_0(y) &= \mathbb{E}\{v_0(y)\} \\ &= \mu(y^+)\mathbb{E}\{y + \delta V_0(y) | y^+\} \\ &= \frac{1}{1 - \delta\mu(y^+)} \mu(y^+)\mathbb{E}\{y | y^+\}, \end{aligned} \tag{2}$$

where $\mathbb{E}\{y|y^+\}$ is the conditional expectation of y given event the set y^+. Equation (2) yields a formula for the valuation of a currently solvent project that has a constant probability $\mu(y^-)$ of becoming bankrupt.

Unconstrained refinancing for a single project

Let us suppose that the project has access to refinancing in states where its current benefit is negative, $y(s) < 0$. Refinancing the project makes economic sense if its continuation value is greater than its current liquidity requirement $-y(s)$. Thus, current and future refinancing can increase the value of the project. This implies that the continuation value itself is affected by the availability of refinancing in the future. Hence, the probability that the project fails again in the future is not necessarily $\mu(y^-)$, and $V_0(y)$ is no longer the expected future value of the project.

In states s where $y(s) < 0$, the project needs at least $-y(s)$ in order to face its liquidity requirement and survive until next period. The maximal amount of liquidities the firm can raise is equal to the expected discounted value of all future benefits, again taking into account future possibilities of bankruptcy and refinancing.

Define by S^* the set of states in which the firm is either not distressed or is successfully refinanced, and, therefore, solvent. Since the decision to refinance is independent of current financial liabilities and benefits are stationary, the set S^* is time independent. Using similar computations as those in the previous section, the expected discounted value of all future benefits is given by

$$\frac{\delta}{1-\delta\mu(S^*)}\mu(S^*)\mathbb{E}\{y|S^*\}.$$

This is the maximum amount of financial capital the firm can raise. Hence, the firm is solvent in state s only if its net present value is non negative, that is, if:

$$y(s) + \frac{\delta}{1-\delta\mu(S^*)}\mu(S^*)\mathbb{E}\{y|S^*\} \geq 0. \tag{3}$$

The set S^* is the set of states s for which condition (3) is satisfied. It is easy to see that, if $s \in S^*$, then all states s' such that $y(s') \geq y(s)$ are also in S^*. This implies that there exists some lower bound y^* below which the firm is optimally bankrupt.

The lower bound y^* must be negative, because it is never optimal to declare bankruptcy when the current benefit is positive. The set of solvency states is given by $S^* = \{s \in S|y(s) \geq y^*\} = (y - y^*)^+$. The lower bound y^* solves

$$y^* + \frac{\delta}{1-\delta\mu((y-y^*)^+)}\mu((y-y^*)^+)\mathbb{E}\{y|(y-y^*)^+\} = 0. \tag{4}$$

This equality implicitly defines the set S^*.

We can now compute the expected value of the project, using $y(s) < y^*$ as the bankruptcy condition. In any period and state s, we have

$$v_{y^*}(y)(s) = \begin{cases} y(s) + \delta V_{y^*}(y) & \text{if } s \in S^*, \\ 0 & \text{if } s \in S \setminus S^*. \end{cases} \tag{5}$$

Taking expectations on (5) yields

$$\begin{aligned} V_{y^*}(y) &= \mathbb{E}\{v_{y^*}(y)\}, \\ &= \mu(S^*)\mathbb{E}\{y + \delta V_{y^*}(y)|S^*\}, \\ &= \frac{1}{1-\delta\mu(S^*)}\mu(S^*)\mathbb{E}\{y|S^*\}. \end{aligned} \tag{6}$$

Equation (6) gives the expected value of the project in an environment without liquidity constraints. This value depends on the time-independent survival policy function which characterizes the efficient bankruptcy rule. For $y(s) \geq y^*$, it is profitable to keep the project operating. Bankrupting it would destroy value since its future value is larger than the amount of liquidity required to keep it solvent. For $y(s) < y^*$, it is optimal to bankrupt the project since its future value is smaller than the amount of liquidity required to keep it solvent.

Without an aggregate liquidity constraint, a project can raise funds up to its discounted expected value taking into account the probability of bankruptcy. The value $V_{y^*}(y)$ can be compared to the autarkic value $V_0(y)$, which corresponds to the case $y^* = 0$. It is easily shown that $V_{y^*}(y) \geq V_0(y)$, and therefore the availability of outside liquidity raises the value of the project.

Refinancing firms in the face of aggregate liquidity constraints

From now on, we relax the assumption that there is no aggregate liquidity constraint. We suppose instead that liquidities have to be supplied by existing projects and hence cannot exceed the sum of positive benefits in the economy, $\Sigma\, y(s)^+$. Therefore, a project must rely on other projects' liquidities to refinance a negative benefit. However, the availability of refinancing for a project also depends on the demand for liquidity by other projects. This means that there might be some states where a given project should optimally be refinanced but may not be, due to aggregate liquidity being insufficient. The survival of a project now depends on the aggregate liquidity of the economy. That means that the value of a project y is no longer equal to $V_{y^*}(y)$.

For example, there may be states s and s' such that $y(s) = y(s')$ but the project is solvent in state s and bankrupt in state s' although its current liquidity requirement and future expected value are *the same* in both states.[3] *Liquidity constraints* may bind at the aggregate level so that states s and s' differ in the sense that it is easier for the project to get refinancing in state s than in state s'. Hence, liquidity constraints increase the probability that a project fails and reduce its value.[4] This is important since when a project goes bankrupt, the aggregate flow of liquidity in the future is reduced. This could jeopardize the solvency of other projects in the future.

The determination of which distressed projects go bankrupt when there is not enough aggregate liquidity for all of them depends on the allocation mechanism. In the next section, we compute project values when aggregate liquidities are optimally allocated by a central planner. The optimal allocation maximizes the value of the group of projects surviving in each possible state of nature. In section 4, we decentralize the allocation of funds so that projects can obtain funds from a liquidity market at a competitive price.

[3]Since a state s is a description of the whole economy, it is conceivable that a project may have the same benefit in two different states, while benefits of other projects differ in these two states.

[4]To a large extent, our model fits this story: exogenous shocks on the total supply of funds affect the "effective" discount rate different projects face since they affect their probability of bankruptcy. This can be contrasted with standard macroeconomic models where changes in the "effective" discount rate are driven by exogenous technological shocks.

3 A centralized model of refinancing

The ability of projects to obtain refinancing is limited by the aggregate constraint on the supply of liquidity. We derive a recursive formula to compute the value of a *coalition* of projects. A *coalition* is a finite set of projects belonging to a network and that provide each other with liquidities. Our approach is to maximize the current expected value of the coalition's liquidities. This is done through a complex financial "contract" that optimally assigns realized liquidities to a surviving coalition.

3.1 The coalition model

We take the convention that y denotes the current coalition before the realization of the state of nature in any period. Since there is no entry of new projects and not all projects survive from one period to the other, the existing population may decrease with time. A coalition y faces a *liquidity constraint* in a given state, if the sum of all positive liquidities in the coalition is lower than the sum of requirements by distressed projects that are worth saving.[5] In this case, only a sub-coalition of y can survive and some projects must disappear. The centralized mechanism optimally designs a survival policy that determines which project should be refinanced and which should be bankrupted. The coalition z that survives after coalition y, and realization of state s, is *feasible* if and only if it satisfies the following two properties:

Admissibility (AD)**:** If a project y has a non-negative benefit in state s, then it must belong to the surviving coalition in state s. Equivalently, if z is solvent in state s, then $y_s^+ \subseteq z$.

Budget Balance (BB)**:** If coalition z survives in state s, then

$$-\Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s).$$

[5] The decision of which projects are worth refinancing will be explicitly analysed and this decision depends on the allocation mechanism we assume.

In any given state s, admissibility requires that all projects in the set y_s^+ survive. Budget Balance ensures that the surviving coalition satisfies the aggregate liquidity constraint. This is possible if and only if the total liquidity requirement of these distressed projects in the surviving coalition, z, does not exceed the total liquidity generated by the projects with positive benefits.

The optimal survival policy maximizes the value of the surviving coalition. It is thus necessary to compute the value of all possible coalition of projects. Suppose that we know how to compute the expected value of an arbitrary coalition of projects z of size less than or equal to $M \geq 1$. Let $V(z)$ be this expected value. In what follows, we show how to compute the value of an arbitrary coalition y of $M + 1$ projects. Let 2^y be the power set of sub-coalitions of y. Assume that the current set of active projects is y. In state s, an optimal survival policy selects a coalition that solves

$$\text{Program 1}: \quad \max_{z \in 2^y} \quad \Sigma\, z(s) + \delta V(z),$$

$$\text{s.t.} \quad y_s^+ \subseteq z, \tag{AD}$$

$$-\Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s). \tag{BB}$$

This problem is well defined by assumption, up to $V(y)$ which is unknown. That is, the expected value $V(z)$ of all subcoalitions z of no more than M projects is known by assumption, but the expected value of the (current) coalition y of $M + 1$ projects is unknown.

By admissibility (AD), for all states s such that y_s^- is empty, the set of instruments contains only y and Program 1 reduces to

$$\Sigma\, y(s) + \delta V(y). \tag{7}$$

Consider now the states for which y_s^- is not empty. The following restricted program (for which y is not a solution) is now well defined,

$$\begin{aligned} \text{Program 1a}: \quad & \max_{z \in 2^y} \quad \Sigma\, z(s) + \delta V(z), \\ & \text{s.t.} \qquad y_s^+ \subseteq z, && \text{(AD)} \\ & \qquad\qquad - \Sigma\, z_s^-(s) \leq \Sigma\, z_s^+(s), && \text{(BB)} \\ & \qquad\qquad z \neq y. \end{aligned}$$

By construction, we know how to solve Program 1a since $V(y)$ need not be evaluated.

Program 1 can be represented as a dynamic program where, if y_s^- is non-empty, one decides first if y should survive and, in the case where it should not, which coalition z should survive. Define the random variable $\nu : S \to \mathbb{R}$ that takes the value of Program 1a. The value $v(s)$ of Program 1 then becomes

$$v(s) = \begin{cases} \Sigma\, y(s) + \delta V(y), & \text{if } y_s^- = \emptyset, \\ \max\left\{\Sigma\, y(s) + \delta V(y), \nu(s)\right\}, & \text{if } y_s^- \neq \emptyset \text{ and } \Sigma\, y(s) \geq 0, \\ \nu(s), & \text{otherwise.} \end{cases}$$

Since this is a stationary value, $V(y) = \mathbb{E}(v)$.

Now let

$$S^* = \left\{s \in S |\, \Sigma\, y(s) + \delta V(y) \geq \nu(s) \text{ and } \Sigma\, y(s) \geq 0\right\}.$$

This is the set of states where the full coalition y survives, either because y_s^- is empty, or because it is feasible and profitable to refinance all distressed projects. In what follows, we assume that $\mu(S^*) \in (0, 1)$. The following lemmas describe the solution.

Lemma 1. $\{s \in S | y_s^- = \emptyset\} \subseteq S^* \subseteq \{s \in S |\, \Sigma\, y(s) \geq 0\}$.

All proofs are relegated to the Appendix.

Lemma 2. *Monotonicity. Let $s \in S^*$ and consider s'. If, for all projects, $y(s') \geq y(s)$, then $s' \in S^*$.*

For any given $\mathbf{y}$, the value of the coalition $\mathbf{y}$, is the real number $V(\mathbf{y})$ that solves (8):

$$V(\mathbf{y}) = \max_{S^*} \quad \mu(S^*)(\mathbb{E}(\Sigma\, \mathbf{y}|S^*) + \delta V(\mathbf{y})) + (1 - \mu(S^*))\mathbb{E}(\nu|\neg S^*). \tag{8}$$

We have shown in section 2 that a coalition composed of a single project ($\mathbf{y} = \{y\}$) has an expected value of $V(\mathbf{y}) = V_0(y)$. We have shown that if we know how to compute the expected value of M projects or less, we may compute the value of $M + 1$ projects. By induction, we can therefore compute the expected value of an arbitrary but finite coalition of projects. In the next section, we do so explicitly for a coalition of 2 projects.

3.2 A two-project coalition

Let $\mathbf{y} = \{y_1, y_2\}$. We know that $V(y_1) = V_0(y_1)$ and that $V(y_2) = V_0(y_2)$. We want to compute $V(\mathbf{y})$. To do so, we need to identify S^*.

By Lemma 1, we need only to identify those states where only one project is distressed and it makes economic sense to refinance it. If $y_j(s) > 0 > y_i(s)$ and $y_i(s) + y_j(s) \geq 0$, then project y_i will be rescued if

$$y_i(s) + y_j(s) + \delta V(\mathbf{y}) \geq y_j(s) + \delta V_0(y_j),$$

that is, if

$$y_i(s) \geq \delta(V_0(y_j) - V(\mathbf{y})) \equiv y_i^{**}.$$

Hence, both projects remain solvent as long as $\Sigma\, \mathbf{y}(s) \geq 0$ and each $y_i(s)$ is at least equal to some endogenous stationary value y_i^{**} that depends on $V(\mathbf{y})$. $V(\mathbf{y})$ may be obtained as the solution to (8) where

$$S^* = \{s \in S | y_1(s) \geq y_1^{**}, y_2(s) \geq y_2^{**} \text{ and } y_1(s) + y_2(s) \geq 0\}.$$

Notice that the fact that y_i^{**} is independent of s is an artifact of the two-project coalition. In general, this threshold value depends on the state s. For example, suppose there are three projects. Further assume that only one project is solvent (say project 1) and that it can only refinance one of the two distressed projects. Whether say project 2 is refinanced or not depends not only on the net future payoff of doing so (as it is the case with two projects), but also on the cost of bankrupting project 3. This cost depends on the current amount of liquidity needed to refinance project 3. Hence, the survival rules depends on the state s for a coalition of three or more projects.

Finally, it is now possible to isolate the individual value of a single project within coalition $\boldsymbol{y}$. Denote the value of project $y_i \in \boldsymbol{y}$ by

$$V^{y_i}(\boldsymbol{y}) = \frac{\mu(S^* \cup S_{y_i})\mathbb{E}\left(y_i \mid S^* \cup S_{y_i}\right) + \mu(S_{y_i})\delta V_0(y_i)}{1 - \delta\mu(S^*)},$$

where S_{y_i} is the set of states for which only project y_i is solvent. The value of project y_i depends implicitly on the value of the whole coalition through its dependence on the set S^*. Individual values are such that $V^{y_1}(\boldsymbol{y}) + V^{y_2}(\boldsymbol{y}) = V(\boldsymbol{y})$. However, the individual value must be distinguished from the *contributory* value of y_i to coalition $\boldsymbol{y}$. The contributory value is the difference of values between the coalition $\boldsymbol{y}$ with the project y_i and the coalition without it, that is,

$$CV^{y_i}(\boldsymbol{y}) = V(\boldsymbol{y}) - V(\boldsymbol{y} \setminus y_i) = V(\boldsymbol{y}) - V_0(y_j) = -\delta^{-1} y_i^{**}$$

where $\boldsymbol{y} \setminus y_i$ is the remaining coalition after removing project y_i from the coalition $\boldsymbol{y}$. The sum of the two contributory values in the coalition $\boldsymbol{y}$ exceed the value of the coalition, or,

$$CV^{y_1}(\boldsymbol{y}) + CV^{y_2}(\boldsymbol{y}) = 2V(\boldsymbol{y}) - V_0(y_2) - V_0(y_1) > V(\boldsymbol{y}).$$

The contributory value of a project exceeds its individual value: $CV(y_i) = V(\boldsymbol{y}) - V_0(y_j) > V(\boldsymbol{y}) - V^{y_j}(\boldsymbol{y}) = V^{y_i}(\boldsymbol{y})$ since $V_0(y_j) < V^{y_j}(\boldsymbol{y})$. Each project, therefore, has a shadow value that reflects its externality on the value of the other project.

4 Decentralization

We now decentralize our coalition economy to examine the characteristics of the surviving set of projects when refinancing can be obtained from other projects at a market price. A project that realizes a positive benefit in any period is a supplier of funds at that period. A distressed project has a demand for liquidity whenever its realized benefit is negative.

4.1 The liquidity market

There is a spot market for funds on which the size of supply and demand determines the equilibrium price of liquidity, that is, the lending rate. Funds are measured in units of the consumption good. However, the liquidity market excludes the demand for final consumption of the good. Once funds have been allocated on the liquidity market, the residual funds, if any, are transferred to the consumption market, where they are consumed.

A project may enter a period with an obligation to repay a debt or a claim on the debt repayment from its participation in the liquidity market in the previous period. Suppose that a project with a negative benefit today has lent the amount $x > -y(s)/R$ in a previous period that entitles it to receive Rx today. In that case, the project's net liquidity is $y(s) + Rx > 0$. Nevertheless, if the project's owners decide to use the amount Rx to rescue their project, they are lending the liquidity to themselves. An alternative option would be to let the project die and invest Rx on the liquidity market. A negative benefit indicates that an investment in the project is required in order for it to survive, this does not mean that the project's owners owe $-y(s)$ and are obliged to pay for it out of their loans portfolio. Hence, whether it is used by the project to refinance itself or invested in other project, the amount Rx is part of the supply of fund. For the same reason, the amount $-y(s)$ potentially becomes part of the demand for funds.

The financial instrument exchanged by projects on this market is generic. It could be a share in the project or a promise of a future payment. Since all agents are risk-neutral, the equilibrium risk premium is necessarily zero. Consequently the value of every financial instrument is equal to its expected discounted payoff measured in units of the good.

Supply of funds

The supply of funds in state s is driven by the constant marginal rate of substitution of consumers, δ, since consumers supply funds on the liquidity market through their ownership of the various projects. The liquidity generated by projects that are not distressed in state s is $\Sigma y(s)^+$. Let R be the market interest rate. Then, consumers agree to provide liquidity on the market as long as $R \geq \delta^{-1}$. If R is too low, consumers prefer to consume the funds rather than to invest it on the liquidity market. The *gross supply of liquidity* is then

$$Y^S(s,R) \in \begin{cases} 0 & \text{if } R < \delta^{-1}, \\ [0, \Sigma y(s)^+] & \text{if } R = \delta^{-1}, \\ \Sigma y(s)^+ & \text{if } R > \delta^{-1}. \end{cases}$$

Demand for funds

The demand for funds comes from distressed projects. A project with benefit $y(s) < 0$ needs $-y(s)$ to be rescued. Other projects will agree to finance $-y(s) > 0$ as long as they receive a financial instrument that is worth at least $-y(s)$. Since they are the residual claimants of the project's portfolio, the owners of the project are willing to issue a claim worth $-y(s)$ on future benefits if keeping the project alive has more value than $-y(s)$. Let $V_m(y)$ be project y's market value if rescued. The project is said to be solvent if

$$V_m(y) \geq -y(s)R.$$

The owners of a project that is not solvent will never choose to seek refinancing for it.[6] Hence, the demand for funds from insolvent projects is zero.

[6] Another way to state this is that a project that is not solvent is not allowed on the market.

The demand for funds by a solvent project is

$$d_y(s,R) \in \begin{cases} \{0, -y(s)\} & \text{if } V_m(y) = -y(s)R, \\ -y(s) & \text{if } V_m(y) > -y(s)R. \end{cases}$$

When the market rate of return is R, the set of distressed but solvent projects is

$$\chi \equiv \left\{ y \in \mathcal{y}_s^- | V_m(y) \geq -y(s)R \right\}.$$

The aggregate demand for funds is then

$$Y^D(s,R) = \sum_{y \in \chi} d_y(s,R).$$

Equilibrium

Because the demand for funds by a solvent project is insensitive to the rate of return and because there is a finite number of projects (all with positive measure), the demand for funds is discontinuous since a marginal raise in the rate of return can turn a solvent project into an insolvent one and reduce its demand from some positive amount,$-y(s)$, to zero. As a result, an equilibrium in which demand and supply are equated may fail to exist for some configurations of the parameters. As we shall see, the set of parameters where that happens is of measure zero. Nevertheless, we adapt our description of the market mechanism to preclude the non existence of an equilibrium.

Demand is non-increasing, zero if the market price R is sufficiently high, and equal to $-\Sigma \mathcal{y}(s)^- \geq 0$ if the market price is zero. Likewise, supply is non-decreasing, zero if R is low and equal to $\Sigma \mathcal{y}(s)^+ \geq 0$ if R is high. An equilibrium would surely exist if demand was continuous. At a very high price such that demand is zero and supply is $\Sigma \mathcal{y}(s)^+$, excess supply equals $\Sigma \mathcal{y}(s)^+ \geq 0$. At a price of zero, excess supply equals $\Sigma \mathcal{y}(s)^- \leq 0$. By the theorem of the mean value, there is a value of R for which this difference is equal to zero.

Discontinuities in demand imply that there may be excess supply at some price R and excess demand at some price R' which is marginally lower that R. Our notion of an equilibrium must thus account for an excess supply at the "equilibrium" price R. As we shall see, the behavior of the model depends crucially on the total demand that can be accommodated by the market. An excess supply of funds only affects the timing of consumption but has no effect on the overall performance of the economy. Consequently, we devise a rationing device that regulates an excess supply to yield an equilibrium on the market.

The device works as follows. Suppliers in the market are told that if there is a strictly positive demand, an unspecified strictly-positive fraction α, $\alpha \in (0, 1]$, of their supply will be channelled through the market. The rest of their supply will be returned for consumption. Notice that the supply of funds is unaffected by this device: if supplying z was optimal when the price is R and $\alpha = 1$, then supplying z is still optimal as long as $\alpha > 0$. Once the demand and the supply of funds have been expressed at an equilibrium price (to be defined below), the parameter α is set by the market operator to a value that clears the market: $\alpha = Y^D / \Sigma y^+$.

An equilibrium is reached when the price R clears the market with $\alpha = 1$ or when there is excess supply and a reduction of the price R would lead to an excess demand. This definition captures the idea that competition among suppliers should drive R down. Formally, given the demand and supply schedules Y^D and Y^S, we have an equilibrium price R^* when either

1. $Y^D(s, R^*) = Y^S(s, R^*)$.

2. $Y^D(s, R) < Y^S(s, R)$ for all $R \geq R^*$ and $Y^D(s, R) > Y^S(s, R)$ for all $R < R^*$.

The equilibrium allocation is defined to be $Y^D(s, R^*)$. Various possible equilibrium configurations are represented in Figure 1 in Appendix B.

4.2 A two-project economy

Suppose there currently exist two projects on the market: y_1 and y_2. Each project has a continuation value that is equal to the expected discounted value of all its future benefits. The discount rate must be the equilibrium interest rate. Notice that in a case with only two projects, the equilibrium rate is $R^* = \delta^{-1}$. If supply exceeds demand, either the distressed project has $-V_m(y_i)/y_i(s) \geq \delta^{-1}$ and the project is refinanced at rate $R^* = \delta^{-1}$ or the project fails and there is no exchange. If demand exceeds supply, the distressed project cannot be refinanced, regardless of its value, and the market collapses. In any case, there is no room for an equilibrium lending rate higher than δ^{-1}.

The expected value of future benefits depends on the probability of failure of the projects in the future. Moreover, continuation values depend on the state prevailing today, namely, a project can have three possible continuation values: (1) either the project fails and the continuation value is 0, (2) or the other project fails and y_i's continuation value is its autarkic value $V_0(y_i)$, or (3) no project fails and y_i's continuation value is the two-project market value $V_m(y_i)$. Notice that in an i.i.d. environment, $V_m(y_i)$ does not evolve in time. Let us describe all the potential market configurations that follow the realization of a state of nature.

1. States S^{all} such that $y_i(s) > 0$ for all i.

   In those states, $Y^D(s, R) = 0$ at all R. Both projects survive with **no exchange** of liquidity and will be on the market next period. Their continuation value is $V_m(y_i)$, $\forall i$.

2. States $S^{\emptyset}$ such that $y_i(s) < 0$ for all i.

   In those states, $Y^S(s, R) = 0$ at all R. Both projects die, and their value reverts to 0. There will be no market open next period.

3. States S_i^{cstr} such that $y_i(s) < 0 < y_j(s)$ with $y_i(s) + y_j(s) < 0$.

   In those states, the **aggregate liquidity constraint binds**. Project y_i fails, its value reverting to 0. Project y_j survives with its autarkic value $V_0(y_j)$.

4. States $\hat{S}_i$ such that $y_i(s) < 0 < y_j(s)$ with $y_i(s) + y_j(s) > 0$.

   In those states, y_i needs refinancing from y_j. Since the **liquidity constraint does not bind**, there is an excess supply of funds that drives the equilibrium interest rate to $R^* = \delta^{-1}$. Hence, project y_i receives refinancing $-y_i(s)$ if its continuation value $V_m(y_i)$ is at least equal to $y_i(s)/\delta$. Let us partition the set of states $\hat{S}_i$ into

4.1 $\hat{S}_i^+$, set of states such that $V_m(y_i) \geq y_i(s)/\delta$ so that project i finds refinancing at the market rate and can, therefore, survive with continuation value $V_m(y_i)$;

4.2 $\hat{S}_i^-$, set of states such that $V_m(y_i) < y_i(s)/\delta$ so that the project's value reverts to 0 and it fails.

The partition of $\hat{S}_i$ is endogenous since it depends on the market value $V_m(y_i)$ itself.

Since the autarkic value $V_0(y_i)$ has been determined in section 2 for any y_i, only the market value $V_m(y_i)$ is left to compute. This value is the continuation value of project y_i following the realization of a state in either S^{all} or $\hat{S}_i^+$. Since the probability of failure of project i in the future depends on its decision to refinance y_j in earlier periods, both values cannot be independent. The market values are, then, equal to

$$\begin{aligned}
V_m(y_1) &= \left(\mu(S^{\emptyset}) + \mu(S_1^{cstr}) + \mu(\hat{S}_1^-)\right) \times 0 + \left(\mu(S_2^{cstr}) + \mu(\hat{S}_2^-)\right) \mathbb{E}\left(y_1(s) + \delta V_0(y_1) | S_2^{cstr} \cup \hat{S}_2^-\right) \\
&\quad + \left(\mu(S^{all}) + \mu(\hat{S}_1^+) + \mu(\hat{S}_2^+)\right) \mathbb{E}\left(y_1(s) + \delta V_m(y_1) | S^{all} \cup \hat{S}_1^+ \cup \hat{S}_2^+\right) \\
V_m(y_2) &= \left(\mu(S^{\emptyset}) + \mu(S_2^{cstr}) + \mu(\hat{S}_2^-)\right) \times 0 + \left(\mu(S_1^{cstr}) + \mu(\hat{S}_1^-)\right) \mathbb{E}\left(y_2(s) + \delta V_0(y_2) | S_1^{cstr} \cup \hat{S}_1^-\right) \\
&\quad + \left(\mu(S^{all}) + \mu(\hat{S}_2^+) + \mu(\hat{S}_1^+)\right) \mathbb{E}\left(y_2(s) + \delta V_m(y_2) | S^{all} \cup \hat{S}_2^+ \cup \hat{S}_1^+\right).
\end{aligned}$$

Consequently, market values are jointly determined with the set of states $\hat{S}_1^+$ and $\hat{S}_2^+$ as the solution of the following system of 4 equations:

$$\begin{aligned}
V_m(y_1) &= \frac{\left(\mu(S_2^{cstr}) + \mu(\hat{S}_2) + \mu(S^{all}) + \mu(\hat{S}_1^+)\right) \mathbb{E}\left(y_1(s) | S_2^{cstr} \cup \hat{S}_2 \cup S^{all} \cup \hat{S}_1^+\right)}{1 - \delta\left(\mu(S^{all}) + \mu(\hat{S}_1^+) + \mu(\hat{S}_2^+)\right)} \\
&\quad + \frac{\left(\mu(S_2^{cstr}) + \mu(\hat{S}_2^-)\right) \delta V_0(y_1)}{1 - \delta\left(\mu(S^{all}) + \mu(\hat{S}_1^+) + \mu(\hat{S}_2^+)\right)} \\
V_m(y_2) &= \frac{\left(\mu(S_1^{cstr}) + \mu(\hat{S}_1) + \mu(S^{all}) + \mu(\hat{S}_2^+))\right) \mathbb{E}\left(y_2(s) | S_1^{cstr} \cup \hat{S}_1 \cup S^{all} \cup \hat{S}_2^+\right)}{1 - \delta\left(\mu(S^{all}) + \mu(\hat{S}_2^+) + \mu(\hat{S}_1^+)\right)} \\
&\quad + \frac{\left(\mu(S_1^{cstr}) + \mu(\hat{S}_1^-)\right) \delta V_0(y_2)}{1 - \delta\left(\mu(S^{all}) + \mu(\hat{S}_2^+) + \mu(\hat{S}_1^+)\right)}
\end{aligned}$$

$$y_1(s) + \delta V_m(y_1) \geq 0,\ y_1(s) < 0,\ \text{and } y_1(s) + y_2(s) > 0 \quad \text{for all } s \in \hat{S}_1^+$$

$$y_2(s) + \delta V_m(y_2) \geq 0,\ y_2(s) < 0,\ \text{and } y_1(s) + y_2(s) > 0 \quad \text{for all } s \in \hat{S}_2^+.$$

The partition of the set S is much more complex in a case with three projects. It encompasses the states in which all projects fail, those in which only two project fail, leaving the other one in autarky, and the case where only one project fail, leaving the two others with those two-project market values. Finally, the projects' market values in the states where all three survive rely on all these less-than-three-project cases and a rate of interest $\mathbb{R}^*$ that can be higher than δ^{-1}. Having computed the three-projects values, it is possible to find the market values in the four-project case with an even more complex partition of the set of states of nature. Appendix C provides a numerical example where market values are computed in a two-project case.

5 Static and dynamic efficiency

Since the value of projects, and hence the survival rule, depends on whether liquidities are allocated by a central planner (centralized mechanism) or through a decentralized liquidity market, the allocation mechanism can condition the fragility of the system. To compare the performance of both mechanisms, we need to be able to compare the set of existing projects in each case, after a given history of realization of the states of nature. A natural point of comparison is the coalitions arrived at in the steady state under the two mechanisms. We will explain the notion of fragility more clearly when we have defined the idea of a steady state.

Let y be a running coalition of projects. As the history of shocks evolves, this coalition shrinks if some projects become bankrupt. Hence, the number of surviving projects weakly decreases through time until a stationary state is reached. Let us define this stationary state with the notion of stable coalition.[7]

[7]Note that the term coalition in this context does not imply that the allocation mechanism is centralized.

Definition 1 (Stable coalitions). *Let y be the existing coalition in the beginning of a period. The coalition y is stable if it is the surviving coalition after any realization of the state of nature in this period.*

A stable coalition defines the stationary state because states of nature are drawn from identical and independent distributions in every period. If a coalition survives through all states in one period, it must survive in any state in the future.

There are two necessary conditions for a coalition y to be stable. One condition is that budget balance holds in every state of nature, or there are no states in which the aggregate liquidity constraint is binding. The other condition is that no project has to be bankrupted in any state of nature. This latter condition differs according to whether the mechanism is centralized or not.

Stable coalition with a centralized mechanism *In a centralized allocation mechanism, y is a stable coalition if y is feasible and if there is no smaller coalition that would have a greater value in any state of the world. Formally, y is stable if*

$$\sum y(s)^{+} \geq -\sum y(s)^{-} \quad \textit{and}$$
$$y = arg\ \max_{z} \sum z(s) + \mathbb{E}\left(V(z)\right) \quad \textit{for all} s \in S.$$

Stable coalition on a decentralized market *In a decentralized mechanism, y is stable if, in all possible states, the supply of funds is greater than the demand for funds and no project is insolvent at the decentralized equilibrium. Formally,*

$$\sum y(s)^{+} \geq -\sum y(s)^{-} \quad \textit{and}$$
$$R^{*} \leq \underline{R} = \min_{y \in y_s^{-}} \{-V_m(y)/y(s)\} \quad \textit{for all } s \in S.$$

If the existing coalition of projects is stable in a market equilibrium, the aggregate liquidity constraint never binds and, thus, the market gross rate of return is $R^{*} = \delta^{-1}$.

The term applies to any group of projects supplying funds to each other as defined earlier.

In accordance with both these definitions, we can say that the empty set is stable. This means that at least one stable coalition exists. Since there is no entry, the number of projects in the economy can only weakly decrease in time. However, the rate at which projects disappear and the stable coalition that is reached depends on the history of states of nature that were realized. This means that project failures that follow temporary liquidity shocks make shocks permanent. With no entry of projects in the system, a failure in period t may trigger further failures in the future. Suppose that the set of stable coalitions achievable by a given allocation mechanism includes sets other than the empty set. We can say that a system is *fragile* because the history of realized states can force the system towards a less valuable stable coaltion. In the extreme, a system can be forced towards the empty set. Furthermore, since all firms that belong to any stable coalition would have had an episode of distress but was refinanced, they all have positive value, both in the individual and contributory sense. Hence, the stable coalition with the larger number of projects is more valuable than one with a smaller number of projects.

The set of states in which all projects survive in a stable coalition is the set S itself since there are no bankruptcies. The value of a coalition is then simply equal to the discounted expected value of the cash-flows of all remaining projects in the centralized as well as in the decentralized mechanism,

$$V(y) = \frac{\mathrm{E}\left(\sum y\right)}{1-\delta}.$$

Projects are also easy to value in a decentralized market when we have a stable coalition, since their individual values can also be expressed as the expected discounted value of their cashflows. Hence,

$$V_m(y) = \frac{\mathrm{E}\left(y\right)}{1-\delta}$$

for all projects y in a stable coalition.

Result 1. *If y is a stable coalition in a decentralized market, then it is stable in a centralized institution.*

Proofs of this Result is given in Appendix A.

Result 2. *If y is a stable coalition in a centralized environment, it may not be stable in a decentralized one.*

A counterexample is enough to prove this last result. Consider the following simple environment with two projects y and z and 4 equiprobable states of nature. The following table gives the benefits of each project in each state of nature:

s :	1	2	3
y :	-1	Y	1
z :	1	0	-1

with $Y > 0$. Note that the economy faces no aggregate liquidity constraint. Suppose that the current state of nature is $s = 3$, so that project z is in financial distress. On a decentralized market, if project z was refinanced in state 3, its market value would be $V_m(z) = 0$. This is a contradiction because the required funds, $-z(3) = 1$, being greater than the discounted market value at all levels of the interest rate, project z cannot find refinancing in state 3. Hence, project z is bankrupt in state 3. However, next period, y could be in financial distress in state 1 with a need of $-y(1) = 1$ of funds. If project z has been bankrupt the period before, no funds are available for project y; there is now an aggregate liquidity constraint due to the disappearance of project z. Project y has, then, to fail also, even if its future value, be it some liquidity available to refinance it, would be $Y/(3(1-\delta)) > 1$ for Y sufficiently high.

A central planner would allocate funds differently. Maximizing the value of coalition $\{y, z\}$, it would allow project z to survive today in state 3 so that project y can be refinanced tomorrow if state 1 occurs.

Note that the contribution of project z in the centralized coalition is not 0, although the centralized value of the coalition is equal to $V(\{y, z\}) = Y/(3(1-\delta))$, that is, the value of y if there is no aggregate constraint. Indeed, in the absence of z, the liquidity constraint binds and y has to go bankrupt in state 1 . The contributory value of z is, then,

$$V^z(\{y,z\}) = V(\{y,z\}) - V_0(y) = \frac{\delta Y - (3-3\delta)}{(3-3\delta)(3-2\delta)} \ > \ 0 \ \text{ for } \ Y > \frac{3-3\delta}{\delta}.$$

This example illustrates Result 2, that is, there may be stable coalitions in the centralized case that cannot be reached on decentralized markets. Decentralized markets continue bankrupting projects that would survive forever through a centralized decision process. This allows us to conclude that a decentralized allocation mechanism on which projects exchange liquidities at a market rate of return, is in general more fragile than a centralized one where bankruptcies are managed in each state of nature in order to maximize the liquidity of the system. Since a stable coaltion with a larger number of projects is more valuable than one with a smaller number of firms, the more fragile system also has a lower expected value in the long run.

This may not be obvious in a static environment. Consider a two-period time horizon in which the liquidity market opens in the first period and debt repayments take place in the second period. In this case, $V_m(y)$ is both the private (market) *and* social values of a project y. Because the decentralized market refinances those projects with the highest market value, subject to the liquidity constraint, it will also pick the "right" projects to refinance and the spot market for liquidities yields an efficient allocation. In that sense, the market is statically efficient. We have shown, however, that a dynamic (infinite-horizon) framework yields a different result with respect to efficiency. In such a context, the decentralized market is, in general, dynamically inefficiency. The problem is that market value only accounts for the *private* value associated with a distressed project, that is, its expected discounted flow of future benefits. It does not account for the fact that this project, if refinanced, may be on the supply side of the market in the future and help relax the constraint on the supply of funds. Hence, the first welfare theorem does not hold in this economy.

In appendix C, we provide a numerical example where centralized and decentralized values are computed and compared. Dynamic inefficiency and the idea of financial fragility associated with Result 2 are made explicit by the example.

6 Conclusion

We show, in this paper, that the efficiency of a liquidity allocation system depends on its ability to measure the value of a project, taking into account its contribution to the liquidity of the economy in future periods. This contribution is not taken into account of by decentralized markets because it results in an externality which increases the values of other projects. In general, the larger the number of projects, the higher is the diversification of liquidity shocks. This means that an economy with a higher number of projects is less fragile. The main result of this study is given in Result 2 and states that a competitive liquidity market may be more fragile than a centralized institution. This has implications on how a public authority could supervise financial markets to make sure that liquidities are properly allocated among productive projects. The existence of a competitive financing rate for liquidity exchanges is necessary to signal the opportunity cost of liquidities and drive the price of capital in the economy. However, intervention by a market regulator to rescue a distressed project that cannot find refinancing on the liquidity market can help ensure that this liquidity market remains sound in the future.

It is interesting to consider the following interpretation to our model. The coalitional model can be related to a financial market with a financial intermediary. The intermediary allocates financing among its firms to maximize the value of its portfolio of firms. A long-lived financial intermediary can therefore endogenize the type of externalities that prevent the market from being efficient, that is, it can take into account the potential future contribution of a financially distressed firm when deciding to refinance it or not.

The only source of financial imperfection we consider is a potential shortage of liquidity at the aggregate level. If markets cannot decentralize the optimal allocation, firms may have to use complicated long-term contracts which would depend on all realized shocks in the economy. It would then be interesting to characterize the nature of these contracts when they suffer from this and other market imperfections such as non-commitment.

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Appendix A

Proof of Lemma 1

The first part comes directly from admissibility (AD). The second part, directly from the budget balance condition (BB). $\square$

Proof of Lemma 2

If $y_s^- = \emptyset$, the result is obvious. If $y_s^- \neq \emptyset$, then the question becomes: *given that we manage to keep all projects solvent, would we want to drop a project now that aggregate liquidity has risen?* The answer is "No". Suppose that in state $s \in S^*$ the coalition z survives, and that projects $w \subset z$ are bankrupt in state $s' \in S^*$. This implies that

$$\Sigma\, z(s) + \delta V(z) \geq \Sigma\, z \setminus w(s) + \delta V(z \setminus w). \tag{9}$$

In state s', y increases for all projects. Given stationarity, this affects only the first term on each side of condition (9). Since there are more projects in z than in $z \setminus w$, this condition must also be satisfied in state s'. Hence, it is not optimal to bankrupt more projects in s' than in s. $\square$

Proof of Result 1

Since budget balance holds, the market rate of return has to be equal to δ^{-1}. The stability of $\boldsymbol{y}$ implies that all y in $\boldsymbol{y}$ are such that $y(s) + \delta V_m(y) \geq 0$ for any possible s. Suppose $\boldsymbol{y}$ is not stable with a centralized institution, then, there is a state s in which sub-coalition $\bar{\boldsymbol{z}}$ must optimally be bankrupt. This also writes:

$$\Sigma\, \boldsymbol{y}(s) + \delta V(\boldsymbol{y}) < \Sigma\, \boldsymbol{z}(s) + \delta V(\boldsymbol{z}),$$

where $\boldsymbol{z}$ is the value maximizing coalition in state s, that is, the complement of $\bar{\boldsymbol{z}}$ in $\boldsymbol{y}$. This implies

$$\Sigma\, \bar{\boldsymbol{z}}(s) + \delta(V(\boldsymbol{y}) - V(\boldsymbol{z})) < 0.$$

By stability on the decentralized market, we must have that

$$\Sigma\, \bar{\boldsymbol{z}}(s) + \delta \sum_{y_j \in \bar{\boldsymbol{z}}} V_m(y_j) \geq 0.$$

This means that the contribution $(V(\boldsymbol{y}) - V(\boldsymbol{z}))$ of $\bar{\boldsymbol{z}}$ to the centralized value of $\boldsymbol{y}$ is smaller than its market value, that is, a contradiction. □

Appendix B

The decentralized equilibrium.

Various possible equilibrium configurations are represented in figure 1. In this figure, the demand is constant but many supply curves (from Y_0^S to Y_4^S) are drawn by changing the values of δ and $\Sigma \boldsymbol{y}^+$. Only Y_1^S is drawn in full for $\delta = \frac{1}{10}$ and $\Sigma \boldsymbol{y}^+ = 9$ (the numbers are arbitrary).

The aggregate demand is that of 7 troubled projects with pairs $(-y(s), V_y)$ equal to

$$\{(2,30),(3,36),(2,20),(3,30),(3,21),(2,6),(2,6)\}.$$

At prices $R > 15$, no project is solvent and all demand zero fund. At price $R = 15$, the first project becomes solvent since $V_y/R = 30/15 \geq -y(s) = 2$. At that price, the owners of the project are indifferent between rescuing or not the project, so that the demand for funds is either 2 or zero. For any lower price, that project is surely financed. For any price $R \in (12, 15]$, this is the only project that can be financed. At price $R = 12$, the second project may be financed since $36/12 \geq 3$. Whether it is or not, the demand for funds is either $5 = 2 + 3$ or 2. A marginal decrease in R sets the demand at 5. At price $R = 10$, two other projects can be financed, both having their ratio V_y/R equal to 10. Whether, zero, one or both projects are rescued, the aggregate demand equals 5, 7, 8 or 10. Then the price must be lowered to 7 to make project $(3, 21)$ solvent, etc.

Let $\overline{R} \equiv \max_{y^-}\{-V/y(s)\}$ and $\underline{R} \equiv \min_{y^-}\{-V/y(s)\}$. The generic equilibrium configurations are

- When $\delta^{-1} > \overline{R}$ no project is ever solvent at a price for which there is a positive supply of funds. Then any $R \in (\overline{R}, \delta^{-1}]$ is an equilibrium price with $Y^D(R^*) = 0$. See the case with Y_6^S in figure 1. The equilibrium set is E_6.
- When $Y^D(R^*) = Y^S(R^*)$ like at the crossing of the demand and the supply curve Y_3^S. Then $R^* \in [\underline{R}, \overline{R}]$. See the case with Y_3^S and the equilibrium point E_3.
- When $\delta^{-1} < \underline{R}$ and all projects are solvent. Then there is a sufficient supply of liquidities on the market to make rescuing all troubled projects profitable. The equilibrium price is then $R^* = \delta^{-1}$ and $Y^D(R^*) = \Sigma y^-$. See the case with Y_5^S and the equilibrium point E^5.

There are many instances of multiple equilibria that involve different equilibrium allocations (like with Y_1^S) but they are non generic cases. The supply curve Y_2^S yields an instance where there is rationing at the equilibrium price.

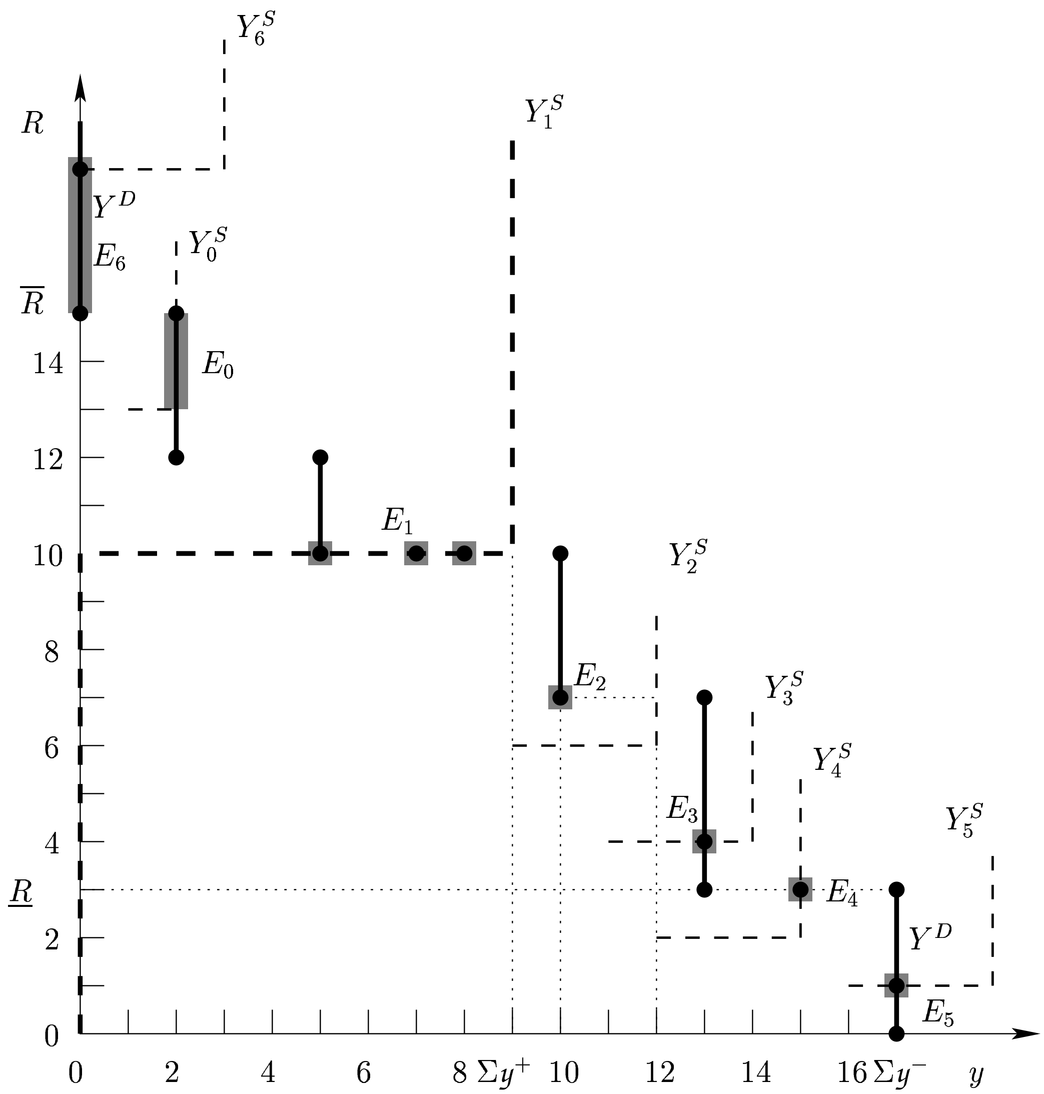

Figure 1: The spot market for liquidities.

Appendix C

Let the state of nature s be uniformly distributed on $S = [0,1]$. Consider a coalition of two projects $y = \{y_1, y_2\}$. Let us assume that project y_i's benefit is linear in s:

$$y_i(s) = (b_i - a_i)s + a_i.$$

Assume without loss of generality that $b_1 - a_1 = 1$ and let $y_i^h = \max\{a_i, b_i\}$ and $y_i^l = \min\{a_i, b_i\}$. Project y_i's random return is uniformly distributed on $[y_i^l, y_i^h]$. Its expected value in one period is

$$\mathbb{E}(y_i) = \frac{y_i^h + y_i^l}{2}.$$

Benefits of projects 1 and 2 are perfectly correlated and one can write : $s = \dfrac{y_1 - a_1}{b_1 - a_1} = y_1 - a_1$, so that

$$\begin{aligned} y_2 &= a_2 - (a_2 - b_2)(y_1 - a_1), \\ &= -\alpha y_1 + (a_2 + \alpha a_1), \end{aligned}$$

with $\alpha = a_2 - b_2$. Notice that total output is given by

$$y(s) = y_1(s) + y_2(s) = a + (1 - \alpha)s,$$

where $a = a_1 + a_2$.

We assume that $\alpha > 0$ so that y_1 and y_2 are perfectly negatively correlated. Total output $y(s)$ is positively correlated with s if $\alpha < 1$. Without loss of generality, we assume that this is the case so that $0 < \alpha < 1$. With this parameterization, variation of the total output reflects mostly variations in y_1.

We also assume that total output is always positive by setting $a > 0$. Our example is, then, specific in the sense that the aggregate liquidity constraint never binds. However, this assumption allows us to have that $\{y_1, y_2\}$ can possibly form a stable coalition.

Our assumptions so far imply that y_1 is distributed on $[a_1, b_1]$ with $y_1^l = a_1 < 0 < b_1 = y_1^h$ and that y_2 is distributed on $[b_2, a_2]$ with $y_2^l = b_2 < 0 < a_2 = y_2^h$. Besides, $y_1^h > 0$ implies that $1 + a_1 > 0$ and $y_2^l < 0$ implies that $\alpha > a_2$.

We partition the set of states of nature $[0, 1]$ into specific events. Let S_i be the event where project i survives and the other project goes bankrupt. The set of states where both projects survive, either because they have positive returns or because they refinance each other, is denoted S_{12}. Note that, since projects' benefits are negatively correlated, low (high) realizations of state s correspond to low (high) realizations of y_1 and high (low) realizations of y_2. Therefore, the three events correspond to ranked sub-intervals of $[0, 1]$. If not empty, S_2 corresponds to low values of s, and S_1 to high values of s, whereas intermediate values of s belong to event S_{12}.

The autarkic (expected) future value of a project, given by equation (2), writes

$$V_0(y_i) = \frac{{y_i^h}^2}{2((1-\delta)y_i^h - y_i^l)},$$

with

$$\mu(y_i^+) = \frac{y_i^h}{y_i^h - y_i^l},$$

$$\mathbb{E}\{y_i | y_i^+\} = \frac{y_i^h}{2}.$$

Finally, let us denote $V_\emptyset(y_i)$, the value of project i that would be realized if it was compelled to get refinanced in any event, that is, the expected discounted value of all possible benefits

$$V_\emptyset(y_i) = \frac{\mathbb{E}\{y_i\}}{1-\delta} = \frac{y_i^l + y_i^h}{2(1-\delta)}.$$

The centralized value of the coalition

Let $\underline{s}$ and $\overline{s}$ define the boundaries of the aforementioned events. Those values are such that $y_1(\underline{s}) = y_1^{**}$ and $y_2(\overline{s}) = y_2^{**}$. The state $\underline{s}$ (resp. $\overline{s}$) is the lowest (highest) state where project 1 (project 2) survives. Hence,

$$S_2 = [0, \underline{s}], \quad S_{12} = [\underline{s}, \overline{s}], \quad S_1 = [\overline{s}, 1].$$

The centralized value of coalition $\boldsymbol{y}$ is given by formula (8) in section 3:

$$V(\boldsymbol{y}) = \max_{S_{12} \in \mathcal{A}} \mu(S_{12})(\mathbb{E}(\Sigma\, \boldsymbol{y}|S_{12}) + \delta V(\boldsymbol{y})) + \mu(\neg S_{12})\mathbb{E}(\nu|\neg S_{12}), \tag{10}$$

where the solution S_{12} is the event S^* that maximizes the coalition value. For simplicity, let us now write $V(\boldsymbol{y}) = V$, $V_0(y_i) = V_i^0$ and $V_\emptyset(y_i) = V_i^\emptyset$. Hence, finding the value of the coalition amounts to find the highest fixed point V of

$$V - M^*(V) = 0, \tag{11}$$

where $M^*(V)$ is the r.h.s. of (10).

$$\begin{aligned} M^*(V) = \max_{\{\underline{s}, \overline{s}\}} \quad & \int_0^{\underline{s}} (y_2(s) + \delta V_2^0) ds + \int_{\underline{s}}^{\overline{s}} (y(s) + \delta V) ds + \int_{\overline{s}}^1 (y_1(s) + \delta V_1^0) ds, \\ \text{s.t.} \quad & \underline{s} \geq 0, && (12) \\ & 1 - \overline{s} \geq 0. && (13) \end{aligned}$$

The solution to that constrained problem is

$$\begin{aligned} \underline{s} &= \max\{0, -a_1 - \delta(V - V_2^0)\}, \\ &= \begin{cases} 0 & \text{if} \quad V \geq V_2^0 - a_1/\delta, \\ -(a_1 + \delta(V - V_2^0)) & \text{else.} \end{cases} \\ \overline{s} &= \min\{(a_2 + \delta(V - V_1^0))/\alpha, 1\}, \\ &= \begin{cases} 1 & \text{if} \quad V \geq V_1^0 - b_2/\delta, \\ (a_2 + \delta(V - V_1^0))/\alpha & \text{else.} \end{cases} \end{aligned}$$

Notice that since $V(y) \geq V_0(y_1) + V_0(y_2)$, we have that $V > V_i^0$ for $i = 1, 2$. This implies

$$
\begin{aligned}
0 &< a + \delta V + \delta(V - V_1^0 - V_2^0), \\
-a_1 - \delta(V - V_2^0) &< a_2 + \delta(V - V_1^0), \\
&< (a_2 + \delta(V - V_1^0))/\alpha, \\
\max\{0, -a_1 - \delta(V - V_2^0)\} &< \min\{(a_2 + \delta(V - V_1^0))/\alpha, 1\}, \\
\underline{s} &< \overline{s}.
\end{aligned}
$$

In any event, the solution $M^*(V)$ is a piecewise quadratic differentiable function of V.

Next, consider the following parameter values that yield an interior solution in $\underline{s}$ and $\overline{s}$. First set $y_i^l + y_i^h = 0$ so that $V_i^{\emptyset} = 0$ for all i. This ensures that projects have more value in autarky than if they are unconditionally refinanced $V_i^0 > V_i^{\emptyset}$ for all i. Hence, liquidating a project can be an option in some states, that is, $S_i \neq \emptyset$, $i = 1, 2$. This also means that coalition y is not stable.

Let $\alpha = 1/2$. These values imply that project 1's returns are distributed on $[-1/2, 1/2]$ and project 2's returns are distributed on $[-1/4, 1/4]$. The discount rate is set sufficiently low so that the system has interior solutions for the critical states: $\delta = 1/4$. With these values, we compute that

$$V_1^0 = 2/14 \simeq 0.14286, \qquad V_2^0 = 1/14 \simeq 0.07143.$$

The value V of the coalition will be at least as great as the sum $V_1^0 + V_2^0 = 3/14 \simeq 0.21429$ but should not exceed much that value. Conditions for an interior solution are

$$
\begin{aligned}
-a_1 - \delta(V - V_2^0) &> 0 \\
(a_2 + \delta(V - V_1^0))/\alpha &> 1,
\end{aligned}
$$

which yield

$$
\begin{aligned}
V &< V_2^0 - a_1/\delta, \\
V &< V_1^0 - b_2/\delta.
\end{aligned}
$$

Yet we find

$$V_2^0 - a_1/\delta = 29/14 \simeq 2.07,$$
$$V_1^0 - b_2/\delta = 16/14 \simeq 1.14.$$

It seems unlikely that $V \geq 16/14$ or $V \geq 29/14$. Hence assume as an educated guess that $V < 16/14$ so that $M^*(V)$ is computed with an interior solution

$$\underline{s} = \frac{29 - 14V}{56},$$
$$\overline{s} = \frac{24 + 28V}{56},$$
$$M^*(V) = \frac{3}{32}V^2 - \frac{5}{224}V + \frac{1353}{6272}.$$

Solving $V = M^*(V)$ then yields

$$V = \frac{229 - \sqrt{48382}}{42} \simeq 0.21526,$$

which is higher than 0.21429 and lower than 1.14 as previously stated. Besides

$$\underline{s} = \frac{\sqrt{48382} - 142}{168} \simeq 0.46404219,$$
$$\overline{s} = \frac{530 - 2\sqrt{48382}}{168} \simeq 0.536201344.$$

In autarky, project 1 is solvent as long as $y_1(s) \geq 0$, that is for $s \geq -a_1 = 1/2$ and project 2 is solvent as long as $y_2(s) \geq 0$, that is for $s \leq a_2/\alpha = 1/2$. The coalition allows optimal refinancing of project 1 by project 2 over $[\underline{s}, 1/2] \simeq [0.46, 0.50]$ and of project 2 by project 1 over $[1/2, \overline{s}] \simeq [0.50, 0.54]$.

Decentralized values

In a decentralized market, projects are refinanced as long as their continuation value is greater than their cash requirement. In a rational expectations equilibrium, continuation values depend on which of the three possible event, S_1, S_2 or S_{12} is realized. We now denote:

$$S_2 = [0, \hat{s}_1], \quad S_{12} = [\hat{s}_1, \hat{s}_2], \quad S_1 = [\hat{s}_2, 1].$$

The current market value of project 1 in state s depends on which of these three event s belongs to. Let us write $v_i^m(s)$ this value.

$$v_1^m(s) = \begin{cases} 0 & \text{if } s \in [0, \hat{s}_1), \\ y_1(s) + \delta V_1^m & \text{if } s \in [\hat{s}_1, \hat{s}_2], \\ y_1(s) + \delta V_1^0 & \text{if } s \in (\hat{s}_2, 1], \end{cases}$$

$$\begin{aligned} V_1^m &= E(v_1^m(s)), \\ &= (1-\hat{s}_1)E(y_1|s \geq \hat{s}_1) + \delta(\hat{s}_2 - \hat{s}_1)V_1^m + \delta(1-\hat{s}_2)V_1^0, \\ &= \frac{(1-\hat{s}_1)(a_1 + (1+\hat{s}_1)/2) + \delta(1-\hat{s}_2)V_1^0}{1-\delta(\hat{s}_2 - \hat{s}_1)}. \end{aligned}$$

Likewise for project 2

$$v_2^m(s) = \begin{cases} 0 & \text{if } s \in (\hat{s}_2, 1], \\ y_2(s) + \delta V_2^m & \text{if } s \in [\hat{s}_1, \hat{s}_2], \\ y_2(s) + \delta V_2^0 & \text{if } s \in [0, \hat{s}_1), \end{cases}$$

$$\begin{aligned} V_2^m &= E(v_2^m(s)), \\ &= \hat{s}_2 E(y_2|s \leq \hat{s}_2) + \delta(\hat{s}_2 - \hat{s}_1)V_2^m + \delta\hat{s}_1 V_2^0, \\ &= \frac{\hat{s}_2(a_2 - \alpha\hat{s}_2/2) + \delta\hat{s}_1 V_2^0}{1-\delta(\hat{s}_2 - \hat{s}_1)}. \end{aligned}$$

In a decentralized market, both projects will be refinanced as long as $y_i(s) + \delta V_i^m \geq 0$. Critical values $\hat{s}_1$ and $\hat{s}_2$ are such that $y_i(\hat{s}_i) + \delta V_i^m = 0$ or

$$\begin{aligned} y_1(\hat{s}_1) + \delta V_1^m &= 0, \\ y_2(\hat{s}_2) + \delta V_2^m &= 0. \end{aligned}$$

Applying the parameter values used in the centralized case gives the following non-linear system with two unknown, $\hat{s}_1$ and $\hat{s}_2$:

$$\begin{aligned} (1/2)\hat{s}_1^2 + 4\hat{s}_1 - \hat{s}_1\hat{s}_2 + (13/28)\hat{s}_2 - (55/28) &= 0, \\ (1/4)\hat{s}_2^2 - 2\hat{s}_2 - (1/2)\hat{s}_1\hat{s}_2 + (15/56)\hat{s}_1 + 1 &= 0. \end{aligned}$$

And it is (numerically) solved in

$$\hat{s}_1 \simeq 0.464123,$$
$$\hat{s}_2 \simeq 0.535877.$$

With these values, we have

$$V_1^m \simeq 0.143506,$$
$$V_2^m \simeq 0.071753,$$
$$V_1^m + V_2^m \simeq 0.215259.$$

The different values found with this simulation are summarized in the following table.

Centralized Sol.	—	—	$V = 0.21525983$	$[\underline{s}, \overline{s}] = [0.4640, 0.5362]$
Market Solution	$V_1^m = 0.1435$	$V_2^m = 0.0717$	$V_1^m + V_2^m = 0.21525980$	$[\hat{s}_1, \hat{s}_2] = [0.4641, 0.5359]$
Autarky	$V_1^0 = 0.1429$	$V_2^0 = 0.0714$	$V_1^0 + V_2^0 = 0.21429$	$[0.50, 0.50]$

In that simple example with no aggregate liquidity constraint, it appears that the centralized institution can value the externality each firm plays on the value of the other. The total market value, V, is greater than the sum of both market values, $V_1^m + V_2^m$. This is also made clear by the range of states of nature on which both firm can survive. This range is greater with a centralized institution, $[\underline{s}, \overline{s}] = [0.4640, 0.5362]$, than with a decentralized liquidity market $[\hat{s}_1, \hat{s}_2] = [0.4641, 0.5359]$.