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A New Sufficient Condition for Uniqueness in Continuous Games

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Abstract:

Consider the class of games in which each player chooses a strategy from a connected subset of the real line. Many oligopoly models fall into this class. In many of these applications, it would be useful to show that an equilibrium was unique, or at least to have a set of conditions under which uniqueness would hold.

In this paper, we first prove a uniqueness theorem that is slightly less restrictive than the contraction mapping theorem for mappings from the subsets of the real line onto itself, and then show how uniqueness in the general game can be shown by proving uniqueness using an iterative sequence of $\mathbb{R}$-to-$\mathbb{R}$ mappings. This iterative approach works by considering the equilibrium for an *m*-player game holding the strategies of all other players fixed, starting with a two-player game. If one can show that the *m*-player game has a unique equilibrium for all possible values for the remaining players strategies, then one can add one player at a time and consider the $\mathbb{R}$-to-$\mathbb{R}$ mapping from that player's strategy on to the unique equilibrium of the first *m* players and back onto the (*m*+1)th player's strategy.

We then show how a general condition for each one of this sequence of mappings to have a unique equilibrium is that the leading principal minors of a matrix derived from the Jacobean matrix of best-response functions be positive, and how this general condition encompasses and generalises some existing uniqueness theorems for particular games

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A New Sufficient Condition for Uniqueness in Continuous Games

1. Introduction:

Many oligopoly models fall into the class of continuous games in which each player chooses a strategy from a connected subset of the real line. For example, in Cournot models, each firm chooses a quantity to produce, in product differentiation models firms typically choose a price, and so on. Existence of a Nash equilibrium in pure strategies in such models are generally shown by means of the Brouwer or Kakutani fixed-point theorems. In many of these applications, it would be useful to show that an equilibrium was unique, or at least to have a set of easy-to-apply conditions under which uniqueness would hold.

There are a number of uniqueness conditions that have been found for particular subsets of this general class of models.[1] Generally, these involve a trade-off between generality and ease of application. For instance, the classic paper by Rosen (1965) offers a general sufficient condition for uniqueness in a wide class of games that encompasses the class considered in this paper, but the condition is quite opaque and hence difficult to apply directly to particular applications. At the opposite end of the generality/usability continuum, are uniqueness conditions that are specific to particular applications. For instance, conditions for a unique equilibrium in the Cournot quantity-setting oligopoly model have been derived by Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991), and Long and Soubeyran (2000).

One of the most useful general ways of showing uniqueness is through the contraction mapping theorem. The contraction mapping theorem provides a quite general condition for there to be a unique fixed point, with the added benefit that it guarantees existence without the requirements that the space being mapped onto itself be convex or bounded. To be useable, however, one needs to show that the particular mapping for which a fixed point defines an equilibrium constitutes a contraction mapping, a task that is not always straightforward.

In this paper, we derive a variant of the contraction mapping theorem, and present an approach in which this variant is applied iteratively to the class of games outlined above to derive an easy-to-apply uniqueness condition defined in terms of the slopes of the players' best-response functions. This uniqueness condition encompasses and generalises a number of existing uniqueness conditions. Furthermore, the theorems that show that particular games satisfy the general uniqueness condition derived in this paper have relatively simple induction proofs, suggesting that the general uniqueness condition could be easily applied to other specific games.

In the next section, we present the contraction mapping theorem and a related, less-restrictive theorem for the special case of a mapping from a subset of the real line onto itself. Section 3 lays out the general problem and shows by example why the conventional characterisation of an equilibrium as a fixed point of an $\mathbb{R}^n$-to-$\mathbb{R}^n$ mapping is too restrictive. Sections 4 shows how an equilibrium in the general game can be defined in terms of a

[1] A good survey of the existing uniqueness theorems is contained in Cachon and Netessine (2004).

sequence of contraction mappings involving $\mathbb{R}$-to-$\mathbb{R}$ mappings; Section 5 then shows how the uniqueness condition derived iteratively in this way can be represented in terms of the slopes of the best-response functions of each player. Section 6 shows how this general uniqueness condition encompasses and generalises many existing results. Section 7 concludes.

2. The Contraction Mapping Theorem and a Related Result.

A. The Contraction Mapping Theorem in Euclidean Space.

Typically in oligopoly models, the existence of an equilibrium is proved by showing the existence of a fixed point in a mapping from a subset of Euclidean space onto itself. Let $\mathbb{X}$ be a subset of $\mathbb{R}^n$, and let $f:\mathbb{X}\mapsto\mathbb{X}$ be a single-valued function mapping $\mathbb{X}$ onto itself. In this context, the definition of a contraction mapping and the contraction mapping theorem are as follows:

Definition 1:

If there exists $\beta\in(0,1)$ and a norm $\|\boldsymbol{x}\|$ such that

$$\|f(\boldsymbol{y})-f(\boldsymbol{x})\|\le\beta\|\boldsymbol{y}-\boldsymbol{x}\| \qquad \forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{X}, \tag{1}$$

then *f* is a *contraction mapping*.

Theorem 1 (The Contraction Mapping Theorem):

If $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and *f* is a contraction mapping, then

a) (existence and uniqueness) there exists a unique fixed point $x^*\in\mathbb{X}$ such that $f(x^*)=x^*$,

b) (convergence) for any $x\in\mathbb{X}$ and $n\ge 1$, $\|f^n(x)-x^*\|\le\beta^n\|x-x^*\|$.

The contraction mapping theorem has three advantages over the Brouwer or Kakutani fixed point-theorems if a contraction mapping can be shown to exist: First, and most importantly, it shows uniqueness as well as existence; second, it does not require that the set $\mathbb{X}$ be bounded; and third, it has the convergence property.

The convergence property implies that the unique fixed point can easily be found numerically. It can be useful in a game-theoretic context if we imagine the Nash equilibrium to be one iteration of a repeated game, as it suggests that the Nash-equilibrium outcome can be stable in the sense that if every period each player chooses the best response to the previous-period strategies of the other players, the game will converge to the unique equilibrium.

Such dynamic interpretations of a static equilibrium are not always appropriate, however, and numerical solveability is not often important. If we only require existence and uniqueness and not convergence, we can, in principle, relax Condition (1). We do this below for the case of case of $\mathbb{R}$-to-$\mathbb{R}$ mappings.

B. The Contraction Mapping Theorem in $\mathbb{R}^1$ Space.

In this paper, we show how existence of an equilibrium that is a point in Euclidean *n* space, can be represented as a set of fixed points of a sequence of mappings from the real line onto itself. For $\mathbb{R}$-to-$\mathbb{R}$ mappings, the natural norm to use is the absolute value, $\|x\| = |x|$, and the definition of a contraction mapping becomes as follows:

Definition 2:

If there exists $\beta \in (0,1)$ such that

$$\frac{|f(y) - f(x)|}{|y - x|} \le \beta \qquad \forall x, y \in \mathbb{X}, \tag{2}$$

then *f* is a *contraction mapping*.

In words, this says that the straight line between any two points on the graph of the function, must have a slope in the interval (-1,1). If, we don't require the convergence property, we only require that the slope be less than 1. We will define such a function as a "quasi-contraction mapping".

Definition 3:

If there exists $\beta \in (0,1)$ such that

$$\frac{f(x) - f(y)}{x - y} \le \beta \qquad \forall x, y \in \mathbb{X}, \tag{3}$$

then *f* is a *quasi-contraction mapping*.

This gives us the following variant of the contraction mapping theorem:

Theorem 2:

If $\mathbb{X}$ is a closed, connected subset of $\mathbb{R}$ and *f* is a quasi-contraction mapping, then there exists a unique fixed point, $x^* \in \mathbb{X}$, such that $f(x^*) = x^*$.

Proof:

First we show that a fixed point must exist. For any $x_0 \in \mathbb{X}$, we have $f(x_0) \ge x_0$, or $f(x_0) \le x_0$. If $f(x_0) \ge x_0$, then define,

$$y_0 = \begin{cases} \max\{x | x \in \mathbb{X}\} & \text{if } \mathbb{X} \text{ is bounded above} \\ \dfrac{f(x_0) - \beta x_0}{1 - \beta} & \text{otherwise} \end{cases} . \tag{4}$$

If $\mathbb{X}$ is not bounded above, we have, from (3) and (4),

$$f(y_0) - f(x_0) \le \beta(y_0 - x_0)$$

$$\Rightarrow \quad f(y_0) \le \beta y_0 + f(x_0) - \beta x_0$$

$$\Rightarrow \quad f(y_0) \le y_0.$$

If $\mathbb{X}$ is not bounded above, we have directly that

$$f(y_0) \le y_0.$$

By the intermediate value theorem, therefore, there exists $x^* \in [x_0, y_0]$ such that $f(x^*) = x^*$. A similar argument holds if $f(x_0) \le x_0$.

To show uniqueness, let $x^* \in \mathbb{X}$ be a fixed point of *f*. Then $\forall x > x^*$,

$$\frac{f(x) - f(x^*)}{x - x^*} \le \beta < 1$$

$$\Rightarrow \quad f(x) - f(x^*) \le \beta(x - x^*) < x - x^*$$

$$\Rightarrow \quad f(x) - x < f(x^*) - x^* = 0.$$

Thus any fixed point of *f* must be the maximum fixed point, implying that only one can exist.

□

Finally, if, in addition to the above assumptions, we assume that *f* is differentiable almost everywhere, then Condition (2) is equivalent to the following:

a) f is continuous over $\mathbb{X}$,
b) $|f'(x)| \le \beta \quad \forall x \in \mathbb{X}$ where f is differentiable. (5)

Similarly, Condition (3) is equivalent to the following:

a) f is continuous over $\mathbb{X}$,
b) $f'(x) \le \beta \quad \forall x \in \mathbb{X}$ where f is differentiable. (6)

This gives a general uniqueness theorem that we shall use in this paper:

Theorem 3:

Let $\mathbb{X}$ be a closed, connected subset of $\mathbb{R}$, and let *f* be a single-valued continuous function from $\mathbb{X}$ onto itself that is differentiable almost everywhere. Then if, for some $\beta \in (0,1)$

a) $f'(x) < \beta \qquad \forall x \in \mathbb{X}$ where f is differentiable,

there exists a unique fixed point $x^* \in \mathbb{X}$ such that $f(x^*) = x^*$.

If in addition, we have

b) $f'(x) > -\beta \qquad \forall x \in \mathbb{X}$ where f is differentiable,

then the fixed point is stable in the sense that for any $x \in \mathbb{X}$ and $n \ge 1$, $|f^n(x) - x^*| \le \beta^n |x - x^*|$.

In the next section we show how the contraction mapping theorem in $\mathbb{R}^n$ space is used to establish uniqueness in the class of games considered in this paper, and show by example why we seek to reduce the problem to one involving $\mathbb{R}$-to-$\mathbb{R}$ mappings.

3. The General Problem:

A. Notation:

Imagine that there are *n* players. We employ the following notation. The strategy space for each player *i* is $\mathbb{X}_i$, and the set of all possible combinations of strategies for all players is $\mathbb{X}^n$. Let $x_i \in \mathbb{X}_i$ denote a strategy for player *i* and let $\boldsymbol{x} = (x_1, ... x_n) \in \mathbb{X}^n$ denote a strategy combination for all players. Rather than specify the payoffs for each player, we will express everything in terms of the best-response functions. Let $\boldsymbol{x}_{-i}$ denote the (*n*-1)-vector of strategies of all players other than player *i*, and let $f_i(\boldsymbol{x}_{-i})$ be the best-response of player *i* to this combination of strategies of the other players.

We impose the following restrictions on this general set-up:

Assumption 1:

a) For each *i*, $\mathbb{X}_i$ is a connected subset of the real line;

b) for each *i*, f_i is continuous, single-valued, and differentiable almost everywhere over $\mathbb{X}^n$.

We do not require that the f_i be fully differentiable so that the model will be able to handle non-differentiabilities that can arise from boundary solutions to an individual player's optimisation problem. For ease of exposition, however, when presenting expressions involving derivatives we will omit the repeated caveat, "$\forall \boldsymbol{x} \in \mathbb{X}^n$ where $\boldsymbol{f}$ is differentiable", but this is implied.

Proofs of existence of an equilibrium in this class of games typically proceed by defining the aggregate best-response function, $\boldsymbol{f} : \mathbb{X} \mapsto \mathbb{X}$ where $\boldsymbol{f} = (f_1, f_2, ... f_n)$, so that a Nash equilibrium in pure strategies is a fixed point of $\boldsymbol{f}$ and vice versa, and then appealing to the Brouwer fixed-point theorem. We want to find sufficient conditions for $\boldsymbol{f}$ to have a unique fixed point. For this, it would be sufficient to show that $\boldsymbol{f}$ is a contraction mapping. This, however, would be too restrictive, as illustrated by the following simple example.

B. A Numerical Example:

Consider a Cournot game in which the *n* players are firms choosing the quantity to produce taking the quantity produced by each of the other *n*-1 firms as given. Assume that the market inverse demand curve is linear, and that each firm has a constant marginal cost of production. The linear structure satisfies the conditions required for uniqueness by Szidarovszky and Yakowitz (1977) amongst others.

To see whether the aggregate best-response function constitutes a contraction mapping, we need to define a norm. Rather than choosing a particular norm, we will consider any *p*-norm of the form

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} \left(|x_i|\right)^p \right)^{\frac{1}{p}} \text{ for some real number } p \geq 1.$$

Now consider some initial vector of outputs, $\boldsymbol{x} = (x_1, \dots x_n)$, and a second vector, $\boldsymbol{y} = (y_1, \dots y_n)$, where all outputs have been perturbed by the same constant, δ, so that $y_i = x_i + \delta \;\; \forall i.$ The *p*-norm for this perturbation is

$$\|\boldsymbol{y} - \boldsymbol{x}\|_p = n^{1/p}\delta.$$

The linear Cournot game produces linear best-response functions in which, at interior solutions, $\partial x_i / \partial x_j = -0.5 \quad \forall i, j \neq i.$ We therefore have $f_i(\boldsymbol{x}_{-i}) = x_i - 0.5(n-1)\delta,$ and hence

$$\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})\|_p = n^{1/p}0.5(n-1)\delta.$$

For any *n*>2, therefore, we have

$$\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})\|_p \geq \|\boldsymbol{y} - \boldsymbol{x}\|_p$$

and hence ***f*** is not a contraction mapping for any *p*-norm.

What this example shows is that if we wish to use the contraction mapping theorem to show uniqueness in a class of models that encompasses this standard example we will need to use a different function than ***f*** for which a fixed point defines an equilibrium. In the next section, we develop an approach that enables us to transform the problem so that an equilibrium is a fixed point in a mapping from a subset of the real line onto itself.

4. An Alternative Approach:

To transform the problem, we define an equilibrium iteratively, starting with 2 players and then progressively adding more in. To do this, define an *m*-equilibrium as a set of x_i such that

$$x_i = f_i(\boldsymbol{x}_{-i}) \quad \forall i = 1..m,$$

where $m \leq n.$ That is, it is a set of strategies such that the first *m* players' strategies are the best response to the strategies of *all* other players, but the remaining *n-m* players' strategies are unconstrained.

Let $\boldsymbol{x}^m$ be the vector of strategies by the first *m* firms, and let $\boldsymbol{x}^{-m}$ be the strategies of the remaining *n-m* firms when $m < n.$ An *m*-equilibrium when $m < n$ is therefore a Nash equilibrium in $\boldsymbol{x}^m$ taking $\boldsymbol{x}^{-m}$ as given, and the *m*-equilibrium when $m = n$ is a Nash equilibrium of the full game.

Let $h_i^m(\boldsymbol{x}^{-m})$ denote an *m*-equilibrium value of x_i when $m < n$ and let $\boldsymbol{h}^m(\boldsymbol{x}^{-m})$ be an *m*-vector of those values. If there exists a unique *m*-equilibrium for each value of $\boldsymbol{x}^{-m}$, then the function $\boldsymbol{h}^m$ is single valued and defined for all $\boldsymbol{x}^{-m}$.

Our approach to finding sufficient conditions for a unique equilibrium is to find the conditions for a unique 2-equilibrium that holds for all values of $\boldsymbol{x}^{-2}$ and then to extend that by induction by finding conditions for there to exist a unique (*m*+1)-equilibrium that holds for all values of $\boldsymbol{x}^{-(m+1)}$ conditional on there being a unique *m*-equilibrium.

Let *m*=2. Define the mapping $g_2 : \mathbb{R} \mapsto \mathbb{R}$ as

$$g_2(x_2, \boldsymbol{x}^{-2}) \equiv f_2(f_1(x_2, \boldsymbol{x}^{-2}), \boldsymbol{x}^{-2}). \tag{7}$$

There is an equivalence between a 2-equilibrium and a fixed point of g_2. From Theorem 2, a sufficient condition for there to exist a unique 2-equilibrium is that g_2 be a quasi-contraction mapping.

Now imagine that there exists a unique (m-1)-equilibrium for each value of $\boldsymbol{x}^{-(m-1)}$. In this case, define the mapping $g_m : \mathbb{R} \mapsto \mathbb{R}$ as

$$g_m(x_m, \boldsymbol{x}^{-m}) \equiv f_m(\boldsymbol{h}^{m-1}(x_m, \boldsymbol{x}^{-m}), \boldsymbol{x}^{-m}). \tag{8}$$

Again, there is an equivalence between an m-equilibrium and a fixed point of g_m, and so a sufficient condition for there to exist a unique m-equilibrium given $\boldsymbol{x}^{-m}$ is that g_m be a quasi-contraction mapping.

Definition 4:

We say that $\boldsymbol{f}$ exhibits an "iterative quasi-contraction mapping" if g_m exists and is a quasi-contraction mapping for each $m \in \{2...n\}$, and that it exhibits an "iterative contraction mapping" if g_m exists and is a contraction mapping for each $m \in \{2...n\}$.

The main result of this paper is then

Theorem 4:

If the best-response functions $\boldsymbol{f}$ exhibit an iterative quasi-contraction mapping, then the game has a unique Nash equilibrium.

Proof:

As we have shown, if g_2 is a quasi-contraction mapping, there exists a unique 2-equilibrium for all values of $\boldsymbol{x}^{-2}$. If there exists a unique (m-1)-equilibrium for all values of $\boldsymbol{x}^{-(m-1)}$ then g_m exists, and if g_m is a quasi-contraction mapping, there exists a unique m-equilibrium. By induction, then, if g_m is a quasi-contraction mapping for each $m \in \{2...n\}$, then there must exist a unique m-equilibrium for each m, and hence a unique equilibrium for the full game.

□

Now imagine that each of the g_m is a full contraction mapping so that repeated applications of g_m will generate convergence to the unique fixed point. This does not imply that the full equilibrium would be stable in the way it would be if $\boldsymbol{f}$ were a contraction mapping.[2] It does, however, that imply a numerical solvability using the best-response functions in the following sense. First, note that the 2-equilibrium can found iteratively by alternately adjusting player 1's strategy to that of player 2 and vice versa. Then, if the m-equilibrium is iteratively solvable by sequentially adjusting each of the first m player's strategies to be on his best-response functions, and if g_{m+1} is a contraction mapping, then the (m+1)-equilibrium is iteratively solvable by adjusting the m-equilibrium to x_{m+1} and then x_{m+1}

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[2] This can be seen from the example in Section 3, for which the equilibrium is not stable, but for which, as will be shown in Section 6, each of the g_m is a contraction mapping.

to the $\boldsymbol{h}^m$ and so on. Iterative solveability is perhaps not the most useful property one might desire of an equilibrium, but it is essentially a free result.

5. Sufficient Conditions with Calculus.

The analysis of the previous section gives sufficient conditions for uniqueness and iterative solvability that derive from our sequential approach. They are not, however, particularly user friendly. For that, we would like to express the conditions in terms of the slopes of the best-response functions.

To do this let $\boldsymbol{J}_n$ be the *n*x*n* Jacobean matrix of $\boldsymbol{f}$ with elements J_{ij}, so that

$$J_{ij} = \frac{\partial f_i}{\partial x_j}, \quad \forall j \neq i, \forall i, \quad \text{and} \quad J_{ii} = 0 \quad \forall i.$$

That is, $\boldsymbol{J}_n$ is the matrix of slopes of the best-response functions of each player with respect to the strategies of each other player.

Let $\boldsymbol{A}_n$ be the *n*x*n* matrix, $\boldsymbol{A}_n = \boldsymbol{I}_n - \boldsymbol{J}_n$, where $\boldsymbol{I}_n$ is the *n*x*n* identity matrix. Finally, let $\boldsymbol{J}_m$ and $\boldsymbol{A}_m$ be the *m*x*m* submatrices comprising the first *m* rows and first *m* columns of $\boldsymbol{J}_n$ and $\boldsymbol{A}_n$, respectively. The derivatives of the functions g_m can be expressed in terms of the determinants of the $\boldsymbol{A}_m$ as follows:

Theorem 5:

If g_m exists, then

$$\frac{dg_m}{dx_m} = 1 - \frac{|\boldsymbol{A}_m|}{|\boldsymbol{A}_{m-1}|}. \tag{9}$$

Proof:

Note that

$$|\boldsymbol{A}_1| = 1, \text{ and } \quad |\boldsymbol{A}_2| = 1 - \frac{\partial f_2}{\partial x_1} \cdot \frac{\partial f_1}{\partial x_2},$$

and, from Equation (7),

$$\frac{dg_2}{dx_2} = \frac{\partial f_2}{\partial x_1} \cdot \frac{\partial f_1}{\partial x_2},$$

so Equation (9) holds for *m*=2.

For *m*>2, we have from Equation (8) that

$$\frac{dg_m}{dx_m} = \sum_{i=1}^{m-1} \frac{\partial f_m}{\partial x_i} \cdot \frac{\partial h_i^{m-1}}{\partial x_m}. \tag{10}$$

Define $\boldsymbol{A}_{Cm}$ as the column vector containing the first *m*-1 elements of the *m*'th column of $\boldsymbol{A}$, and define $\boldsymbol{A}_{Rm}$ similarly as the row vector containing the first *m*-1 elements of the *m*'th row of $\boldsymbol{A}$, so that $\boldsymbol{A}_m$ is the partitioned matrix

$$\boldsymbol{A}_m = \begin{bmatrix} \boldsymbol{A}_{m-1} & \boldsymbol{A}_{Cm} \\ \boldsymbol{A}_{Rm} & 1 \end{bmatrix}.$$

We can then rewrite Equation (10) as

$$\frac{dg_m}{dx_m} = -\boldsymbol{A}_{Rm} \frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m}. \tag{11}$$

The h_i^{m-1} variables are defined by the fixed point in the *m*-equilibrium

$$h_i^{m-1} \equiv f_i(\boldsymbol{h}_{-i}^{m-1}, x_m, \boldsymbol{x}^{-m}) \qquad \forall i \le m-1.$$

Total differentiation yields

$$\frac{\partial h_i^{m-1}}{\partial x_m} = \sum_{j \in \{1 \ldots m-1\} \setminus \{i\}} \frac{\partial f_i}{\partial h_j} \cdot \frac{\partial h_j^{m-1}}{\partial x_m} + \frac{\partial f_i}{\partial x_m}.$$

In matrix notation this gives

$$\frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} = \boldsymbol{J}_m \frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} - \boldsymbol{A}_{Cm}$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} = -\boldsymbol{A}_{m\text{-}1}^{-1} \boldsymbol{A}_{Cm}$$

so Equation (11) becomes

$$\frac{dg_m}{dx_m} = \boldsymbol{A}_{Rm} \boldsymbol{A}_{m\text{-}1}^{-1} \boldsymbol{A}_{Cm}.$$

Finally, note that

$$\boldsymbol{A}_{Rm} \boldsymbol{A}_{m\text{-}1}^{-1} \boldsymbol{A}_{Cm} = 1 - \frac{|\boldsymbol{A}_m|}{|\boldsymbol{A}_{m-1}|}, \tag{12}$$

which gives us Equation (9).[3]

□

The conditions for an iterative quasi-contraction mapping can now be stated in terms of the determinants of the $\boldsymbol{A}_m$ matrices:

Theorem 6:

a) If there is an ordering of players, indexed by 1..*n*, such that for some $\varepsilon \in (0,1)$

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[3] Equation (12) is a special case of the general result for partitioned matrices that

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{B} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}| \left| \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{C} \right|$$

for square matrices ***A*** and ***D***, where ***A*** is non-singular. For a proof of this result, see, for example, Rao (1965, p28).

$$\left|\boldsymbol{A}_m\right| \geq \varepsilon > 0 \qquad \forall m \in \{2..n\} \tag{13}$$

then the best-response function exhibits an iterative quasi-contraction mapping.

b) If, in addition, we have

$$\frac{\left|\boldsymbol{A}_m\right|}{\left|\boldsymbol{A}_{m-1}\right|} \leq 2 - \varepsilon < 2 \qquad \forall m \in \{2..n\}. \tag{14}$$

then the best-response function exhibits an iterative quasi-contraction mapping.

Proof:

Follows automatically from Theorems 3 and 4.

□

Condition (12) gives the general uniquness condition of this paper—that the game has a unique equilibrium if the leading principal minors of the matrix $\boldsymbol{I}_n - \boldsymbol{J}_n$ are all positive. In the remainder of the paper, we show that this condition encompasses and extends existing uniqueness conditions.

6. Relationship to Other Uniqueness Conditions.

A. Rosen's Theorem:

The best-known paper providing a generic set of sufficient conditions for games of the form analysed here is Rosen (1965). Rosen considers a very general game structure in which the strategy space for any player can be conditional on the strategy chosen by another (as could happen in a coalition game). In the special case, however, where the player's strategy spaces are orthogonal to each other, i.e. the class of games considered in this paper, Rosen's sufficient condition can be written as follows:

*If there exists a diagonal matrix **R**, with diagonal terms $r_{ii} > 0\ \forall i$ such that the symmetric matrix $(\boldsymbol{RA}) + (\boldsymbol{RA})'$ is positive definite, then there is a unique equilibrium.*

The main result of this paper generalises this result in two ways: First, Rosen establishes existence by means of the Kakutani fixed-point theorem, and thus requires each player's strategy space be bounded, which is not required by the contraction mapping theorem used here; second, Rosen's sufficient condition is strictly encompassed by the conditions of Theorem 6 here, as shown by the following result.

Theorem 7:

*For any symmetric nxn matrix, **A**, if there exists a diagonal matrix **R**, with diagonal terms $r_{ii} > 0\ \forall i$ such that the symmetric matrix $(\boldsymbol{RA}) + (\boldsymbol{RA})'$ is positive definite, then the leading principal minors of **A** will be positive, but the reverse is not necessarily true.*

Proof:

$(\boldsymbol{RA})+(\boldsymbol{RA})'$ is positive definite if and only if $\boldsymbol{RA}$ is positive definite, which implies that the principal minors of $\boldsymbol{RA}$ are all positive, and hence that the principal minors of $\boldsymbol{A}$ are positive. The fact that a non-symmetric matrix with positive principal minors is not necessarily positive definite, however, allows one to construct counterexamples in which the conditions for Theorem 6 are met, but not for Rosen's theorem. For one such counterexample, consider the following matrix.

$$\boldsymbol{A}=\begin{bmatrix}1 & 0 & 8\\ 2 & 1 & 0\\ 0 & .5 & 1\end{bmatrix}.$$

$\boldsymbol{A}$ has positive principal minors. Without loss of generality, we can set $r_1=1,$ so that

$$(\boldsymbol{RA})+(\boldsymbol{RA})'=\begin{bmatrix}2 & 2r_2 & 8\\ 2r_2 & 2r_2 & .5r_3\\ 8 & .5r_3 & 2r_3\end{bmatrix}.$$

For the second principal minor to be positive, we need

$$r_2<1.$$

It is easy to show that the determinant of the full matrix is concave in r_3 given $r_2,$ and hence that the determinant-maximising value of r_3 given r_2 is

$$r_3=8(3-r_2)r_2.$$

Substituting in this value of $r_3,$ it is trivial to show that the determinant of the full matrix is negative for all values of $r_2\in(0,1).$

□

B. Cournot Games.

There are many papers giving conditions for uniqueness in a Cournot quantity-setting game. These include Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991), and Long and Soubeyran (2000). All three of these papers provide conditions which imply that the best-response functions of firms are negatively sloped, along with other conditions required to bound the set of prices over which demand is positive. As shown by the following, theorem, by using the general uniqueness theorem in this paper one only needs to require non-positively-sloped best-response functions; the bounding conditions are not necessary.

Theorem 8:

Let $\boldsymbol{A}_n$ be a square matrix with $a_{ii}=1\ \forall i$ and

$$a_{ij}=\alpha_i\in(-1,0]\ \ \forall j\neq i,\ \ \forall I.$$

Then $|\boldsymbol{A}_n|>0.$

Proof:

Given in the Appendix .

□

C. Row-Sum Conditions:

Cachon and Netessine (2004) show that a sufficient condition for the aggregate best-response function, ***f***, to exhibit a contraction mapping is that

$$\sum_{j\neq i}\left|\frac{\partial f_i}{\partial x_j}\right|<1 \quad \text{or} \quad \sum_{i\neq j}\left|\frac{\partial f_j}{\partial x_i}\right|<1.$$

That is, ***f*** exhibits a contraction mapping if the sum of the absolute values of the off-diagonal elements in the Jacobean matrix be less than one in each row or in each column. This result is established by showing that a function has a contraction mapping if the largest eigenvalue of the Jacobean matrix is less than one, and that, using a result of Horn and Johnson (1996), this will hold if the maximum row sum or the maximum column sum is less than one. Although this approach is very different from ours, it is easy to show that this condition meets our requirement for there to be an iterative quasi-contraction mapping. In the notation of this paper, this requirement is that for the matrix $\boldsymbol{A}_n = \boldsymbol{I}_n - \boldsymbol{J}_n$, where $\boldsymbol{J}_n$ is the Jacobean matrix,

$$\sum_{j\neq i}\left|a_{ij}\right|<1, \quad \text{or} \quad \sum_{i\neq j}\left|a_{ij}\right|<1.$$

We can generalise the theorem a bit:

Theorem 9:

Let $\boldsymbol{A}_n$ be a square matrix with $a_{ii} = 1\ \forall i$. If $\boldsymbol{A}_n$ has a dominant diagonal in the sense that there exist positive numbers, $d_1, d_2, \ldots d_n$, such that either

$$\sum_{j\neq i} d_j \left|a_{ij}\right| < d_i, \text{ or}$$

$$\sum_{i\neq j} d_i \left|a_{ij}\right| < d_j,$$

then $\left|\boldsymbol{A}_m\right| > 0\ \forall m \leq n.$

Proof:

Given in the Appendix.

□

The conditions for the Cachon and Netessine result are a special case where $d_i = 1\ \forall i.$

7. Conclusion.

This paper has presented a simple uniqueness condition for continuous games which is both quite general and easy to apply. The condition encompasses and generalises a number of existing uniqueness conditions that were derived using a wide variety of approaches. The condition in this paper, then, provides a unifying framework for presenting those conditions.

As shown by the relative simplicity of the proofs of Theorems 8 and 9, the general condition—that the leading principal minors of the matrix $\boldsymbol{I}_n - \boldsymbol{J}_n$ all be positive—lends itself to reasonably simple induction proofs for demonstrating that the condition holds in particular models. The result therefore has the potential to serve as a source for further uniqueness conditions in specific games.

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Appendix

A. Proof of Theorem 8.

Let Ω_n be the set of *n*x*n* matrices, $\boldsymbol{A}_n$, satisfying the following properties:

a) $a_{ii} = 1 \;\; \forall i$

b) $a_{ij} = \alpha_i \in [0,1) \quad \forall i, i \neq j.$

We want to show that

$$\left|\boldsymbol{A}_n\right| > 0 \qquad \forall \boldsymbol{A}_n \in \Omega_n, \forall n.$$

Proof:

The proof is by induction. The proposition is clearly true for *n*=1 and *n*=2. Now assume that there is some $\bar{n} > 2$ such that the proposition holds for all $n < \bar{n}$. we will show that it then holds for $n = \bar{n}$.

Let $\boldsymbol{A}_n^{-i}$ be the submatrix obtained by removing the *i*'th row and column from $\boldsymbol{A}_n$. First note that if $\alpha_i = 0$ for any *i*, then $\left|\boldsymbol{A}_n\right| = \left|\boldsymbol{A}_n^{-i}\right|$, and, since $\boldsymbol{A}_n \in \Omega_n \Rightarrow \boldsymbol{A}_n^{-i} \in \Omega_{n-1}$, the result holds by the induction assumption. We shall therefore only consider the case where $\alpha_i > 0 \; \forall i$.

The proof follows by considering the matrix derived from $\boldsymbol{A}_n$ by replacing the diagonal terms in the last two rows with the off-diagonal term for those rows. We show that this change unambiguously reduces the determinant of the matrix, but results in a matrix with a determinant of zero.

Formally, define the matrix $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)$, which is derived from some *n*x*n* matrix, $\boldsymbol{A}_n$, by replacing a_{mm} with b_m. The determinant of this matrix is

$$\left|\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)\right| = \left|\boldsymbol{A}_n\right| - (a_{mm} - b_m)\left|\boldsymbol{A}_n^{-(m,m)}\right|. \tag{15}$$

We can show the following result:

Lemma 1:

If $\left|\boldsymbol{A}_m\right| > 0$ for all $\boldsymbol{A}_m \in \Omega_m$, for all $m \leq n$, then $\left|\boldsymbol{B}_m(\boldsymbol{A}_m, \alpha_m)\right| \geq 0 \;\; \forall \boldsymbol{A}_m \in \Omega_m$.

That is, changing the diagonal term in the bottom row from unity to be the same as the other *m*-1 terms in that row will not change the sign of the determinant of the matrix from positive to negative.

Proof:

The proof is by contradiction. Assume that $\left|\boldsymbol{B}_m(\boldsymbol{A}_m, \alpha_m)\right| < 0.$ Note that

$$\left|\boldsymbol{B}_m(\boldsymbol{A}_m, b_m)\right| = \left|\boldsymbol{A}_m\right| - (1 - b_m)\left|\boldsymbol{A}_{m-1}\right|.$$

Since $|\boldsymbol{A}_{m-1}|$ is positive by the induction assumption, $|\boldsymbol{B}_m(A_m, b_m)|$ is continuous and monotone increasing in b_m. Further, since $|\boldsymbol{B}(\boldsymbol{A}_m, 1)| > 0,$ by the intermediate value theorem there must exist some $\hat{b}_m \in (\alpha_m, 1)$ such that $|\boldsymbol{B}(A_m, \hat{b}_m)| = 0.$ Now multiply the *m*th row of $\boldsymbol{B}(A_m, \hat{b}_m)$ by $1/\hat{b}_m$. Since $\hat{b}_m$ is positive, this leaves the sign of the determinant unchanged. But it produces a matrix that is a member of Ω_m and so must have a positive determinant. This establishes the contradiction.

□

Now consider the matrix $\boldsymbol{B}_{n-1}(\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n), \alpha_{n-1})$—that is, the matrix found by replacing the diagonal terms in each of the last two rows with the entry for the off-diagonal terms in those rows. Since the *n*th row is then a multiple of the (*n*-1)th row,

$$|\boldsymbol{B}_{n-1}(\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n), \alpha_{n-1})| = 0.$$

Using Equation (15) above, we can also write

$$\begin{aligned}|\boldsymbol{B}_{n-1}(\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n), \alpha_{n-1})| &= |\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n)| - (1-\alpha_{n-1})|\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n)^{-(m-1)}| \\ &= |\boldsymbol{A}_n| - (1-\alpha_n)|A_n^{-n}| - (1-\alpha_{n-1})|\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n)^{-(m-1)}|.\end{aligned}$$

By the induction base, we have

$$|\boldsymbol{A}_n^{-n}| > 0 \qquad \forall \boldsymbol{A}_n \in \Omega_n,$$

and by the induction base and Lemma 1, we have

$$|\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n)^{-(m-1)}| \geq 0 \qquad \forall \boldsymbol{A}_n \in \Omega_n.$$

We therefore have

$$|\boldsymbol{A}_n| > |\boldsymbol{B}_{n-1}(\boldsymbol{B}_n(\boldsymbol{A}_n, \alpha_n), \alpha_{n-1})| = 0,$$

which establishes the result.

□

B. Proof of Theorem 9.

For ease of exposition, it will be convenient to prove a trivially generalised statement of Theorem 9 in which the diagonal elements of $\boldsymbol{A}_n$ can take any positive values:

Theorem 9a:

Let $\boldsymbol{A}_n$ be a square matrix with $a_{ii} \geq 0 \;\; \forall i$. If $\boldsymbol{A}_n$ has a dominant diagonal in the sense that there exist positive numbers, $d_1, d_2, \ldots d_n$, such that either

$$\sum_{j \neq i} d_j |a_{ij}| < d_i, \text{ or}$$

$$\sum_{i \neq j} d_i |a_{ij}| < d_j,$$

then $|\boldsymbol{A}_m| > 0 \;\forall m \leq n.$

Proof:

Theorem 4.C.1 in Takayama (1985), shows that a dominant diagonal matrix with no constraint on the sign of the diagonal elements must be non-singular. It is then straightforward to show that if the diagonal elements are all positive, the determinant must be positive. The proof is by induction.

Trivially, the 1x1 matrix whose single element is positive has a positive determinant. Now assume that the theorem holds for all matrices of size *m*-1, and let $\boldsymbol{A}_m$ be a dominant diagonal matrix. This implies that

$$\frac{\partial}{\partial a_{mm}}\left|\boldsymbol{A}_m\right| = \left|\boldsymbol{A}_{m-1}\right| > 0. \tag{16}$$

Now let

$$\hat{a}_{mm} = a_{mm} - \frac{\left|\boldsymbol{A}_m\right|}{\left|\boldsymbol{A}_{m-1}\right|}, \tag{17}$$

and let $\hat{\boldsymbol{A}}_m$ be the matrix created by replacing a_{mm} with $\hat{a}_{mm}$. Equations (16) and (17) then imply that

$$\left|\hat{\boldsymbol{A}}_m\right| = 0,$$

which from Takayama's result implies that $\hat{\boldsymbol{A}}_m$ cannot be dominant diagonal and hence that

$$\hat{a}_{mm} < a_{mm}. \tag{18}$$

Since $\left|\boldsymbol{A}_{m-1}\right| > 0$, (17) and (18) together imply that $\left|\boldsymbol{A}_m\right| > 0$.

□