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Working Paper No. 0613
July 2006

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All-pay Auctions with Budget Constraints and Fair Insurance.

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July 11, 2006

Abstract

We study all-pay auctions with budget-constrained bidders who have access to fair insurance before bidding simultaneously over a prize. We characterize a unique equilibrium for the special cases of two bidders and one prize, show existence and a heuristic for finding an equilibrium in the case of multiple bidders and multiple prizes. We end with an example of non-uniqueness of equilibria for the general case of multiple prizes and multiple players.

Keywords: All-pay Auctions, Fair Lotteries, Political Campaigning, Oligopoly, Regional Competition, Patent Races

JEL: C72, D72, D42

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1 Introduction

Suppose you are contesting a prize with a single adversary, where the highest bidder wins the prize and both bidders loose their investment. Investments have no opportunity cost and your adversary has a higher initial budget. If you'd simply both invest your initial budget you'd loose with certainty, so what do you do? You'd look for the possibility of gambling with your initial budget in the hope of increasing it. Knowing this, the adversary will also gamble with his initial budget. This paper solves for the optimal strategy and pay-offs in this all-pay auction game with known asymmetric initial budgets under the assumption of the existence of a fair insurance market. We derive existence and uniqueness for the two player, one slot (prize) game; uniqueness and existence for the two player, T-slot game; existence and generic examples for the n player, 1 slot game; and existence and an example of multiple equilibria for the multiple slots, n player game.

Our most interesting result from the point of view of the auction literature is that the pay-off function is non-standard. The dominant pay-off function in the literature is the contest function for all-pay auctions proposed by Tullock (1980) given by $\frac{f(s_i)}{\sum f(s_j)}$. Here s_i denotes the budget of player i. This pay-off function was subsequently adapted and used by many others (eg. Cornes and Hartley (2005)). The pay-off function in our two-player game with $i = 1$ being the player with the smaller initial budget, turns out to be $\frac{s_1}{2s_2}$. The major difference is the higher expected return to the person with the higher budget over the Tullock (1980) contest. The intuition behind this result is that on a fair insurance market player 1 can write a contract in which she obtains the same budget as player 2 with probability $\frac{s_1}{s_2}$ (conditional on which she'd have a 50% chance of winning the contest) and ends up with zero with probability $(1 - \frac{s_1}{s_2})$. Our main empirical prediction is that players with lower budgets with positive probability make an eventual bid of zero, whereas the player with the highest budget makes positive bids on all slots in all equilibria we look at. This may help explain one of the stylized observation on political lobbying (which is an example of an all-pay auction) that, ex post, some particular lobbying markets appear uncontested (see for instance Katz et al 1990).

By now, a large literature exists detailing the optimal strategy of bidders in an all-pay auction under various valuation, information, budgetary, and pre-commitment constraints. Hillman and Riley (1989) derived the Nash-equilibrium of the 2-player, 1 slot all-pay auction under the assumption of heterogeneous valuations without budget constraints. Extensions to this basic all-pay auction framework have included the possibility of precommittment to lower valuations via delegation (Konrad et al. 2004), incomplete information (Barut et al. 2002), sequential bids rather than simultaneous bids (Leininger 1991), and binding budget constraints (Laffont and Robert, 1996).[1] Baye et al. (1996) provide an early overview of the basic 1-slot all-pay auction and Klemperer (1999) provides a survey of the general auction literature.

The model in our paper differs in two important respects from the existing literature. The main innovation is the presumption of a fair insurance market where individuals can gamble with their initial budgets. This extension can be given three justifications which delimit the interpretation of the results of this paper.

The first interpretation is to take the existence of a fair insurance literally and to interpret the results that way: individuals, departments, and whole organisations can gamble on stock markets, option markets, and on betting markets. Our paper shows how budget-constrained players should

[1]The paper by Laffont and Robert (1996) shares the assumption of our paper that bidders are financially constrained and that those constraints and the ensuing strategies are common knowledge. Unlike them, we assume heterogeous budgets.

The idea to look at optimal bidding behavior in all-pay auctions rather than optimal auction design is the spirit of Konrad et al. (2004). They ask whether delegation in all pay auctions is an optimal strategy and show the optimal two-part contract a buyer will set for an agent bidding on his behalf. This delegation in their model has the benefit of precommting to having lower expected bids in the actual auction. The possibility of delegation in our model would not change anything because players only care about winning and not about the resources lost in the contest and thus there would be no benefit to delegation.

gamble with their initial budgets if the subsequent game is an all-pay auction. The natural starting assumption to study this situation is to presume the existence of a fair insurance market.

A second rationale for the fair insurance assumption is to see the assumption of fair insurance as the limit situation when players are playing over very many slots simultaneously and have to allocate a fixed budget over these slots. Then, on average, each slot is allocated a certain amount of the resource in expectation.

A final rationale for the assumption of a fair insurance market is take fair insurance as describing an intermediary stage. In the intermediary stage each player can choose from a menu of possible investments on which the fixed budget is spent, where individuals can choose the result of that investment strategy if the strategy works whereby the chance of success is inversely related to that result. An example of different strategies in the context of a patent race would be to either thoroughly research everything such that a certain level of innovation in a new product is reached with certainty, or to research only a fraction and gamble that one nevertheless has hit upon a big innovation by chance. The result of that intermediary stage then forms the 'bid' in the outcome stage (the patent request), whereby the player with the highest intermediary result wins. Again, a natural starting point for the analysis of such a contest is to presume that the intermediary process in which one can choose between levels of risks of failure is characterised by fair insurance (i.e. the odds of achieving a result in the intermediary stage is perfectly inversely related to the hight of the result).

A lesser difference between our paper and the existing literature is that in our paper, we presume that players care only about the probability of winning the contest. There are two main rationales for that assumption. One is that it simply recognises that there are many situations where agents are given a fixed budget by a principle to achieve a given objective. Examples would be an R&D department that is given a fixed budget to win a patent race by a principal; sports institutes given a budget to produce as many winners as possible by a government; and a legal firm given a fixed budget to win a particular case. A secondary rationale for the assumption that the players dont care about the invested resource is to fit situations where the bid is not monetary but is in terms of something with little opportunity value to the player, such as campaigning effort by a politician. A natural extension to our work is to look at situations where there is an opportunity cost to the resources put into the contest.

There are many markets that are set up as all-pay auctions and which thereby form potential applications of our model. Konrad et al. (2004) argue that all-pay auctions play an important role in many allocation processes, from lobbying over political campaign spending and spatial competition to sport contests, patent / R&D races, and military campaigns. Cornes and Hartley (2005) additionally note that law cases, competitive research grants, and status games are also all-pay auctions. Formalising these arguments, Baye and Hoppe (2003) show that rent seeking contests, patent races and innovation tournaments are strategically equivalent.

The paper proceeds as follows. We first state the problem formally for the two player case. Section 3 provides a solution for the two player multiple objects case. We show existence and characterize the unique equilibrium. In section 4 we show existence for the case of more than two players. In the final section we conclude and discuss possible extensions.

2 The basic 2-player problem

Consider the following problem: Two players, 1 and 2, simultaneously allocate scarce resources to T locations, which form the set $Z = \{1, .., T\}$. Without loss of generality we assume that each location pays a identical return of 1 to the player who allocates more to the location in question. Both players maximize their return and we require the resource constraint to be fulfilled in expectation. Let Ts_1 and Ts_2 denote the total amount of the resource available to players 1 and 2 and x^z, y^z the amount allocated to slot

z by player $1,2$ respectively. Without loss of generality we assume $s_1 \leq s_2$. Given that we look at mixed strategies, we denote by F_1^z and F_2^z the strategically chosen stochastic distribution functions of x^z, y^z. Then the problem is the following:

$$
\begin{aligned}
(1)\, Ts_1 &\geq \sum_{z=1}^{T} \int_0^\infty x dF_1^z(x) \\
(2)\, Ts_2 &\geq \sum_{z=1}^{T} \int_0^\infty y dF_2^z(y) \\
(3)\quad P_1 &= \frac{1}{T} \sum_{z=1}^{T} \int_0^\infty \int_0^\infty [\mathrm{I}_{\{(x,y)\in\mathbb{R}^2|x>y\}} + \frac{1}{2}\mathrm{I}_{\{(x,y)\in\mathbb{R}^2|x=y\}}] dF_1^z(x) dF_2^z(y) \\
F_1 &= \arg\max_{\widetilde{F}_1} P_1 \text{ for a given } F_2 \\
F_2 &= \arg\max_{\widetilde{F}_2} 1 - P_1 \text{ for a given } F_1
\end{aligned}
$$

where $\mathrm{I}_{\{.\}}$ is the indicator function and F_1^z, F_2^z denote the equilibrium c.d.f.'s of player 1, 2 respectively for the choice of the amount of the resource allocated to slot z and $F_1 = (F_1^1, .., F_1^T)$, $F_2 = (F_2^1, .., F_2^T)$ and similarly for $\widetilde{F}_1$, $\widetilde{F}_2$. These c.d.f.'s are chosen strategically. What makes the problem non-trivial is that equilibrium distributions F_1^{z*} and F_2^{z*} both have to be non degenerate for all z. We intuitively argue this here by supposing the converse. If F_1^z was degenerate in x for all z, then player 2 could overbid player 1 for this z and P_1 would be 0. Indeed, player 2 would overbid player 1 marginally in any region z if x^z was known with certainty. Converse, if y^z was known, then the optimal reaction of player 1 would be to overbid player 2 marginally in those regions with lowest y^z until mass runs out. Knowing this, the optimal y^z would be equal to s_2, in which case P_1 would be $\frac{s_1}{s_2}$ and player 2 could profit by switching to the same strategy as player 1. In equilibrium it thus cannot be that for any $z \in \{1, .., T\}$, the strategies of players are not random.

We continue by solving for the 2 player 1 slot case, after which we extend both the number of players and the number of slots.

3 Existence and Uniqueness for the case of two players

Without loss of generality, we take $0 < s_1 \leq s_2$. The following theorem settles the existence problem.

Theorem 1. *The following strategies form a Nash equilibrium of the game: In each slot z player 1 chooses F_1^{z*} according to*

$$F_1^{z*}(x) = \begin{cases} 0 \textit{ for } x < 0 \\ 1 - \frac{s_1}{s_2} \textit{ for } x = 0 \\ 1 - \frac{s_1}{s_2} + \frac{s_1}{2s_2^2}x \textit{ for } 0 < x \leq 2s_2 \\ 1 \textit{ for } x > 2s_2 \end{cases}$$

and player 2 chooses F_2^{z*} according to

$$F_2^{z*}(y) = \begin{cases} 0 \textit{ for } y < 0 \\ \frac{1}{2s_2}x \textit{ for } 0 \leq y \leq 2s_2 \\ 1 \textit{ for } y > 2s_2\,. \end{cases}$$

In this equilibrium $P_1 = \frac{s_1}{2s_2}$.

Proof. A simple calculation shows that for these strategies the resource constraints (1) and (2) are satisfied.

Now if player 2 chooses F_2^{z*}, $z = 1, \ldots, T$, and player 1 invests an amount of $x \geq 0$ with probability one in slot z, then the expected payoff of player 2 in z is equal to 0 if $x > 2s_2$ and equal to $1 - \frac{1}{2s_2}x$ otherwise. Consequently, if player 2 chooses F_2^{z*}, $z = 1, \ldots, T$, and player 1 chooses F_1^z, $z = 1, \ldots, T$, and respects his resource constraint, then the (average over slots) expected payoff of 2 is equal to

$$
\begin{aligned}
& \frac{1}{T}\sum_{z=1}^{T}\int_0^{2s_2}\left(1-\frac{x}{2s_2}\right)dF_1^z(x) \\
\geq\ & \frac{1}{T}\sum_{z=1}^{T}\int_0^{\infty}\left(1-\frac{x}{2s_2}\right)dF_1^z(x) \\
=\ & 1-\frac{1}{2s_2}\frac{1}{T}\sum_{z=1}^{T}\int_0^{\infty}x\,dF_1^z(x) \\
\geq\ & 1-\frac{1}{2s_2}s_1 \qquad \text{by the resource constraint for player 1.}
\end{aligned}
$$

On the other hand, when player 1 chooses F_1^{z*}, $z = 1, \ldots, T$, and player 2 invests an amount of $y \geq 0$ with probability one in slot z, then the expected payoff of player 1 in z is equal to 0 if $y > 2s_2$, equal to $1-(1-\frac{s_1}{s_2}+\frac{s_1}{2s_2^2}y) \equiv \frac{s_1}{s_2} - \frac{s_1}{2s_2^2}y$ if $0 < y \leq 2s_2$, and equal to $\frac{1}{2}(1-\frac{s_1}{s_2})+\frac{s_1}{s_2}$ if $y = 0$. Consequently, when player 1 chooses F_1^{z*}, $z = 1, \ldots, T$, and player 2 chooses F_2^z, $z = 1, \ldots, T$, and respects his resource constraint, then the (average over slots) expected payoff of player 1 is equal to

$$
\begin{aligned}
& \frac{1}{T}\left(\sum_{z=1}^{T}\left(\frac{1}{2}\left(1-\frac{s_1}{s_2}\right)\right)+\int_0^{2s_2}\left(\frac{s_1}{s_2}-\frac{s_1 y}{2s_2^2}\right)dF_2^z(y)\right) \\
\geq\ & \frac{1}{T}\sum_{z=1}^{T}\int_0^{2s_2}\left(\frac{s_1}{s_2}-\frac{s_1 y}{2s_2^2}\right)dF_2^z(y) \\
\geq\ & \frac{1}{T}\sum_{z=1}^{T}\int_0^{\infty}\left(\frac{s_1}{s_2}-\frac{s_1 y}{2s_2^2}\right)dF_2^z(y) \\
=\ & \frac{s_1}{s_2}-\frac{s_1}{2s_2^2}\frac{1}{T}\sum_{z=1}^{T}\int_0^{\infty}y\,dF_2^z(y) \\
\geq\ & \frac{s_1}{s_2}-\frac{s_1}{2s_2^2}s_2 \qquad \text{by the resource constraint for player 2} \\
=\ & \frac{s_1}{2s_2}.
\end{aligned}
$$

Since for any choices of strategies by players 1 and 2 the (average over slots) expected payoffs sum to one, it follows that the choices F_1^{z*}, $z =$

$1, \ldots, T$, by player 1 and F_2^{z*}, $z = 1, \ldots, T$, by player 2 form an equilibrium. $\square$

The proof establishes in particular that in any equilibrium the expected payoff of player 1 is equal to $\frac{s_1}{2s_2}$ and the expected payoff of 2 equal to $1 - \frac{s_1}{2s_2}$. Thus in any existing equilibrium, the player having less resources wins less than his share of the overall resources in the economy ($\frac{s_1}{2s_2} < \frac{s_1}{s_1+s_2}$). The basic idea of the proof is that, if player 2 uses a strategy to allocate the same expected amount of the resource on each slot and a uniform distribution on the interval $[0, 2s_2]$, then player 1 cannot acquire more than $P_1 = \frac{s_1}{2s_2}$. We now show that the proposed mixing strategies are a unique equilibrium.

Theorem 2. *The equilibrium strategies F_1^{z*}, F_2^{z*}, $z = 1, \ldots, T$, from the statement of Theorem 1 are the only equilibrium strategies.*

Proof. Consider first the case of only one slot. Suppose the pair $(F_1,\ F_2)$ is any equilibrium for this case. Then by the proof of Theorem 1, the pair $(F_1,\ F_2^*)$ is an equilibrium, too. Clearly this implies that $F_1(2s_2) = 1$. Let $g\colon [0, 2s_2] \to [0, 1]$ be given by $g(0) = (1/2)F_1(0)$, and for $x > 0$ by $g(x) = F_1(x)$ if F_1 is continuous at x and by $g(x) = \lim_{x' \uparrow x} F(x') + (1/2)(F_1(x) - \lim_{x' \uparrow x} F(x'))$ otherwise. By Lemma 5 in the Appendix, the equilibrium conditions with respect to F_2^* and the fact that $\operatorname{supp} dF_2^* = [0, 2s_2]$ imply that for some numbers α, β we have $g(x) = \alpha + \beta x$ for all $x \in (0, 2s_2]$. Thus $F_1(x) = \alpha + \beta(x)$ for all $[0, 2s_2]$ (since F_1 is right continuous at 0). An easy calculation shows that the resource constraint for player 1 and the fact that $F_1(2s_2) = 1$ imply that $\alpha = 1 - (s_1/s_2)$ and $\beta = s_1/(2s_2^2)$ must hold. Thus $F_1 = F_1^*$. Similarly it follows that $F_2 = F_2^*$.

Now consider case of multiple slots $z = 1, \ldots, T$ and suppose that the pair $\left(F_1 = (F_1^1, \ldots, F_1^T),\ F_2 = (F_2^1, \ldots, F_2^T)\right)$ is any equilibrium. We have to show that for each z, $F_1^z = F_1^{z*}$ and $F_2^z = F_2^{z*}$.

The following terminology will be used in the rest of this proof. For real numbers $s_1',\ s_2' \geq 0$ and distribution functions $F_1',\ F_1'$, with $F_1'(x) = 0$ for $x < 0$ and $F_2'(y) = 0$ for $y < 0$, we will say that the pair $(F_1',\ F_2')$ is a one slot equilibrium for s_1' and s_2' if F_1' chosen by player 1 and F_2' by player 2 forms an equilibrium in the one slot problem where the resources of player 1 are given by s_1' and that of player 2 by s_2'.

For each $z = 1, \ldots, T$, let $s_1^z = \int_0^\infty x\, dF_1^z(x)$ and $s_2^z = \int_0^\infty y\, dF_2^z(y)$. Clearly, in any equilibrium, for both players 1 and 2 the resource condition must hold with equality. Thus we must have $\sum_{z=1}^T s_1^z = Ts_1$ and $\sum_{z=1}^T s_2^z = Ts_2$.

Now by the arguments in the proof of Theorem 1 and the remarks following that proof, the pair $(F_1^*,\, F_2)$ is also an equilibrium. Clearly, this means in particular that for each z the pair $(F_1^{z*},\, F_2^z)$ is a one slot equilibrium for s_1 and s_2^z. By the uniqueness result for the one slot case established above, it follows that for any z,

if $s_1 < s_2^z$ then

$$F_1^{z*}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{s_1}{s_2^z} & \text{if } x = 0 \\ 1 - \frac{s_1}{s_2^z} + \frac{s_1}{2(s_2^z)^2}x & \text{if } 0 < x \le 2s_2^z \\ 1 & \text{if } x > 2s_2^z\,. \end{cases} \tag{4}$$

and if $s_1 \ge s_2^z$ then

$$F_1^{z*}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2s_1}x & \text{if } x \le 0 \le 2s_1 \\ 1 & \text{if } x > 2s_1\,. \end{cases}$$

On the other hand, by the definition of F_1^{z*},

$$F_1^{z*}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{s_1}{s_2} & \text{if } x = 0 \\ 1 - \frac{s_1}{s_2} + \frac{s_1}{2s_2^2}x & \text{if } 0 < x \le 2s_2 \\ 1 & \text{if } x > 2s_2\,. \end{cases} \tag{5}$$

If $s_1 < s_2$ then it is immediate that (4) and (5) are consistent only if $s_2^z = s_2$. In case $s_1 = s_2$, (4) and (5) imply at least that $s_1 \ge s_2^z$ must hold for all

z, and then the resource constraint $\sum_{z=1}^{T} s_2^z = Ts_2$ implies that also in this case $s_2^z = s_2$ for each z. Thus in any case, $s_2^z = s_2$ for each z (since $s_1 \leq s_2$ by hypothesis). That is, (F_1^{z*}, F_2^z) is a one slot equilibrium for s_1 and s_2. By the uniqueness result for the one slot case, it follows that $F_2^z = F_2^{z*}$ for each z.

Now to see that $F_1^z = F_1^{z*}$ must hold for each z, first note that the pair (F_1, F_2) being an equilibrium implies that $s_1^z > 0$ for each z. For each z define a mapping $\phi^z\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi^z(y^z) = \begin{cases} \frac{1}{2s_1^z} y^z & \text{if } y^z < s_1^z \\ 1 - \frac{s_1^z}{2} \frac{1}{y^z} & \text{if } y^z \geq s_1^z \,, \end{cases}$$

and define a mapping $\phi\colon \mathbb{R}_+^T \to \mathbb{R}$ by

$$\phi(y^1, \ldots, y^T) = \sum_{z=1}^{T} \phi^z(y^z) \,.$$

By the argument of the proof of Theorem 1, the fact that the pair (F_1, F_2) is an equilibrium implies that the pair (F_1, F_2^*) is an equilibrium, too. It follows that for each z, the pair (F_1^z, F_2^{z*}) is a one slot equilibrium for s_1^z and s_2. By the remarks following the proof of Theorem 1, then, the expected payoff of player 2 in the one slot equilibrium (F_1^z, F_2^{z*}) is equal to $\frac{s_2}{2s_1^z}$ if $s_2 < s_1^z$ and is equal to $1 - \frac{s_1^z}{2s_2}$ if $s_2 \geq s_1^z$. Consequently the total expected payoff of player 2 in the equilibrium (F_1, F_2^*) is equal to $\phi(s_2, \ldots, s_2)$.

The mapping ϕ attains a maximum at $(y^1, \ldots, y^T) = (s_2, \ldots, s_2)$ over the set $\{(y^1, \ldots, y^T) \in \mathbb{R}_+^T \colon \sum_{z=1}^{T} y^z = Ts_2\}$. Indeed, suppose there would be an $(y^1, \ldots, y^T) \in \mathbb{R}_+^T$ with $\sum_{z=1}^{T} y^z = Ts_2$ such that $\phi(y^1, \ldots, y^T) > \phi(s_2, \ldots, s_2)$. Again by the proof of Theorem 1, there is a strategy $F' = (F^{1\prime}, \ldots, F^{T\prime})$ such that for each slot z, $\int_0^\infty y \, dF_Y^{z\prime}(y) = y^z$ and such that the expected payoff for z is $\geq \frac{y^z}{2s_1^z}$ if $y^z < s_1^z$ and is $\geq 1 - \frac{s_1^z}{2y^z}$ if $y^z \geq s_1^z$. It follows that the total expected payoff of F' is larger than $\phi(s_2, \ldots, s_2)$, contradicting the fact that the pair (F_1, F_2^*) is an equilibrium.

Observe now that each mapping ϕ^z is differentiable in y^z and that the derivative with respect to y^z is strictly decreasing on the interval $[s_1^z, \infty)$. Also

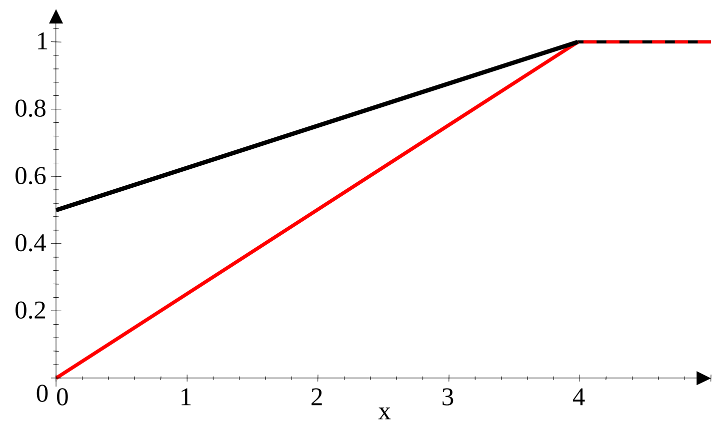

Figure 1: Equilibrium strategies for $s_1 = 1$ and $s_2 = 2$: $F_1^{z*}(x)$ (black) and $F_2^{z*}(x)$ (red).

observe that (F_1^z, F_2^{z*}) being a one slot equilibrium for s_1^z and s_2 means that $s_2 \geq s_1^z$ must hold by the uniqueness result for the one slot case. Consequently the fact that ϕ attains a maximum at $(s_2, \ldots, s_2)$ over the set

$$\{(y^1, \ldots, y^T) \in \mathbb{R}_+^T : \sum_{z=1}^{T} y^z = Ts_2\}$$

implies that the s_1^z 's must coincide whence, by the resource constraint for player 1, $s_1^z = s_1$ must hold for each z. Thus, for each z, the pair (F_1^z, F_2^{z*}) is a one slot equilibrium for s_1 and s_2. By the uniqueness result for the one slot case again, we may conclude that $F_1^z = F_1^{z*}$ for each z. This completes the proof of the theorem. □

The equilibrium allocation follows a clear-cut mixing rule. For player 2 it is optimal to allocate to any region z an independent random draw from the uniform distribution on the range $[0, 2s_2]$. The equilibrium response of player 1 is to choose 0 with probability $1 - \frac{s_1}{s_2}$ and a random draw of the same uniform distribution as the one by player 2 with probability $\frac{s_1}{s_2}$. The pictures below illustrate this result.

Putting this result into the context of a motivating example, say the political game with two parties spending a fixed campaign budget over a continuum of elections: for the bigger party 2, who has on average s_2 to spend in each of the elections it contests, it is optimal to randomly choose an amount y in the range $[0, 2s_2]$ with each amount having equal probability. Simultaneously, party 1 would decide with probability $(1 - \frac{s_1}{s_2})$ not to spend any resources on that particular election at all, and with probability $\frac{s_1}{s_2}$ would spend a positive amount of resources, randomly choosing an amount x to spend from the range $[0, 2s_2]$ with each point having equal probability. This choice process is then repeated in all the elections that these two parties simultaneously contest: in each election the parties take fresh draws from F_1 and F_2. The expected outcome is that party 1 wins a proportion $\frac{s_1}{2s_2}$ of all the elections.

4 Existence with more than two agents.

It is convenient to first treat the special case of only one slot (Section 4.1). Existence of equilibrium in the multiple slots case will follow easily from existence in the one slot case (Section 4.2).

4.1 The special case of one slot

We use the following notation:

- There are n agents $j = 1, \ldots, n$ with resources $s_j > 0$.
- $0 < r_1 < r_2 \ldots < r_m$ are the levels of resources appearing among the n agents.
- n_i, $i = 1, \ldots, m$, denotes the number of agents j with $s_j = r_i$.

The term "distribution function on $\mathbb{R}_+$" means a weakly increasing and right-continuous function $F \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to \infty} F(x) = 1$.

The proof of equilibrium existence is organized in a series of lemmata.

Lemma 1. *Let G_j be a distribution function on $\mathbb{R}_+$ for each $j = 1, \ldots, n$. Suppose the following conditions to hold for each j.*

(a) $\int\limits_{\mathbb{R}_+} x dG_j(x) = s_j$.

(b) There are real numbers $\alpha_j \geq 0$ and $\beta_j > 0$ such that

  (i) $\prod_{j' \neq j} G_{j'}(x) \leq \alpha_j + \beta_j x$ for all $x \in \mathbb{R}_+$;

  (ii) $\prod_{j' \neq j} G_{j'}(x) = \alpha_j + \beta_j x$ for G_j-almost all $x \in \mathbb{R}_+$.

(c) $G_j(0) = 0$ if $\prod_{j' \neq j} G_{j'}(0) > 0$.

(d) The function $\prod_{j' \neq j} G_{j'}(\cdot)$ is continuous at G_j-almost all $x \in \mathbb{R}_+$.

Then $(G_1, \ldots, G_n)$ is an equilibrium.

Proof. Consider any j and let H_j be any distribution function on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} x \, dH_j(x) \leq s_j$ Note that if j sets some amount x with certainty, then his expected payoff is $\leq \prod_{j' \neq j} G_{j'}(x)$, and is $= \prod_{j' \neq j} G_{j'}(x)$ if $x > 0$ and $G_{j'}$ is continuous at x for each $j' \neq j$. Thus the expected payoff for H_j is

$$\begin{aligned} &\leq \int_{\mathbb{R}_+} \prod_{j' \neq j} G_{j'}(x) \, dH_j(x) \\ &\leq \int_{\mathbb{R}_+} (\alpha_j + \beta_j x) \, dH_j(x) \\ &\leq \alpha_j + \beta_j s_j \, , \end{aligned}$$

while the expected payoff from G_j is

$$\begin{aligned} \int_{\mathbb{R}_+} \prod_{j' \neq j} G_{j'}(x) \, dG_j(x) &= \int_{\mathbb{R}_+} (\alpha_j + \beta_j x) \, dG_j(x) \\ &= \alpha_j + \beta_j s_j \, . \end{aligned}$$

This completes the proof. □

The next two lemmata essentially give a heuristic for finding equilibria. Figure 2 provides an illustration of how strategies generically look like for this heuristic. In this example $s_1 = 1, s_2 = 2$ and $s_3 = 3$. The stratagies are characterized by two critical values, a maximum bid c_0, in this case

$c_0 = 6.9358$ and a critical value c_1, in this example $c_1 = 3.8456$. [2] All players distribute their bids over the interval $[0; c_0]$. The strategies of all players are continuous piecewise defined distribution functions. Players 1 and 2 have a positive probability mass on 0. Player 1 never bids any positive amount below c_1 and concentrates all his budget on high bids. Player 2 and 3 have a positive probability for all bids between 0 and the maximum bid c_0.[3]

Given these strategies every player is indifferent between any distribution of bids that fulfill his or her budget requirement.

Lemma 2. *Suppose $n > 2$ and $n_m > 1$. Let F_i, $i = 1, \ldots, m$, be distribution functions on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} x\, dF_i(x) = r_i$ for each $i = 1, \ldots, m$. Suppose there are real numbers $\beta > 0$ and c_i, $i = 1, \ldots, m$, such that the following conditions hold.*

(i) $(F_m(x))^{n_m - 1} \cdot \prod_{i<m} (F_i(x))^{n_i} = \beta x$ *for all* $0 \le x \le 1/\beta$*;*

(ii) $c_1 > c_2 \ldots > c_{m-1} > c_m = 0$*;*

(iii) $F_i(x) = \max\{F_m(x), F_m(c_i)\}$ *for each* $i = 1, \ldots, m-1$*;*

Then $(G_1, \ldots, G_n)$ with $G_j = F_i$ if $s_j = r_i$ is an equilibrium.

[2] In a general version with players with m different levels of budgets there are $m-1$ critical values, each with the feature that a player with a budget r_k has no probability mass on bids below c_k.

[3] The exact functions are

$$F_1(x) = \begin{cases} \sqrt{a + bc_1} \text{ for } 0 \le x \le c_1 \\ \sqrt{a + bx} \text{ for } c_1 < x \le c_0 \end{cases},$$

$$F_2(x) = \begin{cases} \frac{a+bx}{\sqrt{a+bc_1}} \text{ for } 0 \le x \le c_1 \\ \sqrt{a + bx} \text{ for } c_1 < x \le c_0 \end{cases} \quad \text{and}$$

$$F_3(x) = \begin{cases} \frac{xb}{(1-a)\sqrt{a+bc_1}} \text{ for } 0 \le x \le c1 \\ \frac{xb}{(1-a)\sqrt{a+bx}} \text{ for } c_1 < x \le c_0 \end{cases},$$

where $c_0 = \frac{1-a}{b} = 6.9358$, $a = 0.237754$, $b = 0.109901$and $c_1 = 3.8456$.

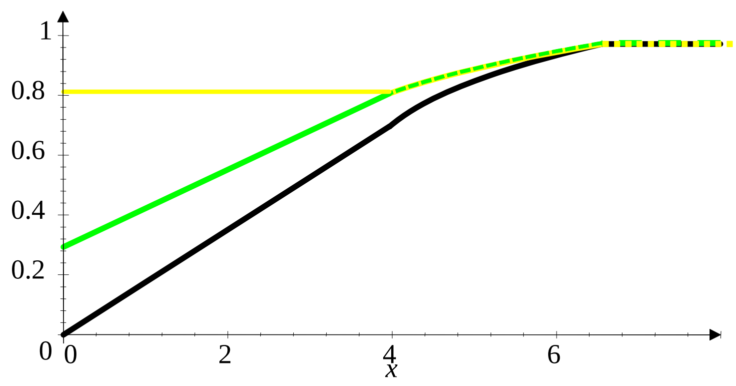

Figure 2: Equilibrium strategies for the case $s_1 = 1$ (yellow), $s_2 = 2$ (green), $s_3 = 3$ (black).

Proof. Conditions (a) to (d) of Lemma 1 hold, with $\alpha_j = 0$ and $\beta_j = \beta$ for each $j = 1, \ldots, n$. $\square$

Lemma 3. *Suppose $n > 2$ and $n_m = 1$. Let F_i, $i = 1, \ldots, m$, be distribution functions on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} x\, dF_i(x) = r_i$ for each $i = 1, \ldots, m$. Suppose there are real numbers $\alpha > 0$, $\beta > 0$, and c_i, $i = 1, \ldots, m-1$, such that the following conditions hold.*

(i) $\prod_{i<m}(F_i(x))^{n_i} = \alpha + \beta x$ *for all* $0 \le x \le (1-\alpha)/\beta$;

(ii) $F_m(x) \cdot (F_{m-1}(x))^{n_{m-1}-1} \cdot \prod_{i<m-1}(F_i(x))^{n_i} = \frac{\beta}{1-\alpha} x$ *for all* $0 \le x \le \frac{1-\alpha}{\beta}$;

(iii) $c_1 > c_2 \ldots > c_{m-2} > c_{m-1} = 0$;

(iv) $F_i(x) = \max\{F_{m-1}(x), F_{m-1}(c_i)\}$ *for each* $i = 1, \ldots, m-2$;

Then $(G_1, \ldots, G_n)$ with $G_j = F_i$ if $s_j = r_i$ is an equilibrium.

Proof. Conditions (a) to (d) of Lemma 1 hold, with $\alpha_j = \alpha$ and $\beta_j = \beta$ for that j with $s_j = r_m$, and $\alpha_j = 0$ and $\beta_j = \beta/(1-\alpha)$ for the other j's. $\square$

The next lemma provides the main tool to establish that strategies as given by the previous lemmata indeed exist.

Lemma 4. *Let $p \geq 2$ be an integer, let $q_i \geq 1$ be an integer for each $i = 1, \ldots, p$, and let $0 < t_1 < t_2 \ldots < t_p$ be real numbers. Then:*
(I) *Given any number $a \in [0, 1)$ there are distribution functions F_i on $\mathbb{R}_+$, $i = 1, \ldots, p$, such that*

(i) $\int_{\mathbb{R}_+} x dG_i(x) = t_i$ *for each* $i = 1, \ldots, p$

and such that for some real numbers $b(a) > 0$ and $c_1 > c_2 \ldots > c_p = 0$

(ii) $\prod_{i=1}^{p} (F_i(x))^{q_i} = a + \frac{1-a}{b(a)} x$ *for all* $0 \leq x \leq b(a)$, *and*

(iii) $F_i(x) = \max\{F_p(x), F_p(c_i)\}$ *for each* $i = 1, \ldots, p-1$.

(II) *Moreover, given any number $\overline{r} > t_p$, the number a can be chosen in such a way that the distribution function $\overline{F}$ on $\mathbb{R}_+$ given by*

$$(*) \qquad \overline{F}(x) = \frac{\frac{1}{b(a)} x}{a + \frac{1-a}{b(a)} x} F_p(x) \quad \text{for} \quad 0 \leq x \leq b(a).$$

satisfies

$$(**) \qquad \int_{\mathbb{R}_+} x d\overline{F}(x) = \overline{r}.$$

Proof: see Appendix. In Lemma 2 and Lemma 3 we proposed candidates for equilibrium strategies. With the help of Lemma 4 we show in the proof of the following theorem the existence of these strategies.

Theorem 3. *An equilibrium exists in the one slot case with any finite number of agents.*

Proof. Suppose first that $n_m \geq 2$. Let $F_1, \ldots, F_m$ be distribution functions on $\mathbb{R}_+$, chosen according to Lemma 4(I), with $a = 0$, $p = m$, $t_i = r_i$ for all $i = 1, \ldots, p$, $q_i = n_i$ for $i = 1, \ldots, p-1$, but(!) $q_p = n_m - 1$. Let $G_1, \ldots, G_n$ be defined as in the statement of Lemma 2. Then, by Lemma 2, $(G_1, \ldots, G_n)$ is an equilibrium. For the case $n_m = 1$, let $F_1, \ldots, F_{m-1}$ be distribution functions on $\mathbb{R}_+$ chosen according to Lemma 4, with $p = m - 1$,

$t_i = r_i$ and $q_i = n_i$ for $i = 1, \ldots, m-1$, such that the number a is so that $(**)$ of that lemma holds for $\overline{r} = r_m$ and the distribution function $\overline{F}$ determined by $(*)$. Then for each j with $s_j = r_i < r_m$, set $G_i = F_i$, and for that j with $r_j = r_m$ set $G_j = \overline{F}$. where $\overline{F}$ is the distribution function according to part (II) of Lemma 4. A glance at Lemma 3 reveals that $(G_1, \ldots, G_n)$ is an equilibrium. □

4.2 The case of multiple slots

As before, there are n agents $j = 1, \ldots, n$ with resources $Ts_j > 0$ for each j, where T denotes the number of slots. By Theorem 3, there is an n-tuple $(\overline{G}_1, \ldots, \overline{G}_n)$ of distribution functions on $\mathbb{R}_+$ which constitutes a *partial* equilibrium in any single slot when each agent invests an amount s_j of his resources in each slot. For each $j = 1, \ldots, n$, let F_j be the T-tuple of distribution functions on $\mathbb{R}_+$ given by $F_j^z = \overline{G}_j$ for each $z = 1, \ldots, T$. We claim that the n-tuple $(F_1, \ldots, F_n)$ constitutes an equilibrium for the T slots problem. To see this, consider any agent j. By the definition of F_j we have

$$\sum_{z=1}^{T} \int_{\mathbb{R}_+} x \, dF_j^z(x) = \sum_{z=1}^{T} \int_{\mathbb{R}_+} x \, d\overline{G}_j(x) = Ts_j$$

i.e. the resource constraint holds for F_j. Observe that since each agent $i \neq j$ chooses the same strategy $\overline{G}_i$ in each slot, j is confronted with the same payoff function in each single slot. Let us denote this payoff function common for all slots by π_j. (That is, $\pi_j(x)$, $x \in \mathbb{R}_+$, is the expected payoff when j invests an amount of x with certainty in any of the single slots.) Now suppose there is a T-tuple $H_j = (H_j^1, \ldots, H_j^T)$ of distribution functions on $\mathbb{R}_+$, i.e. of strategies for the single slots $z = 1, \ldots, T$, such that the resource constraint holds for H_j and such that H_j yields a expected payoff larger than that of F_j, i. e. such that

$$\sum_{z=1}^{T} \int_{\mathbb{R}_+} x \, dH_j^z(x) \leq Ts_j$$

and

$$\sum_{z=1}^{T} \int_{\mathbb{R}_+} \pi(x)\, dH_j^z(x) > \sum_{z=1}^{T} \int_{\mathbb{R}_+} \pi(x)\, dF_j^z((x)\,.$$

Set $\overline{H}_j = \frac{1}{T} \sum_{z=1}^{T} H_j^z$. Then $\overline{H}_j$ is a distribution function on $\mathbb{R}_+$ and we have

$$\int_{\mathbb{R}_+} x\, d\overline{H}_j(x) = \frac{1}{T} \sum_{z=1}^{T} \int_{\mathbb{R}_+} x\, dH_j^z(x) \le s_j$$

as well as

$$\begin{aligned} \int_{\mathbb{R}_+} \pi_j(x)\, d\overline{H}_j(x) = \frac{1}{T} \sum_{z=1}^{T} \int_{\mathbb{R}_+} \pi_j(x)\, dH_j^z(x) &> \frac{1}{T} \sum_{z=1}^{T} \int_{\mathbb{R}_+} \pi_j(x)\, dF_j^z(x) \\ &= \int_{\mathbb{R}_+} \pi_j(x)\, d\overline{G}_j(x)\,. \end{aligned}$$

But this amounts to a contradiction to the fact that $(\overline{G}_1, \ldots, \overline{G}_n)$ is a partial equilibrium in any single slot for the resources s_j. We may conclude that $(F_1, \ldots, F_n)$ is an equilibrium for the T slots problem. Thus we have shown:

Theorem 4. *An equilibrium exists for the multiple slots problem.*

This theorem establishes existence for the multi player multi slot problem. A simple example shows that equilibria in this case are not unique. Take as an example the case of 2 slots and 3 players, with the first two players having an equal amount of resources equal to $s_1 = s_2 = b$, and the third player having double the amount of resources of the other players, i.e. $s_3 = 2b$. In one equilibrium, player 1 allocates b to the first market, and 0 to the second market; player 2 allocates 0 to the first market and b to the second market. The third player allocates b to market 1 and b to market 2. On each market, the players allocating positive expected resources play as in the 2-player game. Hence, the highest observable bid in this equilibrium is given as $c_1 = 2b$ and given the equilibrium allocation an additional marginal unit spend on any of the slots yields the same payoff equal to $\frac{1}{2b}$ for every player on every market, making all players indifferent between the equilibrium and any alternative allocation in that equilibrium. Hence the proposed first equilibrium is indeed an equilibrium. Payoffs are then proportional to the share of each agent of the total budget, i.e. player 3's payoff (expected proportion of all markets won) is 0.5 and that of player 1 and 2 is 0.25 each. Consider now a second equilibrium where player 1 and player 2 each allocate $\frac{b}{2}$ to both market 1 and market 2, and player 3 allocates b to market 1 and b to market 2. On each market, players play according to the multiple player on one market allocation of Lemma 4. In that case, the highest bid with positive probability is $c_1 \approx b * 2.12$ and the payoff of player 3 is 0.588 whereas the payoff of player 1 and player 2 is 0.206 each. Marginal payoffs to player 1 and 2 in each market is now $\frac{1}{c_1}$ whereas it equals $\frac{1}{c_1} * (1 - 0.22)$ for player 3 on both markets, whereby marginal payoffs are non-increasing. Thus, no player can improve their payoff and the proposed second equilibrium holds. Interestingly enough, this example shows that expected payoffs are not unique in the multiple slot case whilst they are unique in the N slots, 2 player game. In the shown equilibrium of the 1 slot game, there is a positive bonus (≡expected payoff -(share of resources)) to being the single biggest player. In the multiple slot case this bonus may disappear if the biggest player overall is not the biggest (i.e. highest amount of resources) in any of the individual slots. This is exactly the situation in the first equilibrium given above, where player 1 and 2 both increased their payoffs by concentrating their resources on one market, thereby denying the the bonus to the biggest player of being the biggest in either of the two markets. Unfortunately, we cannot find an example for multi player, one slot game of multiple equilibria and conjecture that the 1 slot game equilibrium is unique.

5 Conclusions, applications and extensions

The theorems in this paper have straightforward applications. For one, they are prescriptive theorems that can be used in practise by budget constrained players facing an all-pay auction who have access to a fair insurance market. They should thus for instance be useful to campaign managers of politicians, and the managers of R&D departments engaged in patent races. The theorems in this paper can also be used as predictive theorems of the allocation of resources we should observe in practise in these instances. The prediction that the smaller party doesn't contest some regions at all whilst the bigger party always contests all regions is particularly suited for empirical applications.

An extension is to vary the payoff over slots from winning. We can accommodate this into our framework by interpreting a 'large slot' as nothing more than having a higher weight. Specifically, we can introduce a weighting function $w(t)$ with t denoting the slot where $w(t)$ is always positive and has expectation 1 over all slots. The payoff in t gets multiplied with this weight. All the reasoning of the theorem goes through. For instance, in the two-player case we get exactly the same mixing distributions F_A^* and F_B^* as long as we multiply the randomly chosen X^z and Y^z for the slot with pay-off 1 by $w(z)$, i.e. the allocated expected budgets and bids are proportional to the profit of the slot.

Another natural extension is to introduce the assumption that bidders care about the resources lost in the auction. We'd anticipate that this would have to mean an increase in the marginal probability of winning the contest for an additional bid, which in turn would suggest that the average bids would reduce. Introducing an opportunity costs of bids also reintroduces the questions of optimal auction design and entry issues (with zero opportunity costs, all available resources will be spent in any auction where additional resources imply a higher probability of winning the contest, which makes the issue of design trivial). A promising area of future research in the line we have opened lies in the multiple slots case, where our investigation has shown the possibility of non-unique pay-offs and strategies. It would be interesting to further investigate the case with one big player and various small ones where tacit collusion between small players by not bidding on each others' market can bring them benefits whilst remaining optimal for each smaller player. Such a situation would especially seem to be important for military and political markets where groups of smaller players sometimes tacitly collude against a single big player and sometimes do not. A thought would be to see what happens if one would allow for contractable side-payments.

6 Appendix

Proof of **Lemma 4**:

Proof. Fix any numbers $a \in [0, 1)$ and $b > 0$. Inductively define distribution functions $F_1^{a,b}, \dots, F_p^{a,b}$ on $\mathbb{R}_+$ and real numbers $b \geq c_1^{a,b} \geq c_2^{a,b} \dots \geq c_p^{a,b} \geq 0$ in the following way. Let $(F_1^{a,b}, c_1^{a,b})$ be the (uniquely determined) pair, where $F_1^{a,b}$ is a distribution function and $0 \leq c_1^{a,b} \leq b$ is a real number, such that

1. $\left(F_1^{a,b}(x)\right)^{q_1+\dots+q_p} = a + \frac{1-a}{b}x$ for all $c_1^{a,b} \leq x \leq b$;
2. $F_1^{a,b}(x) = F_1^{a,b}\big(c_1^{a,b}\big)$ for $0 \leq x < c_1^{a,b}$;
3. $\int_{\mathbb{R}_+} x \, dF_1^{a,b}(x) \leq t_1$;
4. there is no real number $0 \leq c < c_1^{a,b}$ such that (a) to (c) hold with c substituted for $c_1^{a,b}$.

Given that functions $F_1^{a,b}, \dots, F_k^{a,b}$ and numbers $c_1^{a,b} \geq c_2^{a,b} \dots \geq c_k^{a,b}$ have been defined for $1 \leq k < p$, define $F_{k+1}^{a,b}$ and $c_{k+1}^{a,b}$ by

1. $F_{k+1}^{a,b}(x) = F_k^{a,b}(x)$ for all $c_k^{a,b} \leq x \leq b$;
2. $\prod_{i=1}^{k} \left(F_i^{a,b}\big(c_i^{a,b}\big)\right)^{q_i} \cdot \left(F_{k+1}^{a,b}(x)\right)^{q_{k+1}+\dots+q_p} = a + \frac{1-a}{b}x$ for $c_{k+1}^{a,b} \leq x < c_k^{a,b}$;
3. $F_{k+1}^{a,b}(x) = F_{k+1}^{a,b}\big(c_{k+1}^{a,b}\big)$ for $0 \leq x < c_{k+1}^{a,b}$;
4. $\int_{\mathbb{R}_+} x \, dF_{k+1}^{a,b}(x) \leq t_{k+1}$;
5. there is no real number $0 \leq c < c_{k+1}^{a,b}$ such that (a) to (d) hold with c substituted for $c_{k+1}^{a,b}$.

(This construction can be done because $0 < t_1 < t_2 \ldots < t_p$ by hypothesis.) Observe that the numbers $c_i^{a,b}$ and the integrals $\int_{\mathbb{R}_+} x\, dF_i^{a,b}(x)$, $i = 1, \ldots, p$, depend continuously on (a, b).[4] In particular, if $(a_n, b_n) \to (a, b)$ in $[0, 1) \times \mathbb{R}_{++}$, then $F_i^{a_n,b_n}(x) \to F_i^{a,b}(x)$ for every $x \in \mathbb{R}_+$ and each i. Moreover, for any fixed $a \in [0, 1)$, if b is sufficiently large then $c_p^{a,b} > 0$, while if $b > 0$ is sufficiently small then $c_p^{a,b} = 0$. But if $c_p^{a,b} > 0$ then $\int_{\mathbb{R}_+} x\, dF_p^{a,b}(x) = t_p$ by construction. Hence, by continuity, there is a number $b(a) > 0$ such that both $c_p^{a,b(a)} = 0$ and $\int_{\mathbb{R}_+} x\, dF_p^{a,b(a)}(x) = t_p$. But $\int_{\mathbb{R}_+} x\, dF_p^{a,b(a)}(x) = t_p$ implies $\int_{\mathbb{R}_+} x\, dF_i^{a,b(a)}(x) = t_i$ for each $1 \leq i \leq p-1$, again by construction. Thus (I) of the lemma follows in view of the way the functions $F_i^{a,b}$ were defined. As for part (II), note first that our construction guarantees that for any given $a \in [0, 1)$, whenever $b' > b > 0$ and both $c_p^{a,b} = 0$ and $\int_{\mathbb{R}_+} x\, dF_p^{a,b}(x) = t_p$ then $c_p^{a,b'} > 0$, so the number $b(a)$ from the previous paragraph is uniquely determined. It is evident that if $a_n \to a$ in $[0, 1$ then the sequence $(b(a_n))$ must be bounded. Combining these facts with the fact mentioned above that the numbers $c_i^{a,b}$ and the integrals $\int_{\mathbb{R}_+} x\, dF_i^{a,b}(x)$, $i = 1, \ldots, p$, depend continuously on (a, b) one finds that the mapping $a \mapsto b(a)$ is continuous on $[0, 1)$. In particular it follows that if $a_n \to a$ in $[0, 1$ then $F_p^{a_n,b(a_n)}(x) \to F_p^{a,b(a)}(x)$ for every $x \in \mathbb{R}_+$. Consequently, for the distribution functions $\overline{F}^a$ on $\mathbb{R}_+$ defined for each $a \in [0, 1)$ by

$$\overline{F}^a(x) = \frac{\frac{1}{b(a)}x}{a + \frac{1-a}{b(a)}x} F_p^{a,b(a)}(x) \quad \text{for} \quad 0 \leq x \leq b(a)$$

we have that if $a_n \to a$ in $[0, 1)$ then $\overline{F}^{a_n}(x) \to \overline{F}^a(x)$ for every $x \in \mathbb{R}_+$ whence $\int_{\mathbb{R}_+} x\, d\overline{F}^{a_n}(x) \to \int_{\mathbb{R}_+} x\, d\overline{F}^a(x)$. Evidently, for $a = 0$ we have $\int_{\mathbb{R}_+} x\, d\overline{F}^a(x) = \int_{\mathbb{R}_+} x\, dF_p^{a,b(a)}(x)$, while if $a_n \to 1$ we must have $b(a_n) \to \infty$ implying that $\overline{F}^{a_n}(x) \to 0$ for each $x > 0$ and hence that $\int_{\mathbb{R}_+} x\, d\overline{F}^{a_n}(x) \to \infty$. Thus (II) of the lemma follows. □

[4] To see this and some of the points mentioned in the sequel, use for instance the fact that if F is any distribution function on $\mathbb{R}_+$ then $\int_{\mathbb{R}_+} x\, dF(x) = \int_{\mathbb{R}_+} (1 - F(x))\, d(x)$.

In the following lemma, z is a real number > 0 and P is the set of all (Borel-) probability measures on $[0, z]$.

Lemma 5. *Let $g\colon [0, z] \to \mathbb{R}_+$ be an increasing function, let $\mu \in P$ with $\operatorname{supp}\mu = [0, z]$, and suppose $(*)$ $\int_{[0,z]} x\, d\mu(x) = c$ for some real number c. Suppose*

$$\infty > \int_{[0,z]} g\, d\mu \geq \int_{[0,z]} g\, d\mu' \quad \text{for all } \mu' \in P \text{ with } \int_{[0,z]} x\, d\mu'(x) = c.$$

Then for some numbers α, β we must have $g(x) = \alpha + \beta x$ for all $x \in [0, z]$ with $x > 0$. That is, on $(0, z]$, g is the restriction of an affine function.

Proof. Since g is increasing and $\operatorname{supp}\mu = [0, z]$, it is readily seen that $(*)$ and $(**)$ imply that g must be continuous at every $x \in [0, z]$ with $x > 0$. It follows that we may assume that g is also continuous at 0. Let M be the vector space of all bounded signed Borel measures on $[0, z]$ and let I_μ be the order ideal in M generated by μ; that is,

$$I_\mu = \{\mu' \in M\colon -n\mu \leq \mu' \leq n\mu \text{ for some } n \in \mathbb{N}\}.$$

Let $\mathbf{1}$ be the function from $[0, z]$ to $\mathbb{R}$ that is constant equal to 1, and let q_1, q_2, and q_3 be the linear mappings from M to $\mathbb{R}$ given by

$$q_1(\mu') = \int_{[0,z]} g\, d\mu'$$
$$q_2(\mu') = \int_{[0,z]} \mathbf{1}\, d\mu'$$
$$q_3(\mu') = \int_{[0,z]} x\, d\mu'(x)\ .$$

Let $\overline{q}_1$, $\overline{q}_2$, and $\overline{q}_3$ denote the restrictions to I_μ of q_1, q_2, and q_3, respectively. Let ker denote the kernel of a linear mapping. Evidently $(*)$ and $(**)$ imply that $\ker \overline{q}_2 \cap \ker \overline{q}_3 \subset \ker \overline{q}_1$. According to a standard fact from linear algebra this means that $\overline{q}_1 = \alpha\overline{q}_2 + \beta\overline{q}_3$ for some real numbers α and β. Now since $\operatorname{supp}\mu = [0, z]$, I_μ is weak* dense in M. Since q_1, q_2, and q_3 all are weak* continuous it follows that $q_1 = \alpha q_2 + \beta q_3$ whence g has the form $g(x) = \alpha + \beta x$, $x \in [0, z]$. □

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