

On the Uniformly Most Powerful Invariant Test for the Shoulder Condition in Line Transect Sampling

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Summary

In wildlife population studies one of the main goals is estimating the population abundance. Line transect sampling is a well established methodology for this purpose. The usual approach for estimating the density or the size of the population of interest is to assume a particular model for the detection function (the conditional probability of detecting an animal given that it is at a given distance from the observer). Two common models for this function are the half-normal model and the negative exponential model. The estimates are extremely sensitive to the shape of the detection function, particularly to the so-called shoulder condition, which ensures that an animal is almost certain to be detected if it is at a small distance from the observer. The half-normal model satisfies this condition whereas the negative exponential does not. Therefore, testing whether such a hypothesis is consistent with the data is a primary concern in every study aiming at estimating animal abundance. In this paper we propose a test for this purpose. This is the uniformly most powerful test in the class of the scale invariant tests. The asymptotic distribution of the test statistic is worked out by utilising both the half-normal and negative exponential model while the critical values and the power are tabulated via Monte Carlo simulations for small samples.

Keywords: Line Transect Sampling, Shoulder Condition, Uniformly Most Powerful Invariant Test, Asymptotic Critical Values, Monte Carlo Critical Values.

1. INTRODUCTION

Many studies of wildlife populations aim at estimating the population abundance. Transect sampling provides an effective approach for the estimation of the population size ν or the density $\delta = \nu / A$, where A is the area of the study region. A thorough review of this methodology is given for instance by Barabesi (2000).

The Line transect design (Buckland et al., 2001) in particular assumes that k not overlapping lines are randomly chosen within the study area and, then at each of the selected lines, an observer measures the distance from the line to any animal detected. Since the number of animals observed from each line is quite small in many contexts where this sample scheme is adopted, sampled distances are pooled together to increase the sample size.

Let z_1, \dots, z_n be the sample of size n obtained by pooling together the distances measured at each of the k lines. Let f be the probability density function (pdf) of the observed distances and let g be the detection function, that is to say $g(y)$ is the conditional probability of detecting an animal given that it is at distance y from the line. The relation:

$$f(z) = \frac{g(z)}{\int_0^{+\infty} g(y)dy} \quad (1)$$

holds for every distance z . The estimator of the population density δ is:

$$\hat{\delta} = \frac{n}{2l} \hat{f}(0)$$

where l is the total length of the considered lines and $\hat{f}(0)$ is an estimator of f at 0 which satisfies the fundamental identity:

$$f'(0) = \frac{1}{\int_0^{+\infty} g(y)dy}$$

(Buckland et al., 2001). The basic problem for estimating δ , or equivalently ν , is therefore to estimate $f(0)$.

We consider two popular families of detection functions (Zhang, 2001; Eidous, 2005):

the half-normal family:

$$g(y) = \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad (\sigma > 0) \quad (2)$$

and the negative exponential family:

$$g(y) = \exp\left(-\frac{y}{\sigma}\right) \quad (\sigma > 0). \quad (3)$$

The former satisfies the shape criterion:

$$g'(0) = 0 \quad (4)$$

whereas the latter does not. This property, also known as the shoulder condition, ensures that animal detection is nearly certain at small distances from the observer (Buckland et al., 2001; Buckland et al., 2004). However, such a condition fails when detectability decreases sharply around the observation lines because of low or inexistent visibility (e.g. in presence of fog or dense vegetation) and it cannot hold for transect data of many wildlife species (Mack and Quang, 1998; Mack et al., 1999).

In the line transect framework, Eidous (2005) reported some simulation results suggesting that the usual estimators of δ are extremely sensitive to departures from the shape criterion (4).

Hence evaluating whether the shape criterion is consistent with the data should be a preliminary step for any attempt to estimate wildlife population density via line transect sampling (Zhang, 2003; Eidous, 2005). This problem has been previously addressed by Mack (1998) and Zhang (2001).

In this paper we propose a procedure for testing the shoulder condition (4).

As this condition is independent from the choice of the measure unit for the distance, the scale invariance seems to be quite a natural restriction for a statistical test. Particularly we consider a scale invariant test for discriminating between the two families (2) and (3). Because of (1) this turns out to be equivalent to testing that the distance probability density function (pdf) belongs to one of the two families:

$$F_0 = \left\{ f(z) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right) : \sigma > 0 \right\} \quad (5)$$

and:

$$F_1 = \left\{ f(z) = \frac{1}{\sigma} \exp\left(-\frac{z}{\sigma}\right) : \sigma > 0 \right\}. \quad (6)$$

The test proposed herein is the uniformly most powerful (UMP) in the class of the scale invariant tests. This is discussed in the following section where the asymptotic distributions of the test statistic under (5) and (6) are calculated.

In section 3 the critical values and the powers of the test are tabulated by Monte Carlo simulations for several typical α -levels and small sample sizes n . Conclusions are provided in section 4.

2. THE UMP SCALE INVARIANT TEST

Given n independent observations z_1, \dots, z_n from an unknown pdf f , we consider the problem of testing:

$$H_0 : f \in F_0 \quad \text{vs.} \quad H_1 : f \in F_1, \quad (7)$$

where F_0 is the family of half-normal distributions with scale parameter σ , and F_1 is the family of Gamma distributions with shape parameter 1 and scale parameter σ , specified

in (5) and (6) respectively. This problem is invariant under the group of scale transformations:

$$G = \{\gamma(z) = rz : r > 0\}.$$

A maximal invariant under G (Lehmann and Romano, 2005, pp. 214-215) is:

$$(z_1/z_n, z_2/z_n, \dots, z_{n-1}/z_n). \quad (8)$$

It can be proved (see the Appendix) that the UMP test among all of the functions of this maximal invariant rejects the null hypothesis for large values of the likelihood ratio:

$$\lambda = \frac{(n-1)!(\pi/n)^{n/2}}{2^{n-1}\Gamma(n/2)} \left[\frac{\frac{1}{n} \sum_{i=1}^n z_i^2}{\left(\frac{1}{n} \sum_{i=1}^n z_i\right)^2} \right]^{n/2}. \quad (9)$$

As λ is a monotonically increasing function of the statistic:

$$Q_n = \frac{\frac{1}{n} \sum_{i=1}^n z_i^2}{\left(\frac{1}{n} \sum_{i=1}^n z_i\right)^2}, \quad (10)$$

the critical region of the UMP scale invariant test for the hypotheses (7) is:

$$Q_n \geq q_{n,\alpha}, \quad (11)$$

where α denotes the level of significance and $q_{n,\alpha}$ is the corresponding critical value so that:

$$P(Q_n \geq q_{n,\alpha} | H_0) = \alpha.$$

It may be observed that the test procedure is equivalent to the likelihood ratio test proposed by Zhang (2001) although the UMP invariant property was not considered in that paper.

Furthermore, the asymptotic normal distribution under H_0 is:

$$\sqrt{n}(Q_n - \pi/2) \xrightarrow{d} N\left(0, \frac{\pi^2(\pi-3)}{2}\right), \quad (12)$$

and under H_1 :

$$\sqrt{n}(Q_n - 2) \xrightarrow{d} N(0,4),$$

which are derived from the bivariate central limit theorem and the delta method. For large n the approximate critical value and the power are given respectively by:

$$q_{n,\alpha} \cong 1.57 + 0.84 z_{1-\alpha} / \sqrt{n}$$

and:

$$1 - \beta = P(Q_n \geq q_{n,\alpha} | H_1) \cong 1 - \Phi(0.42 z_{1-\alpha} - 0.21 \sqrt{n}),$$

where $z_{1-\alpha}$ is the $(1-\alpha)$ th quantile and Φ is the cumulative distribution function of the standard normal distribution. Hence, the proposed test is consistent (Lehmann, 2001, p. 158).

3. TABLES OF CRITICAL VALUES AND POWERS

In this section Monte Carlo simulations are performed in order to obtain the empirical critical values and powers for small sample sizes.

The simulation design consists of randomly drawing n distances from the distribution (5) setting $\sigma = 1$. We can make this without loss of generality as the distribution of the test statistic under (5) or (6) does not depend on the scale parameter.

The statistic (10) is then applied to each of the simulated samples and the procedure is repeated 5000 times. The critical value $q_{n,\alpha}$ for a considered significance level α is obtained as $100 \times (1-\alpha)$ -th percentile of the Monte Carlo replicates. We obtained the power of the test analogously by simulating each sample according to the alternative distribution (6). Monte Carlo approximations of the critical values $q_{n,\alpha}$ and powers are reported in Table 1 and in Table 2. The power obtained under (6) is good even in the case of a small sample and low α .

TABLE 1 about here

TABLE 2 about here

The test performs reasonably well in terms of the power in the case of the data generated from a mixture of (5) and (6) too. In particular we consider the case where the sample is drawn from the following pdf :

$$p \sqrt{\frac{2}{\pi}} \exp(-z^2/2) + (1-p) \exp(-z),$$

where p is the average proportion of the observed distances simulated from a population distributed according to the alternative hypothesis. Table 3 shows the power of the test of level $\alpha = 0.05$ for a range of mixture proportions p and some sample sizes.

TABLE 3 about here

It can be observed that the proposed procedure performs well even in the case of a sample of moderate size drawn from mixture model with a large p .

4. CONCLUSIONS

In transect sampling the problem of testing the shoulder condition of a detection function is invariant under the group of scale transformations. Hence, the scale invariance is a natural restriction on the statistical procedure one has to use. In the case of the half-normal and the negative exponential family, two commonly used models of detection functions, the above problem is reduced to testing (7). In this paper we proposed the UMP scale invariant test for the abovementioned problem and the limiting normal distribution of the test statistic is provided. For small samples we tabulated the critical values and related powers via Monte Carlo simulations for a range of different sample sizes and significant levels. It turned out that the simulated critical values and powers are very similar to those obtained by the asymptotic distribution for a sample size of 100 or more. For example, in the case of a sample size equal to 100, the empirical and asymptotic critical values at the 5% level were 1.71 and 1.73 respectively; also looking at the power, the asymptotic and the empirical approaches provides similar values: 0.93 and 0.95 respectively.

APPENDIX

The pdf of the sample (z_1, \dots, z_n) can be written as:

$$L(z_1, \dots, z_n) = \frac{1}{\sigma^n} \prod_{i=1}^n f\left(\frac{z_i}{\sigma}\right),$$

where:

$$f(z) = \begin{cases} \sqrt{\frac{2}{\pi}} \exp(-z^2/2) & : \text{ under } H_0 \\ \exp(-z) & : \text{ under } H_1 \end{cases} .$$

Hence the maximal invariant (8) is expressed as:

$$(r_1, \dots, r_{n-1}) = (z_1/z_n, z_2/z_n, \dots, z_{n-1}/z_n)$$

and has pdf given by:

$$\int_0^{+\infty} z_n^{n-1} L(z_n r_1, \dots, z_n r_{n-1}, z_n) dz_n = z_n^n \int_0^{+\infty} u^{n-1} \prod_{i=1}^n f(z_i u) du$$

$$= \begin{cases} \frac{2^{n-1} \Gamma(n/2)}{\left(\pi \sum_{i=1}^n z_i^2\right)^{n/2}} & : \text{ under } H_0 \\ \frac{(n-1)!}{\left(\sum_{i=1}^n z_i\right)^n} & : \text{ under } H_1 \end{cases}$$

from which the likelihood ratio (9) follows.

By the Neyman-Pearson Lemma the most powerful test rejects the null hypothesis when (9) is too large. Given that its critical region does not depend on σ , the test is UMP among all invariant tests.

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TABLES

Table 1. Critical values $q_{n,\alpha}$ of the UMP scale invariant test

$q_{n,\alpha}$	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n=100$
$\alpha = 0.01$	1.998	1.946	1.879	1.840	1.788
$\alpha = 0.05$	1.815	1.793	1.768	1.754	1.708
$\alpha = 0.10$	1.748	1.734	1.709	1.703	1.674

Table 2. Powers $1-\beta$ of the UMP scale invariant test

$1-\beta$	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n = 100$
$\alpha = 0.01$	0.336	0.439	0.581	0.667	0.859
$\alpha = 0.05$	0.601	0.682	0.774	0.815	0.953
$\alpha = 0.10$	0.706	0.776	0.859	0.887	0.976

Table 3. Powers $1-\beta$ of the UMP scale invariant test for different mixture proportions

$\alpha = 0.05$	$n = 40$	$n = 50$	$n = 100$
$p = 0.25$	0.580	0.656	0.887
$p = 0.50$	0.442	0.517	0.740
$p = 0.75$	0.261	0.299	0.454