FINITE GROUPS IN AXIOMATIC INDEX NUMBER THEORY

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ABSTRACT. In this paper we adopt Group Theory to investigate the symmetry and invariance properties of price index numbers. An alternative treatment is given to the study of the reversibilty axioms, that clarifies their meaning and allows for a conceptual unification of this topic, within the framework of Axiomatic Index Number Theory.

1. INTRODUCTION

Axiomatic Index Number Theory is probably the most developed formal framework for the study of price and quantity index numbers. Its deductive approach is based on the identification of a set of axioms that a function of prices and quantities has to satisfy, to be actually considered a price (or a quantity) index. These axioms can be subdivided into two main classes. The first class contains the axioms of *proportionality, commensurability, homogeneity* and *monotonicity* that establish constraints on the behaviour of the index number when the prices (or the quantities) assume specific values or undergo particular changes. The second class contains the axioms of *basis reversibility* and *factor reversibility* that establish constraints on the behaviour of the index number when the price and quantity vectors are exchanged each other in various ways[1]. In the following, we focus on the reversibility axioms only (a full account of the axiomatic approach to price index numbers can be found in [1], [5] and in the references there cited). The essence of the reversibility axioms is to define internal simmetries that price and quantity index numbers should satisfy. By "internal symmetry" we mean some invariance property of an index number under the action of a suitable transformation involving the price and quantity vectors. When interpreting reversibility axioms as symmetry (or invariance) properties, we are naturally led to study the set of transformations that induce them. It turns out that the set of symmetry transformations of a price

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[1]Actually, proposals of other axioms exist, but the ones we have cited are those almost universally accepted. For details see [1] and [5].

(or quantity) index number satisfies the axioms of an algebraic object called a *group*.

Group Theory was originally developed in connection with the study of the roots of algebraic equations, in the nineteenth century, but modern Group Theory was founded on a sound basis in the last '800s, when the fundamental axioms were identified by great mathematicians like S. Lie and H. Weber [2]. Since then, the theory had an enormous development, becoming one of the most important branches of pure mathematics with application in a lot of other disciplines (other branches of mathematics, physics, social sciences, just to mention some).

As far as we know, the first to apply group theory to the study of price index numbers was Vogt [5]. Later, and independently, Fattore [3] and Fattore&Quatto [9] proposed similar arguments. When applying group theory to a topic such as Axiomatic Index Number Theory, some results are expected. First, a conceptual simplification and unification. Second, a simplification of proofs of already known results. Third, the chance to obtain new results, that can be difficult to derive from a pure analytical perspective, but that are much simpler to derive in a group theoretical context. In this paper we just utilize Group Theory to clarify the deep structure of the simmetries of index numbers. Nevertheless, we feel sure that Group Theory can help in developing Axiomatic Index Number Theory in other directions, allowing for new results to be obtained.

2. Elements of Group theory

In this paragraph we collect some fundamental definitions of Group Theory and give some examples that will be relevant for the subsequent discussion (details can be found in [2], [8], [6], [4]).

2.1. **Basic definitions.** A *group* is a pair $(G, \circ)$, where G is a set and $\circ$ is a binary associative operation defined on $G \times G$ (usually called a *multiplication*), such that the following axioms are satisfied:

(1) $g_1 \circ g_2 \in G \quad \forall g_1, g_2 \in G$
(2) $\exists\, e \in G : e \circ g = g \circ e = g \quad \forall g \in G$
(3) $\forall g \in G \quad \exists h \in G : g \circ h = h \circ g = e.$

The first axiom requires G to be closed under the action of $\circ$, the second axiom defines e as the *identity* of the group and the third axiom requires that for every element of the group its *inverse* exists in G. Usually, the inverse of an element g is denoted by g^{-1}. If, in addition, it is verified that $g_1 \circ g_2 = g_2 \circ g_1$ for every pair g_1, g_2 of elements of G, then the group is called *commutative* or *abelian*. When there is no risk of ambiguity, the multiplication $\circ$ is often

omitted and $g_1 g_2$ is written instead of $g_1 \circ g_2$. In the following, we will adopt such a notation. If the set G is composed of a finite number of elements, then the group is called a *finite group* and its *order* is the cardinality of the set G.

A finite group is completely specified by its *Cayley table* that reports, for each ordered pair g_1, g_2, the product $g_1 g_2$. For a generic group G of order n, the Cayley table has the following form:

(2.1)

$$
\begin{array}{c|ccccc}
G & g_1 & \cdots & \cdots & g_n \\
\hline
g_1 & g_1 g_1 & \cdots & \cdots & g_1 g_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
g_n & g_n g_1 & \cdots & \cdots & g_n g_n
\end{array}
$$

A *subgroup* G' of G is a subset of G that is a group with respect to the restriction of $\circ$ to $G' \times G'$.

A finite group is called *cyclic* if there is an element $a \in G$ and a positive integer n such that:

(2.2)
$$G = \{e, a, a^2, \ldots, a^{n-1}\}$$

with $a^n = e$. Obviously, cyclic groups are abelian. A cyclic group of order n will be denoted by Z_n. Given an element $a \in G$, the set $\{e, a, a^2, \ldots, a^{n-1}\}$ is the cyclic subgroup generated by a. The *order* of an element of a group G is the order of the cyclic subgroup it generates.

Let G be a group and G_1 and G_2 be two subgroups of G such that

(1) $g_1 g_2 = g_2 g_1$ for every $g_1 \in G_1$ and $g_2 \in G_2$
(2) every $g \in G$ can be uniquely represented as $g = g_1 g_2 = g_2 g_1$, with $g_1 \in G_1$ and $g_2 \in G_2$.

The group G is said to be the *direct product* of G_1 and G_2.

Let $(G, \circ_g)$ and $(H, \circ_h)$ be two groups and let φ be a mapping $G \to H$ such that

(2.3)
$$\varphi(g_1 \circ_g g_2) = \varphi(g_1) \circ_h \varphi(g_2) \quad \forall\, g_1, g_2 \in G.$$

The mapping φ is said a *homomorphism* and the groups G and H are said to be *homomorphic*. If φ is bijective, then it is said to be an *isomorphism* and G and H are said to be *isomorphic*. Two isomorphic groups share the same structure and can be essentially identified.

The nature of the elements of a group is completely arbitrary. In the following we will deal with finite groups composed of invertible operators acting on a suitable set of functions, where the multiplication $\circ$ stands for the ordinary law of composition of operators. Such groups are called *group of transformations* (or *group of operators*).

Let G be a group of transformations and let us indicate with X the set on which the elements of G act. For each $x \in X$ and $g \in G$, let us indicate with $g \cdot x$ the action of the element g on x. The set $\{g \cdot x, \ g \in G\}$ is called the *orbit* of x under the action of G. The relation R defined on $X \times X$ by

$$x_1 R \, x_2 \Leftrightarrow x_1 \text{ and } x_2 \text{ belong to the same orbit}$$

is an equivalence relation. Thus two orbits either coincide or have no element in common.

2.2. **Examples of finite groups.** In the following, we present some finite groups that are relevant for the theory of index numbers.

The group Z_2. The group $Z_2 = \{e, a\}$ having the following Cayley table:

(2.4)

Z_2	e	a
e	e	a
a	a	e

is a cyclic group of order 2. It is the only group of order 2.

The group Z_4. The group $Z_4 = \{e, a, a^2, a^3\}$ with the following Cayley table

(2.5)

Z_4	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

is a cyclic group of order 4. It has three subgroups: $\{e\}$, $\{e, a^2\}$ and Z_4 itself.

The Klein group. The Klein group V_4 is an abelian group of order 4. Its Cayley table is given by:

(2.6)

V_4	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The Klein group is isomorphic to the group of the symmetries of a rectangle (i.e. the group of the transformations leaving a rectangle invariant). It has five

subgroups, precisely: $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$ and V_4 itself. The Klein group can be easily seen to be isomorphic to the group $Z_2 \times Z_2$. As a matter of fact we have

$$(2.7) \qquad V_4 = \{e, a\} \times \{e, b\} = \{e, b\} \times \{e, c\} = \{e, a\} \times \{e, c\}.$$

The group $Z_2 \times Z_2 \times Z_2$. The group $Z_2 \times Z_2 \times Z_2$ is an abelian group of order 8, defined as the direct product of three cyclic groups of order 2. Its Cayley table has the following structure:

(2.8)

Z_2^3	e	a	b	c	ab	ac	bc	abc
e	e	a	b	c	ab	ac	bc	abc
a	a	e	ab	ac	b	c	abc	bc
b	b	ab	e	bc	a	abc	c	ac
c	c	ac	bc	e	abc	a	b	ab
ab	ab	b	a	abc	e	bc	ac	c
ac	ac	c	abc	a	bc	e	ab	b
bc	bc	abc	c	b	ac	ab	e	a
abc	abc	bc	ac	ab	c	b	a	e

$Z_2 \times Z_2 \times Z_2$ has 16 subgroups: $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, ab\}$, $\{e, bc\}$, $\{e, ac\}$, $\{e, abc\}$, $\{e, a, b, ab\}$, $\{e, a, c, ac\}$, $\{e, b, c, bc\}$, $\{e, b, ac, abc\}$, $\{e, ab, c, abc\}$, $\{e, a, bc, abc\}$, $\{e, ab, bc, ac\}$ and $Z_2 \times Z_2 \times Z_2$ itself. Note that all the subgroups of order 4 are isomorphic to the Klein group.

The group D_4. The dihedral group D_4 is the group of the simmetries of a square. It is not abelian and has 8 elements. Its Cayley table is:

(2.9)

D_4	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	c	b	f	d	h	g
b	b	c	e	a	g	h	d	f
c	c	b	a	e	h	g	f	d
d	d	f	h	g	e	a	c	b
f	f	d	g	h	a	e	b	c
g	g	h	d	f	b	c	a	e
h	h	g	d	f	c	b	e	a

D_4 has 10 subgroups: $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, d\}$, $\{e, f\}$, $\{e, g, a, h\}$, $\{e, a, d, f\}$, $\{e, a, b, c\}$ and D_4 itself. The subgroup $\{e, g, a, h\}$ is isomorphic to the cyclic group Z_4, while both the other subgroups of order 4 are isomorphic to the Klein group.

The group $D_4 \times Z_2$. The group $D_4 \times Z_2$ is a non abelian group of 16 elements with the following Cayley table:

(2.10)

$D_4 \times Z_2$	e	a	b	c	d	f	g	h	l	m	n	p	q	r	s	t
e	e	a	b	c	d	f	g	h	l	m	n	p	q	r	s	t
a	a	e	c	b	m	l	n	p	f	d	g	h	t	s	r	q
b	b	c	e	a	l	m	p	n	d	f	h	g	s	t	q	r
c	c	b	a	e	f	d	h	g	m	l	p	n	r	q	t	s
d	d	g	h	f	e	c	a	b	q	r	s	t	l	m	n	p
f	f	h	g	d	c	e	b	a	r	q	t	s	m	l	p	n
g	g	d	f	h	r	q	s	t	c	e	a	b	p	n	m	l
h	h	f	d	g	q	r	t	s	e	c	b	a	n	p	l	m
l	l	p	n	m	b	a	c	e	s	t	q	r	d	f	h	g
m	m	n	p	l	a	b	e	c	t	s	r	q	f	d	g	h
n	n	m	l	p	s	t	r	q	b	a	e	c	h	g	d	f
p	p	l	m	n	t	s	q	r	a	b	c	e	g	h	f	d
q	q	t	s	r	h	g	f	d	n	p	l	m	e	c	b	a
r	r	s	t	q	g	h	d	f	p	n	m	l	c	e	a	b
s	s	r	q	t	n	p	m	l	h	g	d	f	b	a	e	c
t	t	q	r	s	p	n	l	m	g	h	f	d	a	b	c	e

This group has 35 subgroups (we will not provide the full list here): 11 of order 2, 15 of order 4 and 7 of order 8, plus the two trivial subgroups $\{e\}$ and $D_4 \times Z_2$ itself. Details of this group and further references can be found in [5].

3. Price and quantity index numbers

Let $\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b$ be the price and quantity vectors of a set of n different goods in two temporal or spatial situations a and b. Let us consider situation b as the reference situation (the *basis*) for the comparison. We define the *value index* V_{ab} as:

$$(3.1) \qquad V_{ab} = \frac{\sum_{i=1}^{n} p_{ai} q_{ai}}{\sum_{i=1}^{n} p_{bi} q_{bi}}.$$

The goal of index number theory is to decompose V_{ab} as the product of two functions, P and Q, the first accounting for the variation in the prices between b and a, and the second accounting for the variation in the quantities:

$$(3.2) \qquad V_{ab} = \frac{\sum_{i=1}^{n} p_{ai} q_{ai}}{\sum_{i=1}^{n} p_{bi} q_{bi}} = P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \cdot Q(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b).$$

When there is no ambiguity, we will write:

$$(3.3) \qquad V_{ab} = P_{ab} \cdot Q_{ab}$$

where $P_{ab} = P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$ and $Q_{ab} = Q(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$.

Suppose we have identified a price index P_{ab}. A quantity index Q_{ab} can thus be naturally obtained as V_{ab}/P_{ab}. This index is called the *cofactor* of P_{ab} and it is indicated as $cof(P_{ab})$. A different quantity index naturally associated with P_{ab} is obtained exchanging the vectors $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ with the vectors $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$, in the function defining P_{ab} itself. This index is called the *correspondent* of P_{ab} and it is indicated as $cor(P_{ab})$:

$$(3.4) \qquad cor(P) = P(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b).$$

Cofactor and correspondent of a price index have a central role in the following. Being both induced by the choice of P_{ab}, wether they coincide or not is a condition for the internal consistency of the price comparison itself, as the following section discusses.

4. Reversibility axioms

To guarantee the logical and economical consistency of a price or a quantity index number, Axiomatic Index Number Theory requires P and Q to satisfy a list of mathematical properties or axioms [1]. Here, we focus only on the basis reversibility and factor reversibility properties which aim at guaranteeing the internal consistency of the formulas adopted as price or quantity index numbers (in the following, we focus ourselves on price indexes, but the discussion could have been developed in terms of quantity indexes as well).

Basis reversibility. Given a price index P_{ab}, basis reversibility requires that when exchanging the situation b and a (i.e. assuming a as the basis of the comparison), P_{ab} turns into P_{ab}^{-1}:

$$(4.1) \qquad P(\boldsymbol{p}_b, \boldsymbol{p}_a, \boldsymbol{q}_b, \boldsymbol{q}_a) = \frac{1}{P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)}$$

or, with a compact notation:

$$(4.2) \qquad P_{ba} = \frac{1}{P_{ab}}.$$

The index P_{ba}^{-1} is called the *basis anthitesis* of P_{ab} and the axiom of basis reversibility states that the index P_{ab} and its basis anthitesis have to coincide.

Factor reversibility. Given a price index P_{ab}, factor reversibility requires that the two naturally quantity indexes associated with P_{ab} coincide, i.e.:

$$(4.3) \qquad cof(P_{ab}) = cor(P_{ab}).$$

Alternatively, (4.3) can be stated as [7]:

$$(4.4) \qquad P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = \frac{V_{ab}}{P(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b)}.$$

The right hand side of (4.4) is called the *factor antithesis* of P_{ab} and the factor reversibility axiom states that the index P_{ab} and its factor antithesis have to coincide.

5. The group of the four antitheses

Let Π be the set of strictly positive functions of the four vectors $\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b$. Let $\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}$ be four operators acting on Π with values in Π, defined as:

$$(5.1) \qquad \boldsymbol{I}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)$$

$$(5.2) \qquad \boldsymbol{B}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = \frac{1}{f(\boldsymbol{p}_b, \boldsymbol{p}_a, \boldsymbol{q}_b, \boldsymbol{q}_a)}$$

$$(5.3) \qquad \boldsymbol{F}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = \frac{V_{ab}}{f(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b)}$$

$$(5.4) \qquad \boldsymbol{D}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = V_{ab} \cdot f(\boldsymbol{q}_b, \boldsymbol{q}_a, \boldsymbol{p}_b, \boldsymbol{p}_a)$$

It is immediately checked that $\boldsymbol{D} = \boldsymbol{B} \circ \boldsymbol{F} = \boldsymbol{F} \circ \boldsymbol{B}$.

By means of the operators $\boldsymbol{B}$ and $\boldsymbol{F}$, the reversibility axioms can be stated as the invariance properties $\boldsymbol{B}(P_{ab}) = P_{ab}$ (basis reversibility) and $\boldsymbol{F}(P_{ab}) = P_{ab}$ (factor reversibility).

The operators $\boldsymbol{B}$ and $\boldsymbol{F}$ can thus be called *basis antithesis operator* and *factor antithesis operator* respectively and the operator $\boldsymbol{D}$ can be called *double antithesis* operator (for alternative denominations, see [5]).

The set $G_1 = \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}\}$ is an abelian group with respect to the composition of operators. This group will be called the *group of the four anthiteses* and its Cayley table is given by:

$$(5.5)$$

G_1	$\boldsymbol{I}$	$\boldsymbol{B}$	$\boldsymbol{F}$	$\boldsymbol{D}$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{B}$	$\boldsymbol{F}$	$\boldsymbol{D}$
$\boldsymbol{B}$	$\boldsymbol{B}$	$\boldsymbol{I}$	$\boldsymbol{D}$	$\boldsymbol{F}$
$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{D}$	$\boldsymbol{I}$	$\boldsymbol{B}$
$\boldsymbol{D}$	$\boldsymbol{D}$	$\boldsymbol{F}$	$\boldsymbol{B}$	$\boldsymbol{I}$

The Cayley table of G_1 shows that this group is isomorphic to the Klein group. As a consequence, it has five subgroups, precisely: $\{\boldsymbol{I}\}$, $\{\boldsymbol{I}, \boldsymbol{B}\}$, $\{\boldsymbol{I}, \boldsymbol{F}\}$, $\{\boldsymbol{I}, \boldsymbol{D}\}$ and G_1 itself.

From the Cayley table, the following proposition follows easily:

Proposition 5.1. P_{ab} *is invariant under the action of* $\boldsymbol{D}$, *if and only if* $\boldsymbol{B}(P_{ab}) = \boldsymbol{F}(P_{ab})$.

Proof. If $\boldsymbol{B}(P_{ab}) = \boldsymbol{F}(P_{ab})$, then from the Cayley table of G_1:

$$(5.6) \qquad \boldsymbol{D}(P_{ab}) = (\boldsymbol{BF})(P_{ab}) = \boldsymbol{B}^2(P_{ab}) = P_{ab}.$$

On the other hand, if $\boldsymbol{D}(P_{ab}) = P_{ab}$, again from the Cayley table, we have:

$$\boldsymbol{B}(P_{ab}) = \boldsymbol{B}\big(\boldsymbol{D}\,(P_{ab})\big) = (\boldsymbol{BD})(P_{ab}) = \boldsymbol{F}\,(P_{ab})\,.$$

$\square$

Similarly, it can be shown that $\boldsymbol{F}(P_{ab}) = P_{ab}$ if and only if $\boldsymbol{B}(P_{ab}) = \boldsymbol{D}(P_{ab})$ and that $\boldsymbol{B}(P_{ab}) = P_{ab}$ if and only if $\boldsymbol{F}(P_{ab}) = \boldsymbol{D}(P_{ab})$.

The definition of the antithesis operators involves some permutations on the arguments of the index numbers P_{ab}. A direct inspection reveals that such permutations are the exchange $\boldsymbol{\sigma}_{ab}$ of the situation a and b, the exchange $\boldsymbol{\sigma}_{pq}$ of the vectors $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ with $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ respecitvely and what can be called *double exchange* $\boldsymbol{\sigma}_d$, that is given by the composition of $\boldsymbol{\sigma}_{ab}$ and $\boldsymbol{\sigma}_{pq}$ (these permutations commutes, so their order in the definition of $\boldsymbol{\sigma}_d$ is irrelevant). The set $G_1^* = \{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{pq}, \boldsymbol{\sigma}_d\}$ ($\boldsymbol{\sigma}_e$ is the permutation that leaves all the arguments in the original order) is an abelian group, and its Cayley table is given by:

$$(5.7)$$

G_1^*	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$
$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$
$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$
$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$
$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$

Thus, also the group G_1^* is isomorphic to the Klein group. Its five subgroups are the following: $\{\boldsymbol{\sigma}_e\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{pq}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_d\}$ and G_1^* itself.

6. THE STRUCTURE OF THE FACTOR ANTITHESIS OPERATOR

The second group we are interested in is another abelian group of order four. Let $\boldsymbol{H}$ and $\boldsymbol{K}$ be two operators acting on the set Π as follows:

$$(6.1) \qquad \boldsymbol{H}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = \frac{V_{ab}}{f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)}$$

$$(6.2) \qquad \boldsymbol{K}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = f(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_a, \boldsymbol{p}_b)$$

or, more sintethically:

$$(6.3) \qquad\qquad \boldsymbol{H}(f) \;=\; cof(f)$$

$$(6.4) \qquad\qquad \boldsymbol{K}(f) \;=\; cor(f).$$

The four operators $\boldsymbol{I}, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{F}$ form an abelian group G_2 with the following Cayley table:

(6.5)

G_2	$\boldsymbol{I}$	$\boldsymbol{H}$	$\boldsymbol{K}$	$\boldsymbol{F}$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{H}$	$\boldsymbol{K}$	$\boldsymbol{F}$
$\boldsymbol{H}$	$\boldsymbol{K}$	$\boldsymbol{I}$	$\boldsymbol{F}$	$\boldsymbol{K}$
$\boldsymbol{K}$	$\boldsymbol{H}$	$\boldsymbol{F}$	$\boldsymbol{I}$	$\boldsymbol{H}$
$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{K}$	$\boldsymbol{H}$	$\boldsymbol{I}$

As it can be seen, G_2 has the same structure of the Klein group and its five subgroups are: $\{\boldsymbol{I}\}$, $\{\boldsymbol{I}, \boldsymbol{H}\}$, $\{\boldsymbol{I}, \boldsymbol{K}\}$, $\{\boldsymbol{I}, \boldsymbol{F}\}$ and G_2 itself.

The group G_2 clarifies the structure of the factor antithesis operator and its elements will have an important role in the subsequent discussion.

7. The ACC group

When we get a price index number P_{ab}, we can generate other index numbers by means of the antithesis operators[2], i.e. by means of the action of the group G_1 on P_{ab}. The images of P_{ab} under G_1 define an equivalence class of index numbers, that is the orbit of P_{ab}. Starting from P_{ab} we can also generate quantity index numbers, by means of the operators $\boldsymbol{H}$ (the cofactor) and $\boldsymbol{K}$ (the correspondent). Obviously, we could start generating quantity index numbers from any element of the orbit of P_{ab}. Similarly, when we get a quantity index Q_{ab}, we can generate other quantity index numbers by means of G_1 and we can also generate price index numbers applying $\boldsymbol{H}$ and $\boldsymbol{K}$ to Q_{ab} and to the elements of its orbit. The following question naturally arises: what is the relationship between the price and quantity indexes generated as above? From a group theoretical point of view, this is the same as asking about the relationship between the action of G_1 on $\boldsymbol{H}(P_{ab})$ and $\boldsymbol{K}(P_{ab})$ and the action of $\boldsymbol{H}$ and $\boldsymbol{K}$ on the orbit of P_{ab}. This leads to the study of the set of operators $S = \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{K}\}$.

The set S is not a group, since the operators $\boldsymbol{BH}$, $\boldsymbol{HB}$, $\boldsymbol{BK}$ and $\boldsymbol{KB}$ do not belong to S, as can be easily verified. By a direct computation, we have $\boldsymbol{BH} = \boldsymbol{HB}$ and $\boldsymbol{BK} = \boldsymbol{KB}$, where:

$$(7.1) \qquad \boldsymbol{BH}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = V_{ab} \cdot f(\boldsymbol{p}_b, \boldsymbol{p}_a, \boldsymbol{q}_b, \boldsymbol{q}_a)$$

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[2]Even if we do not deal with this problem, we stress the fact that in order to be actually accepted as index numbers, the images of P_{ab} under the antithesis operators have to satisfy the same list of axiomatic properties that are to be satisfied by the P_{ab} itself.

and

(7.2) $$\boldsymbol{BK}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = \frac{1}{f(\boldsymbol{q}_b, \boldsymbol{q}_a, \boldsymbol{p}_b, \boldsymbol{p}_a)}.$$

If we put $\boldsymbol{J} = \boldsymbol{BH}$ and $\boldsymbol{L} = \boldsymbol{BK}$, the set $G_3 = \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{J}, \boldsymbol{L}\}$ does form a group of order 8 with respect to the usual operator law of composition. We can call it the *ACC* (*Antithesis, Cofactor* and *Correspondent)* group. Its Cayley table is:

(7.3)

G_3	I	B	F	D	H	K	J	L
I	I	B	F	D	H	K	J	L
B	B	I	D	F	J	L	H	K
F	F	D	I	B	K	H	L	J
D	D	F	B	I	L	J	K	H
H	H	J	K	L	I	F	B	D
K	K	L	H	J	F	I	D	B
J	J	H	L	K	B	D	I	F
L	L	K	J	H	D	B	F	I

As it can be seen, G_3 is an abelian group containing G_1 and G_2 and it is isomorphic to the group $Z^2 \times Z^2 \times Z^2$ (to see this, put $\boldsymbol{B} = a, \boldsymbol{F} = b$ and $\boldsymbol{H} = c$). Its 16 subgroups are the following: $\{\boldsymbol{I}\}, \{\boldsymbol{I}, \boldsymbol{B}\}, \{\boldsymbol{I}, \boldsymbol{F}\}, \{\boldsymbol{I}, \boldsymbol{H}\}, \{\boldsymbol{I}, \boldsymbol{D}\}, \{\boldsymbol{I}, \boldsymbol{K}\}, \{\boldsymbol{I}, \boldsymbol{J}\}, \{\boldsymbol{I}, \boldsymbol{L}\}, \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}\} = G_1, \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{H}, \boldsymbol{J}\}, \{\boldsymbol{I}, \boldsymbol{F}, \boldsymbol{H}, \boldsymbol{K}\} = G_2, \{\boldsymbol{I}, \boldsymbol{F}, \boldsymbol{J}, \boldsymbol{L}\}, \{\boldsymbol{I}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{L}\}, \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{K}, \boldsymbol{L}\}, \{\boldsymbol{I}, \boldsymbol{D}, \boldsymbol{K}, \boldsymbol{J}\}$ and G_3 itself.

8. New axioms and possible extensions

Axiomatic Index Number Theory is now a well established deductive framework, for the study of price and quantity index number formulas. However, the discussion about the axioms that should be included is not over and from time to time different authors propose new properties that index numbers should satisfy, enlarging and modifing the axiomatic structure of the theory. Here we give an example that is relevant for our discussion about finite groups.

In 1978, Funke and Voeller proposed two new axioms that we can directly state in terms of invariance under the action of a new pair of operators. Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two operator on Π, defined as follows:

(8.1) $$\boldsymbol{U}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_b, \boldsymbol{q}_a)$$

(8.2) $$\boldsymbol{V}\big(f(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b)\big) = \frac{1}{f(\boldsymbol{p}_b, \boldsymbol{p}_a, \boldsymbol{q}_a, \boldsymbol{q}_b)}.$$

Funk and Voeller require that:

$$(8.3) \qquad\qquad \boldsymbol{U}(P_{ab}) = P_{ab}$$
$$(8.4) \qquad\qquad \boldsymbol{V}(P_{ab}) = P_{ab}.$$

If we add $\boldsymbol{U}$ and $\boldsymbol{V}$ to the antithesis operators we have considered so far, we get the set $W = \{\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{D}, \boldsymbol{U}, \boldsymbol{V}\}$. W is not a group, not being closed under the composition of operators, nor it is a subset of any finite group, as can be easily seen[3] (note that the *Fisher Group* introduced by Vogt, is based on a different definition of some of the antithesis operators. This group is isomorphic to $D_4 \times Z_2$. For details, see [5]).

Nevertheless, the set of permutations involved in the definition of the elements of W do generate a finite group. A simple inspection of the operators $\boldsymbol{U}$ and $\boldsymbol{V}$ shows that the set of permutations we are dealing with is $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{pq}, \boldsymbol{\sigma}_d, \boldsymbol{\sigma}_{p_a p_b}, \boldsymbol{\sigma}_{q_a q_b}\}$, where $\boldsymbol{\sigma}_{p_a p_b}$ and $\boldsymbol{\sigma}_{q_a q_b}$ exchange the price vectors and the quantity vectors respectively. This set is not a group, since it is not closed under the composition of permutations. Particularly, we have:

$$(8.6) \qquad \boldsymbol{\sigma}_{pq} \circ \boldsymbol{\sigma}_{p_a p_b} P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = P(\boldsymbol{q}_b, \boldsymbol{q}_a, \boldsymbol{p}_a, \boldsymbol{p}_b)$$
$$(8.7) \qquad \boldsymbol{\sigma}_{p_a p_b} \circ \boldsymbol{\sigma}_{pq} P(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) = P(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{p}_b, \boldsymbol{p}_a)$$

(in the following we will write $\boldsymbol{\sigma}_{pq} \circ \boldsymbol{\sigma}_{p_a p_b} = \boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_{p_a p_b} \circ \boldsymbol{\sigma}_{pq} = \boldsymbol{\sigma}_v$.). A direct computation shows that $\boldsymbol{\sigma}_{q_a q_b} \circ \boldsymbol{\sigma}_{pq} = \boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_{pq} \circ \boldsymbol{\sigma}_{q_a q_b} = \boldsymbol{\sigma}_v$ and that $\boldsymbol{\sigma}_u^2 = \boldsymbol{\sigma}_v^2 = \boldsymbol{\sigma}_{ab}$, so the set $G_4^* = \{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{pq}, \boldsymbol{\sigma}_d, \boldsymbol{\sigma}_{p_a p_b}, \boldsymbol{\sigma}_{q_a q_b}, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ is a non abelian group of order 8, with the following Cayley table:

G_4^*	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_v$
$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_v$
$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_u$
$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{q_a q_b}$
$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_{p_a p_b}$
$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$
$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$
$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{ab}$	$\boldsymbol{\sigma}_e$
$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_v$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\sigma}_{p_a p_b}$	$\boldsymbol{\sigma}_{q_a q_b}$	$\boldsymbol{\sigma}_d$	$\boldsymbol{\sigma}_{pq}$	$\boldsymbol{\sigma}_e$	$\boldsymbol{\sigma}_{ab}$

From the multiplication table, we see that the group G_4^* is isomorphic to the group D_4. Its ten subgroups are: $\{\boldsymbol{\sigma}_e\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{pq}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_d\}$,

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[3]Some computations shows that

$$(8.5) \qquad\qquad \boldsymbol{FU}(P_{ab}) = w(\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{q}_a, \boldsymbol{q}_b) \cdot P_{ab}$$

where $w(\cdot, \cdot, \cdot, \cdot)$ ia a suitable function. So no finite group can contain the set W.

$\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{p_a p_b}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{q_a q_b}\}$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{pq}, \boldsymbol{\sigma}_d\} = G_1^*$, $\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$,
$\{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_{ab}, \boldsymbol{\sigma}_{p_a p_b}, \boldsymbol{\sigma}_{q_a q_b}\}$ and G_4^* itself.

9. CONCLUSIONS

In this paper we have briefly presented some sets of symmetry transformations that are relevant in the framework of Axiomatic Index Number Theory, discussing their algebraic properties from the point of view of Group Theory. The scope of the paper was limited to the analysis of the reversibility axioms. A natural extension is thus the study of the conditions that guarantee an antitheses of a price index to satisfy all the axioms of Axiomatic Index Number Theory. This leads to the study of the "antitheses of the axiomatic properties", an interesting idea introduced by Vogt ([5]) that deserves more study, in order to fully understand the action of the antithesis groups on the set of price indexes and the chance to utilize Group Theory to obtain new results in Axiomatic Index Number Theory.

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