Elasticity of Substitution and Growth: Normalized CES in the Diamond Model

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Abstract

Klump and de La Grandville (2000) used the "normalized" Constant Elasticity of Substitution (CES) specification to prove that the Solow growth model exhibits a positive relationship between per capita output and the elasticity of substitution both in transition and in steady state. This paper shows that their result does not extend to the Diamond overlapping generations model. In particular, their result is reversed when capital and labor are relatively substitutable; countries with a higher elasticity of substitution have lower per capita output and growth.

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1 Introduction

In a recent paper, Klump and de La Grandville (2000) utilized the "normalized" Constant Elasticity of Substitution (CES) production function in the Solow (1956) growth model and found that a country endowed with a greater elasticity of substitution experiences greater capital and output per worker both in transition and in steady state. The objective of this paper is to examine whether their result carries over to the Diamond (1965) overlapping-generations model. Such examination is warranted because the Diamond model has increasingly been used in recent years to study economic growth as an alternative to the Solow model. Our main finding is that the Klump-de La Grandville result does not hold in the Diamond model; in particular, their result is reversed if the elasticity of substitution is sufficiently large.

2 The Normalized CES Production Function in the Solow Model

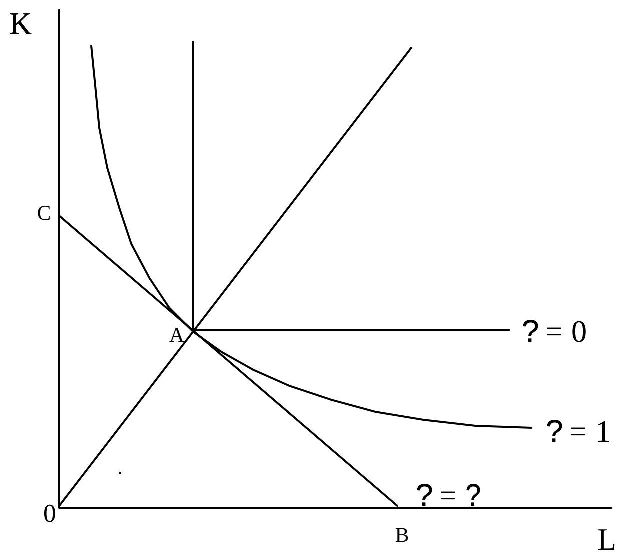
Oliver de La Grandville (1989) suggested that a meaningful examination of the properties of different members of the same family of CES production functions requires the following normalization. Given the standard intensive-form CES production function $f(k_t) = A[\delta k_t^{\rho} + (1-\delta)]^{\frac{1}{\rho}}$, where k_t is the capital per worker at time t, choose arbitrary baseline values for capital per worker ($\bar{k}$), output per worker ($\bar{y}$) and the marginal rate of substitution between capital and labor defined by $\bar{m} = [f(\bar{k}) - \bar{k}f'(\bar{k})]/f'(\bar{k})$ (primes denote derivatives with respect to k). Then, use those baseline values to solve for the normalized efficiency parameter $A(\sigma) = \bar{y}\left(\frac{\bar{k}^{1-\rho}+\bar{m}}{\bar{k}+\bar{m}}\right)^{1/\rho}$, and the normalized distribution parameter $\delta(\sigma) = \frac{\bar{k}^{1-\rho}}{\bar{k}^{1-\rho}+\bar{m}}$ as a function of $\sigma = \frac{1}{1-\rho}$, the elasticity of substitution. Substituting these normalized parameters into the initial equation yields the normalized CES production function:[1]

$$f_{\sigma}(k_t) = A(\sigma)\left\{\delta(\sigma)k_t^{\rho} + [1-\delta(\sigma)]\right\}^{\frac{1}{\rho}}: \tag{1}$$

Figure 1 illustrates the de La Grandville normalization. Despite disparate values for σ, all the isoquants for a given initial level of output ($\bar{y}$) are shown to go through the common point (point A) defined by $\bar{k}$ (given by ray OA) and $\bar{m}$ (given by line BAC). As shown by Pitchford (1960), an increase in σ without the normalization causes not only an increase in the curvature of the

[1] For extensive discussions on the normalized CES function see de La Grandville (1989, p.476), and Klump and Preissler (2000, pp.44-45).

Figure 1: Illustration of de La Grandville's normalized CES production function



isoquant for a given level of output; it also causes the isoquant to shift inward by making factors more e±cient. The de La Grandville normalization prevents such dispersions.

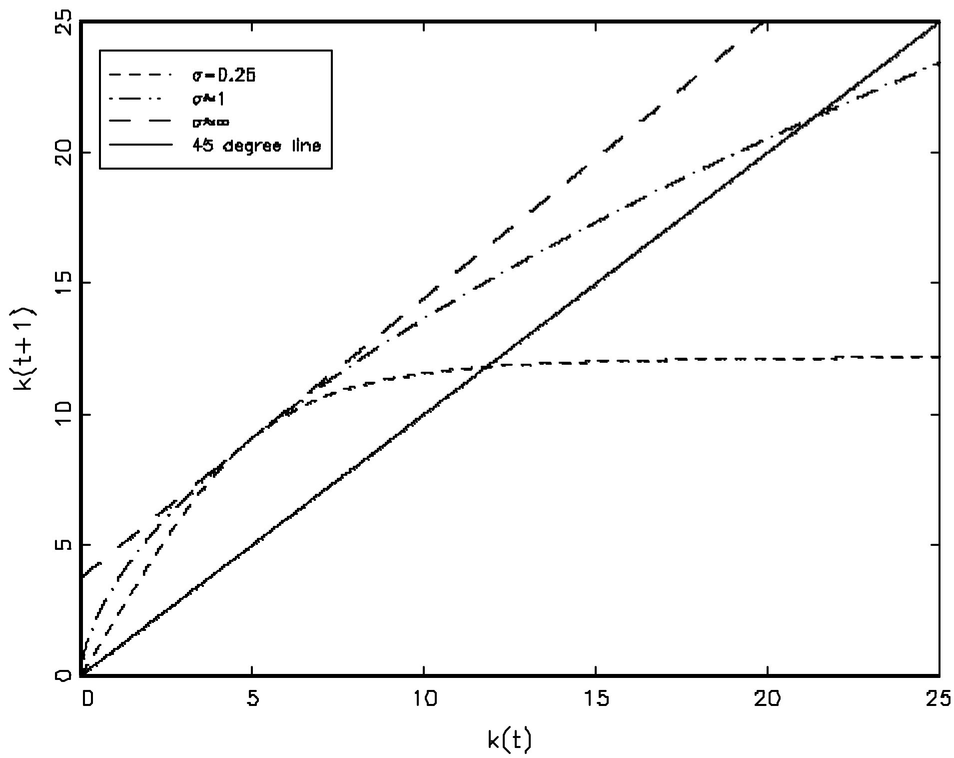
The de La Grandville normalization generates a family of dynamical paths in the Solow growth model that depend only on the value of ¾. Paths of capital per worker for three values of ¾ are shown in Figure 2. Figure 2 di®ers from Figure 1 in Klump and de la Grandville (2000 p.284) because here the Solow model is recast in a discrete-time setting to facilitate comparison with the Diamond model below. More speci¯cally, the paths shown in the ¯gure are generated by the dynamical equation

$$k_{t+1} = \frac{°}{1+n} f_{¾}(k_t);$$

where ° is the exogenous saving rate out of output per worker, n is the exogenous labor growth rate and where for simplicity capital is assumed to depreciate fully at the end of each period.

Despite the translation into the discrete-time setting, the Klump-de la Grandville result is evident; a country having a greater value of ¾ clearly has more capital per worker in transition and in steady state than a country endowed with a lower value of ¾. It follows that, the greater the value of ¾, the greater income per worker is both in transition and in steady state.

Figure 2: Transitional paths of per capita capital for different ES in the Solow model



3 The Normalized CES Production Function in the Diamond Model

In the Diamond (1965) overlapping-generations model a new generation is born at the beginning of every period. Agents are identical and live for two periods. In the first period each agent supplies a unit of labor inelastically and receives a competitive wage

$$w_{\sigma,t} = f_\sigma(k_t) - k_t f'_\sigma(k_t) = [1 - \delta(\sigma)][A(\sigma)]^{\psi}[f_\sigma(k_t)]^{1-\psi}.$$

To make the model consistent with the Solow model, assume that agents save a fixed proportion γ of the wage income to finance consumption in the second period of their lives. All savings are invested as capital to be used in the next period's production; that is

$$k_{t+1} = \frac{\gamma}{1+n} w_\sigma(k_t) = \frac{\gamma}{1+n}[1 - \delta(\sigma)][A(\sigma)]^{\psi}[f_\sigma(k_t)]^{1-\psi} \equiv h_\sigma(k_t); \quad (2)$$

where n is the exogenous labor growth rate and where capital depreciates fully.[2] Equations (2)

[2] Alternatively, we could assume that agents have preferences over consumption in the two periods of their lives

determines the dynamical path of capital per worker. Then, the dynamical path of output per worker is obtained from (1).

Steady states for k (denoted by $*$) are solutions to the polynomial equation

$$k - h_\sigma(k) = 0. \tag{3}$$

If $\sigma \geq 1$ ($\rho \in [0, 1]$), there always exists one unique positive steady state for k^*, since $\lim_{k \to 0} h'_\sigma(k) > 1$ and $\lim_{k \to +\infty} h'_\sigma(k) = 0$. If $\sigma < 1$ ($\rho < 0$), there are either zero or two positive and distinct steady-state values for k^*, depending on the value of the scale factor $A(\sigma)$.[3]

We now turn to our two main findings. (All proofs are in the Appendix.)

Theorem 1 Suppose that a country is represented by the one-sector Diamond model with a normalized CES aggregate production function. If $\rho \geq \frac{\bar{m}}{\bar{k}}$, for any $k_t > \bar{k}$,

(A) the higher the elasticity of substitution the lower the level of capital and output per worker at any stage of the transition path, and

(B) the higher the elasticity of substitution the lower the growth rates of capital and output per worker along the transitional path.

Theorem 2 Suppose that a stable steady state exists in the one-sector Diamond model with a normalized CES aggregate production function. If $\rho \geq \frac{\bar{m}}{\bar{k}}$, the higher the elasticity of substitution, the lower the steady-state level of capital and output per worker.

Figure 3 illustrates the dynamical paths of capital per worker in the Diamond model for $\rho \geq \frac{\bar{m}}{\bar{k}}$, where we set $\bar{m} = 1$ and $\bar{k} = 5$. As σ increases from 1.25 to 5 and to ∞, the level of capital per worker falls both in transition and in steady state for any $k_t > \bar{k}$, thereby reversing the Klump-de La Grandville result.[4]

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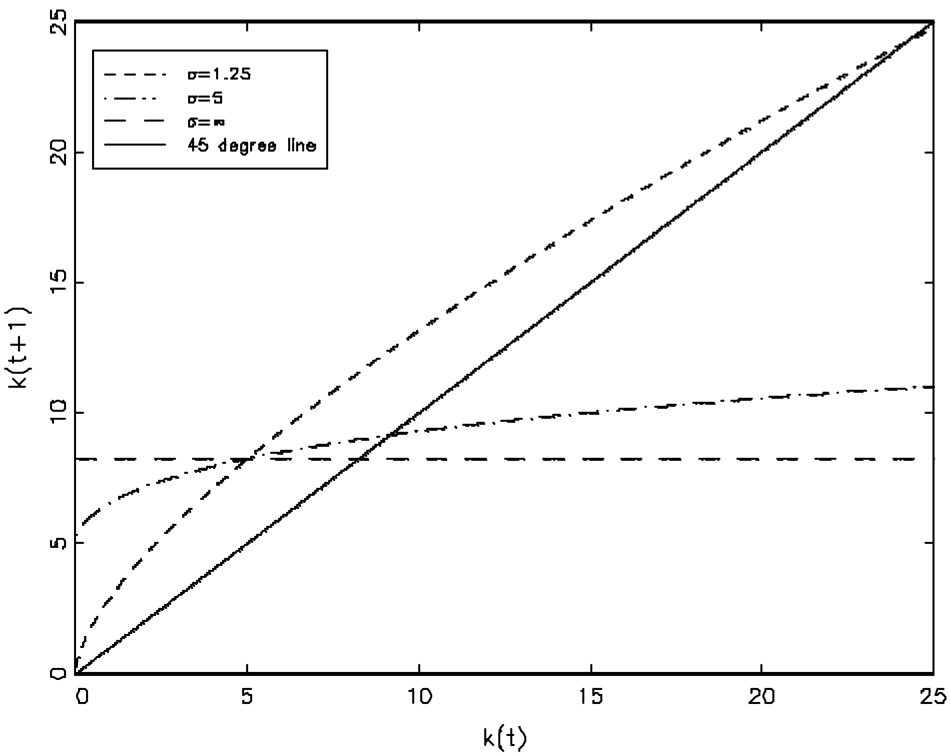
given by $U(c_t^1, c_{t+1}^2) = (1 - \gamma) \ln c_t^1 + \gamma \ln c_{t+1}^2$, where c_{t+j}^i is period i consumption by the representative agent in period $t + j$, $j = 0, 1$. The representative agent maximizes $U(c_t^1, c_{t+1}^2)$ subject to the constraint, $c_t^1 + \frac{c_{t+1}^2}{R_{\sigma,t+1}} \leq w_{\sigma,t}$, where $w_{\sigma,t}$ and $R_{\sigma,t+1}$ represent the returns to labor and capital, respectively. Maximization yields the transition equation, $k_{t+1} = \frac{\gamma}{1+n} w_\sigma(k_t)$, which is equivalent to equation (2).

[3]When there are two positive steady states, the larger of the two is locally asymptotically stable. In this case, the trivial steady state ($k^* = 0$) is also locally asymptotically stable. The domains of attraction of the two stable steady states are distinct, and depend on whether the initial capital stock lies above or below the locally unstable equilibrium. The conditions for and characterization of multiple equilibria in the Diamond (1965) model (see e.g. Azariadis 1993, pp.198-204) remain unaffected by the normalization.

[4]Parametric examples of the dynamic relationship between y_{t+1} and y_t are available upon request.

Figure 3: Transitional paths of per capita capital in the Diamond model when $\rho \geq \frac{\bar{\mu}}{\bar{k}}$



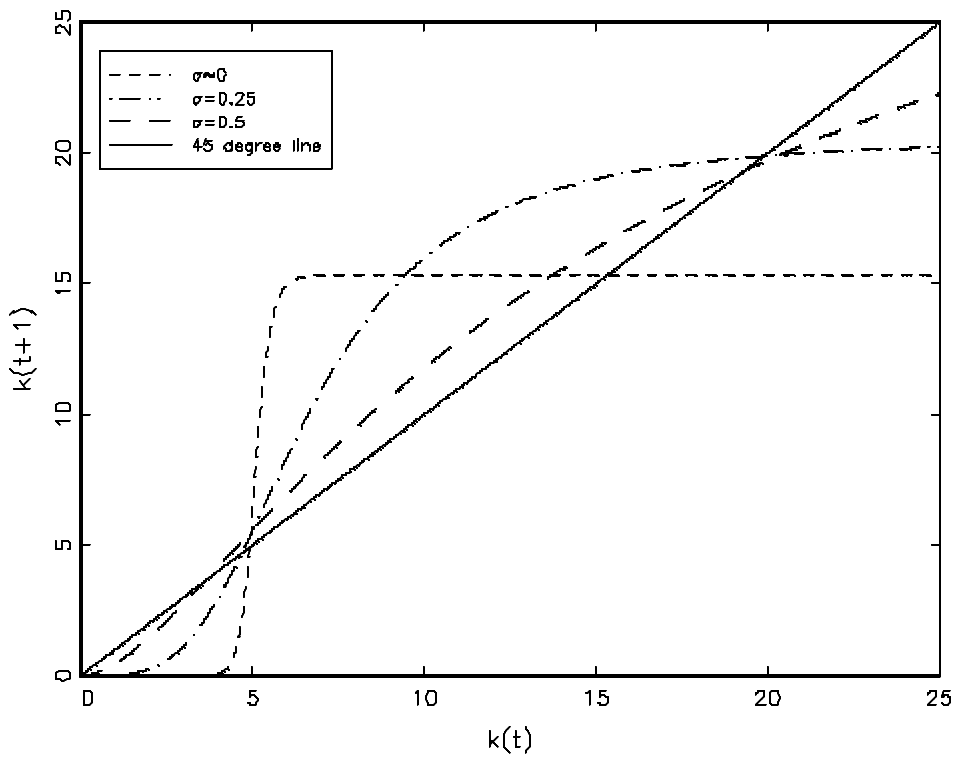
Moreover, if $\rho < \frac{\bar{\mu}}{\bar{k}}$ the relationship between the σ and the level of capital per worker is not unique because, as shown in Figure 4, the dynamical paths of k for different values of σ cross each other at some $k_t > \bar{k}$.[5]

Why do our results contrast with those of Klump and de La Grandville? The Diamond model differs from the Solow model in one important respect: individual savings come out of wage income in the former and out of total (wage and rental) income in the latter. A useful way to demonstrate the difference is offered by Galor (1996). Suppose that the fraction saved out of wage income, γ^w, differs from the fraction saved out of rental income, γ^r, possibly because of differences in preferences or endowments among agents. Then the law of motion for capital per worker in the normalized CES production function is

$$k_{t+1} = \frac{\gamma^w}{1+n}\left[f_{\sigma}(k_t) - f'_{\sigma}(k_t)k_t\right] + \frac{\gamma^r}{1+n} f'_{\sigma}(k_t)k_t. \quad (4)$$

[5] In constructing Figure 4, we set $\bar{\mu} = 3$, $\bar{k} = 5$ to keep the diagram from getting cluttered.

Figure 4: Transitional paths of per capita capital in the Diamond model when ½ < $\frac{1}{k}$



Since °w = °r = ° in the Solow model while °w = ° and °r = 0 in the Diamond model, the dynamical path in the former contains the additional term, $\frac{°}{1+n} f'_{¾}(k_t)k_t$, that represents savings out of rental income.

For example, when ¾ = 1 (capital and labor are perfect substitutes), equation (4) reduces to

$$k_{t+1} = \frac{°ỳm̄}{(1+n)(k̀ + m̄)} + \frac{°ỳ}{(1+n)(k̀ + m̄)} k_t;$$

in the Solow model. Thus, k_{t+1} is a linear positive function of k_t with the vertical intercept at $\frac{°ỳm̄}{(1+n)(k̀+m̄)}$ and the slope $\frac{°ỳ}{(1+n)(k̀+m̄)}$. On the other hand, in the Diamond model equation (4) reduces to

$$k_{t+1} = \frac{°m̄ỳ}{(1+n)(k̀ + m̄)}:$$

Thus, k_{t+1} is a horizontal line at $\frac{°m̄ỳ}{(1+n)(k̀+m̄)}$.[6] Then, as k grows from the common initial value k̀, the entire capital intensity path of the Solow model lies above the path of the Diamond model.

[6]The former line is depicted by the parametric curve ¾ ¼ 1 in Figure 2, while the latter line is depicted by the parametric curve ¾ = 1 in Figure 3.

Now, to get an intuition of our result differentiate equation (4) with respect to σ to obtain:

$$\frac{\partial k_{t+1}}{\partial \sigma} = \frac{\gamma^w}{1+n} \underbrace{\left[(1-\pi_t)\frac{\partial f_\sigma(k_t)}{\partial \sigma} - f_\sigma(k_t)\frac{\partial \pi_t}{\partial \sigma} \right]}_{\frac{\partial w_t}{\partial \sigma}\ (?)} + \frac{\gamma^r}{1+n} \underbrace{\left[\pi_t \frac{\partial f_\sigma(k_t)}{\partial \sigma} + f_\sigma(k_t)\frac{\partial \pi_t}{\partial \sigma} \right]}_{\frac{\partial r k_t}{\partial \sigma}\ (+)}; \qquad (5)$$

where $\pi_t \equiv \frac{f'_\sigma(k_t)k_t}{f_\sigma(k_t)}$ is the rental income share. The first and the second term on the RHS of equation (5) show the change in wage and rental incomes, respectively, due to a change in σ. The second expression is clearly positive while the first is generally ambiguous in sign. No matter what, the second expression must dominate the first in the Solow model (where $\gamma^w = \gamma^r = \gamma$) since $\frac{\partial k_{t+1}}{\partial \sigma} > 0$ as shown by Klump and de La Grandville (2000).

In the Diamond model (where $\gamma^w = \gamma$ and $\gamma^r = 0$) the second expression in equation (5) is absent. Within the first expression, $(1-\pi_t)\frac{\partial f_\sigma(k_t)}{\partial \sigma}$ represents a positive effect of σ on wage income due to an increase in labor productivity for a given wage income share $(1-\pi_t)$. The second term $-f_\sigma(k_t)\frac{\partial \pi_t}{\partial \sigma}$ represents a negative effect of σ on wage income due a decrease in the wage income share triggered by substitution of capital for labor. If the wage income share $(1-\pi_t)$ is sufficiently small, then the negative effect dominates and our result follows. Indeed $\rho \geq \frac{\bar{n}}{\bar{k}}$ implies $(1-\bar{\pi}) < 1/2$ which is sufficient to obtain our result.[7]

4 Conclusion

In this paper we have shown that the positive relationship between the elasticity of substitution and economic growth discovered recently by Klump and de la Grandville does not carry over to the Diamond model. Thus, whether the elasticity of substitution has a positive or negative effect on economic growth depends on our view of the world, that is, on the particular framework (Solow vs. Diamond) we believe as a better representation of the world.

Both our work and that of Klump and de la Grandville take the elasticity of substitution as exogenous. However, as pointed out by Hicks (1932), the aggregate elasticity of substitution itself is likely to be influenced by factors that also affect economic growth. Thus, endogenizing the elasticity of substitution in the context of a growth model seems like a natural next step in this line of research.

[7]Since $\bar{\pi} = \frac{\bar{k}}{\bar{k}+\bar{n}}$ implies that $\frac{\bar{n}}{\bar{k}} \equiv \frac{1-\bar{\pi}}{\bar{\pi}}$; substitution into $\rho \geq \frac{\bar{n}}{\bar{k}}$ yields $\bar{\pi} \geq \frac{1}{1+\rho}$. Given that $\rho \in (0;1]$ under $\rho \geq \frac{\bar{n}}{\bar{k}}$, $\rho \geq \frac{\bar{n}}{\bar{k}}$ finally implies $\bar{\pi} \geq 1/2$.

Appendix

Proof of Theorem 1

Rewrite equation (2) as

$$\begin{aligned} k_{t+1} &= \frac{\gamma}{1+n}\left[f_\sigma(k_t) - k_t f'_\sigma(k_t)\right] \\ &= \frac{\gamma}{1+n}\left[f_\sigma(k_t)(1-\pi_t)\right]; \end{aligned}$$

where $\pi_t \equiv \frac{f'_\sigma(k_t)k_t}{f_\sigma(k_t)}$ is the rental income share. Differentiating with respect to σ yields

$$\frac{\partial k_{t+1}}{\partial \sigma} = \frac{\gamma}{1+n}\left[(1-\pi_t)\frac{\partial f_\sigma(k_t)}{\partial \sigma} - f_\sigma(k_t)\frac{\partial \pi_t}{\partial \sigma}\right]:$$

Substituting $\frac{\partial f_\sigma(k_t)}{\partial \sigma} = -\frac{1}{\sigma^2}\frac{1}{\rho^2}y_t\left[\pi_t \ln\frac{\bar{\pi}}{\pi_t} + (1-\pi_t)\ln\frac{1-\bar{\pi}}{1-\pi_t}\right]$ and $\frac{\partial \pi_t}{\partial \sigma} = \frac{1}{\sigma^2}(1-\pi_t)\pi_t \ln\frac{k_t}{\bar{k}}$ from Klump and de La Grandville (2000, pp.284-285) yields

$$\begin{aligned} \frac{\partial k_{t+1}}{\partial \sigma} &= -\frac{\gamma}{1+n}\left\{\frac{(1-\pi_t)y_t}{\sigma^2\rho^2}\left[\pi_t \ln\frac{\bar{\pi}}{\pi_t} + (1-\pi_t)\ln\frac{1-\bar{\pi}}{1-\pi_t}\right] + \frac{y}{\sigma^2}(1-\pi_t)\pi_t \ln\frac{k_t}{\bar{k}}\right\} \\ &= -\frac{\gamma}{1+n}\frac{(1-\pi_t)y_t}{\sigma^2\rho^2}\left[\pi_t \ln\frac{\bar{\pi}}{\pi_t} + (1-\pi_t)\ln\frac{1-\bar{\pi}}{1-\pi_t} + \rho^2\pi_t \ln\frac{k_t}{\bar{k}}\right]; \qquad \text{(A1)} \end{aligned}$$

where $\bar{\pi} = \frac{\bar{k}}{\bar{k}+\bar{m}}$. From

$$\begin{aligned} \frac{\bar{\pi}}{\pi_t} &= \left(\frac{\bar{k}}{k_t}\frac{y_t}{\bar{y}}\right)^{\rho}; \\ \frac{1-\bar{\pi}}{1-\pi_t} &= \left(\frac{y_t}{\bar{y}}\right)^{\rho}; \end{aligned}$$

we obtain $\frac{k_t}{\bar{k}} = \frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}}$. Substituting this into the last term in the brackets of equation (A1) gives

$$\frac{\partial k_{t+1}}{\partial \sigma} = -\frac{\gamma}{1+n}\frac{(1-\pi_t)y_t}{\sigma^2\rho^2}\left[\pi_t \ln\frac{\bar{\pi}}{\pi_t} + (1-\pi_t)\ln\frac{1-\bar{\pi}}{1-\pi_t} - \rho\pi_t \ln\left(\frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}}\right)\right]:$$

Since the logarithmic function is strictly concave, we have that

$$\ln\frac{\pi_t}{\bar{\pi}} < \frac{\pi_t}{\bar{\pi}} - 1 \Rightarrow -\ln\frac{\pi_t}{\bar{\pi}} > 1 - \frac{\pi_t}{\bar{\pi}} \Rightarrow \ln\frac{\bar{\pi}}{\pi_t} > 1 - \frac{\pi_t}{\bar{\pi}}; \qquad \text{(A2)}$$

$$\ln\frac{1-\pi_t}{1-\bar{\pi}} < \frac{1-\pi_t}{1-\bar{\pi}} - 1 \Rightarrow -\ln\frac{1-\pi_t}{1-\bar{\pi}} > 1 - \frac{1-\pi_t}{1-\bar{\pi}} \Rightarrow \ln\frac{1-\bar{\pi}}{1-\pi_t} > 1 - \frac{1-\pi_t}{1-\bar{\pi}}; \qquad \text{(A3)}$$

$$\ln\left(\frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}}\right) < \frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}} - 1 \Rightarrow -\ln\left(\frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}}\right) > 1 - \frac{\bar{\pi}}{\pi_t}\frac{1-\pi_t}{1-\bar{\pi}}: \qquad \text{(A4)}$$

Assume that $\sigma > 1$ ($\rho \in (0, 1]$) and $k_t > \bar{k}$. Multiplying both sides of the final inequalities in (A2), (A3) and (A4) by π_t, $(1 - \pi_t)$ and $\rho\pi_t$, respectively, yields the following inequality:

$$
\begin{aligned}
& \pi_t \ln \frac{\bar{\pi}}{\pi_t} + (1 - \pi_t) \ln \frac{1 - \bar{\pi}}{1 - \pi_t} - \rho\pi_t \ln \left(\frac{\bar{\pi}}{\pi_t} \frac{1 - \pi_t}{1 - \bar{\pi}} \right) \\
> \;& \pi_t \left(1 - \frac{\pi_t}{\bar{\pi}} \right) + (1 - \pi_t) \left(1 - \frac{1 - \pi_t}{1 - \bar{\pi}} \right) + \rho\pi_t \left(1 - \frac{\bar{\pi}}{\pi_t} \frac{1 - \pi_t}{1 - \bar{\pi}} \right) \\
= \;& \frac{\pi_t}{\bar{\pi}} (\bar{\pi} - \pi_t) + \frac{1 - \pi_t}{1 - \bar{\pi}} (\pi_t - \bar{\pi}) + \frac{\rho}{1 - \bar{\pi}} (\pi_t - \bar{\pi}) \\
= \;& (\pi_t - \bar{\pi}) \left(\frac{1 - \pi_t}{1 - \bar{\pi}} + \frac{\rho}{1 - \bar{\pi}} - \frac{\pi_t}{\bar{\pi}} \right) \\
= \;& \left(\frac{\pi_t - \bar{\pi}}{1 - \bar{\pi}} \right) \left(1 - \pi_t + \rho - \frac{\pi_t (1 - \bar{\pi})}{\bar{\pi}} \right) \\
= \;& \left(\frac{\pi_t - \bar{\pi}}{1 - \bar{\pi}} \right) \left(1 + \rho - \frac{\pi_t}{\bar{\pi}} \right) \\
= \;& \left(\frac{\pi_t - \bar{\pi}}{1 - \bar{\pi}} \right) \left[1 + \rho - \left(\frac{\bar{k} + \bar{m}}{\bar{k}} \right) \frac{k_t^{\rho} \bar{k}^{1-\rho}}{k_t^{\rho} \bar{k}^{1-\rho} + \bar{m}} \right];
\end{aligned}
\qquad \text{(A5)}
$$

where the last equality comes from the fact that $\pi_t = \frac{k_t^{\rho} \bar{k}^{1-\rho}}{k_t^{\rho} \bar{k}^{1-\rho} + \bar{m}}$. The function

$$
\phi(k_t) \equiv \left[1 + \rho - \left(\frac{\bar{k} + \bar{m}}{\bar{k}} \right) \frac{k_t^{\rho} \bar{k}^{1-\rho}}{k_t^{\rho} \bar{k}^{1-\rho} + \bar{m}} \right];
$$

is monotonically decreasing with the horizontal asymptote at $\rho - \frac{\bar{m}}{\bar{k}}$. Therefore, if $\rho \geq \frac{\bar{m}}{\bar{k}}$, $\phi(k_t) \geq 0$. Then since $\frac{\pi_t - \bar{\pi}}{1 - \bar{\pi}} \geq 0$, the last expression in (A5) is non-negative. Consequently, $\frac{\partial k_{t+1}}{\partial \sigma} < 0$.

To prove that output per worker is a decreasing function of the σ when $\rho \geq \frac{\bar{m}}{\bar{k}}$ and $k > \bar{k}$, rewrite $\frac{\partial y_{t+1}}{\partial \sigma}$ as

$$
\frac{\partial y_{t+1}}{\partial \sigma} = \frac{\partial y_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial \sigma}.
$$

We have just shown that $\frac{\partial k_{t+1}}{\partial \sigma} < 0$ for $\rho \geq \frac{\bar{m}}{\bar{k}}$ and $k > \bar{k}$. Given that $\frac{\partial y_{t+1}}{\partial k_{t+1}}$ is positive for all $k_{t+1} > 0$, then $\frac{\partial y_{t+1}}{\partial \sigma} < 0$. This completes the proof of Theorem 1A.

To prove Theorem 1B, define the growth rate of capital per worker by $g_k = \frac{k_{t+1}}{k_t} - 1$ and the growth rate of output per worker by $g_y = \frac{y_{t+1}}{y_t} - 1$. Differentiation yields

$$
\begin{aligned}
\frac{\partial g_k}{\partial \sigma} &= \frac{1}{k_t} \frac{\partial k_{t+1}}{\partial \sigma} < 0, \; \forall \rho \geq \frac{\bar{m}}{\bar{k}}; \text{ and } k > \bar{k}; \\
\frac{\partial g_y}{\partial \sigma} &= \frac{1}{y_t} \frac{\partial y_{t+1}}{\partial \sigma} < 0, \; \forall \rho \geq \frac{\bar{m}}{\bar{k}}; \text{ and } k > \bar{k}.
\end{aligned}
$$

This completes the proof.

Proof of Theorem 2

At steady state, $k_t = k_{t+1} = k^*$ and therefore equation (2) reduces to the polynomial equation (3). Differentiating k^* with respect to σ yields

$$\frac{\partial k^*_\sigma}{\partial \sigma} = \frac{\frac{\gamma}{1+n}(w^*_\sigma)'}{1 - \frac{\gamma}{1+n}(w^*_k)'}; \tag{B1}$$

where $(w^*_\sigma)' = \left[(1 - \pi^*)\frac{\partial f^*}{\partial \sigma} - f^*\frac{\partial \pi^*}{\partial \sigma}\right]$ and $(w^*_k)' = \left[(1 - \pi^*)(f^*)' - f^*\frac{\partial \pi^*}{\partial k^*}\right]$. By Theorem 2, $(w^*_\sigma)' < 0$ for $\rho \ge \frac{\eta}{\tilde{k}}$ and $k^* > k$, so the numerator is negative. To show that the denominator is positive, solve the definition of $\pi^* = \frac{(f^*)'k^*}{f^*}$ and the steady-state polynomial equation $k^* = \frac{\gamma}{1+n}[f^* - (f^*)'k^*]$ simultaneously for f^* and $(f^*)'$ to obtain

$$f^* = \frac{(1+n)}{\gamma}\frac{k^*}{1 - \pi^*}; \tag{B2}$$

$$(f^*)' = \frac{(1+n)}{\gamma}\frac{\pi^*}{1 - \pi^*}: \tag{B3}$$

Substituting equations (B2), (B3) and $\frac{\partial \pi^*}{\partial k^*} = \frac{\rho\pi^*(1-\pi^*)}{k^*}$ into $(w^*_k)'$ gives

$$\begin{aligned}(w^*_k)' &= (1 - \pi^*)\frac{(1+n)}{\gamma}\frac{\pi^*}{1-\pi^*} - \frac{(1+n)}{\gamma}\frac{k^*}{1-\pi^*}\frac{\rho\pi^*(1-\pi^*)}{k^*}\\ &= \frac{(1+n)}{\gamma}(1-\rho)\pi^*:\end{aligned}$$

Then $\frac{\gamma}{1+n}(w^*_k)' = (1-\rho)\pi^*$ and hence $1 - \frac{\gamma}{1+n}(w^*_k)' = 1 - (1-\rho)\pi^* > 0$. Therefore $\frac{\partial k^*}{\partial \sigma} < 0$.

To prove that the steady-state output per worker is a decreasing function of the σ when $\rho \ge \frac{\eta}{\tilde{k}}$ and $k > \tilde{k}$; once again rewrite $\frac{\partial y^*}{\partial \sigma}$ as

$$\frac{\partial y^*}{\partial \sigma} = \frac{\partial y^*}{\partial k^*}\frac{\partial k^*}{\partial \sigma}:$$

Given that $\frac{\partial y^*}{\partial k^*} > 0$ for all $k^* > 0$; and $\frac{\partial k^*}{\partial \sigma} < 0$ for $\rho \ge \frac{\eta}{\tilde{k}}$ and $k > \tilde{k}$ as shown above, $\frac{\partial y^*}{\partial \sigma} < 0$ as desired.

References

Azariadis, C., Intertemporal Macroeconomics. Cambridge, MA: Blackwell Publishers, 1993.

Diamond, P.A., \National Debt in a Neoclassical Growth Model," American Economic Review, 55:1126-1150, 1965.

de La Grandville, O., \In Quest of the Slutsky Diamond," American Economic Review, 79:468-481, 1989.

Galor, O., \Convergence? Inferences from Theoretical Models," Economic Journal, 106:1056-1069, 1996.

Hicks, J.R., The Theory of Wages, London: MacMillan, 1932.

Klump, R. and de La Grandville, O., \Economic Growth and Elasticity of Substitution: Two Theorems and Some Suggestions," American Economic Review, 90:282-291, 2000.

Klump, R. and Preissler, H., \CES Production Functions and Economic Growth," Scandinavian Journal of Economics, 102:41-56, 2000.

Pitchford, J.D, \Growth and the Elasticity of Substitution," Economic Record, 36:491-504, 1960.

Solow, R.M., \A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, 70:65-94, 1956.