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Two-Level CES Production Technology in the Solow and Diamond Growth Models*

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Abstract

The two-level CES aggregate production function - that nests a CES into another CES function - has recently been used extensively in theoretical and empirical applications of macroeconomics. This paper examines its theoretical properties and establishes existence and stability conditions of equilibria under the Solow and Diamond growth models. In addition, it examines the effect of changes in substitution parameters on transitional growth and steady states. It is shown that in the Solow model, the sufficient condition for a steady state is fulfilled for a larger range of substitution parameter values than with the basic CES function. In the Diamond model an increase in substitution parameters results in higher transitional growth under weaker conditions than with the basic CES function.

Keywords: Two-level CES production functions, normalization, Solow model, Diamond model, economic growth.

JEL classification: *E13, E23, O40, O47.*

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1 Introduction

The two-level "nested" CES production technology, pioneered by Sato (1967), has recently been used widely in macroeconomics. Its flexibility coming from the substitution parameters and the inclusion of an additional input makes it an attractive choice for many applications in economic theory and empirics. Researchers interested in issues such as Griliches' capital-skill complementarity, or the wage differential between skilled and unskilled workers, have made extensive use of the two-level CES production technology with capital, skilled labor and unskilled labor as inputs.[1]

The function has been introduced to study primarily distributional aspects of the aggregate economy. Surprisingly, little has been done in exploring growth aspects of the economy using this function. We explore the properties of the two-level CES function and its effect on the Solow and Diamond models, the basic workhorses of growth theory. In particular, we study the effects of changes in the substitution parameters. We hope that our work will prompt other researchers to (a) fully study growth models using this rich functional form (b) further explore how changes in substitution parameters affect growth.

The paper establishes existence and stability conditions of equilibria under the Solow (1956) and Diamond (1965) growth models. Moreover it examines how changes in the input-substitution parameters underlying capital-skill complementarity can affect economic growth in transition and steady state. We take advantage of recent contributions by Klump and de La Grandville (2000), Miyagiwa and Papageorgiou (2003) and Palivos and Karagiannis (2004) that examine the effect of the elasticity of substitution between capital and labor on growth under the Solow and Diamond models with two inputs.[2]

We obtain the following results: For a given fraction of unskilled labor we can express the

[1]At the empirical front, Krusell et al. (2000) estimate a variant of the nested CES function with exogenous technical progress for the U.S. between 1962 and 1993. An elasticity between the capital-skill aggregate and unskilled labor above one and an elasticity within the aggregate below one indicate strong capital-skill complementarity. In addition, using the nested CES function and data on a panel of 73 countries Duffy et al. (2004) conclude that neither the presence of capital-skill complementarity nor the skill level for which it matters are universal. In the theoretical literature, models of capital-skill complementarity or biased technical change generally focus on technical progress that affects efficiency parameters. For example Goldin and Katz (1998) model the transition between four technologies. Each is characterized by a nested Leontief-Cobb-Douglas function, a special case of the nested CES function. Acemoglu (1998) considers skill-biased technical progress that raises the efficiency of skilled labor. Finally, Caselli and Coleman (2004) use the two-level CES production technology in a model that examines the notion of a world technology frontier.

[2]We consider changes in substitution parameters as exogenous. Benabou (forthcoming) and Miyagiwa and Papageorgiou (2004) model endogenous changes in substitution parameters.

substituability between total labor (skilled and unskilled) and capital by a single aggregate elasticity of substitution. This aggregate elasticity of substitution changes with capital accumulation. In the Solow model, the sufficient condition for a steady state is fulfilled for a larger range of substitution parameter values than with the basic CES function. In addition, an increase in substitution parameters has a positive impact on transitional growth and the steady state. In the Diamond model unstable equilibria occur when the elasticity of substitution is lower than the capital share. In addition, an increase in substitution parameters results in higher transitional growth under weaker conditions than with the basic CES function.

In the next section we extend the Klump-de La Grandville "CES normalization" for the two-level CES function. We also discuss capital-skill complementarity and define the aggregate elasticity of substitution as implied by the two-level CES function. In sections 3 and 4 we analyze the effect of the input-substitution parameters of the two-level CES function under the Solow and Diamond growth models, respectively. Section 5 concludes.

2 Building Blocks

2.1 Klump-de La Grandville CES normalization

Normalization of basic CES function

Production functions with two inputs, constant returns to scale and a constant elasticity of substitution between capital and labor are characterized by three parameters: an efficiency parameter, a distribution parameter (or alternatively by two non-neutral efficiency parameters) and a substitution parameter. The substitution parameter determines the curvature of the isoquant. As shown by Klump and de La Grandville (2000, henceforth KL), normalization makes it possible to "straighten" the isoquant in an arbitrary point without shifting it, while holding the efficiency and the distribution parameters constant. As shown in the k-y-diagram of Figure 1, this means that any point on a CES function can be chosen as a baseline value for a family of functions that are tangent to it. More precisely, a family of normalized CES functions is defined by baseline levels of per capita capital, k_0, per capita output, y_0, and wage to the interest rate ratio, $\mu_0 = \frac{w_0}{r_0}$. The functions belonging to any family differ only in the elasticities of substitution σ, where $\sigma = \frac{1}{1-\psi}$ and $-\infty \leq \psi \leq 1$.

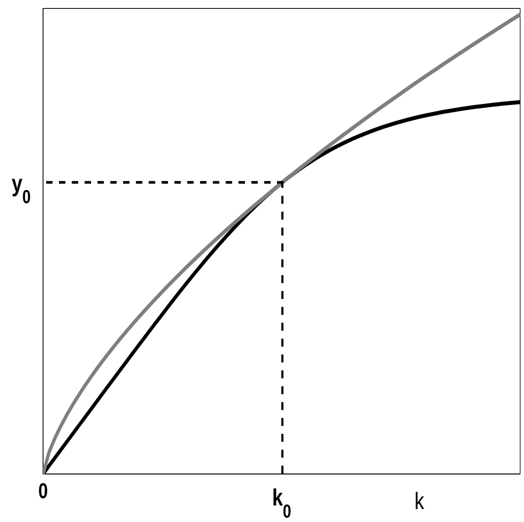

Figure 1: CES functions with a common baseline point

Normalization of two-level CES function

We consider a three-factor two-level production technology with capital (K), skilled labor (L_s) and unskilled labor (L_u) as inputs. The "first level" of the two-level CES function is given by a CES function

$$X = B[\beta K^{\theta} + (1-\beta)L_s^{\theta}]^{\frac{1}{\theta}}. \tag{1}$$

This CES function is then nested into another CES function, representing the "second level" given by

$$Y = A[\alpha X^{\psi} + (1-\alpha)L_u^{\psi}]^{\frac{1}{\psi}}. \tag{2}$$

Substituting (1) into (2) yields the two-level CES function

$$Y = A[\alpha B^{\psi}(\beta K^{\theta} + (1-\beta)L_s^{\theta})^{\frac{\psi}{\theta}} + (1-\alpha)L_u^{\psi}]^{\frac{1}{\psi}}. \tag{3}$$

There are two points worth making about equation (3). First, our formulation includes, in addition to the standard technology parameter A, a second technology parameter, B. This is done because it makes normalization easier to handle, facilitating the application of the chain rule for derivatives. Second, although there are two other possibilities of nesting the two-level CES function, we prefer the formulation given by equation (3) which is consistent with the rest of literature. Fallon and Layard (1975) and Krusell et al. (2000) present empirical evidence in support of (3) and explain why including skilled labor and capital in the first level aggregate is the most plausible variant of nesting.[3]

Next, we apply the KL normalization to the two-level nested CES function. The baseline point is defined according to KL by a set of baseline values given by $\{Y_0,\ X_0,\ K_0,\ L_{u0},\ L_{s0}\}$. The population is composed of skilled and unskilled workers ($N = L_s + L_u$, $N_0 = L_{s0} + L_{u0}$, $u = \frac{L_u}{N}$). The baseline values in intensive form are given by: $\tilde{y}_0 = \frac{Y_0}{L_{u0}}$, $\tilde{x}_0 = \frac{X_0}{L_{u0}}$, $\hat{x}_0 = \frac{X_0}{L_{s0}}$, $\hat{k}_0 = \frac{K_0}{L_{s0}}$, $y_0 = \frac{Y_0}{N_0}$, $x_0 = \frac{X_0}{N_0}$ and $k_0 = \frac{K_0}{N_0}$. Lowercase variables designate per capita variables, the tilde denotes values per unskilled worker, and the hat denotes values per skilled worker. It is assumed throughout the paper that $k > k_0$ and $x > x_0$. The factor prices of capital, skilled and unskilled labor are r, w_s and w_u. The price of a unit of aggregate X in terms of output is p_X. Moreover we define the income shares $\pi_X = \frac{p_X \tilde{x}}{\tilde{y}}$ and $\pi_K = \frac{r\hat{k}}{\hat{x}}$. The baseline values of relative factor prices are μ_0 and ν_0.

The normalization of the parameters of the production function is obtained from the following conditions:

$$\frac{w_{s0}}{r_0} = \nu_0 = \frac{1-\beta}{\beta}\hat{k}_0^{1-\theta}, \tag{4}$$

$$X_0 = B[\beta K_0^\theta + (1-\beta)L_{s0}^\theta]^{\frac{1}{\theta}}, \tag{5}$$

$$\frac{w_{u0}}{p_{X0}} = \mu_0 = \frac{1-\alpha}{\alpha}\tilde{x}_0^{1-\psi}, \tag{6}$$

$$Y_0 = A[\alpha X_0^\psi + (1-\alpha)L_{u0}^\psi]^{\frac{1}{\psi}}. \tag{7}$$

The normalized parameters are

$$A = \tilde{y}_0\left(\frac{\tilde{x}_0^{1-\psi} + \mu_0}{\tilde{x}_0 + \mu_0}\right)^{\frac{1}{\psi}}, \tag{8}$$

$$\alpha = \frac{\tilde{x}_0^{1-\psi}}{\tilde{x}_0^{1-\psi} + \mu_0}, \tag{9}$$

and

$$B = \hat{x}_0\left(\frac{\hat{k}_0^{1-\theta} + \nu_0}{\hat{k}_0 + \nu_0}\right)^{\frac{1}{\theta}}, \tag{10}$$

$$\beta = \frac{\hat{k}_0^{1-\theta}}{\hat{k}_0^{1-\theta} + \nu_0}. \tag{11}$$

[3]For more discussion on the nesting of the two-level CES function see Fallon and Layard (1975).

The parameters of each CES function with two arguments depend only on their own baseline values and their substitution parameters. This is a consequence of the strong separability of the nested CES function (see, Sato 1967). We now turn to our first proposition. (All proofs are in the Appendix.)

Proposition 1 *At given input values an increase in each substitution parameter in the two-level CES function (3) has a positive impact on output per capita.*

This proposition builds on KL theorems for the two-factor case. A summary of KL results for normalized CES functions, which are used throughout the paper, is given in the ((24)-(23)). Notice that Proposition 1 is independent of any model assumption and therefore holds for both the Solow and Diamond growth models.[4]

2.2 Capital-skill complementarity and the two-level CES function

Using the two-level CES function the skill premium is governed by the following equation:

$$\ln\left(\frac{w_s}{w_u}\right) = \ln\left(B\frac{\alpha(1-\beta)}{1-\alpha}\right) + \frac{\psi-\theta}{\theta}\ln\left[\beta\hat{k}^{\theta} + (1-\beta)\right] + (\psi-1)\ln\left(\frac{L_s}{L_u}\right). \tag{12}$$

Defining the value of capital-skill complementarity as the percentage increase in the skill premium resulting from a one percent increase in capital per skilled worker yields

$$\frac{\partial \ln\left(\frac{w_s}{w_u}\right)}{\partial \ln \hat{k}} = (\psi-\theta)\pi_K, \tag{13}$$

where $\psi - \theta > 0$ implies capital-skill complementarity, and π_K determines its magnitude additionally. Thus capital-skill complementarity means relatively higher complementarity between capital and skilled labor than between the capital-skill aggregate and unskilled labor.

2.3 Aggregate elasticity of substitution

Capital-skill complementarity compares the ease of substitution of both kinds of labor (skilled and unskilled) with capital. It is an important concept for explaining income distribution between the three inputs. Transitional and long-run growth, however, depend less on capital-skill complementarity than on what we call the aggregate elasticity of substitution. Instead of comparing

[4] This proposition was independently shown by Dupuy (2004) and Dupuy and de Grip (2004). Their proof without normalization is more complex.

the substitution parameters of both kinds of labor, this elasticity aggregates them into a single value. For a given fraction of unskilled labor u we aggregate skilled and unskilled labor to the total number of workers. We then compute the aggregate elasticity of substitution between capital and the number of workers. It corresponds to the usual elasticity of substitution of a function with two arguments. While this elasticity is constant for the basic CES function, it is variable for the two-level CES function.[5] The following lemmas describe formally two of its properties that are important for our subsequent investigation of the Solow and Diamond growth models:

Lemma 1 *For a given fraction of unskilled workers u the elasticity of substitution between capital K and the number of workers N in the two-level CES function (3) is an harmonic mean of the two-factor elasticities within each CES function, $\frac{1}{1-\psi}$ and $\frac{1}{1-\theta}$:*

$$\begin{aligned}\sigma = \left[\frac{1}{\sigma}\right]^{-1} &= \left[(1-\theta)\frac{1-\pi_K}{1-\pi_X\pi_K} + (1-\psi)\frac{\pi_K(1-\pi_X)}{1-\pi_X\pi_K}\right]^{-1} \\ &= \left[(1-\theta)(1-g) + (1-\psi)g\right]^{-1}.\end{aligned}$$

Lemma 2

(i) If θ and ψ have opposing signs or if $|\theta| > |\psi|$, $\lim_{k\to 0}\sigma = \max[\frac{1}{1-\theta}, \frac{1}{1-\psi}]$ *and* $\lim_{k\to\infty}\sigma = \min[\frac{1}{1-\theta}, \frac{1}{1-\psi}]$.

(ii) If $\psi > \theta > 0$ or $0 > \theta > \psi$, both limits are equal to $\frac{1}{1-\theta}$.

[5] We note that the aggregate elasticity of substitution is declining in k in most cases. Other theoretical and empirical studies point to an increasing elasticity of substitution. But keeping in mind that we assume a constant technology and constant skill-levels, the result of a decreasing elasticity of substitution appears less surprising. Hicks (1932) already speculated about a declining elasticity of substitution that "may be counteracted by invention" (p. 132).

3 The Solow Model

3.1 Existence and stability of steady states

We now introduce the two-level CES function into the basic Solow model. We assume that total population, skilled and unskilled labor grow at the same rate n, what leaves the fraction of unskilled labor u constant. Also, for simplicity of exposition we assume $\theta \neq \psi$ and $\theta, \psi \neq 0$. The savings ratio is s, the depreciation rate δ.

The equation of capital accumulation is as usual:

$$\dot{k} = sy - (n+\delta)k. \tag{14}$$

The condition for a steady state is

$$\begin{aligned} sy^* &= (n+\delta)k^* \\ \Leftrightarrow sA[\alpha B^{\psi}(\beta k^{*\theta} + (1-\beta)(1-u)^{\theta})^{\frac{\psi}{\theta}} + (1-\alpha)u^{\psi}]^{\frac{1}{\psi}} &= (n+\delta)k^*, \end{aligned} \tag{15}$$

where (*) denotes steady-state values. As with two inputs, the economy can experience continuous decline, converge to a constant steady state or grow endogenously in the long-run.

Proposition 2 *Under the Solow model with the two-level CES function (3) the following holds:*
(i) If θ and ψ have not the same sign, a steady state $k^* > 0$ always exists.
(ii) For ψ and θ both positive, a steady state $k^* > 0$ exists iff $A\alpha^{\frac{1}{\psi}} B\beta^{\frac{1}{\theta}} \leq \frac{n+\delta}{s}$, otherwise $k^* \to \infty$.
(iii) For ψ and θ both negative, a steady state $k^* > 0$ exists iff $A\alpha^{\frac{1}{\psi}} B\beta^{\frac{1}{\theta}} > \frac{n+\delta}{s}$, otherwise $k^* = 0$.
(iv) All positive steady states are unique and stable.

Alternatively to our proof, the result on endogenous growth *(ii)* follows from studying the limiting behavior of the aggregate elasticity of substitution (see Palivos and Karagiannis 2004).

If both parameters are positive or negative the results for the two-level CES function correspond to the results for the basic CES function shown by Klump and Preissler (2000). A notable difference arises if the parameters have opposing signs. In the Solow model with the basic CES function the only value of the substitution parameter which is sufficient to guarantee a positive steady state is zero (Cobb-Douglas function). With the two-level CES function we obtain a much weaker sufficient condition. Whenever the substitution parameters have opposing signs a positive steady state exists.

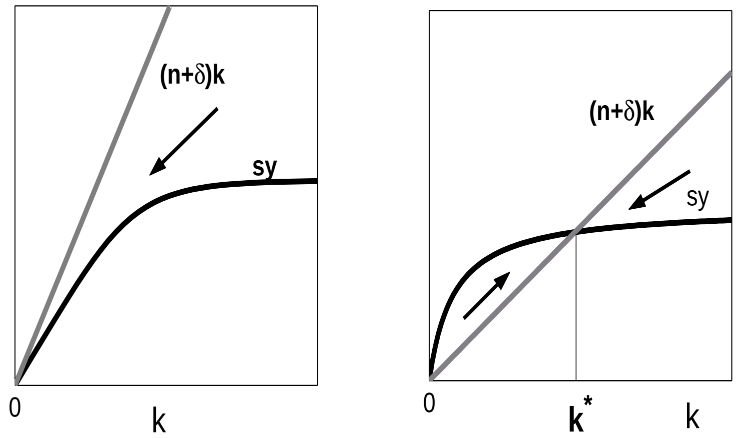

Figure 2: Both ψ and $\theta < 0$

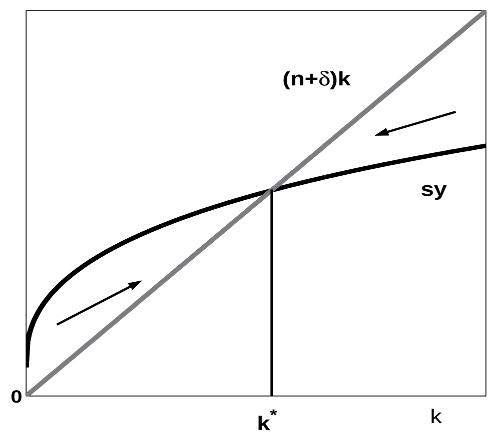

Figure 3: ψ and θ have opposing signs

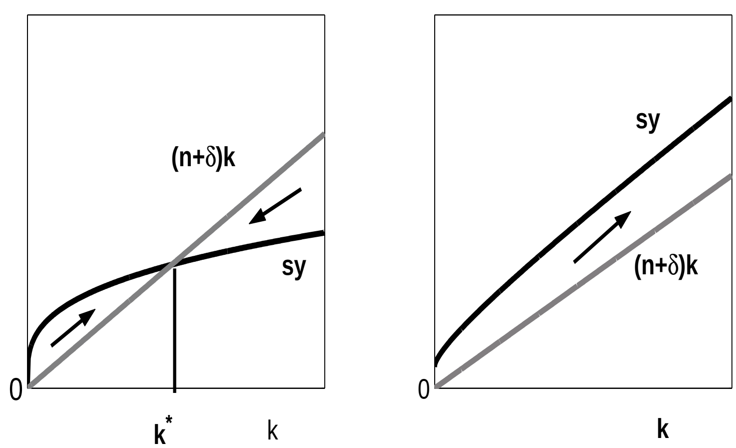

Figure 4: Both ψ and $\theta > 0$

3.2 Effects of substitution parameters on transition and steady state

Under the Solow model the effects of an increase in any of the substitution parameters carry over from the basic CES function to the two-level CES function:

Corollary 1 *Given k with $k > k_0$, an increase in any of the substitution parameters ψ and θ has a positive effect on the growth rate of capital $\frac{\dot{k}}{k}$ under the Solow model.*
Looking at (14) this follows from Proposition 1.

Proposition 3 *Under the Solow model with the two-level CES function (3) an increase in any of the two substitution parameters ψ and θ has a positive effect on the steady state k^*.*

Goldin and Katz (1998) argue that during early industrialization capital became more complementary to unskilled labor, whereas in the twentieth century capital became more complementary to skilled labor. Our results show in a simple way why both changes can have spurred economic growth. With reference to the definition of capital-skill complementarity in section 2.2., we can read the change of technology in the first phase as an increase in θ and the change in the second phase as an increase in ψ. Both increases have a positive impact on transitional growth in the Solow model. The effects are not as unambiguous if capital accumulation depends on the distribution of income.

4 The Diamond model

In the Diamond model with two inputs the effects of the elasticity of substitution on growth and the steady state differ from the Solow model in two ways. First the elasticity of substitution affects the uniqueness and stability of steady states, second a higher elasticity of substitution does not always have a positive effect on growth and the steady state. We show to what extent the results carry over to the model with skilled and unskilled labor.

The reason for the differences to the Solow model is that growth now depends on the distribution of income. In the Diamond model with logarithmic utility, savings turn out to be a constant fraction of wage income. We restrict our attention to this case. As in the Solow model, we assume $\theta \neq \psi$ and $\theta, \psi \neq 0$ as well as constant population growth and a constant fraction of unskilled labor.

4.1 Existence and stability of steady states

With the two-factor CES function the model has one stable positive steady state if the elasticity of substitution is greater or equal to one ($\Leftrightarrow \psi \geq 0$). Endogenous growth does not occur. If the elasticity of substitution is smaller than one the model has either one stable and one unstable positive steady state or it does not have any (Azariadis 1996 p.203.). With three inputs the average wage w is the weighted sum of the two wage rates. The equation of capital accumulation is

$$k_{t+1} = sy_t(1 - \pi_{Xt} + \pi_{Xt}(1 - \pi_{Kt})) = sw_t, \tag{16}$$

and the condition for a steady state

$$sw^* = k^*, \tag{17}$$

where s is now the savings ratio out of wages only.
The stability of a steady state hinges on the derivative $\partial w/\partial k$.

Proposition 4 *In the Diamond model with constant savings out of wages and the two-level CES function (3) the following holds:*
(i) If θ and ψ are both positive, exactly one positive steady state exists and it is stable.
(ii) If θ and ψ have not the same sign, at least one positive steady state exists. The lowest and the highest are stable.
(iii) If ψ and θ are both negative, there are either multiple positive steady states or none (except for $s\frac{\partial w}{\partial k}$ only once tangent to $k_t = k_{t+1}$). In the case of multiple steady states the lowest is unstable and the highest is stable.
(iv) Unstable steady states only occur if the aggregate elasticity of substitution σ is lower than the capital share $\pi_X\pi_K$.

As $\pi_X\pi_K$ and $\sigma < 1$ decline jointly in most cases, it is difficult to exclude more than one unstable equilibrium analytically. But with plausible parameter values the condition for instability will only be fulfilled for a small range of the capital stock. Unstable equilibria become impossible as soon as the capital share has fallen below the lower bound of the elasticity of substitution. In simulations we have never found more than one unstable equilibrium.

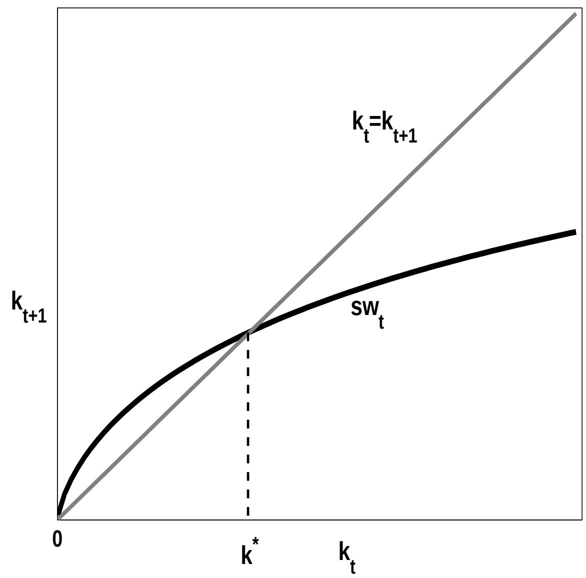

Figure 5: If σ always exceeds $\pi_X \pi_K$, a unique and stable steady state exists in the Diamond model.

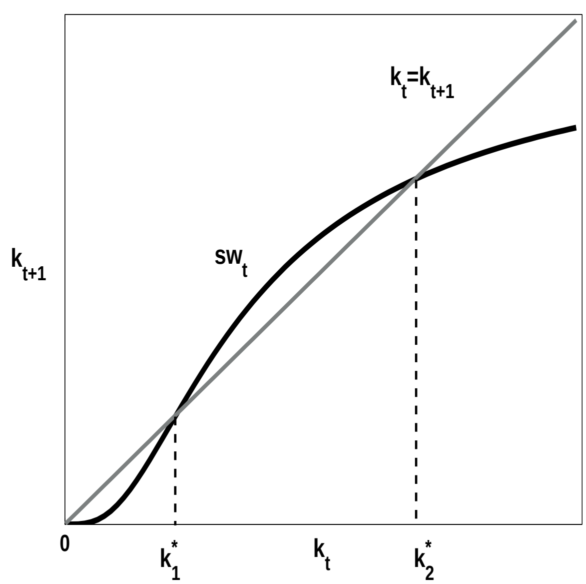

Figure 6: For $\sigma < \pi_X \pi_K$, unstable steady states are possible.

4.2 Effects of substitution parameters on transition and the steady state

With two factors of production Miyagiwa and Papageorgiou (2003) and independently Irmen (2001) have shown that the elasticity of substitution has a threshold above one ($\Leftrightarrow \psi > 0$) for which its impact on wages is always negative. It is moreover possible to show that irrespective of σ the impact is negative for a certain range of k with $k > k_0$.

With a fixed savings ratio capital accumulation depends only on the average wage w. Given a capital stock $k_t > k_0$ the influence of a higher substitution parameter ψ on next period's capital stock k_{t+1} is

$$\frac{\partial k_{t+1}}{\partial \psi} = s\frac{\partial w_t}{\partial \psi}, \tag{18}$$

and analogously

$$\frac{\partial k_{t+1}}{\partial \theta} = s\frac{\partial w_t}{\partial \theta}. \tag{19}$$

Using results (24)-(23) from KL and omitting the time subscript we obtain from (16)

$$\begin{aligned}\frac{\partial w}{\partial \psi} &= \frac{\partial y}{\partial \psi}(1-\pi_X\pi_K) - y\pi_K\frac{\partial \pi_X}{\partial \psi}\\ &= -\frac{y}{\psi^2}\left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right)(1-\pi_X\pi_K)\\ &\quad -y\pi_K\pi_X(1-\pi_X)\ln\left(\frac{x}{x_0}\right),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{\partial y}{\partial \theta}(1-\pi_X\pi_K) - y\pi_X\frac{\partial \pi_K}{\partial \theta} - y\pi_K\frac{\partial \pi_X}{\partial \theta}\\ &- \frac{y}{\theta^2}\pi_X\left[((1-\pi_K)+\pi_K(1-\pi_X)(1-\psi))\left(\pi_K\ln\left(\frac{\pi_{K0}}{\pi_K}\right)+(1-\pi_K)\ln\left(\frac{1-\pi_{K0}}{1-\pi_K}\right)\right)\right]\\ &\quad -y\pi_X\pi_K(1-\pi_K)\ln\left(\frac{k}{k_0}\right).\end{aligned} \tag{21}$$

Proposition 5 *In the Diamond model with constant savings out of wages and the two-level CES function (3),* $\frac{\partial k_{t+1}}{\partial \psi}$ *and* $\frac{\partial k_{t+1}}{\partial \theta}$ *are always negative in an interval* $(k_0, k_0+\epsilon]$, ϵ *being an arbitrarily small positive number.*

Proposition 6a

(i) For $k \to \infty$, $\frac{\partial k_{t+1}}{\partial \psi}$ *is positive if* $\psi < 0$ *or* $\theta < 0$, *or if* $\psi > \theta > 0$.

(ii) For $k \to \infty$, $\frac{\partial k_{t+1}}{\partial \psi}$ *is negative if* $\theta > \psi > 0$.

Proposition 6b

(i) For $k \to \infty$, $\frac{\partial k_{t+1}}{\partial \theta}$ *is positive if* $\psi < 0$ *or* $\theta < 0$, *or if* $\theta > \psi > 0$.

(ii) For $k \to \infty$, $\frac{\partial k_{t+1}}{\partial \theta}$ *is negative if* $\psi > \theta > 0$.

We are not able to exclude multiple changes in the sign of the derivatives in Proposition 6 analytically. But in simulations we have not obtained multiple changes for $k > k_0$. So far the immediate impact of substitution parameters on the transitional growth rate $\frac{k_{t+1}}{k_t}$ has been examined, the results are easily extended to the impact on the steady state.

Proposition 7a *An increase in* ψ *has a positive effect on a stable steady* $k^* > k_0$ *state and a negative effect on an unstable steady state if*

$$\frac{\partial w}{\partial \psi}_{|k=k^*} > 0.$$

An increase in ψ *has a negative effect on a stable steady state* $k^* > k_0$ *and a positive effect on an unstable steady state if*

$$\frac{\partial w}{\partial \psi}_{|k=k^*} < 0.$$

Proposition 7b *An increase in* θ *has a positive effect on a stable steady state* $k^* > k_0$ *and a negative effect on an unstable steady state if*

$$\frac{\partial w}{\partial \theta}_{|k=k^*} > 0.$$

An increase in θ *has a negative effect on a stable steady state* $k^* > k_0$ *and a positive effect on an unstable steady state if*

$$\frac{\partial w}{\partial \theta}_{|k=k^*} < 0.$$

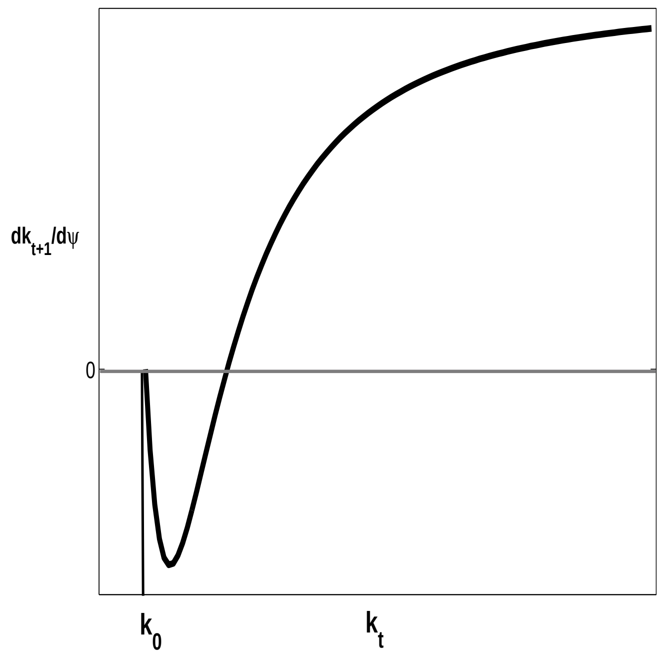

Figure 7: Effects of higher substitution of *unskilled* labor if θ or ψ is negative

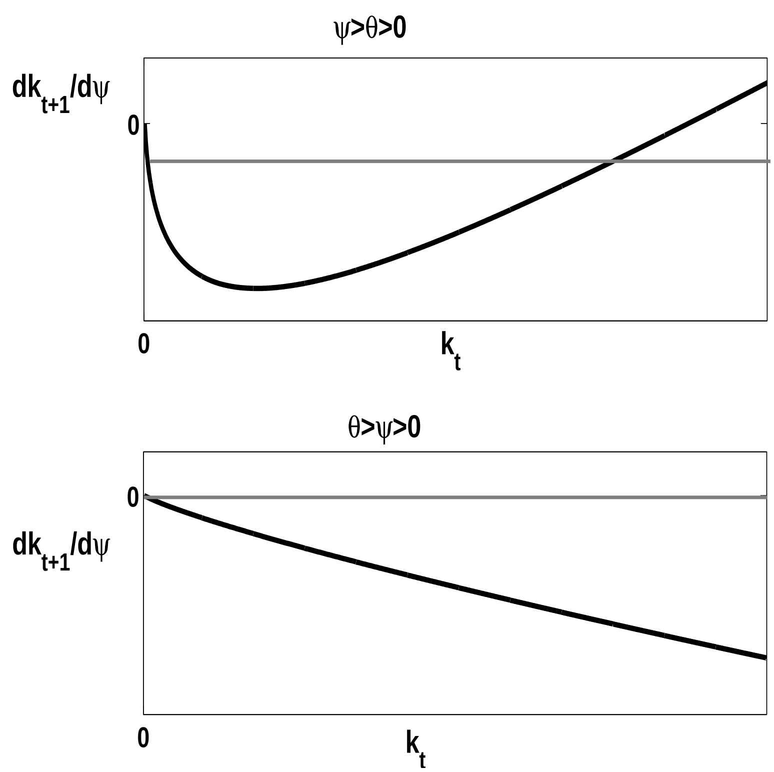

Figure 8: Effects of higher substitution of *unskilled* labor if θ and ψ are positive

Introducing the two-level CES function narrows down the range of parameter values for which an increase in one of them can have a negative effect on transitional growth and stable steady states independently of k. With two inputs a sufficient condition for this is $\sigma > \frac{1}{\pi_0}$ (Irmen 2001). With three inputs, a more restricted necessary condition is obtained: Both parameters have to be positive and the one that is increased has to be lower than the one that remains constant.

Under capital-skill complementarity there is always a capital stock k above which higher substituability of unskilled labor rises transitional growth and stable steady states, even if both substitution parameters are high. The reason is that wages of skilled labor remain high enough to support capital accumulation. We can conclude that in most cases not the substitution parameters alone but a low capital stock will be the reason for negative effects of higher substitution.

5 Conclusion

Motivated by revived interest in flexible aggregate production functions, we considered the Solow and Diamond growth models under a two-level CES function. Existence and stability conditions for steady states were derived. In addition, the effect of substitution parameters on transitional growth and steady states was examined. Our results show that beyond using different substitution parameters for skilled and unskilled labor to analyze distributional aspects, such as wage differentials, we should also consider their effect on growth.

Appendix

Results on the CES function with two factors

KL show the following results for the basic CES function. They are written down for both levels of our function:

$$\frac{\tilde{y}}{\tilde{y}_0} = \frac{y}{y_0} = \left(\frac{1-\pi_{X0}}{1-\pi_X}\right)^{\frac{1}{\psi}}, \qquad \frac{\hat{x}}{\hat{x}_0} = \frac{x}{x_0} = \left(\frac{1-\pi_{K0}}{1-\pi_K}\right)^{\frac{1}{\theta}} \tag{22}$$

$$\frac{\tilde{x}}{\tilde{x}_0} = \frac{x}{x_0} = \left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)^{\frac{1}{\psi}}, \qquad \frac{\hat{k}}{\hat{k}_0} = \frac{k}{k_0} = \left(\frac{\pi_K(1-\pi_{K0})}{\pi_{K0}(1-\pi_K)}\right)^{\frac{1}{\theta}}. \tag{23}$$

$$\frac{\partial \pi_X}{\partial x} = \frac{\psi}{x}\pi_X(1-\pi_X), \qquad \frac{\partial \pi_K}{\partial k} = \frac{\theta}{k}\pi_K(1-\pi_K) \tag{24}$$

$$\frac{\partial \pi_X}{\partial \psi} = \pi_X(1-\pi_X)\ln\left(\frac{x}{x_0}\right), \qquad \frac{\partial \pi_K}{\partial \theta} = \pi_K(1-\pi_K)\ln\left(\frac{k}{k_0}\right) \tag{25}$$

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial \psi} &= -\frac{\tilde{y}}{\psi^2}\left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right) \\ \frac{\partial \hat{x}}{\partial \theta} &= -\frac{\hat{x}}{\theta^2}\left(\pi_K \ln\left(\frac{\pi_{K0}}{\pi_K}\right) + (1-\pi_K)\ln\left(\frac{1-\pi_{K0}}{1-\pi_K}\right)\right) \end{aligned} \tag{26}$$

Proof of Lemma 1

The aggregate elasticity of substitution is defined as

$$\sigma = \frac{w/r}{k\frac{\partial w/r}{\partial k}}. \tag{27}$$

As in the two-factor case

$$\frac{w}{r} = \frac{1-\pi_X\pi_K}{\pi_X\pi_K}k. \tag{28}$$

We obtain the derivative of the capital share making use of (24):

$$\begin{aligned} \frac{\partial \pi_X\pi_K}{\partial k} &= \pi_K\frac{\partial \pi_X}{\partial x}\frac{\partial x}{\partial k} + \pi_X\frac{\partial \pi_K}{\partial k} \\ &= \pi_K\frac{\psi}{x}\pi_X(1-\pi_X)\pi_K\frac{x}{k} + \pi_X\frac{\theta}{k}\pi_K(1-\pi_K) \\ &= \frac{\pi_X\pi_K}{k}(\psi\pi_K(1-\pi_X) + \theta(1-\pi_K)). \end{aligned} \tag{29}$$

With (28) and (29) we obtain

$$\sigma = \frac{1-\pi_X\pi_K}{(1-\psi)\pi_K(1-\pi_X) + (1-\theta)(1-\pi_K)}. \blacksquare \tag{30}$$

Proof of Lemma 2

With $g = \frac{\pi_K(1-\pi_X)}{1-\pi_X\pi_K}$ as the weight in the harmonic mean we rewrite the result from Lemma 1 as

$$\frac{1}{\sigma} = (1-\theta) + (\theta - \psi)g. \tag{31}$$

We rewrite g as

$$g = \frac{\pi_K}{\frac{1-\pi_K}{1-\pi_X} + \pi_K}. \tag{32}$$

If ψ and θ have opposing signs, the limits are straightforward. Note that for $\psi > 0 > \theta$ the limit of π_X for $k \to \infty$ is lower than one because x is bounded. In this case

$$\lim_{k\to 0} g = 1 \qquad \lim_{k\to\infty} g = 0. \tag{33}$$

For $\theta > 0 > \psi$:

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 1. \tag{34}$$

If the substitution parameters have the same sign, we evaluate the limit of $\frac{1-\pi_K}{1-\pi_X}$ using (22) and (23):

$$\frac{1-\pi_K}{1-\pi_X} = (1-\pi_{K0})\left(\frac{\pi_{X0}}{1-\pi_{X0}}\right)^{\frac{\theta}{\psi}}\left(\frac{1}{\pi_X}\right)^{\frac{\theta}{\psi}}(1-\pi_X)^{\frac{\theta}{\psi}-1} \tag{35}$$

Plugging the result into (32) we obtain: For $\psi > \theta > 0$ and $0 > \theta > \psi$

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 0, \tag{36}$$

for $\theta > \psi > 0$

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 1, \tag{37}$$

and for $0 > \psi > \theta$

$$\lim_{k\to 0} g = 1 \qquad \lim_{k\to\infty} g = 0. \tag{38}$$

Plugging the results (33)-(38) into (31) yields Lemma 2 *(i)* and *(ii)*.■

Proof of Proposition 1

As L_u and L_s do not depend on ψ and θ, it follows from (26) that

$$\frac{\partial y}{\partial \psi} > 0 \tag{39}$$

and

$$\frac{\partial x}{\partial \theta} > 0. \tag{40}$$

The impact of θ on y is obtained as

$$\frac{\partial y}{\partial \theta} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial \theta} > 0. \blacksquare \tag{41}$$

Proof of Proposition 2

For $\theta, \psi > 0$ we show the condition for endogenous growth

$$\begin{aligned}\lim_{k\to\infty}\frac{\dot{k}}{k} = \lim_{k\to\infty} sA\left[\alpha B^{\psi}\left(\beta + (1-\beta)\left(\frac{1-u}{k}\right)^{\theta}\right)^{\frac{\psi}{\theta}} + (1-\alpha)\left(\frac{u}{k}\right)^{\psi}\right]^{\frac{1}{\psi}} - (n+\delta) > 0 \\ \Leftrightarrow A\alpha^{\frac{1}{\psi}}B\beta^{\frac{1}{\theta}} > \frac{n+\delta}{s}\end{aligned} \tag{42}$$

If $\theta, \psi < 0$ the condition for the existence of a positive steady state k^* is

$$\begin{aligned}\lim_{k\to 0} sA\left(\frac{1}{\alpha B^{\psi}\left(\beta + (1-\beta)\left(\frac{k}{1-u}\right)^{-\theta}\right)^{\frac{\psi}{\theta}} + (1-\alpha)\left(\frac{k}{u}\right)^{-\psi}}\right)^{-\frac{1}{\psi}} - (n+\delta) > 0 \\ A\alpha^{\frac{1}{\psi}}B\beta^{\frac{1}{\theta}} > \frac{n+\delta}{s}.\end{aligned} \tag{43}$$

If $\psi < 0$ and $\theta > 0$, we see that the condition for endogenous growth is never fulfilled

$$\lim_{k\to\infty} sA\left(\frac{1}{\frac{1}{\alpha B^{\psi}\left(\beta+(1-\beta)\left(\frac{1-u}{k}\right)^{\theta}\right)^{-\frac{\psi}{\theta}}} + (1-\alpha)\left(\frac{k}{u}\right)^{-\psi}}\right)^{-\frac{1}{\psi}} - (n+\delta) = -(n+\delta), \tag{44}$$

and a steady state k^* always exists

$$\lim_{k\to 0} sA\left(\frac{1}{\frac{1}{\alpha B^{\psi}\left(\beta+(1-\beta)\left(\frac{1-u}{k}\right)^{\theta}\right)^{-\frac{\psi}{\theta}}}+(1-\alpha)\left(\frac{k}{u}\right)^{-\psi}}\right)^{-\frac{1}{\psi}} - (n+\delta) = \infty - (n+\delta) > 0. \tag{45}$$

In an analogous way it is shown that a steady state always exists for $\psi > 0$ and $\theta < 0$. From (44) and (45) follows easily that endogenous growth never occurs if both parameters are negative and that $k^* > 0$ if at least one parameter is positive.

Part *(iv)* follows from the declining marginal product of capital.■

Proof of Proposition 3

$$sy^* = (n+\delta)k^* \tag{46}$$

$$\Leftrightarrow I(\theta,\psi,k^*) = y^* - \frac{n+\delta}{s}k^* = 0. \tag{47}$$

Because (46) is fulfilled for every k^*, dI=0 for any variation of θ or ψ.

$$\frac{\partial I}{\partial \theta} + \frac{\partial I}{\partial k^*}\frac{\partial k^*}{\partial \theta} = 0$$

$$\Leftrightarrow \frac{\partial y}{\partial x}\frac{\partial x}{\partial \theta}_{|k=k^*} + \left[\frac{\partial y}{\partial x}\frac{\partial x}{\partial k}_{|k=k^*} - \frac{n+\delta}{s}\right]\frac{\partial k^*}{\partial \theta} = 0 \tag{48}$$

and

$$\frac{\partial k^*}{\partial \theta} = -\left(\frac{\frac{\partial y}{\partial x}\frac{\partial x}{\partial \theta}}{\frac{\partial y}{\partial x}\frac{\partial x}{\partial k} - \frac{n+\delta}{s}}\right)_{|k=k^*} \tag{49}$$

It follows from the positive marginal product of y and (26) that $\frac{\partial y}{\partial x}\frac{\partial x}{\partial \theta}$ is positive. The condition for $\frac{\partial k^*}{\partial \theta} > 0$ is therefore

$$\frac{\partial y}{\partial x}\frac{\partial x}{\partial k}_{|k=k^*} < \frac{n+\delta}{s}. \tag{50}$$

Replacing with (46) yields

$$\frac{\partial y}{\partial x}\frac{\partial x}{\partial k}_{|k=k^*} < \frac{y^*}{k^*}. \tag{51}$$

It means that the marginal product is lower than the average product. As the two-level CES function has a declining marginal product for $\psi < 1$ and $\theta < 1$, it is true. The derivative with respect to ψ follows analogously from

$$\frac{\partial I}{\partial \psi} + \frac{\partial I}{\partial k^*}\frac{\partial k^*}{\partial \psi} = 0.\blacksquare \tag{52}$$

Proof of Proposition 4

From the steady state condition (17) follows

$$s(1-\pi_X\pi_K^*)y^* = k^* \Leftrightarrow \frac{1}{s} = (1-\pi_X\pi_K^*)\frac{y^*}{k^*} \tag{53}$$

A steady state is stable if and only if

$$\begin{aligned} \left| s\frac{\partial w}{\partial k}\right|_{k=k^*} &< 1 \\ \Leftrightarrow \left(s\pi_X\pi_K\frac{y}{k}\left[(1-\psi)(1-\pi_X)\pi_K + (1-\theta)(1-\pi_K)\right]\right)_{|k=k^*} &< 1. \end{aligned} \tag{54}$$

To obtain (54) we used $\frac{\partial y}{\partial x}\frac{\partial x}{\partial k} = \pi_X\pi_K\frac{y}{k}$ and (29).

To proof *(i)-(iii)* we study the limiting behavior of $\partial w/\partial k$. From Proposition 2 follows: If at least one parameter is negative, the marginal product of capital $\pi_X\pi_K\frac{y}{k}$ converges to a positive finite value for $k \to 0$ and to zero for $k \to \infty$. If both parameters are positive, it converges to infinity for $k \to 0$ and to a positive finite value for $k \to \infty$. Evaluating the marginal product and the income shares in (54) we obtain $\lim_{k\to\infty}\left(s\frac{\partial w}{\partial k}\right) = 0$ irrespective of ψ and θ. If $\theta, \psi < 0$ $\lim_{k\to 0}\left(s\frac{\partial w}{\partial k}\right) = 0$. If at least one parameter is positive $\lim_{k\to 0}\left(s\frac{\partial w}{\partial k}\right) = \infty$. As $\frac{\partial w}{\partial k}$ is continuous, part *(i)-(iii)* follows.■

To show *(iv)* we plug (53) and the aggregate elasticity of substitution from Lemma 1 into (54) and obtain the following necessary and sufficient condition for stability:

$$\frac{\pi^*}{\sigma^*} < 1 \Leftrightarrow \pi^* < \sigma^*. \tag{55}$$

Part *(iv)* follows.■

Proof of Proposition 5

At $k = k_0$ and $x = x_0$, $\frac{\partial w}{\partial \psi}$ and $\frac{\partial w}{\partial \theta}$ are 0 because the logarithms in (20) and (21) are 0.

With $\Phi_X = \pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)$ (20) is rewritten as

$$\frac{\partial w}{\partial \psi} = -\frac{y}{\psi^2}(1-\pi_X)(\pi_X\pi_K)\left[\Phi_X \frac{1-\pi_K\pi_X}{\pi_K\pi_X(1-\pi_X)} + \psi^2 \ln\left(\frac{x}{x_0}\right)\right] \tag{56}$$

The term $-\frac{y}{\psi^2}(1-\pi_X)$ remains negative as k increases. As Irmen (2001) we differentiate the expression in brackets with respect to k.

$$\begin{aligned}\frac{\partial[...]}{\partial k} &= \frac{\partial \Phi_X}{\partial \pi_X}\frac{\partial \pi_X}{\partial k}\left(\frac{1-\pi_K\pi_X}{\pi_K\pi_X(1-\pi_X)}\right) + \Phi_X \frac{\partial \frac{1-\pi_X\pi_K}{\pi_X\pi_K(1-\pi_X)}}{\partial k} + \frac{\psi^2}{x}\frac{\partial x}{\partial k} \\ &= \ln\left(\frac{\pi_{X0}(1-\pi_X)}{\pi_X(1-\pi_{X0})}\right)\frac{\partial \pi_X}{\partial k}\left(\frac{1-\pi_K\pi_X}{\pi_K\pi_X(1-\pi_X)}\right) + \Phi_X \frac{\partial\left(\frac{1-\pi_X\pi_K}{\pi_X\pi_K}\right)}{\partial k} + \frac{\psi^2}{x}\frac{\partial x}{\partial k}\end{aligned} \tag{57}$$

For $k = k_0, x = x_0$ follows

$$\frac{\partial[...]}{\partial k} = 0 + 0 + \frac{\psi^2}{x_0} > 0 \Leftrightarrow \frac{\partial w}{\partial \psi} < 0. \tag{58}$$

The results for a change in θ are obtained in the same way. Again $\frac{\partial w}{\partial \theta}$ is zero at the baseline point. As the derivatives of π_X, π_K and Φ_K with respect to k are zero at this point, $\frac{\partial^2 w}{\partial \theta \partial k}$ is negative. For $k_0 + \epsilon$, $\frac{\partial w}{\partial \theta}$ is thus negative.■

Proof of Proposition 6 a and b

Using (23) we rewrite (20) as

$$\begin{aligned}\lim_{k\to\infty}\frac{\partial w}{\partial \psi} = \lim_{k\to\infty}\left(\frac{-y}{\psi^2}\right) & \Bigg[\left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right)(1-\pi_X\pi_K) \\ & + \psi\pi_K\pi_X(1-\pi_X)\ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)\Bigg].\end{aligned} \tag{59}$$

For $\theta < 0$ or $\psi < 0$, y has an upper bound and $\pi_X\pi_K$ converges to zero. In functions of the type $z \ln z$ the logarithm converges more slowly, $\lim_{z\to 0} z\ln z = 0$. The expression in square brackets converges thus to $\ln \pi_0$ or $\ln(1-\pi_0)$.

If both parameters are positive, the aggregate elasticity of substitution plays a central role through its weighting variable $g = \frac{\pi_K(1-\pi_X)}{1-\pi_X\pi_K}$.

For the case that $\psi > \theta > 0$ we rewrite (59)

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = \lim_{k\to\infty} \left(\frac{-y}{\psi^2}\right)(1-\pi_X\pi_K) \quad \Bigg[\quad \left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right) + g\pi_X \ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)\Bigg]. \tag{60}$$

With (22) we see that in $y(1-\pi_X)$ the convergence of y to infinity dominates for $1 > \theta, \psi > 0$. Because $(1-\pi_X\pi_K) = 1-\pi_X+\pi_X-\pi_X\pi_K$ also $y(1-\pi_X\pi_K)$ converges to infinity. From Lemma 1 and 2 *(ii)* follows that for $\psi > \theta > 0$, g converges to 0. Because of the properties of the natural logarithm it converges faster than $\ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)$. Thus for $\psi > \theta > 0$

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = -\infty[\ln \pi_0 + 0] = \infty.\blacksquare \tag{61}$$

For $\theta > \psi > 0$ we rewrite

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = \lim_{k\to\infty} \left(\frac{-y}{\psi^2}\right)(1-\pi_X)\pi_K \Bigg[\left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right)\frac{1}{g} + \pi_X \ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)\Bigg]. \tag{62}$$

From Lemma 1 and 2 *(i)* follows that for $\theta > \psi > 0$, g converges to 1. Thus Proposition 6 a *(ii)* follows:

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = -\infty[\ln \pi_0 * 1 + \infty] = -\infty.\blacksquare \tag{63}$$

Proposition 6 b for $\frac{\partial w}{\partial \theta}$ is obtained in an analogous way. With (23) we rewrite (21)

$$\frac{\partial w}{\partial \theta} = -\frac{y}{\theta^2}\pi_X\Bigg[((1-\pi_K)+\pi_K(1-\pi_X)(1-\psi))\left(\pi_K\ln\left(\frac{\pi_{K0}}{\pi_K}\right)+(1-\pi_K)\ln\left(\frac{1-\pi_{K0}}{1-\pi_K}\right)\right)$$
$$+\theta\pi_K(1-\pi_K)\ln\left(\frac{\pi_K}{\pi_{K0}}\frac{1-\pi_{K0}}{1-\pi_K}\right)\Bigg] \tag{64}$$

For ψ or $\theta < 0$ one sees that $\lim_{k\to\infty}\left(\frac{\partial w}{\partial \theta}\right) > 0$.

For $\psi, \theta > 0$ the proof is analogous to 6 a. The behavior of $\frac{g}{1-g}$ is considered.■

Proof of Proposition 7 a and b

From the steady state condition (17) we obtain with the implicit function theorem

$$\frac{\partial k^*}{\partial \psi} = \left(\frac{s\frac{\partial w}{\partial \psi}}{1-s\frac{\partial w}{\partial k}}\right)_{|k=k^*}. \tag{65}$$

and the analogous derivative for θ. Propositions 7a and b follow using Propositions 5, 6a and 6b.■

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