Heterogeneous Borrowers, Liquidity, and the Search for Credit

Zsolt Becsi
Federal Reserve Bank of Atlanta and Louisiana State University

Victor Li
U.S. Naval Academy

Ping Wang
Vanderbilt University and NBER

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ABSTRACT

What happens when liquidity increases in credit markets? We examine this question in a general equilibrium search model where borrowers (firms) and lenders (households) are matched randomly or assortively and where the composition of borrowers adjusts to satisfy equilibrium entry conditions. We find that liquidity enhances entry by all borrowers and tends to benefit low quality borrowers disproportionately. However, liquid credit markets may or may not be associated with higher social output. The result depends on whether the effect of higher market participation outweighs that of lower average quality. We show that the net effect depends crucially on the source of the liquidity shock.

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Address for Correspondence: Ping Wang, Department of Economics, Vanderbilt University, 214 Calhoun Hall, Nashville, TN 37235; 615-322-2388 (phone), 615-343-8495 (fax), ping.wang@vanderbilt.edu (e-mail).

1. Introduction

Entrepreneurs, especially new entrants and small business owners, are constantly on the lookout for funds to finance their activities.[1] Lenders, particularly individual investors, smaller, less established financial institutions, and loan brokers, devote significant resources to identifying viable borrowers. Given that the process of matching lenders and borrowers takes time and resources, how are loan contracts and the composition of borrowers determined in equilibrium? And what are the effects of additional liquidity when credit is allocated through time-consuming search? A thorough answer to these questions may shed light on the divergent and sometimes surprising credit market outcomes that we observe in reality.

In his pivotal work, Diamond (1990) studies search in credit markets with pairwise meeting and matching of *ex ante* identical borrowers and *ex ante* identical lenders.[2] Our paper extends Diamond's framework in three significant ways. First, we allow market participation and the matching probabilities to be determined endogenously.[3] This endogenizes the "tightness" of the credit market and allows us to investigate the importance of search and entry frictions for the effects of liquidity. Second, we consider heterogeneous borrowers who have different risk, productivity and cost profiles.[4] This enables us to examine how liquidity affects the composition of loans when entry and exit of heterogeneous borrowers is determined in equilibrium. Third, we compare the benchmark random matching technology to an alternative framework with "assortive" matching where matchmakers give priority to matches with high quality borrowers. Giving matchmakers an active role in alleviating credit market frictions allows

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[1] For instance, Evans and Jovanovic (1989) find many such entrepreneurs in the U.S. have been denied a loan but continued searching for funding. Blanchflower and Oswald (1998) report that in interview surveys, potential entrepreneurs find that raising capital is their principal problem.

[2] Townsend (1978) compares bilateral and centralized trading and shows the existence of an efficiency-enhancing role for middlemen. However, the matching process by which trading agents are paired is not explicitly modeled. Den Haan, Ramey, and Watson (1999) consider matching between borrowers and lenders, but focus on the possibility of long-term credit relationships.

[3] Laing, Palivos and Wang (1995) model search with endogenous matching probabilities.

[4] Such heterogeneity is discussed by Gertler and Gilchrist (1994) and many others.

comparisons with results obtained in the middleman literature.[5]

We develop a general equilibrium model where search frictions arise because seeking out trades and bargaining over contracts takes time and entry frictions arise because market participation for borrowing firms is costly. For illustrative purposes, we only consider productive credit. That is, households would like to lend a part of their endowment to firms who need to finance their productive activity. Although firms differ along several dimensions, these attributes are common knowledge. Hence, there does not exist an adverse-selection or a moral hazard problem and the resulting simplicity enables us to highlight the role of search and entry frictions in credit markets.

In the benchmark case, borrowers and lenders are brought together by a random matching technology similar to Diamond (1990) and Rubinstein and Wolinsky (1987). Once a borrower and a lender are paired, a loan contract is mediated. The contract determines the loan rate paid by firms by dividing the net surplus of the match according to the agents' bargaining power. As a consequence of search and entry frictions, equilibrium is reflected by the number of active and inactive credit relationships. Equilibrium is also reflected by credit market tightness as measured by the share of unmatched projects from the credit applicant pool, which we interpret as frictional capital unemployment.

We prove the existence and the uniqueness of a nondegenerate steady-state search equilibrium with endogenous entry. Then we analyze the effects of shocks to credit markets and firms' profitability and show that the comparative statics depend crucially on the response of endogenous matching rates and differential entry. Generally, any shock that enhances matching rates causes aggregate liquidity to rise. While an increase in liquidity increases market participation by all firms, we show that low quality firms benefit disproportionately and the average quality of firms falls (unless the shock also raises the relative

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[5] The middleman literature focuses on the emergence and equilibrium pattern of intermediated trade. Rubinstein and Wolinsky (1987) consider random matching under complete information, where middlemen emerge and capture a share of the matching surplus. Biglaiser (1993) constructs a bargaining model with asymmetric information about quality, where middlemen reduce adverse selection related inefficiencies. Yavas (1994) assumes that matching between demanders, suppliers and middlemen is immediate, thus giving rise to active middlemen when random matching is too ineffective.

profitability of low quality firms). Thus, liquid credit markets may or may not be associated with high output and welfare, depending on whether the market participation effect outweighs the composition effect on average quality. Welfare may fall when the composition effect is large, so that it is possible to have outcomes in a complete information framework that resemble results found in models with asymmetric information (cf. Biglaiser and Friedman, 1999).

Our analysis demonstrates that the balance of market participation and composition effects depends on the source of increased market liquidity. Firm profitability shocks (or changes in productivity and entry costs) that enhance aggregate liquidity usually have strong market participation effects. By contrast, credit-market shocks such as enhanced matching efficacy increase market liquidity and participation, but because of a strong composition effect output and welfare may rise or fall. When the credit-market shock is due to lower contract quit rates, the outcome is enhanced liquidity and strong market participation effects. Because lower contract quit rates tend to lengthen contractual relationships, long-term relationships are positively related to output and loan rates. We also show that the behavior of loan rate spreads depend on the source of the shock and the type of matching technology assumed.

2. The Basic Model

Time is continuous. The economy is populated with a continuum of identical lenders (or households) of unit mass and a continuum of identical borrowers (or firms) of mass I. The benchmark framework is similar to that of Diamond (1990) and features lenders and borrowers who are brought together by an anonymous random matching technology. Upon a successful match, bilateral credit arrangements are made. Specifically, symmetric Nash bargaining between lenders and borrowers determines the terms of the credit contracts. In contrast to Diamond, the entry of firms and the rates at which borrowers contact lenders and lenders contact borrowers are determined endogenously.

Each household is endowed with a tree that generates flow income normalized to one unit. What households do with their income depends on the environment. Because there are many borrowers and lenders, the probability of rematches in a random environment is zero. Thus, IOUs from borrowers are not possible because they have no value for lenders. Autarky is the natural outcome under these circumstances and each household is limited to consuming a flow value of one. More interesting outcomes are possible, if households can make credit arrangements with firms that have access to a production technology. In this case, the households' flow income can earn positive returns provided that households' saving can be converted into productive uses that yield a gross rate of return of $R > 1$.

Firms enter the loanable funds market after paying a setup (or entry) cost, v_o. There they search for loanable funds to implement the investment project.[6] When matched with a household, a firm produces flow output A with probability p and zero with probability (1-p).

Because the loanable funds market features spatially separated borrowers and lenders, pairwise meetings are not instantaneous. We assume Poisson arrivals and designate : as the household's contact rate (or flow probability of meeting a firm) and 0 as the firm's contact rate (or flow probability of meeting a household). For finite contact rates, individual households or firms may either be matched or unmatched. Let H be the mass of searching and unmatched households and F be the searching firms. The mass of matched households is denoted by S, which by construction equals the mass of matched firms. Thus, if we set the mass of households equal to one, we have S = 1-H.

Unmatched households consume their flow endowment when lending is unprofitable, or R #1. If lending is profitable, or $R > 1$, households search for a match in the loanable funds market. Once matched with a firm, they lend their endowment and consume the returns of their savings, R. For simplicity, the length of the lending contract is fixed at 1/*. Thus, the flow probability of separation of a matched lender-borrower pair is * and the repayment can be computed as R/*. Upon separation, households and firms go back to the anonymous matching process in the absence of long-term enduring relationships.

We now formalize the flow value associated with searching and matched households of type i=1,2. Denote J_u as the value associated with an unmatched household and J_m as the value associated with a

[6]Unlike firms, households do not have entry costs. Because we fix the mass of households, allowing households an entry choice implies that all households either enter the credit market or stay away.

household matched with a firm. We then have:

$$rJ_u = 1 + \mu(J_m - J_u) \qquad (1a)$$

$$rJ_m = pR + \delta(J_u - J_m) \qquad (1b)$$

Equation (1a) says that the flow value associated with an unmatched household is the sum of the flow rate of consumption of the endowment good and the net values gained from being matched with a firm (J_m - J_u) which arrives at rate :. Equation (1b) says that the flow value of a household matched with a firm is the sum of the expected returns to the match generated from the loan contract *R* and the net value of terminating the lending contract and re-entering the unmatched state.[7]

Similarly for firms, let A_u and A_m denote, respectively, the unmatched and matched value associated with a firm. These asset values can be specified as:

$$r\Pi_u = \eta(\Pi_m - \Pi_u) \qquad (2a)$$

$$r\Pi_m = p(A - R) + \delta(\Pi_u - \Pi_m) \qquad (2b)$$

Equation (2a) gives the flow value of an unmatched firm as the product of the rate by which firms contact searching households, 0, and the net value of becoming matched (A_m - A_u). Equation (2b) specifies the flow value of a matched firm as the sum of the net expected productivity of the investment project made possible by the loan contract, less the interest costs, and the net value of terminating the lending contract.

Using (1b) and (2b), the potential (ex ante) gains that accrues from a successful match become:

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[7] If repayment is delayed until the time of contract termination, (1b) changes to $rJ_m = p\delta R + \delta(J_u - J_m)$ and *R becomes the appropriate measure of the gross interest rate. This entire analysis would go through with any substantive changes.

$$J_m - J_u = \frac{pR - rJ_u}{r+\delta} \tag{3a}$$

$$\Pi_m - \Pi_u = \frac{p(A - R) - r\Pi_u}{r+\delta} \tag{3b}$$

Since both borrowers and lenders are atomistic, they will take their threat points, the market-determined unmatched value J_u , as parametric in the process of bargaining.[8]

Consider a cooperative Nash bargain which gives a share \$ of the matched surplus to households and 1-\$ to firms. Bargaining entails solving $\max_R \; (J_m - J_u)^\beta (\Pi_m - \Pi_u)^{1-\beta}$ subject to (3a) and (3b), taking both J_u and A_u as given. Thus, the bargaining outcome must satisfy the following first-order condition:

$$\frac{\Pi_m - \Pi_u}{1-\beta} = \frac{J_m - J_u}{\beta} \tag{4}$$

Because firms base their entry decisions on the expected bargaining outcome, we must solve for R with A_u treated parametrically. However, we can substitute (1a) into (3a) to eliminate J_u. Thus, the Nash bargaining interest offer function is given by:

$$pR - 1 = \frac{\beta(r+\delta+\mu)(PA-1) - r\Pi_u}{r+\delta+\beta\mu} \tag{5}$$

The interest offer increases with the household contact rate, : , and with productivity, A, but it decreases with the unmatched value of firms, A_u , and the separation (or quit) rate, *.

Steady-state matching in the loanable funds market requires that the flow of firms seeking loanable funds equals the flow of households providing loanable funds. This implies:

$$\mu H = \eta F = m_0 \tilde{M}(H,F) \tag{6}$$

where $\tilde{M}(,)$ is the random funds matching function that satisfies the following properties: strictly

[8] For a detailed description of the atomistic bargaining process, see Pissarides (1984) and related work cited in Laing, Palivos and Wang (1995).

increasing and strictly concave in each argument, homogeneity of degree one, standard Inada conditions and boundary conditions [i.e., $\tilde{M}(0,) = \tilde{M}(,0) = 0$]. Dividing through by the second argument in the matching function and substituting for *H/F* yields:

$$\eta = m_0 M\left(\frac{\eta}{\mu}\right) \tag{7}$$

This relationship describes 0 as a negative function of : which is a "Beveridge curve" for the loanable funds market. Under the above-specified properties, we have:

Lemma 1: (Beveridge Curve) *The Beveridge curve is downward-sloping in (:, 0)-space and convex, asymptotes to both axes, and shifts away from the origin as the matching parameter, m_0, increases.*

Next, we consider endogenous entry, taking heed that flows into the loanable funds market must equal flows out of the market. Because the household population in steady state is fixed at unity, the inflow of households who enter to search for projects (after having been separated from other projects) must equal the outflow from the market, or

$$\delta S = \mu H \tag{8}$$

Entry of firms causes the unmatched values of firms to be driven down to their entry cost:[9]

$$\Pi_u = \nu_0 \tag{9}$$

Substituting (3b) into (2a) and combining the result with (9) yields the zero-profit (ZP) condition:

$$\eta^{ZP} = \frac{r\nu_0(r+\delta)}{p(A-R) - r\nu_0} \tag{10}$$

Straightforward differentiation implies:

Lemma 2: (Unrestricted Entry) *The firm contact rate that satisfies the zero profit condition rises with the*

[9]This, of course, requires that the ex-ante population of firms *I* is sufficiently large.

*entry cost, v_o, the interest offer, R, and the quit rate, *; it falls when the expected productivity, pA, rises.*

The underlying intuition is clear-cut once we keep in mind that zero profit requires a negative relationship between the net gains of firms accrued from a successful match and the firm contact rates. As net gains rise, more firms tend to participate in the credit market (to restore zero profit). However, having more firms lowers the probability that an individual firm will locate a household.

Consider,

Definition: *A **steady-state equilibrium** is a tuple {R, ∴, 0, H, S, F, A_u} that satisfies: (i) Nash bargaining, (5); (ii) steady-state matching and separation, (6), (7) and (8); (iii) free entry and zero profit, (9) and (10); and, (iv) population identity, S + H = 1.*

Note that the free entry conditions immediately pin down A_u at v_o, whereas (5) gives *R* as an increasing function of : . The latter relationship can be substituted into (10) to yield an upward-sloping ZP locus in (: ,0) space, which together with (7) determines the equilibrium contact rates in steady state, { : *,0*}. Straightforward comparative-static analysis using Figure 1 shows that the equilibrium household contact rates depends positively on matching efficacy (m_o) and productivity (*A*), whereas the effects of the entry cost (v_o) and the quit rate (*) on : * are ambiguous.

One can then use these equilibrium contact rates with (6), (8) and (9) to solve for the equilibrium masses of firms and households:

$$H^* = \frac{\delta}{\delta + \mu^*} \tag{11a}$$

$$S^* = 1 - H^* = \frac{\mu^*}{\delta + \mu^*} \tag{11b}$$

$$F^* = \frac{\delta \mu^*}{(\delta + \mu^*)\eta^*} \tag{11c}$$

These three equations imply that the mass of searching households is negatively related to the household contact rate. Also, the mass of matched household-firm pairs depends positively on the household contact rate. Moreover, the mass of searching firms is increasing in the household contact rate but decreasing in the firm contact rate. Finally, it is useful to point out that equation (11b) pins down the equilibrium value of social output, $Y^* = S^*A$. Social output is increasing in the productivity and the household contact rate but decreasing in the quit rate. In summary, we have:

Proposition 1: (Equilibrium with Homogeneous Borrowers) *Provided that $(pA-1) - rv_o > 0$, a unique nondegenerate steady-state equilibrium with homogeneous borrowers exists, where the interest rate, the household contact rate and social output are increasing in matching efficacy and productivity.*

3. From Homogeneous to Heterogenous Borrowers

There are two types of firms indexed by i with mass I^i. Firms have different riskiness, productivity, and set-up costs. While a firm's type is known to all, the number of firms of each type is determined by unrestricted entry with differential costs. Without loss of generality, the type 1 firm has access to a low-risk, low-return investment project, whereas the type 2 firm has access to a high-risk, high-return project. Membership in the population set I^i is determined by a random lottery. Thus, it is necessary to identify firm types by adding superscripts *i* to the notation defined previously. Let N^i $(N^1 + N^2 = 1)$ denote the (endogenous) fraction of type i firms entering the loanable funds market, which need not be the same as the ex ante population share I^i. Thus, N^iF represents the population of type i firms searching for funds.

Specifically, we assume throughout the paper that the high-type pays a higher entry fee, that is, $\nu_0^2 > \nu_0^1$. Moreover, we assume that the high-type firms are more productive both in absolute terms and on average, but face a lower success rate:

(A1) (Productivity) $A^2 > A^1$

(A2) (Success Rate) $p^2 < p^1$

(A3) (Expected Productivity) $p^2 A^2 > p^1 A^1 > 1.$

With heterogenous borrowers, the value functions remain the same with superscript *i* added to all relevant variables. The only exception is the value function for unmatched households:

$$rJ_u = 1 + \mu^1(J_m^1 - J_u) + \mu^2(J_m^2 - J_u) \tag{12}$$

Also, when there are different types of firms, the household contact rates must be proportional to the relative masses of the firms, or $:^i = :N^i$ for i = 1,2. Under these circumstances, Nash bargaining implies an interest offer function that is analogous to (5):

$$p^i[A^i - R^i] - r\Pi_u^i = \frac{1-\beta}{\beta}\frac{r+\delta}{r+\delta+\mu}\left\{(p^iR^i - 1) + \frac{\mu N^j}{r+\delta}[(p^iR^i - 1) - (p^jR^j - 1)]\right\} \tag{13}$$

This expression clearly illustrates that the outcomes of both types of firms are interdependent, because households' threat points depend on the expected returns of both types .

We characterize the "interest offer function" by totally differentiating equation (13):

Proposition 2: (Interest Offer) *The interest offer function R^i (∴, N^i ; A^i, A_u^i , *) is increasing in the household contact rate,* ∴*, and the own-type productivity, A^i, but decreasing in the fraction of low-type firms, N^1, the own-type unmatched value of firms, A_u^i , and the quit rate,* *.

While the results are straightforward, the response of the interest offer to the (endogenous) composition of firms deserves further comment. If more type-1 firms enter the loanable funds market, the share of type-1 firms rises. According to (12), a rise in N^1 lowers the unmatched value of households (J_u), which is their bargaining threat point. Because households' bargaining power falls, the interest offer falls.

Next, we note that all the steady-state conditions also remain valid. However, the zero-profit conditions for type-i firms (i = 1,2) can be written as:

$$\eta^{ZPi} = \frac{r\nu_0^i(r+\delta)}{p^i[A^i - R^i] - r\nu_0^i} \tag{14}$$

This equation has identical properties to those described in Lemma 2.

We have already defined the most important features of steady-state equilibrium in the homogeneous case. With heterogeneous borrowers, the definition needs to be modified slightly. In particular, the composition of types is summarized by N^i and is determined by $\eta^{ZPi} = \eta$ and $N^1 + N^2 = 1$. The latter condition states that all firms face the same contact rate in the anonymous random matching environment. Although the population masses can still be solved recursively using (11a)-(11c), the remaining equations that define an equilibrium all involve the composition variable, N^1. The remainder of this section will use two bargaining equations, two zero profit conditions (with identical η) and the Beveridge curve to jointly solve for two interest rates, two contact rates, and N^1.

For notational convenience, let $B \equiv [(1-\beta)/\beta](r+\delta)/(r+\delta+\mu)$, $\bar{a}^i \equiv p^iA^i - r\nu_0^i$, $a^i \equiv \bar{a}^i + B$, $u^i \equiv N^iu$, and $u \equiv \mu/(r+\delta) = u^1 + u^2$ where $dB/d\mu<0$, $du^1/dN^1>0$, $du^2/dN^1<0$ and $du/d\mu>0$. Then we can rewrite the Nash bargaining conditions (13) to give:

$$a^i = (1+B+Bu^j)p^iR^i - Bu^jp^jR^j \quad i=1,2, \quad i\neq j$$

This system yields the solution

$$p^iR^i = \frac{\bar{a}^i + B}{1+B} + \frac{(1-\beta)\beta\mu}{r+\delta+\beta\mu}\left(\bar{a}^j - \bar{a}^i\right)N^j \tag{15}$$

To ensure sensible results, we impose:

Condition D: (Net production gain and entry cost differentials) $p^2A^2 - p^1A^1 > r(v_0^2 - v_0^1) > 0.$

Moreover, to allow low type firms to operate actively , we need $p^1R^1 > 1$, which is given by $\bar{a}^1 > 1$, or,[10]

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[10] This is clear by rewriting (15) as: $p^1R^1 - 1 = \dfrac{\bar{a}^1 - 1}{1+B} + \dfrac{(1-\beta)\beta\mu}{r+\delta+\beta\mu}(1-N^1)(\bar{a}^2 - \bar{a}^1).$

Condition F: (Active Operation) $p^1 A^1 - 1 > r\nu_0^1$.

Obviously, given Condition D, Condition F is stronger than Assumption (A3) and it is sufficient but not necessary for active operation. Conditions D and F imply $\bar{a}^2 > \bar{a}^1 > 1$ and both are sufficient to guarantee active operation for both high and low type firms (i.e., $J_m^i > J_u$).

In order to work out the comparative statics, it is useful to note:

Lemma 3: (Expected Interest Rates) *Under Assumptions (A1) and (A3), we have the following: (i)* $Mp^1R^1/MN^1 = Mp^2R^2/MN^1 < 0 < Mp^1R^1/MN^2 = Mp^2R^2/MN^2$; *(ii)* $Mp^1R^1/M: = Mp^2R^2/M: > 0$; *(iii)* $Mp^1R^1/M* = Mp^2R^2/M* < 0$; *and, (iv)* $Mp^iR^i/M<_0^i < 0$ *for i = 1,2.*

Proof: See Appendix.

Intuitively, a change in the proportion of types of firms in the market, N^i, impacts both the threat point of households and firms. An increase in N^1 lowers the threat point of a matched low type firm and tends to increase the expected returns to the household implied by the bargaining solution. However, it also lowers the threat point of households matched to low type firms by lowering their unmatched value J_u. Because this latter effect dominates, $MR^1/MN^1 < 0$. An increase in N^1 also lowers J_m but increases the threat point of high type firms, both effects leading to $MR^2/MN^1 < 0$. Similarly, an increase in the fraction of high type firms, N^2, strengthens the relative bargaining position of households matched with both high and low type firms so that $MR^1/MN^2 > 0$ and $MR^2/MN^2 > 0$. By contrast, an increase in $<_0^i$ lowers the net gains to a match for both the household and firm. However, firms lose disproportionately more and hence a reduction in the expected returns to the household is required to satisfy the bargaining rule. Entry costs effects are entirely symmetric (and opposite to the effects of changes in productivity).

Using (15), we can compute the expected interest rate spread between high and low type firms and the actual interest rate spread. Thus,

$$p^2 R^2 - p^1 R^1 = \beta(\bar{a}^2 - \bar{a}^1) \tag{16}$$

$$R^2 - R^1 = \beta\left(\frac{\bar{a}^2}{p^2} - \frac{\bar{a}^1}{p^1}\right) + \frac{1-\beta}{r+\delta+\beta\mu}\left[\beta\mu(N^1\bar{a}^1 + N^2\bar{a}^2) + r + \delta\right]\left(\frac{1}{p^2} - \frac{1}{p^1}\right) \tag{17}$$

where $\lim_{m_0\to\infty, v_o^i\to 0} R^2 - R^1 = \beta(A^2 - A^1) + (1-\beta)(1/p^2 - 1/p^1)(N^1p^1A^1 + N^2p^2A^2 - 1) > 0$, depending on the productivity as well as risk differential. We can show:

Proposition 3: (Interest Rate Spreads) *Under Assumption (A1) and Condition D, both the expected (p^2R^2 - p^1R^1) and the actual (R^2 - R^1) interest rate spreads are positive. The expected interest rate spread is driven by the expected profitability differential $(\bar{a}^2 - \bar{a}^1)$. While the actual interest rate spread in a frictionless economy (with $m_0\to\infty$ and $v_0^i\to 0$) is determined by productivity as well as risk differentials, such a spread is smaller in an economy with search and entry frictions.*

Proof: See Appendix.

Condition D says that a positive expected and actual interest spread between high and low types requires that the productivity differential between high and low types be sufficiently large relative to the flow entry cost differential so that $\bar{a}^2 > \bar{a}^1$. Hence, interest paid by type 2 always exceeds the interest paid by type 1 in both expectations and realization. Both interest rate spreads generally depend positively on the expected net productivity differential. The difference between the two is that the actual or realized interest rate spread also depends negatively on the share of low quality firms and positively on the household contact rate. Both the composition effect and contact-rate effect on the actual rate spread diminish as firms' bargaining power (1-β) decreases. In a frictionless economy where matching is instantaneous (as in Yavas (1994)) and firm entry is costless (i.e., $m_0\to\infty$ and $v_0^i\to 0$), productivity and risk differentials pin down the actual rate spread. When matching is not instantaneous (as in Rubinstein and Wolinsky (1987)) and when there are entry frictions, the actual rate spread becomes smaller because of the composition effect.

Given the properties of the interest rate function considered above, we are now ready to consider the determination of steady-state equilibrium. From Proposition 2, we can write $R^i = R^i(\because, 0, N^1)$ where $R^i_{:}$

> 0, $R^i_* < 0$, $R^i_{N^1} < 0$. Given the interest functions R^i, steady-state $\{\mu^*,\theta^*,N^{1*}\}$ thus satisfy (7) and (14).

Lemma 4: (Equilibrium Zero-Profit Trace) *Both the ZP^1 and ZP^2 loci from (14) are downward sloping in (N^1,θ)-space with $|d\theta^*/dN^1|_{ZP}^{\,1} > |d\theta^*/dN^1|_{ZP}^{\,2}$. Furthermore, there exists a unique and upward sloping equilibrium zero-profit trace EZ in (N^1,θ)-space, $\theta = \theta^Z(N^1)$ that satisfies (14) for a given μ such that there is a $N^1_{min} > 0$ yielding $\theta^Z(N^1_{min}) = 0$ and $\infty > \theta^{max} \geq \theta^Z(1) > 0$.*

Proof: See Appendix.

The ZP^1 and ZP^2 loci and the EZ trace in (N^1,θ)-space are shown in Figure 2. How these curves relate to the Beveridge Curve (denoted BC) in (μ,θ)-space is also shown. These relationships together pin down the steady-state equilibrium $\{\mu^*,\theta^*,N^{1*}\}$.

Satisfying both ZP conditions in (14) would necessarily imply:

$$D \equiv \frac{p^2[A^2 - R^2]}{r\nu_0^2} - \frac{p^1[A^1 - R^1]}{r\nu_0^1} = 0$$

The term D measures the expected net surplus differential between high and low type firms, or, in short, the firm surplus differential. The firm surplus differential is less than the profitability differential because high type firms must pay a larger entry cost to get it.[11] When D is positive, the share of low type firms must rise to drive the firm surplus differential back down to equilibrium. To see this, when N^1 rises for a given μ^*, households are more likely to contact low type firms. This weakens the ability of high type firms to extract a higher surplus from households and D falls. Thus, any changes resulting in $D > 0$ would require

[11] Note that D is less than the net profitability differential $((1-\beta)(\bar{a}^2 - \bar{a}^1)/(rv_0^2))$ because entry costs differ. This can be seen by using $p^2R^2 - p^1R^1 = \beta(\bar{a}^2 - \bar{a}^1)$ to rewrite D as

$$D = \left\{(1-\beta)(\bar{a}^2 - \bar{a}^1) - \left[p^1(A^1 - R^1) - r\nu_0^1\right](v_0^2 - v_0^1)/v_0^1\right\}/(rv_0^2).$$

Also, N^1 affects D (through R^1) only when relative entry costs differ. Finally, note that

$$\frac{dD}{d\delta} = \frac{v_0^2 - v_0^1}{rv_0^1 v_0^2}\frac{dp^iR^i}{d\delta} < 0 < \frac{dD}{d\mu} = \frac{v_0^2 - v_0^1}{rv_0^1 v_0^2}\frac{dp^iR^i}{d\mu}.$$

an increase in N^1 to restore zero profit. For example, an increase in :* or a decrease in * will ultimately increase N^1. These arguments are useful for understanding the comparative statics derived in Section 5.

The above arguments lead to the following theorem:

Theorem: (Existence and Uniqueness) *Under Assumptions (A1) and (A2) and Conditions D and F, there exists a unique, non-degenerate steady-state equilibrium with full information if the expected production gains are sufficiently high such that* 0^* , *(0,* 0^{max} *).*

Proof: The existence and uniqueness will be proved in two steps. First, we claim that the BC and EZ loci uniquely determine steady-state {:*,0*,N^{1*}}. It is clear from the proof of Lemma 3 and expression (14) that as long as the expected production gains are sufficiently high such that 0^* , (0, 0^{max}), N^1* is bounded in the interval (0,1]. Then as the determinant of the pre-multiplied matrix of system (7) and (14) is strictly positive, the implicit function theorem implies unique determination of steady-state {:*,0*,N^{1*}}. Thus, for a given pair {:,0} satisfying (BC), there exists a unique pair {0,N^1} which satisfies (EZ). Once we obtain the equilibrium matching rates :*,0* and fraction of low to high type firms, N^1*, we can use (11a)-(11c) to solve for the equilibrium masses {H*, S*, F*} and so F^1* = N^1F* and F^2* = (1-N^1)F*. Since (11a)-(11c) are all well-defined monotone functions, the determination of these masses is also unique.

4. Comparative Statics under Differential Entry of Heterogeneous Borrowers

We are now prepared to characterize the steady-state equilibrium with heterogenous borrowers. In addition to examining the determinants of the equilibrium matching rates, the composition and the mass of the matched firms, and the gross interest rates, we are also interested in the percentage of unmatched projects, the aggregate output of the matched firms, and welfare.

From (17) we note that the equilibrium number of matches S* is positively related to :*/ * . This term reflects the market's liquidity because it is also equal to the aggregate share of household funds that is channeled to firms. The size of the credit market is measured by S*+F*. This sum adds market participants that are matched to market participants that are unmatched and still searching. Also, we define U* =

F*/(F*+S*), which is the share of unmatched projects in the credit applicant pool. Because U* measures the tightness of the credit market much like the unemployment rate in the labor market, we will call it the "capital-unemployment rate." Because F*=S*(*/0*), we find that U* = 1/(1+(0*/*)). Thus, our measure of capital-unemployment depends on search and entry frictions solely through the factor */0*.

Next, we compute social output, based on the steady-state masses of matched firms, S^*N^i (i = 1,2):

$$Y^* = S^* [N^{1*}A^1 + (1-N^{1*})A^2] \tag{18}$$

The aggregate output measure can be decomposed into two components. First, S* reflects aggregate matches and enhanced market liquidity (and enhanced market participation). Second, the square bracket term reflects the composition of output and can be interpreted as the average output over all matched firms. Because the two components need not always move in the same direction, the comparative statics with respect to the responses of interest rates and social output are generally ambiguous.

Tedious but straightforward comparative-static analysis yields:

Proposition 4: (Credit Market Shocks) *Under the circumstances described in the Theorem, the effects of matching efficiency (m_0) and separation rate (*) on steady-state {0*, :*, N^{I*}, R^{i*}, S*, U*, Y*, J_u^*} are given by:*

(i) *An improvement in matching efficiency generates more matches and leads to a greater fraction of low-type firms.*

(ii) *An increase in the contract quit rate will raise the firm contact rate and reduce the capital unemployment rate but lower household contact rates and market liquidity. Also, the share of low type firms will fall. When the market participation effect dominates the composition effect, interest rates and output fall; otherwise, the effect is uncertain.*

Proof: See Appendix.

Table 1 summarizes the comparative statics results.[12] First, we discuss what happens when bank matching efficiency increases. Intuitively, an increase in matching efficiency increases the contact rate for households :* and encourages firm's entry. The overall level of matchmaking activity increases (as captured by a rise in S*) because of higher household contact rates. From Proposition 3 we know that a rise in the household contact rate raises the interest offer to each firm by an equal amount. Because the firm surplus differential (*D*) widens, low type firms enter disproportionately and thus N^1* rises.

This composition effect puts downward pressure on loan rates and is sufficiently strong that rates return to where they originally were. Thus, there is no net effect on loan rates and on the firm matching rate 0*. Because 0* is unchanged, the capital unemployment rate is unaffected. Recall from the discussion of Proposition 1 that N^1* and :* have opposing effects on the unmatched value of households. Because of the presence of the composition effect, an increase in matching efficiency creates two offsetting forces on output. More matchmaking means higher output because of greater market participation, but this effect is offset by the fact that there are relatively more low type firms in the economy.

Next we ask, what happens following an increase in the contract quit rate *? An increase in * reduces the unmatched value of all firms relative to their entry cost, lowers their threat points, and induces firm exit. Thus, firm contact rates of surviving firms rise by the zero-profit condition which reinforces the negative direct effect of contract quits on capital unemployment. From the Beveridge curve relationship, household contact rates fall which causes a reduction in market liquidity and a fall in the overall level of matchmaking, S*. While fewer productive firms means output falls, output could rise if the average productivity of the remaining firms rises. Average productivity is determined by the firm composition effect. As explained previously, low types enter relative to high types when the firm surplus differential (going to high type) firms is excessive. Contract quits have a negative direct effect on loan rates and a negative indirect effect on loan rates because household contact rates are reduced. Because lower loan rates

[12]While Figure 2 is useful in illustrating the uniqueness of steady state equilibrium, the comparative statics cannot be easily captured graphically since there are significant feedback effects between the matching rates and the fraction of firm types in the market.

reduce the surplus differential, N^1 falls. The net effect on output and welfare balances a negative effect from reduced market liquidity with a positive composition effect from an increase in the average productivity of remaining firms. Under the assumption that the market participation effect dominates (or production normality) and output fall with a rise in contract quits. Interestingly, our results suggest that long-term relationships (i.e., lower contract quit rates) increase aggregate output and raise loan rates to all firms (although disproportionately more for high quality firms).

In order to derive sensible comparative dynamic results for entry cost shocks, we assume,

Condition Q: (Credit-market Matching Efficacy) There exists a m_o such that

$$Q \equiv r\frac{r+\delta}{\eta} + r(1-\beta)(1 - \frac{\mu}{r+\delta+\beta\mu}) \gg 0$$

The restriction on credit-market matching efficacy is met by imposing a sufficient condition on m_o such that $\mu < (r+\delta)/(1-\beta)$. Intuitively, we want to entry of firms to cause interest rates to fall. Without search frictions it is possible that interest rates rise. Thus, we require that matching frictions are sufficiently strong to avoid this possibility.

We can show:

Proposition 5: (Firm Profitability Shocks) *Under the circumstances described in the Theorem, the effects of productivity (A^i) and entry costs ($<_0^i$) on steady-state {O*, :*, N^{1*}, R^{i*}, S*, U*, Y*, J_u^*} are given by:*

(i) *Productivity and entry cost shocks that raise the profitability of high types increase household contact rates, market liquidity, and the share of low type firms, but lower firm contact rates which raises the capital unemployment rate. Productivity and entry cost shocks that raise the profitability of low types will have opposite effects on these variables.*

(ii) *When the market participation effect dominates the composition effect, productivity shocks raise loan rates and output. Cost shocks tend to have the opposite effect (except for output where the outcome is open), if the market participation effect dominates the composition effect and credit-*

market matching efficacy is not too high to violate Condition Q.

Proof: See Appendix.

To understand the effects of shocks that increase firm profitability $\bar{a}^i$ (due to either an increase in A^i or a reduction in v_o^i), recall that there are two mechanisms at work. First, when $\bar{a}^1$ (or $\bar{a}^2$) rises, $\bar{a}^1 - p^1R^1$(or $\bar{a}^2 - p^2R^2$) rises less (or more) than proportionately because of differences in net expected productivity.[13] Zero-profit thus requires η to rise (or fall) which causes the capital unemployment rate to fall (rise). The Beveridge Curve translates the change of η into a fall (or rise) of μ so that both market liquidity and market participation fall (rise). Following the discussion of Proposition 1, the direct effect of an increase in $\bar{a}^1$ (or $\bar{a}^2$) raises loan rates, but the indirect effect of lower (or higher) μ causes them to fall (or rise). Loan rates rise when $\bar{a}^1$ or $\bar{a}^2$ rises, where production normality guarantees that indirect effects are not too large when the profitability shock benefits high types. Second, N^1 rises whenever shocks cause the firm surplus differential to be excessive. From Propositions 1 and 3, an increase in $\bar{a}^1$ leads to higher profitability for low type firms (A^1-R^1) and higher loan rate for high types (R^2) - the latter results in lower profitability for high type firms. As a consequence, the firm surplus differential decreases, implying a fall in N^1 so as to restore zero profit. By similar arguments, an increase in $\bar{a}^2$ gives rise to a higher N^1. As before, the effect of profitability shocks on output balances market participation effects and average productivity effects, whereby the latter is a sum of individual productivity effects and the composition effect. Productivity shocks tend to enhance average productivity directly, while their indirect effects tend to be offsetting. Similar results apply to changes in entry costs.

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[13]A critical relationship for understanding how the zero profit conditions respond to shocks is given by $\bar{a}^i - p^iR^i = \frac{B}{1+B}(\bar{a}^i - 1) - \frac{(1-\beta)\beta\mu}{r+\delta+\beta\mu}N^j(\bar{a}^j - \bar{a}^i)$, where the second term of the righthand side is negative (or positive) for i = 1 (or 2).

Overall, any shock that enhances matching rates causes aggregate liquidity to rise. While an increase in liquidity increases market participation by all firms, low quality firms enter disproportionately and the average quality of firms falls (unless the shock raises the profitability of low quality firms). Thus, liquid credit markets may or may not be associated with high output and welfare, depending on whether the composition effect on average quality outweighs the effect on market participation. Profitability shocks (to productivity or entry costs) that benefit high type firms will enhance aggregate liquidity but create a negative composition effect. By assuming production normality, positive productivity shocks are generally associated with higher output and welfare. By contrast, positive credit market shocks will increase market liquidity and market participation, but because of a strong composition effect social output may rise or fall.

Finally, we observe that because of the presence of composition effects, the world with heterogeneous borrowers differs greatly from the simpler world with homogeneous borrowers. Composition effects give rise to results in a world without informational asymmetry that are similar to those found in economies with adverse selection (cf. Biglaiser and Friedman, 1999). Moreover, because of the presence of differential entry, an improvement in credit-market matching efficacy no longer generates an unambiguously positive effect on social output. The distinctive effects of changes in composition versus changes in market participation provide a fertile ground for future applied work in the area of credit markets.

5. Assortive Financial Matchmaking

Until now we have assumed that matching is random even though there is full knowledge about firm types. Suppose this knowledge is used to improve the performance of the loanable funds market. Specifically, we allow financial matchmakers to undertake assortive matching and give priority to loans to high-type firms. To justify such an active role, it is necessary to assume that there is excess demand for funds (i.e., $H < F$). Moreover, we assume that the supply of funds exceeds the demand for funds by high-type firms. If we did not assume excess supply for high quality firms, assortive matching would yield a corner solution with only high types receiving loans. This latter restriction requires $N^2F < H$, which together with $H < F$ implies $N^1F > (H - N^2F)$.

Under these regularity conditions, all high-type firms receive loans. The mass of low types who receive loans is equal to the residual $H - N^2F$. Therefore, assortive matching can be thought of as matching in two segmented markets, yet lenders do not undertake directed search *ex ante*.

For notational convenience, we define the tightness of the loanable funds market as $J = F/H > 1$. We also reinterpret the contact rates as the arrival rates of the opportunities for funds. We thus have the following relationships: $:^1 = :(H - N^2F)/H = [1\text{-}J(1\text{-}N^1)]$, $:^2 = :(N^2F)/H = J(1\text{-}N^1)$, and $0^1 = 0^2 = 0$. The last expression indicates that arrivals of opportunities are non-discriminating even though matches are assortive. With the modifications to the household contact rates, we find that the analysis of Sections 3 and 4 remains valid. Rather than going through the entire analysis again, it is therefore sufficient to simply focus on the main difference between assortive financial matchmaking and random matching.

To compare assortive and random matching, we examine the contact rates and find $:^1 = [1\text{-}J(1\text{-}N^1)] < N^1$ and $:^2 = JN^2 > N^2$. That is, lenders are more likely to meet with high-type borrowers under assortive matching than under random matching. This has two immediate consequences. First, as a result of the greater rate of contact with high-type firms, the loan rate of high types increases, which leads to a widening of the interest rate differential between the high quality firms and the low quality firms. Second, due to assortive matching, more high-type firms are granted loans and hence the social output increases unambiguously. This highlights the funds-allocation role of financial matchmakers in search markets.

6. Summary and Extensions

This paper has studied a dynamic general equilibrium search model of credit markets with heterogeneous borrowers and endogenous rates of entry and contact. The analysis identifies channels through which decentralized matching and assortive financial matchmaking affect the size and quality of credit flows. Our results suggest that shocks that increase credit market liquidity also lead to increased market participation by firms and a composition effect whereby the participation of low quality firms rises disproportionately. However, more liquid markets only increase output and welfare when the market participation effect dominates the composition effect. This generally is the case when shocks enhance the profitability of firms or when they make contracts less fragile (and financial relationships longer lasting). By contrast, structural shocks that make matching more efficient may have large composition effects.

Two natural extensions come to mind. First, one could allow financial matchmakers have a more active role than just sorting borrowers by quality. In addition, they could choose matchmaking effort by maximizing their output net of a real resource cost. It would be useful to compare the credit market outcomes with those discussed in Section 5 and those under the middlemen framework developed by Rubinstein and Wolinsky (1987). Second, one could introduce asymmetric information about the firm's type. There are two possibilities. When firms make their investment project selection (high or low type) *prior* to bank loan approval, the adverse selection problem may exist as in the middleman theory developed by Biglaiser (1993). Alternatively, when firms select projects *ex post*, the moral hazard problem may arise. In either case, equilibrium credit rationing may be present with incentive compatible financial contracts. This additional source of capital unemployment will then compound the frictional capital unemployment considered in this paper.

Figure 1: Steady-State Search Equilibrium with Homogeneous Borrowers

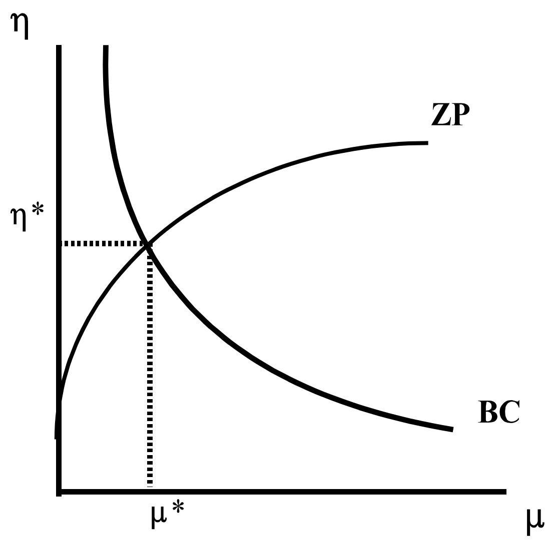

Figure 2: Steady-State Search Equilibrium with Heterogenous Borrowers

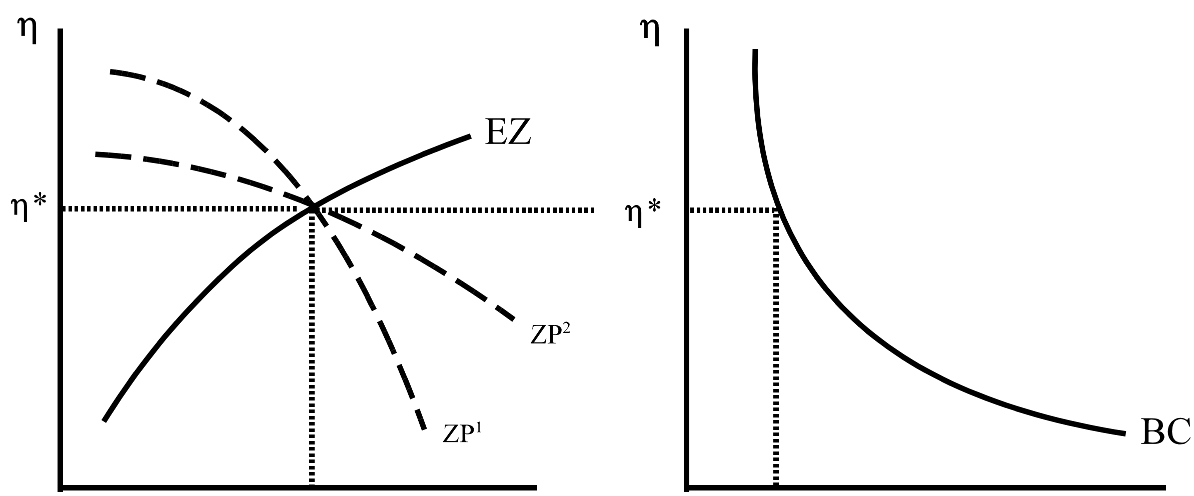

Table 1: Summary of Comparative Statics

Effect of / Effect on	**Credit Market Shocks**		**Firm Profitability Shocks**			
	Credit Matching Efficacy m_0	Contract-Quit Rate *	Type 1 Productivity A^1	Type 2 Productivity A^2	Type 1 Entry Costs[1] $<_0^1$	Type 2 Entry Costs[1] $<_0^2$
1. Contact Rates and Population Masses						
0*	0	+	+	-	-	+
U*	0	-	-	+	+	-
:*	+	-	-	+	+	-
S*	+	-	-	+	+	-
2. Composition of Low-Type Borrowing Firms						
N^1*	+	-	-	+	+	-
3. Interest Rates and Differential						
R^i*	0	- [2]	+	+ [2]	-	- [2]
$R^2 - R^1$	0	- [2]	?	+ [2]	?	- [2]
4. Social Output						
Y*	?	- [2]	+ [2]	+ [2]	?	?

Note 1: To sign some of the effects of entry costs we require that m_0 is sufficiently small to satisfy Condition Q.

Note 2: Assume that the composition effects are comparatively small.

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Appendix

This Appendix contains proofs of Lemma 3 and 4 and Propositions 3-5 in the paper.

Proof of Lemma 3:

(i) Differentiating (15) with respect to N^i gives

$$\frac{\partial(p^iR^i)}{\partial N^i} = -\left[\frac{\beta(1-\beta)\mu}{r+\delta+\beta\mu}\right](\bar{a}^j - \bar{a}^i) = \frac{\partial(p^jR^j)}{\partial N^i}$$

For i = 1 we have $Mp^1R^1/MN^1 = Mp^2R^2/MN^1 < 0$ and for i = 2 we have $Mp^1R^1/MN^2 = Mp^2R^2/MN^2 > 0$.

(ii) Differentiating (15) with respect to : gives

$$\frac{\partial(p^iR^i)}{\partial\mu} = \frac{\beta(1-\beta)(r+\delta)}{(r+\delta+\beta\mu)^2}[(1-N^i)(\bar{a}^j-\bar{a}^i) - (1-\bar{a}^i)] = \frac{\partial(p^jR^j)}{\mu} > 0 \; for \; all \; i,j$$

(iii) Notice that $MB/M* > 0$, and given $\bar{a}^i > 1$, the first term in (15) is strictly decreasing in *. For i = 1, it is clear that the second term in (15) is also strictly decreasing in *, implying $Mp^1R^1/M* < 0$. Thus, manipulating (15), we have $p^2R^2 = p^1R^1 + \beta(\bar{a}^2 - \bar{a}^1)$, implying $Mp^2R^2/M* = Mp^1R^1/M*$.

(iv) Since $Mp^iR^i/M<_0^i$ % - $Mp^iR^i/MA^i < 0$, the result is immediate.

Proof of Proposition 2:

From (16), we can derive $R^2 - R^1 = \frac{\beta}{p^2}(\bar{a}^2 - \bar{a}^1) + \frac{p^1-p^2}{p^1p^2}p^1R^1$. Thus both spreads are positive under Condition D that ensures $\bar{a}^2 - \bar{a}^1 > 0$. Utilizing Proposition 2, we can see implies that $R^2 - R^1$ rises with μ and $\bar{a}^2$, falls with N^1 and δ, may rise or fall with $\bar{a}^1$, and is immune to m_0. Moreover, from (17) we obtain:

$$R^2 - R^1 = \lim_{\mu\to\infty, v_o^i\to 0}(R^2 - R^1) - \beta r\left(\frac{v_0^2}{p^2} - \frac{v_0^1}{p^1}\right) - (1-\beta)\left(\frac{1}{p^2} - \frac{1}{p^1}\right)\Theta$$

where $\Theta = \frac{r+\delta}{r+\delta+\beta\mu}(N^1p^1A^1+N^2p^2A^2-1) + \frac{\beta\mu}{r+\delta+\beta\mu}(N^1v_0^1+N^2v_0^2)$ is a weighted sum of aggregate net outputs and aggregate entry costs, which is positive under (A3) and Condition F. Thus, given (A2), the actual interest rate spread is smaller than that in the absence of search and entry frictions.

Proof of Lemma 4:

From Proposition 3, it is immediate that ZP1 and ZP2 are downward sloping in ($0,N^1$) space.

Differentiating (14) gives

$$\left.\frac{d\eta^*}{dN_1^*}\right|_{zp1} = \frac{d\eta^{zp1}}{dp^1R^1}\frac{dp^1R^1}{dN_1} = -\frac{(\eta^*)^2}{r\nu_0^1(r+\delta)}\frac{\beta(1-\beta)}{r+\delta+\beta\mu}\mu(\overline{a^2}-\overline{a^1}) < 0$$

$$\left.\frac{d\eta^*}{dN_1^*}\right|_{zp2} = \frac{d\eta^{zp2}}{dp^2R^2}\frac{dp^2R^2}{dN_1} = -\frac{(\eta^*)^2}{r\nu_0^2(r+\delta)}\frac{\beta(1-\beta)}{r+\delta+\beta\mu}\mu(\overline{a^2}-\overline{a^1}) < 0$$

since $\nu_0^1 < \nu_0^2$, we have that the pair $\{\eta^*, N^{1*}\}$ satisfying (14) given μ occur where $|d\eta^*/dN^{1*}|_{zp1} > |d\eta^*/dN^{1*}|_{zp2}$. Since both locus' are downward sloping, this pair is unique. To characterize the (EZ) locus, equate ZP1 and ZP2 from (14):

$$\nu_0^2 p^1[R^1(N^1,\mu)-1] - \nu_0^1 p^2[R^2(N^1,\mu)-1] = \nu_0^2 p^1(A^1-1) - \nu_0^1 p^2(A^2-1) \qquad \text{(P1)}$$

Notice that from Proposition 3, $\partial p^iR^i/\partial\mu = \partial p^jR^j/\partial\mu > 0$ and $\partial p^iR^i/\partial N^1 = \partial p^jR^j/\partial N^1 < 0$. Consider now that μ increases. Since p^iR^i is higher (for i = 1,2), (14) implies η must be higher. However, this changes the LHS of (P1) away from the RHS: the LHS increases (decreases) iff $\nu_0^1p^2 - \nu_0^2p^1 <$ (>) 0. In either case, N must rise to restore the equality in (P1), implying $d\eta/dN^1|_{EZ} > 0$.

To characterize the limit points of the EZ locus, consider the case where $\mu \to \infty$ which implies $\eta \to 0$ from (7). From (15) $p^iR^i \to [(1-\beta)N^i(\bar{a}^j - \bar{a}^i) + \bar{a}^i] > 0$. The LHS of (P1) can be written as

$$LHS = (\nu_0^1p^2-\nu_0^2p^1)(1-\beta)+\beta\{(1-\beta)(\overline{a^2}-\overline{a^1})[\nu_0^2-N^1(\nu_0^2-\nu_0^1)]+(\nu_0^2\overline{a^1}-\nu_0^1\overline{a^2})+(\nu_0^1p^2-\nu_0^2p^1)\}$$

Since $RHS = (\nu_0^2\bar{a}^1 - \nu_0^1\bar{a}^2) + (\nu_0^1p^2 - \nu_0^2p^1)$, equating LHS with RHS and solving for N^1 yields

$$N^1 = \frac{\beta(\overline{a^2}-\overline{a^1})\nu_0^2 - (\nu_0^2\overline{a^1}-\nu_0^1\overline{a^2})}{\beta(\overline{a^2}-\overline{a^1})(\nu_0^2-\nu_0^1)}$$

Thus, a condition for $N^1 \in [0,1)$ is given by

$$\beta(\overline{a^2}-\overline{a^1})\nu_0^1 < (\nu_0^2\overline{a^1}-\nu_0^1\overline{a^2}) \leq \beta(\overline{a^2}-\overline{a^1})\nu_0^2 \qquad \text{(P2)}$$

Now consider the limiting case where $\mu \to 0$ which implies $\eta \to \infty$ from (14). From (15) $p^iR^i \to 1 + \beta(\bar{a}^i -$

1) > 1. Thus, there exists an upper bound for 0 such that $\sup_{N1}$ 0(N^1) << 4. Furthermore, there exists a finite 0^{max} << 4 at $N^1 = 1$.

Proof of Propositions 4 and 5:

Totally differentiating (7) and (14) yields

$$C\begin{bmatrix} d\eta \\ d\mu \\ dN^1 \end{bmatrix} = \begin{bmatrix} M \\ 0 \\ 0 \end{bmatrix} dm_0 + \begin{bmatrix} 0 \\ r\nu_0^1 + \dfrac{dp^iR^i}{d\delta} \\ r\nu_0^2 + \dfrac{dp^iR^i}{d\delta} \end{bmatrix} d\delta + \begin{bmatrix} 0 \\ -p^1\left(1 - \dfrac{dR^1}{dA^1}\right) \\ p^2\dfrac{dR^2}{dA^1} \end{bmatrix} dA^1 + \begin{bmatrix} 0 \\ p^1\dfrac{dR^1}{dA^2} \\ -p^2\left(1 - \dfrac{dR^2}{dA^2}\right) \end{bmatrix} dA^2$$

$$+ \begin{bmatrix} 0 \\ \dfrac{r(r+\delta+\eta)}{\eta} + \dfrac{dp^1R^1}{d\nu_0^1} \\ \dfrac{dp^2R^2}{d\nu_0^1} \end{bmatrix} d\nu_0^1 + \begin{bmatrix} 0 \\ \dfrac{dp^1R^1}{d\nu_0^2} \\ \dfrac{r(r+\delta+\eta)}{\eta} + \dfrac{dp^2R^2}{d\nu_0^2} \end{bmatrix} d\nu_0^2$$

where

$$C = \begin{bmatrix} 1 - \dfrac{m_0M'}{\mu} & \dfrac{m_0M'\eta}{\mu^2} & 0 \\ \dfrac{r(r+\delta)\nu_0^1}{\eta^2} & -\dfrac{d(p^1R^1)}{d\mu} & -\dfrac{d(p^1R^1)}{dN^1} \\ \dfrac{r(r+\delta)\nu_0^2}{\eta^2} & -\dfrac{d(p^2R^2)}{d\mu} & -\dfrac{d(p^2R^2)}{dN^1} \end{bmatrix} \equiv \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Note that $C_{22} = C_{32} = -\dfrac{dp^iR^i}{d\mu} < 0 < -\dfrac{dp^iR^i}{dN^1} = C_{23} = C_{33}$ and $C_{31} - C_{21} = \dfrac{r(r+\delta)}{\eta^2}(\nu_0^2 - \nu_0^1) > 0$.

Thus, $|C| = -C_{12}(C_{21} - C_{31})C_{33} > 0$. We also compute comparative static effects:

1. Matching Efficiency:

$$\frac{d\eta^*}{dm_0} = \frac{M}{|C|}(C_{22}C_{33} - C_{23}C_{32}) = 0 \quad , \quad \frac{d\mu^*}{dm_0} = \frac{M}{|C|}(C_{31} - C_{21})C_{33} > 0 \quad ,$$

$$\frac{dN^{1*}}{dm_0} = -\frac{M}{|C|}(C_{31} - C_{21})C_{22} > 0$$

$$\frac{dp^iR^i}{dm_0} = \left(\frac{dp^iR^i}{d\mu}\right)\frac{d\mu}{dm_0} + \left(\frac{dp^iR^i}{dN^1}\right)\frac{dN^1}{dm_0} = C_{i2}\frac{d\mu}{dm_0} + C_{i3}\frac{dN^1}{dm_0} = 0 \quad \text{after substituting terms from above.}$$

Using this last result and (4) and (5), one sees that $\Pi_m^i - \Pi_u^i$ and $J_m^i - J_u$ also are independent of m_0. Then (1b) implies J_m^i and J_u are independent of m_0.

2. Separation Rate:

$$\frac{d\eta^*}{d\delta} = \frac{C_{12}C_{33}}{|C|}r(\nu_0^2 - \nu_0^1) > 0 \quad , \quad \frac{d\mu^*}{d\delta} = -\frac{C_{11}C_{33}}{|C|}r(\nu_0^2 - \nu_0^1) < 0$$

$$\frac{dN^{1*}}{d\delta} = \frac{1}{|C|}\frac{r(\nu_0^2 - \nu_0^1)}{\eta^2}\left((r+\delta)C_{12}\frac{dp^iR^i}{d\delta} - \eta^2 C_{11}\frac{dp^iR^i}{d\mu}\right) < 0$$

3. Productivity:

$$\frac{d\eta^*}{dA^1} = \frac{C_{12}C_{33}}{|C|}\left(p^1(1 - \frac{dR^1}{dA^1}) + p^2\frac{dR^2}{dA^1}\right) > 0 \quad , \quad \frac{d\mu^*}{dA^1} = -\frac{C_{11}C_{33}}{|C|}\left(p^1(1 - \frac{dR^1}{dA^1}) + p^2\frac{dR^2}{dA^1}\right) < 0$$

$$\frac{dN^{1*}}{dA^1} = \frac{C_{11}C_{22}}{|C|}\left(p^1(1 - \frac{dR^1}{dA^1}) + p^2\frac{dR^2}{dA^1}\right) - \frac{C_{12}}{|C|}\left(C_{31}p^1(1 - \frac{dR^1}{dA^1}) + C_{21}p^2\frac{dR^2}{dA^1}\right) < 0$$

Also, the impact of A^2 is inversely related to that of A^1 :

$$\frac{d\eta^*}{dA^2} = -\frac{C_{12}C_{33}}{|C|}\left(p^2(1 - \frac{dR^2}{dA^2}) + p^1\frac{dR^1}{dA^2}\right) < 0 \quad , \quad \frac{d\mu^*}{dA^2} = \frac{C_{11}C_{33}}{|C|}\left(p^2(1 - \frac{dR^2}{dA^2}) + p^1\frac{dR^1}{dA^2}\right) > 0$$

$$\frac{dN^{1*}}{dA^2} = -\frac{C_{11}C_{22}}{|C|}\left(p^2(1 - \frac{dR^2}{dA^2}) + p^1\frac{dR^1}{dA^2}\right) + \frac{C_{12}}{|C|}\left(C_{21}p^2(1 - \frac{dR^2}{dA^2}) + C_{31}p^1\frac{dR^1}{dA^2}\right) > 0$$

4. Entry Costs: The effects are more difficult to sign:

$$\frac{d\eta^*}{d\nu_0^1} = -\frac{C_{12}C_{33}}{|C|}\left(\frac{r(r+\delta+\eta)}{\eta} + \frac{dp^1R^1}{d\nu_0^1} - \frac{dp^2R^2}{d\nu_0^1}\right) \quad , \quad \frac{d\mu^*}{d\nu_0^1} = \frac{C_{11}C_{33}}{|C|}\left(\frac{r(r+\delta+\eta)}{\eta} + \frac{dp^1R^1}{d\nu_0^1} - \frac{dp^2R^2}{d\nu_0^1}\right)$$

$$\frac{dN^{1*}}{d\nu_0^1} = -\frac{C_{11}C_{22}}{|C|}\left(\frac{r(r+\delta+\eta)}{\eta} + \frac{dp^1R^1}{d\nu_0^1} - \frac{dp^2R^2}{d\nu_0^1}\right) + \frac{C_{12}}{|C|}\left(-C_{21}\frac{dp^2R^2}{d\nu_0^1} + C_{31}[\frac{r(r+\delta+\eta)}{\eta} + \frac{dp^1R^1}{d\nu_0^1}]\right)$$

From (15) one obtains $\frac{dp^iR^i}{d\nu_0^i} - \frac{dp^jR^j}{d\nu_0^i} = -r\beta < 0$ and $\frac{dp^jR^j}{d\nu_0^i} = -r\frac{\beta\mu(1-\beta)N^i}{r+\delta+\beta\mu} < 0$ which can be substituted into the above relationships. Therefore, if a temporary variable related to the two contact rates is defined by $Q \equiv r\left(\frac{r+\delta}{\eta} + (1-\beta)(1 - \frac{\mu}{r+\delta+\beta\mu})\right)$, we can rewrite the above expressions as:

$$\frac{d\eta^*}{d\nu_0^1} = -\frac{C_{12}C_{33}}{|C|}Q \quad , \quad \frac{d\mu^*}{d\nu_0^1} = \frac{C_{11}C_{33}}{|C|}Q \quad , \quad \frac{dN^{1*}}{d\nu_0^1} = \frac{C_{12}C_{31} - C_{11}C_{22}}{|C|}Q + \frac{C_{12}(C_{31} - C_{21})}{|C|}\frac{dp^2R^2}{d\nu_0^1}$$

Thus when $Q > 0$, it follows that $\frac{d\eta^*}{d\nu_0^1} < 0 < \frac{d\mu^*}{d\nu_0^1}$. With a strengthening of this "Q-condition" it is also possible to get $\frac{dN^{1*}}{d\nu_0^1} > 0$. Note that these were the qualitative effects that we had solved for originally.

Using the same approach as above yields:

$$\frac{d\eta^*}{d\nu_0^2} = \frac{C_{12}C_{33}}{|C|}Q \quad , \quad \frac{d\mu^*}{d\nu_0^2} = -\frac{C_{11}C_{33}}{|C|}Q \quad , \quad \frac{dN^{1*}}{d\nu_0^2} = \frac{C_{11}C_{22} - C_{12}C_{21}}{|C|}Q + \frac{C_{12}(C_{31} - C_{21})}{|C|}\frac{dp^1R^1}{d\nu_0^2}$$

The simple Q-condition results in $\frac{d\eta^*}{d\nu_0^2} > 0 > \frac{d\mu^*}{d\nu_0^2}$. Also, because $C_{22} < 0$, $\frac{dN^{1*}}{d\nu_0^2} < 0$ without having to make any further assumptions beyond the simple Q-condition (in contrast to the assumptions needed to get $\frac{dN^{1*}}{d\nu_0^1} > 0$).